Can one define the analogs of Mandelbrot and Julia sets in the TGD framework?

April 16, 2024

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Abstract

The stimulus for this article was the question of whether higher-dimensional analogs of Mandelbrot and Julia sets exist, in particular the 3-D analogs of these sets exist. The notion complex analyticity plays a key role in the definition of these notions and it is not all clear whether one can define these analogs in higher dimensions, in particular for space-time surfaces as 4-D objects.

Holography=holomorphy principle relies on a generalization of holomorphy by fusing hypercomplex structure as a Minkowskian variant of complex structure and applies to 4-D situation and space-time surfaces as analogs of Bohr orbits for particles as 3-D surfaces as generalization of point-like particles would be holomorphic surfaces in generalized sense. In this framework, one can imagine various generalizations of Mandelbrot and Julia sets.

Contents

1 Introduction

The stimulus to this contribution came from the question related to possible higher-dimensional analogs of Mandelbrot and Julia sets (see this). The notion complex analyticity plays a key role in the definition of these notions and it is not all clear whether one can define these analogs.

1.1 Standard definitions of Mandelbrot and Julia sets

Consider first the ordinary Mandelbrot and Julia sets.

1. The simplest example of the situation is the map $f : z \to z^2 + c$. One can consider the iteration of f by starting from a selected point z and look for various values of complex parameter c whether the iteration converges or diverges to infinity. The interface between the sets of the complex c-plane is 1-D Mandelbrot set and is a fractal. One can generalize the iteration to an arbitrary rational function f, in particular polynomials.

2. For polynomials of degree n also consider $n-1$ parameters c_i, $i = 1, ..., n$, to obtain $n-1$ complex-dimensional analog of Mandelbrot set as boundaries of between regions where the iteration lead or does not lead to infinity. For $n = 2$ one obtains a 4-D set.

3. One can also fix the parameter c and consider the iteration of f. Now the complex z-plane decomposes to two a finite region with a finite number of components and its complement, Fatou set. The iteration does not lead out from the finite region but diverges in the complement. The 1-D fractal boundary between these regions is the Julia set.

I have already earlier considered the iteration of polynomials in the TGD framework [**?**] suggesting the TGD counterparts of these notions. These considerations however rely on a view of $M^8 - H$ duality which is replaced with dramatically simpler variant and utilizing the holography=holomorphy principle [L2] so that it is time to update these ideas.

This principle states that space-time surfaces are analogous to Bohr orbits for particles which are 3-D surfaces rather than point-like particles. Holography is realized in terms of space-time surfaces which can be regarded as complex surfaces in $H = M^4 \times CP_2$ in the generalized sense. This means that one can give H 4 generalized complex coordinates and 3 such generalized complex coordinates can be used for the 4-surface. These surfaces are always minimal surfaces irrespective of the action defining them as its extermals and the action makes itself visible only at the singularities of the space-time surface.

1.2 Holography= holomorphy principle

The generalization to the TGD framework relies heavily on holography=holomorphy principle.

1. In the recent formulation of TGD, holography required by the realization of General Coordinate Invariance is realized in terms of two functions f_1, f_2 of 4 analogs of generalized complex coordinates, one of them corresponds to the light-like (hypercomplex) M^4 coordinate for a surface $X^2 \subset M^4$ and the 3 complex coordinates to those of Y^2 orthogonal to X^2 and the two complex coordinates of CP_2.

   Space-time surfaces are defined by requiring the vanishing of these two functions: $(f_1, f_2) = (0, 0)$. They are minimal surfaces irrespective of the action as long it is general coordinate invariant and constructible in terms of the induced geometry.

2. In the number theoretic vision of TGD, $M^8 - H$-duality [L2] maps the space-time as a holomorphic surface $X^4 \subset H$ is mapped to an associative 4-surface $Y^4 \subset M^8$. The condition for holography in M^8 is that the normal space of Y^4 is quaternionic.

   In the number theoretic vision, the functions f_i are naturally rational functions or polynomials of the 4 generalized complex coordinates. I have proposed that the coefficients of polynomials are rationals or even integers, which in the most stringent approach are smaller than the degree of the polynomial. In the most general situation one could have analytic functions with rational Taylor coefficients.

   The polynomials $f_i = P_i$ form a hierarchy with respect to the degree of P_i, and the iteration defined is analogous to that appearing in the 2-D situation. The iteration of P_i gives a hierarchy of algebraic extensions, which are central in the TGD view of evolution as an increase of algebraic complexity. The iteratikon would also give a hierarchy of increasingly complex space-time surface and the approach to chaos at the level of space-time would correspond to approach of Mandelbrot or Julia set.

3. In the TGD context, there are 4-complex coordinates instead of 1 complex coordinate z. The iteration occurs in H and the vanishing conditions for the iterates define a sequence of

4-surfaces. The initial surface is defined by the conditions $(f_1, f_2) = 0$. This set is analogous to the set $f(z) = 0$ for ordinary Julia sets.

One could consider the iteration as $(f_1, f_2) \rightarrow (f_1 \circ f_1, f_2 \circ f_2)$ continued indefinitely. One could also iterate only f_1 or f_2. Each step defines by the vanishing conditions a 4-D surface, which would be analogous to the image of the $z = 0$ in the 2-D iteration. The iterates form a sequence of 4-surfaces of H analogous to a sequence of iterates of z in the complex plane.

The sequence of 4-surfaces also defines a sequence of points in the "world of classical worlds" (WCW) analogous to the sequence of points $z, f(z), ...$ This conforms with the idea that 3-surface is a generalization of point-like particles, which by holography can be replaced by a Bohr orbit-like 4-surface.

4. Also in this case, one can see whether the iteration converges to a finite result or not. In the zero energy ontology (ZEO), convergence could mean that the iterates of X^4 stay within a causal diamond CD having a finite volume.

2 Various ways to identify analogs of Mandelbroot and Julia sets in the TGD framework

One can imagine several TGD analogs of Mandelbrot and Julia sets in the framework provided by holography=holomorphy principle.

2.1 The counterparts of Mandelbrot and Julia sets at the level of WCW

What the WCW analogy of the Mandelbrot and Julia sets could look like?

1. Consider first the Mandelbrot set. One could start from a set of roots of $(f_1, f_2) = (c_1, c_2)$ equivalent with the roots of $(f_1 - c_1, f_2 - c_2) = (0, 0)$. Here c_1 and c_2 define complex parameters analogous to the parameter c of the Mandelbrot sent. One can iterate the two functions for all pairs (c_1, c_2). One can look whether the iteration converges or not and identify the Mandelbrot set as the critical set of parameters (c_1, c_2). The naive expectation is that this set is 3-D dimensional fractal.

2. The definition of Julia set requires a complex plane as possible initial points of the iteration. Now the iteration of $(f_1, f_2) = 0$ fixes the starting point (not necessarily uniquely since 3-D surface does not fix the Bohr orbit uniquely: this is the basic motivation for ZEO). The analogy with the initial point of iteration suggests that we can assume $(f_1, f_2) = (c_1, c_2)$ but this leads to the analog of the Mandelbrot set. The notions coincide at the level of WCW.

3. Mandelbrot and Julia sets and their generalizations are critical in a well-defined sense. Whether iteration could be relevant for quantum dynamics is of course an open question. Certainly it could correspond to number theoretic evolution in which the dimension of the algebraic extension rapidly increases. For instance, one could one consider a WCW spinor field as a wave function in the set of converging iterates. Quantum criticality would correspond to WCW spinor fields restricted to the Mandelbrot or Julia sets.

Could the 3-D analogs of Mandelbrot and Julia sets correspond to the light-like partonic orbits defining boundaries between Euclidean and Minkowskian regions of the space-time surface and space-time boundaries? Can the extremely complex fractal structure as sub-manifold be consistent with the differentiability essential for the induced geometry? Could light-likeness help here.

2.2 Do the analogs of Mandelbrot and Julia sets exist at the level of space-time?

Could one identify the 3-D analogs of Mandelbrot and Julia sets for a given space-time surface? There are two approaches.

1. The parameter space (c_1, c_2) for a given initial point h of H for iterations of $f_1 - c_1, f_2 - c_2)$ defines a 4-D complex subspace of WCW. Could one identify this subset as a space-time surface and interpret the coordinates of H as parameters? If so, there would be a duality, which would represent the complement of the Fatou set (the thick Julia set) defined as a subset of WCW as a space-time surface!

2. One could also consider fixed points of iteration for which iteration defines a holomorphic map of space-time surface to itself. One can consider generalized holomorphic transformations of H leaving X^4 invariant locally. If they are 1-1 maps they have interpretation as general coordinate transformations. Otherwise they have a non-trivial physical effect so that the analog of the Julia set has a physical meaning. For these transformations one can indeed find the 3-D analog of Julia set as a subset of the space-time surface. This set could define singular surface or boundary of the space-time surface.

2.3 Could Mandelbrot and Julia sets have 2-D analogs in TGD?

What about the 2-D analogs of the ordinary Julia sets? Could one identify the counterparts of the 2-D complex plane (coordinate z) and parameter space (coordinate c).

1. Hamilton-Jacobi structure defines what the generalized complex structure is [L1] and defines a slicing of M^4 in terms of integrable distributions of string world sheets and partonic 2-surfaces transversal or even orthogonal to each other. Partonic 2-surface could play the role of complex plane and string world sheet the role of the parameter space or vice versa.

   Partonic 2-surfaces *resp.* and string world sheet having complex *resp.* hyper-complex structures would therefore be in a key role. $M^8 - H$ duality maps these surfaces to complex *resp.* co-complex surfaces of octonions having Minkowskian norm defined as number theoretically as $Re(o^2)$.

2. In the case of Julia sets, one could consider generalized holomorphic transformations of H mapping X^4 to itself as a 4-surface but not reducing to 1-1 maps. If f_2 (f_1) acts trivially at the partonic 2-surface Y^2 (string world sheet X^2), the iteration reduces to that for f_1 (f_2). Within string world sheets and partonic 2-surfaces the iteration defines Julia set and its hyperbolic analog in the standard way. One can argue that string world sheets and partonic 2-surfaces should correspond to singularities in some sense. Singularity could mean this fixed point property.

   The natural proposal is that the light-like 3-surfaces defining boundaries between Euclidean and Minkowskian regions of the space-time surface define light-like orbits of the partonic 2-surface. And string world sheets are minimal surfaces having light-like 1-D boundaries at the partonic 2-surface having physical interpretation as world-lines of fermions.

   One could also iterate only f_1 or f_2 allow the parameter c of the initial value of f_1 to vary. This would give the analog of Mandelbrot set as a set of 2-D surfaces of H and it might have dual representation as a 2-surface.

3. The 2-D analog of the Mandelbrot set could correspond to a set of 2-surfaces obtained by fixing a point of the string world sheet X^2. Also now one could consider holomorphic maps leaving the space-time surface locally but not acting 1-1 way. The points of Y^2 would define the values of the complex parameter c remaining invariant under these maps. The convergence of the iteration of f_1 in the same sense as for the Mandelbrot fractal would define the Mandelbrot set as a critical set. For the dual of the Mandelbrot set X^2 and Y^2 would change their roles.

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Books related to TGD

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