About the TGD based notions of mass, of twistors and hyperbolic counterpart of Fermi torus

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Contents

Abstract

The notion of mass in the TGD framework is discussed from the perspective of $M^8 - H$ duality. Also the TGD based notion of twistor space is considered at concrete geometric level. This discussion justifies the proposed concrete solution of a technical problem related to the proposed identification of scattering amplitudes reducing particle reactions to a re-arrangement of the fermions forming Galois singlets to new Galois singlets. The third topic of this article is the hyperbolic generalization of the Fermi torus to hyperbolic 3-manifold H^3/Γ. Here $H^3 = SO(1,3)/SO(3)$ identifiable the mass shell $M^4 \subset M^8$ or its $M^8 - H$ dual in $H = M^4 \times CP_2$. Γ denotes an infinite subgroup of $SO(1,3)$ acting completely discontinuously in H^3. For virtual fermions also complexified mass shells are required and the question is whether the generalization of H^3/Γ, defining besides hyperbolic 3-manifold also tessellation of H^3 analogous to a cubic lattice of E^3.

1 Introduction

The notion of mass in the TGD framework is discussed from the perspective of $M^8 - H$ duality [L2, L3, L13, L8].

1. In TGD, space-time regions are characterized by polynomials P with rational coefficients [L2, L3]. Galois confinement defines a universal mechanism for the formation of bound states. Momenta for virtual fermions have components, which are algebraic integers in an extension of rationals defined by a polynomial P characterizing a space-time region. For the physical many fermion states, the total momentum as the sum of fermion momenta has components, which are integers using the unit defined by the size of the causal diamond (CD) [L1, L5, L9].

2. This defines a universal number theoretical mechanism for the formation of bound states as Galois singlets. The condition is very strong but for rational coefficients it can be satisfied since the sum of all roots is always a rational number as the coefficient of the first order term.

3. Galois confinement implies that the sum of the mass squared values, which are in general complex algebraic numbers in E, is also an integer. Since the mass squared values correspond to conformal weights as also in string models, one has conformal confinement: states are conformal singlets. This condition replaces the masslessness condition of gauge theories [L13].

Also the TGD based notion of twistor space is considered at concrete geometric level.

1. Twistor lift of TGD means that space-time surfaces X^4 is $H = M^4 \times CP_2$ are replaced with 6-surfaces in the twistor space with induced twistor structure of $T(H) = T(M^4) \times T(CP_2)$ identified as twistor space $T(X^4)$. This proposal requires that T(H) has Kähler structure and this selects $M^4 \times CP_2$ as a unique candidate [A2] so that TGD is unique.

2. One ends up to a more precise understanding of the fiber of the twistor space of CP_2 as a space of "light-like" geodesics emanating from a given point. Also a more precise view of the induced twistor spaces for preferred extremals with varying dimensions of M^4 and CP_2 projections emerges. Also the identification of the twistor space of the space-time surface as the space of light-like geodesics itself is considered.

3. Twistor lift leads to a concrete proposal for the construction of scattering amplitudes. Scattering can be seen as a mere re-organization of the physical many-fermion states as Galois singlets to new Galois singlets. There are no primary gauge fields and both fermions and bosons are bound states of fundamental fermions. 4-fermion vertices are not needed so that there are no divergences.

4. There is however a technical problem: fermion and antifermion numbers are separately conserved in the simplest picture, in which momenta in $M^4 \subset M^8$ are mapped to geodesics of $M^4 \subset H$.The led to a proposal for the modification of $M^8 - H$ duality [L2, L3]. The modification would map the 4-momenta to geodesics of X^4. Since X^4 allows both Minkowskian and Euclidean regions, one can have geodesics, whose M^4 projection turns backwards in time. The emission of a boson as a fermion-antifermion pair would correspond to a fermion turning backwards in time. A more precise formulation of the modification shows that it indeed works

2 Conformal confinement

The notion of mass distinguishes TGD from QFT. As in string models, mass squared corresponds to a conformal weight in TGD. However, in the TGD framework tachyonic states are not a curse but an essential part of the physical picture and conformal confinement, generalizing masslessness condition, states that the sum of conformal weights for physical states vanishes. This view conforms with the fact that Euclidean space-time regions are unavoidable at the level of H. Positive *resp.* negative *resp.* vanishing conformal weights can be assigned with Minkowskian *resp.* Euclidean space-time regions *resp.* light-like boundaries associated with them.

2.1 Mass squared as conformal weight, conformal confinement and its breaking

At the level of M^8, the momentum components for momenta as points of $H_c^3 \subset M_c^4 \subset M_c^8$ are (in general complex) algebraic integers in an extension of rationals defined by the polynomial P defining the space-time region. For physical states the momentum components for the sum of the momenta are ordinary integers when the momentum unit is defined by the size scales of causal diamond (CD). This scale corresponds to a p-adic length scale for p-adic prime, which is a ramified prime of the extension of rationals defined by the polynomial P.

For virtual many-fermion states the mass squared is an algebraic integer but an ordinary integer for the physical states [L13]. The question is whether the mass squared for the physical states can be negative so that one would have tachyons. The p-adic mass calculations require the presence of tachyonic mass squared values and the proposal is conformal confinement in the sense that the sum of mass squared values for the particles present in state and identifiable as conformal weights sum up to zero. Conformal confinement would generalize the masslessness condition of gauge field theories.

The observed mass squared values would correspond to the Minkowskian non-tachyonic parts of the mass squared values assignable to states, which in general are entangled states formed from tachyonic and non-tachyonic states. p-Adic thermodynamics would describe the entanglement in terms of the density matrix and observed mass squared would be thermal average. p-Adic thermodynamics leads to a breaking of the generalized conformal invariance and explains why different values of the Virasoro scaling generator L_0 are involved. Since complex mass squared values with a negative real part are allowed as roots of polynomials, the condition is highly non-trivial.

2.2 Association of mass squared values to space-time regions

$M^8 - H$ duality [L2, L3] would make it natural to assign tachyonic masses with CP_2 type extremals and with the Euclidean regions of the space-time surface. Time-like masses would be assigned with time-like space-time regions. In [L11] it was found that, contrary to the beliefs held hitherto, it is possible to satisfy boundary conditions for the action action consisting of the Kähler action, volume term and Chern-Simons term, at boundaries (genuine or between Minkowskian and Euclidean space-time regions) if they are light-like surfaces satisfying also $detg_4 = 0$. Masslessness, at least in the classical sense, would be naturally associated with light-like boundaries (genuine or between Minkowskian and Euclidean regions).

2.3 Riemann zeta, quantum criticality, and conformal confinement

The assumption that the space-time surface corresponds to rational polynomials in TGD is not necessary. One can also consider real analytic functions f [L8]. The condition that momenta of physical states have integer valued momentum components implies integer valued conformal weights poses extremely strong conditions on this kind of functions since the sum of the real parts of the roots of f must be an integer as a conformal weight identified as the sum of in general complex virtual mass squared values.

There are strong indications Riemann zeta (`https://cutt.ly/iVTV1kqs`) has a deep role in physics, in particular in the physics of critical systems. TGD Universe is quantum critical. What quantum criticality would mean at the space-time level is discussed in [L11]. This raises the question whether Riemann zeta could have a deep role in TGD.

First some background relating to the number theoretic view of TGD.

1. In TGD, space-time regions are characterized by polynomials P with rational coefficients [L2, L3]. Galois confinement defines a universal mechanism for the formation of bound states. Momenta for virtual fermions have components, which are algebraic integers in an extension

of rationals defined by a polynomial P characterizing space-time region. For the physical many fermion states, the total momentum as the sum of fermion momenta has components, which are integers using the unit defined by the size of the causal diamond (CD).

This defines a universal number theoretical mechanism for the formation of bound states. The condition is very strong but for rational coefficients it can be satisfied since the sum of all roots is always a rational number as the coefficient of the first order term.

2. Galois confinement implies that the sum of the mass squared values, which are in general complex algebraic numbers in E, is also an integer. Since the mass squared values correspond to conformal weights as also in string models, one can have conformal confinement if negative conformal weights are possible. Also conformal confinement is possible: states would be conformal singlets. This condition replaces the masslessness condition of gauge theories [L13].

Riemann zeta [A1] (https://cutt.ly/oVNSltD) is not a polynomial but has infinite number of roots. How could one end up with Riemann zeta in TGD? One can also consider the replacement of the rational polynomials with analytic functions with rational coefficients or even more general functions [L8].

1. For real analytic functions roots come as pairs but building many-fermion states for which the sum of roots would be a real integer, is very difficult and in general impossible.

2. Riemann zeta and the hierarchy of its generalizations to extensions of rationals (Dedekind zeta functions) is however a complete exception! If the roots are at the critical line as the generalization of Riemann hypothesis assumes, the sum of the root and its conjugate is equal to 1 and it is easy to construct many fermion states as 2N fermion states, such that they have integer value conformal weight.

   Since zeta has also trivial zeros for even negative integers interpretable in terms of tachyonic states, also conformal confinement with vanishing net conformal weight for physical states is possible. The trivial zeros would be associated with Euclidean space-time regions and non-trivial ones to Minkowskian ones.

   One can wonder whether one could see Riemann zeta as an analog of a polynomial such that the roots as zeros are algebraic numbers. This is however not necessary. Could zeta and its analogies allow it to build a very large number of Galois singlets and they would form a hierarchy corresponding to extensions of rationals. Could they represent a kind of second abstraction level after rational polynomials?

A possible interpretation is that in TGD, rational polynomials give discrete cognitive representations as approximations for physics. Cognitive representations are in the intersection of p-adicities and reality defined by the intersection of reals and extension of p-adics defined by the algebraic extension of the polynomial P defining a given space-time surface. Continuum theory would represent real numbers as a factor of the adele.

One can ask whether the various zeta functions consistent with the integer spectrum for the conformal weights and possibly also with conformal confinement, appear at the continuum limit and provide representations for the space-time surfaces at this limit? In this framework, it would be natural for the roots of zeta to be algebraic numbers [K2]. Also in the case of ζ, the virtual momenta of fermions would be algebraic integers for virtual fermions and integers for the physical states. This makes sense if the notions of Galois group and Galois confinement are sensible for ζ.

As noticed, the notion of ζ generalizes. The so-called global L-functions (https://cutt.ly/3VNPYmp) are formally similar to ζ and the extended Riemann Hypothesis (RH) could be true for them. The physical motivation for RH would be that it would allow a fermion with any conformal weight to appear in a state which is conformal singlet. Algebraic integers for a finite extension of rationals replace integers in the ordinary ζ and one has an entire hierarchy of L-functions. Could one think that the global L-functions could define preferred extremals at the continuum limit?

3 About the notion of twistor space

For the twistor lift of TGD, twistor space $T(X^4)$ of the space-time surface X^4 is identified an S^2 bundle over X^4 obtained by the induction of the twistor bundle $T(H) = T(M^4) \times T(CP_2)$. The definition of the $T(X^4)$ as 6-surface in $T(H)$ identifies the twistor spheres of $T(M^4)$ and $T(CP_2)$ and identifies it as a twistor sphere of $T(X^4)$.

3.1 The notion of twistor space for different different types of preferred extremals

I have not previously considered the notion of the induced twistor space for the different types of preferred extremals. Here some technical complications emerge.

1. Since the points of the twistor spaces $T(M^4)$ and $T(CP_2)$ are in 1-1 correspondence, one can use either $T(M^4)$ or $T(CP_2)$ so that the projection to M^4 or CP_2 would serve as the base space of $T(X^4)$. One could use either CP_2 coordinates or M^4 coordinates as space-time coordinates if the dimension of the projection is 4 to either of these spaces. In the generic case, both dimensions are 4 but one must be very cautious with genericity arguments, which turned out to fail at the level of M^8 [L2, L3].

2. There are exceptional situations in which genericity fails at the level of H. String-like objects of the form $X^2 \times Y^2 \subset M^4 \subset CP_2$ is one example of this. In this case, X^6 would not define 1-1 correspondence between $T(M^4)$ or $T(CP_2)$.

   Could one use partial projections to M^2 and S^2 in this case? Could $T(X^4)$ be divided locally into a Cartesian product of 3-D M^4 part projecting to $M^2 \subset M^4$ and of 3-D CP_2 part projected to $Y^2 \subset CP_2$?

3. One can also consider the possibility of defining the twistor space $T(M^2 \times S^2)$. Its fiber at a given point would consist of light-like geodesics of $M^2 \times S^2$. The fiber consists of direction vectors of light-like geodesics. S^2 projection would correspond to a geodesic circle $S^1 \subset S^2$ going through a given point of S^2 and its points are parametrized by azimuthal angle Φ. Hyperbolic tangent $tanh(\eta)$ with range $[-1, 1]$ would characterize the direction of a time like geodesic in M^2. At the limit of $\eta \to \pm\infty$ the S^2 contribution to the S^2 tangent vector to length squared of the tangent vector vanishes so that all angles in the range $(0, 2\pi)$ correspond to the same point. Therefore the fiber space has a topology of S^2.

   There are also other special situations such as $M^1 \times S^3$, $M^3 \times S^1$ for which one must introduce specific twistor space and which can be treated in the same way.

To deal with these special cases in which the dimensions of both M^4 and CP_2 are not equal to 4, one must allow also 6-surfaces X^6 which can have dimension of M^4 and CP_2 projections which are different from the canonical value 4. For CP_2 type extremals the dimension of CP_2 projection would be 6 and the dimension of M^4 projection would be 1. For cosmic strings the dimensions of M^4 projection and CP_2 projection would be 2.

3.2 The concrete definition of the twistor space of H as the space of light-like geodesics

During the writing of this article I realized that the twistor space of H defined geometrically as a bundle, which has as H as base space and fiber as the space of light-like geodesic starting from a given point of H, need not be equal to $T(M^4) \times T(CP_2)$, where $T(CP_2)$ is identified as $SU(3)/U(1) \times U(1)$ characterizing the choices of color quantization axes. Is this really the case?

1. The definition of $T(CP_2)$ as the space of light-like geodesics from a given point of CP_2 is not possible. One could also define the fiber space of $T(CP_2)$ geometrically as the space of geodesics emating from origin at $r = 0$ in the Eguchi-Hanson coordinates [K1] and connecting it to the homologically non-trivial geodesic sphere S^2_G $r = \infty$. This relation is symmetric.

   In fact, all geodesics from $r = 0$ end up to S^2. This is due to the compactness and symmetries of CP_2. In the same way, the geodesics from the North Pole of S^2 end up to the South Pole. If only the endpoint of the geodesic of CP_2 matters, one can always regard it as a point S^2_G.

The two homologically non-trivial geodesic spheres associated with distinct points of CP_2 always intersect at a single point, which means that their twistor fibers contain a common geodesic line of this kind. Also the twistor spheres of $T(M^4)$ associated with distinct points of M^4 with a light-like distance intersect at a common point identifiable as a light-like geodesic connecting them.

2. Geometrically, a light-like geodesic of H is defined by a 3-D momentum vector in M^4 and 3-D color momentum along CP_2 geodesic. The scale of the 8-D tangent vector does not matter and the 8-D light-likeness condition holds true. This leaves 4 parameters so that $T(H)$ identified in this way is 12-dimensional.

   The M^4 momenta corresponds to a mass shell H^3. Only the momentum direction matters so that also in the M^4 sector the fiber reduces to S^2. If this argument is correct, the space of light-like geodesics at point of H has the topology of $S^2 \times S^2$ and $T(H)$ would reduce to $T(M^4) \times T(CP_2)$ as indeed looks natural.

3.3 The twistor space of the space-time surface

The twistor lift of TGD allows to identify the twistor space of the space-time surface X^4 as the base space of the S^2 bundle induced from the 12-D twistor space $T(8) = T(M^4) \times CP_(2)$ to the 6-surface $X^6 \subset T(H)$ by a local dimensional reduction to $X^4 \times S^2$ occurring for the preferred extremals of 6-D Kähler action existing only in case of $H = M^4 \times CP_2$.

Could the geometric definition of $T(X^4)$ as the space of light-like geodesics make sense in the Minkowskian regions of X^4?

1. By their definition, stating that the length of the tangent vector of the geodesic is conserved, the geodesic equations conserve the value of the velocity squared so that light-likeness can be forced via the initial values. This allows the assignment of a twistor sphere to a given point of a Minkowskian space-time region. Whether this assignment can be made global is not at all trivial and the difficulties related to the definition of twistor space in general relativity probably reflects this problem. If this is the case, then the direct geometric definition might not make sense unless the very special properties of the PEs come to rescue.

2. The twistor lift of TGD is proposed to modify the definition of the twistor space so that one can assign twistor structure to the space-time surface by inducing the twistor structure of H just as one can assign spinor structure with the space-time surface by inducing the spinor structure of H.

   Could the generalized holomorphic structure, implying that PEs are extremals of both volume and of 4-D Kähler action, make possible the existence of light-like geodesics and even allow to assign to a given point of the space-time surface sphere parametrizing light-like geodesics?

3. The light-like 3-surfaces X^3 representing partonic orbits carry fermionic lines as light-like geodesics and are therefore especially interesting. They are metrically 2-D and boundary conditions for the field equations force the vanishing of the determinant $det(g_4)$ of the induced metric at them so that the dimension of the tangent space is effectively reduced. Light-like 3-surfaces allow a generalization of isometries such that conformal symmetries accompanied by scaling of the light-like radial coordinate depending on transversal complex coordinates is isometry.

   It seems that to a given point of the space-like intersection, only a single light-like geodesic can be assigned so that the twistor space at a given point would consist of a single light-like geodesic. This would be caused by the light-likeness of X^3.

3.4 The geometric definition of the twistor space for CP_2

In the case of the Euclidean regions, the notion of a light-like geodesic does not make sense. The closed geodesics and the presence of pairs of points analogous to North pole-South pole pairs, where diverging geodesics meet, would be required. This condition is very strong and the minimal requirement is that the space has a positive curvature so that the geodesics do not diverge. Also symmetries seem to be necessary. Clearly, something new is required.

1. The addition of Kähler coupling term equal to an odd multiple of the induced Kähler gauge potential A to the spinor connection is an essential element in the definition of a generalized spinor structure of CP_2.

2. Should one replace the light-like geodesics with orbits of Kähler charged particles for which CP_2 has been replaced with $p - q_K A$. For the counterparts of light-like geodesics $p - q_K A$ would vanish and the analog of mass squared would vanish but one would have a line. For a geodesic p would be constant.

   Is it possible to have $A = constant$ along a closed geodesic? In the case of sphere, the Kähler gauge potential in the spherical coordinates is $(A_\theta = A_\phi = kcos(\theta)$ and is constant along the geodesics going through South and North Poles. Something like this could happen in the case of CP_2 but it seems that a special pair of homological non-trivial spheres S^2 invariant under $U(2) \subset SU(3)$ is selected. One might perhaps speak of symmetry breaking.

   To obtain entire S^2 of light-like geodesics in this sense, the geodesics must emanate from a coordinate singularity, the origin of Eguchi-Hanson coordinates at $r = 0$, where the values of the coordinates (θ, ϕ, ψ) correspond to the same point. The space for the light-like geodesics must be 2-D rather than 3-D. This must be forced by the $p - A = 0$ condition. For the homologically trivial geodesic sphere $r = \infty$, Ψ coordinate is redundant so that the conserved value of A_ψ must vanish for the light-like geodesics and the associated velocities cannot have component in the direction of Ψ.

3. Note that this definition could apply also in Minkowskian regions of space-time surface.

3.5 The description of particle reactions without vertices

In standard field theory, particles are point-like and particle reactions are described using vertices assignable to non-linear interaction terms in the action.

1. In the TGD framework, particles are replaced with 3-surfaces and elementary particles are assigned to partonic 2-surface whose orbits correspond to light-like 3-surfaces identifiables as the boundary regions between Minkowskian and Euclidean space-time regions and modelled as wormhole contacts between two space-time sheets with a Minkowskian signature. Vertices are replaced with topological vertices at which incoming partonic 2-surfaces, whose orbits are light-like 3-surfaces, meet at partonic 2-surfaces.

2. In TGD, all particles are composites of fundamental fermions assignable to the wormhole throats identified as partonic orbits. In particular, bosons consist of fermions and antifermions assignable to the throats of wormholes. Since wormhole contact contains homologically trivial 2-surface of CP_2, there is a monopole flux throwing out of the throat and one must have at least two wormhole contacts so that one obtains a closed monopole flux flowing between the sheets and forming a closed flux tube.

3. The light-like orbits of the partonic 2-surfaces contain fermionic lines defined at the ends of string world sheets connecting different partonic orbits. In QFT description, this would require a 4-fermion vertex as a fundamental vertex involving dimensional coupling constant and leading to a non-renormalizable QFT. Therefore there can be no vertices at the level of fermion lines.

In the number theoretic vision based on Galois confinement [L7, L8], the interactions correspond at the level of M^8 to re-arrangements of virtual fermions, having virtual momentum components in the extension of rationals defined by P, to new combinations required to be Galois singlets and therefore having momentum components, which are ordinary integers. Note that P fixes by holography the 4-surface in M^8 in turn defining the space-time surface in H by $M^8 - H$ duality based on associativity.

There is however a problem. If the particle reactions are mere re-arrangements of fundamental fermions and antifermions, moving along light-like geodesic lines in fixed time direction, the total numbers of fermions and antifermions are separately conserved. How can one overcome this problem without introducing the disastrous 4-fermion vertex?

Consider FFB vertex describing boson emission by fermion as a concrete example.

1. B is described as a pair of partonic surfaces containing at least one fermion-antifermion pair, which must be created in the vertex. Incoming particles for the topolocal FFB 3-vertex correspond to partonic orbits for incoming F and outgoing F, each containing one fermion line and possibly a pair of fermion and antifermion.

2. The idea is that boson emission as a pair creation could be described geometrically as a turning of fermion backwards in time. This forces us to reconsider the definition of $M^8 - H$ duality. The simplest view of $M^8 - H$ duality is that momenta of $M^4 \subset M^8$ are mapped to the geodesic lines of M^4. Tachyonic momenta in $M^4 \subset M^8$ would be mapped to space-like geodesics in H emanating from the center of CD which is a sub-CD of a larger CD in general. It seems that this definition does not allow us to understand boson emission by fermion in the way proposed in [L8].

3. This led to a proposal that the images of momenta could be geodesics of the space-time surface X^4, rather than H. Since X^4 allows also Euclidean regions and the interiors of the deformed CP_2 type extremals are Euclidean, one ends up with the idea that the geodesics lines of X^4 can have M^4 projections, which turn backwards in the time direction [L2, L3, L6].

   This would allow us to interpret the emission of a boson as a fermion-antifermion pair as the turning of a fermionic line backwards in time. Fermions lines would be identified as the boundaries of string world sheets. Sub-manifold gravitation would play a key role in the elimination of 4-fermion vertex and thus of QFT type divergences.

4. But is it possible to have a light-like geodesic arriving at the partonic 2-surface and continuing as a light-like geodesic in the Euclidean wormhole contact and returning back? The problem is that in Euclidean regions, ordinary light-like geodesics degenerate to points. The generalization of the light-like geodesics satisfying $p = qA$ implying $(p-qA)^2 = 0$ is possible. At the space-time level, these conditions could be true quite generally and give as a special case light-like geodesics with $p^2 = 0$ in the Minkowskian regions.

4 About the analogies of Fermi torus and Fermi surface in H^3

Fermi torus (cube with opposite faces identified) emerges as a coset space of E^3/T^3, which defines a lattice in the group E^3. Here T^3 is a discrete translation group T^3 corresponding to periodic boundary conditions in a lattice.

In a realistic situation, Fermi torus is replaced with a much more complex object having Fermi surface as boundary with non-trivial topology. Could one find an elegant description of the situation?

4.1 Hyperbolic manifolds as analogies for Fermi torus?

The hyperbolic manifold assignable to a tessellation of H^3 defines a natural relativistic generalization of Fermi torus and Fermi surface as its boundary. To understand why this is the case, consider first the notion of cognitive representation.

1. Momenta for the cognitive representations [L12] define a unique discretization of 4-surface in M^4 and, by $M^8 - H$ duality, for the space-time surfaces in H and are realized at mass shells $H^3 \subset M^4 \subset M^8$ defined as roots of polynomials P. Momentum components are assumed to be algebraic integers in the extension of rationals defined by P and are in general complex.

   If the Minkowskian norm instead of its continuation to a Hermitian norm is used, the mass squared is in general complex. One could also use Hermitian inner product but Minkowskian complex bilinear form is the only number-theoretically acceptable possibility. Tachyonicity would mean in this case that the real part of mass squared, invariant under $SO(1,3)$ and even its complexification $SO_c(1,3)$, is negative.

2. The active points of the cognitive representation contain fermion. Complexification of H^3 occurs if one allows algebraic integers. Galois confinement [L12, L10] states that physical states correspond to points of H^3 with integer valued momentum components in the scale defined by CD.

   Cognitive representations are in general finite inside regions of 4-surface of M^8 but at H^3 they explode and involve all algebraic numbers consistent with H^3 and belonging to the extension of rationals defined by P. If the components of momenta are algebraic integers, Galois confinement allows only states with momenta with integer components favored by periodic boundary conditions.

Could hyperbolic manifolds as coset spaces $SO(1,3)/\Gamma$, where Γ is an infinite discrete subgroup $SO(1,3)$, which acts completely discontinuously from left or right, replace the Fermi torus? Discrete translations in E^3 would thus be replaced with an infinite discrete subgroup Γ. For a given P, the matrix coefficients for the elements of the matrix belonging to Γ would belong to an extension of rationals defined by P.

1. The division of $SO(1,3)$ by a discrete subgroup Γ gives rise to a hyperbolic manifold with a finite volume. Hyperbolic space is an infinite covering of the hyperbolic manifold as a fundamental region of tessellation. There is an infinite number of the counterparts of Fermi torus [L4]. The invariance respect to Γ would define the counterpart for the periodic boundary conditions.

   Note that one can start from $SO(1,3)/\Gamma$ and divide by $SO(3)$ since Γ and $SO(3)$ act from right and left and therefore commute so that hyperbolic manifold is $SO(3)\setminus SO(1,3)/\Gamma$.

2. There is a deep connection between the topology and geometry of the Fermi manifold as a hyperbolic manifold. Hyperbolic volume is a topological invariant, which would become a basic concept of relativistic topological physics (`https://cutt.ly/RVsdNl3`).

   The hyperbolic volume of the knot complement serves as a knot invariant for knots in S^3. Could this have physical interpretation in the TGD framework, where knots and links, assignable to flux tubes and strings at the level of H, are central. Could one regard the effective hyperbolic manifold in H^3 as a representation of a knot complement in S^3?

   Could these fundamental regions be physically preferred 3-surfaces at H^3 determining the holography and M^8-H duality in terms of associativity [L2, L3]. Boundary conditions at the boundary of the unit cell of the tessellation should give rise to effective identifications just as in the case of Fermi torus obtained from the cube in this way.

4.2 De Sitter manifolds as tachyonic analogies of Fermi torus do not exist

Can one define the analogy of Fermi torus for the real 4-momenta having negative, tachyonic mass squared? Mass shells with negative mass squared correspond to De-Sitter space $SO(1,3)/SO(1,2)$ having a Minkowskian signature. It does not have analogies of the tessellations of H^3 defined by discrete subgroups of $SO(1,3)$.

The reason is that there are no closed de-Sitter manifolds of finite size since no infinite group of isometries acts completely discontinuously on de Sitter space: therefore these is no group replacing the Γ in H^3/Γ. (`https://cutt.ly/XVsdLwY`).

4.3 Do complexified hyperbolic manifolds as analogies of Fermi torus exist?

The momenta for virtual fermions defined by the roots defining mass squared values can also be complex. Tachyon property and complexity of mass squared values are not of course not the same thing.

1. Complexification of H^3 would be involved and it is not clear what this could mean. For instance, does the notion of complexified hyperbolic manifold with complex mass squared make sense.

2. $SO(1,3)$ and its infinite discrete groups Γ act in the complexification. Do they also act completely discontinuously? p^2 remains invariant if $SO(1,3)$ acts in the same way on the real and imaginary parts of the momentum leaves invariant both imaginary and complex mass squared as well as the inner product between the real and imaginary parts of the momenta. So that the orbit is 5-dimensional. Same is true for the infinite discrete subgroup Γ so that the construction of the coset space could make sense. If Γ remains the same, the additional 2 dimensions can make the volume of the coset space infinite. Indeed, the constancy of $p_1 \cdot p_2$ eliminates one of the two infinitely large dimensions and leaves one.

   Could one allow a complexification of $SO(1,3)$, $SO(3)$ and $SO(1,3)_c/SO(3)_c$? Complexified $SO(1,3)$ and corresponding subgroups Γ satisfy $OO^T = 1$. Γ_c would be much larger and contain the real Γ as a subgroup. Could this give rise to a complexified hyperbolic manifold H^3_c with a finite volume?

3. A good guess is that the real part of the complexified bilinear form $p \cdot p$ determines what tachyonicity means. Since it is given by $Re(p)^2 - Im(p)^2$ and is invariant under $SO_c(1,3)$ as also $Re(p) \cdot Im(p)$, one can define the notions of time-likeness, light-likeness, and space-likeness using the sign of $Re(p)^2 - Im(p^2)$ as a criterion. Note that $Re(p)^2$ and $Im(p)^2$ are separately invariant under $SO(1,3)$.

   The physicist's naive guess is that the complexified analogies of infinite discrete and discontinuous groups and complexified hyperbolic manifolds as analogies of Fermi torus exist for $Re(P^2) - Im(p^2) > 0$ but not for $Re(P^2) - Im(p^2) < 0$ so that complexified dS manifolds do not exist.

4. The bilinear form in H^3_c would be complex valued and would not define a real valued Riemannian metric. As a manifold, complexified hyperbolic manifold is the same as the complex hyperbolic manifold with a hermitian metric (see https://cutt.ly/qVsdS7Y and https://cutt.ly/kVsd3Q2) but has different symmetries. The symmetry group of the complexified bilinear form of H^3_c is $SO_c(1,3)$ and the symmetry group of the Hermitian metric is $U(1,3)$ containing $SO(1,3)$ as a real subgroup. The infinite discrete subgroups Γ for $U(1,3)$ contain those for $SO(1,3)$. Since one has complex mass squared, one cannot replace the bilinear form with hermitian one. The complex H^3 is not a constant curvature space with curvature -1 whereas H^3_c could be such in a complexified sense.

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