About the relationship of electroweak and strong interactions in the TGD Universe: part III

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Abstract

Recall that in the second part of the article a generalization of the notion of Weinberg rotation to a 3-dimensional rotation involving besides Weinberg angle a second angle parameter is developed. In the TGD framework, strong and weak interactions are very strongly related and the model relates the Weinberg angle and the strong interaction coupling strength to the choice of the monopole harmonic and to a second parameter of the generalized Weinberg mixing.

In the third part of the article a model estimating the masses of weak bosons and their mass ratio is developed. The model relies on the basic ideas of TGD, in particular p-adic thermodynamics and the fact that weak bosons are not fundamental fields in TGD and can be seen as analogs of bound states of fermion pairs coding the decay amplitudes to fermion pairs. Estimates for masses and mass ratio emerge if one accepts the TGD proposal that the official top quark corresponds to the u quark of M_{89} hadron physics, with a d partner of the same mass scale, and the real top quark of M_{107} hadron physics corresponds to a new quark explaining the Aleph anomaly.

In the Appendix various notions appearing in all three parts of the article are discussed. The topics include conformal involution highly relevant to the TGD based topological view about family replication and the connection of TGD view of particle reactions at parton level with quantum homology of Gromov–Witten. Besides holomorphic 4-surfaces predicted by holography = holomorphy vision, Lagrangian minimal surfaces are allowed if the classical action is identified as the sum of volume term and Kähler action. Therefore basic facts about these surfaces are summarized. There is also a summary of exceptional unitary representations of SL(2,C) serving beside scalar monopole harmonics as a possible candidate for the wave functions for the position of the fermion line at 3-D light-like partonic orbits.

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1 Introduction

In the second part of the chapter a generalization of the notion of Weinberg rotation to a 3-dimensional rotation involving besides Weinberg angle a second angle parameter is developed. In the TGD framework, strong and weak interactions are very strongly related and the model relates the Weinberg angle and the strong interaction coupling strength to the choice of the monopole harmonic and to a second parameter of the generalized Weinberg mixing.

In the third part of the chapter a model estimating the masses of weak bosons and their mass ratio is developed. The model relies on several basic ideas of TGD, in particular p-adic thermodynamics and the fact that weak bosons are not fundamental fields in TGD and can be seen as analogs of bound states of fermion pairs coding the decay amplitudes to fermion pairs. The basic assumption is that the p-adic mass squared values of fermions proportional to conformal weights are additive. Since several p-adic primes are involved, the application requires multi-p p-adic thermodynamics [L15, L14] and its precise formulation remains a challenge.

In TGD, slightly non-deterministic holography implies the absence of path integral. The coding of the couplings of bosons to fermion pairs to the structure of bosons as bound states makes possible

an estimate for the TGD counterparts of radiative corrections for m_Z/m_W. Estimates for weak boson masses and mass ratio in terms of fermion masses emerge if one accepts the TGD proposal that the official top quark corresponds to the u quark of M_{89} hadron physics, with a d partner of the same mass scale, and the real top quark of M_{107} hadron physics corresponds to a new quark explaining the Aleph anomaly [L25].

In the Appendix various notions appearing in all three parts of the chapter are discussed. The topics include conformal involution highly relevant to the TGD based topological view about family replication and the connection of TGD view of particle reactions at parton level with quantum homology of Gromov–Witten. Besides holomorphic 4-surfaces predicted by holography = holomorphy vision [L14], also Lagrangian minimal surfaces are allowed if the classical action is identified as the sum of volume term and Kähler action. This motivates the summary of the basic facts about these surfaces. There is also a summary of exceptional unitary representations of SL(2,C) serving beside scalar monopole harmonics as a possible candidate for the wave functions for the position of the fermion line at 3-D light-like partonic orbits.

2 How to predict weak boson masses and questions related to the relationship of weak and strong interactions

In this section questions related to the relationship of weak and strong interactions are discussed. Standard model predicts the ratio of Z^0 and W boson masses together with quark mass dependent radiative corrections. It is far from obvious whether TGD can predict this ratio with radiative corrections included.

2.1 Can one estimate the W/Z mass ratio including radiative corrections?

W/Z mass ratio is one of the basic predictions of the standard model. In tree approximation this ratio is given by $m(W)/m(Z) = cos(\theta_W)$. One can even consider the definition of $cos(\theta_W)$ as $m(W)/m(Z)$. The second approach, natural in TGD, takes as a starting point the coupling parameters associated with the mixing determined apart from the values of p and c^2 and the overall scale of coupling strength, deducible from W boson mass or weak interaction strength $G_F \propto 1/m_W^2$. In this approach, the mass ratio follows as a prediction. In the standard model, radiative corrections change the prediction for the mass ratio and this provides a test of the theory.

If H-H principle is assumed, path integral is replaced by slightly non-deterministic classical time evolution. The basic question is whether there is a TGD counterpart for the radiative corrections to the mass ratio. A more general question is whether TGD allows to understand the evolution coupling constants and expectations values of various observables in terms of H-H.

2.1.1 The overall view

The answer to the questions posed above involves several pieces.

1. By $M^8 - H$ duality [L7, L24], the time evolution for space-time surface $X^4 \subset H$ with respect to the Lorentz invariant proper time a in the half-cone of CD $\subset H$ corresponds to the evolution Y^4 with respect to mass scale m [L9] characterizing mass shells at the level of M^8 inside the counterpart of CD. There are good reasons to assume that it is possible to restrict to the level of partonic 2-surfaces and fermion lines in H and their counterparts in M^8.

   The time evolution of various observables would be associated with partonic level and classical time evolution of partonic 2-surfaces $X^2 \subset H$ would relate to the mass-scale evolutions of 2-surfaces $Y^2 \subset M^8$ by $M^8 - H$ duality and the coupling parameters of the partonic surface related by $M^8 - H$ duality would be identical.

   The time evolution and mass-scale evolution would obey slightly non-deterministic field equations replacing the path integral. Non-determinism would have interpretation in terms of quantum phase transitions.

UV and IR cutoffs are properties of states rather than approximations of states. IR cutoff is due to the finite size of CD in H and UV cutoff is due to the finite size of its counterpart in M^8.

2. p-Adic thermodynamics [L21] predicts the mass squared values of fundamental fermions. The first basic parameter is the p-adic prime p assumed to correspond to satisfy the p-adic length scale hypothesis stating that p is to a power of 2 in particle physics. Powers of 3 can be considered in the general situation. Second basic parameter is p-adic temperature $T = 1/n$ for fermions $T = 1$ is assumed.

3. In the TGD framework bosons are not fundamental particles but bound states of fermions and antifermions and appear only in the initial and final states. For bosons one can consider p-adic temperature $T = 1/2$.

   Mass squared corresponds to conformal weight in p-adic thermodynamics and this motivates the hypothesis that mass squared is additive in many fermion states. Therefore the mass squared operator has as its eigenvalue for a given fermion-antifermion pair the sum of p-adic thermal mass squared values for fermion and antifermion.

   The possibility that intermediate gauge bosons correspond to M_{89} but have p-adic temperature $T = 1/2$ can be considered but this requires additional assumptions. The more promising option is that bound state fermions correspond to $T = 1$, k for free fermion becomes $k + 1$ for bound state fermion, and mass squared is additive.

4. Bosons as bound states of fermion pairs must represent information about their decay amplitudes to fermion pairs. For instance, photon is a superposition of lepton and quark pairs to which it decays with coefficients proportional to eQ_{em} for the member of the pair. In the case Z^0 in which case the charge matrix is the sum of $g_Z I_L^3$ and $-pQ_{em}$. Mass squared would be non-vanishing also in the case of photons but $T < 1/n < 1$ could make possible an arbitrary small mass scale. The space-time counterpart of the massivation is warping [L23] reducing the M^4 light-velocity.

5. The wave mechanics inspired guess is that the boson states are normalized to unit norm. The gauge coupling g^2 and g_Z^2 could be extracted from both mass squared expectation and normalization factor and for normalized states $m_Z^2/m_W^2 = 1$ hold true in lowest order approximation with respect to p. The same problem would appear also at the level of scattering amplitudes in the embedding space description: coupling disappears from the vertex.

   The second problem is that the proposed structure of W and Z bosons would predict that their mass squared values are smaller than the contribution of the heaviest quark to the mass squared unless one assumes that the p-adic length scales associated with the bound state quarks are longer than for free quarks.

6. The estimate of m_W^2 based on the identification of W wave function as a superposition of fermion pairs gives, for normalized states in the M_{89} option of the W test below, with $m(d_{89}) = m(u_{89})$ and p-adic primes near 2^{k+1} instead of 2^k for all bound state fermions, $m(W) = 79.2$ GeV, 1.5 percent below the measured value; a justification of the step from k to $k + 1$ is suggested below.

The problem is that the masses of W, Z and the $U(1)$ strong boson, call it G_s, are predicted to be identical. How could one obtain mass splitting? Therefore the question concerns the identification of the symmetry breaking mechanism. Could Weinberg rotation as a symmetry breaking somehow imply $m_Z^2/m_W^2 = \alpha_Z/\alpha_W$? W boson does not participate in Weinberg rotation, which suggest m_W is the common mass of the W, Z and G_s, before the symmetry breaking and that symmetry breaking induces also $m_{G_s}^2 \to (\alpha_{G_s}/\alpha_W)m_W^2$, where electric weak duality predicts $\alpha_{G_s} = 1/\alpha_s$. $m_{G_s}^2 = m_W^2/(\alpha_s\alpha_W)$ is a testable prediction and will be considered in the sequel. But what does the symmetry breaking correspond to? The question can be considered at several levels.

1. One can look at the situation at the space-time level. The mechanism inducing the mass splitting should be the TGD counterpart for what happens when Higgs vacuum expectation h is generated and leads to $m_i^2 \propto g_i^2 h^2$ from the coupling of the gauge potentials to the components of Higgs.

2. Kähler action does not possess gauge boson couplings to a Higgs candidate but what about the fermionic sector? At the fermionic lines identified as light-like geodesics of CP_2 parametrized by frequency ω there is a coupling to induced gauge potential $A_t = A_\phi \omega R$, where ϕ defines the coordinate of the geodesic line of CP_2 and t corresponds to the time coordinate of M^4. What is remarkble that the value of A_t would be constant along the fermion line. At partonic surface the situation reduces to Abelian. Furthermore, the square A_t^2 is expected to be invariant under the symmetries of the projection of the partonic 2-surface to a geodesic sphere of CP_2: there are two kinds of these.

   The coupling term is analogous to the term defining Higgs coupling in the ordinary Dirac action: $A_t = A_\phi \omega R \leftrightarrow f(F)H$, where $f(F)$ is the coupling of fermion to Higgs guaranteeing that this coupling corresponds to the mass of the fermion: $A_t^2 \leftrightarrow m^2(F)$. Therefore the analog Higgs mechanism occurs at fermion lines without Higgs and without the need to introduce the ad hoc couplings $f(F)$ tailored to produce the fermion masses.

   The appearance of ωR gives a connection to warping giving the reduced M^4 light velocity along the fermion line as $c_\# = 1 - \omega^2 R^2$. However, it would be $A_\phi \omega R$, which would define the mass rather than ωR, which was the first guess.

   Note: A corresponds to gA_{st} for the gauge potential of QFTs and is dimensionless since it has dimension of gradient and in general coordinate invariant formulation coordinates are dimensionless.

Could Z/W mass splitting have an explanation at the level of CP_2 geometry?

1. CP_2 allows two kinds of geodesic spheres: For S_I^2 the induced Kähler form vanishes and homology charge vanishes but W gauge potential does not vanish. S_I^2 is possible for Lagrangian minimal surfaces allowed by action if it is Kähler action plus volume term (see Appendix). They have an inherent holomorphic structure characterized by the induced metric.

2. For S_{II}^2 the induced Kähler form is non-vanishing and one can assign to its homology charge as Kähler magnetic charge. These are allowed by holography= holomorphy principle and now the inherent holomorphic structure due to metric coincides with the complex structure inherited from CP_2. The isometries of these geodesic spheres correspond to the subgroups $SO(3)$ and $U(2)$ of $SU(3)$. Two homologically non-trivial geodesic spheres intersect at a single point or are identical. S_I^2 is homologically trivial and, as Martin Ebner noticed, the spheres S_I^2 and S_{II}^2 used as an example intersect along a geodesic line. This is due to a failure to be in a general position. The action of $SU(3)$ on either sphere moves the spheres to a general position.

   The symmetries of S_i^2 guarantee that its geodesics are equivalent as far as the analog of Higgs expectation is considered. Since color rotations are represented by 2-bein rotations, the change of the geodesics by color rotation means only a different choice 2-bein and by general coordinate invariance this does not affect the mass squared value.

3. The geodesics defining fermion lines associated with W and Z bosons are transversal to the geodesic spheres so that they cannot correspond to geodesics of S_I^2 and S_{II}^2. The fermions and antifermions associated with W *resp.* Z boson could correspond to S_I^2 *resp.* S_{II}^2. If so, the mass squared scales for the fermions in Z and W could be in the ratio α_Z/α_W.

What could the space-time picture correspond at the level of H and WCW?

1. Assume quadratic mass squared formula for bosons as superpositions of fermion pairs as an expectation value and predicting $m_Z/m_W = 1$. What goes wrong?

   The mass squared values of fermions predicted by p-adic thermodynamics are proportional to the inverse square $1/R^2(CP_2)$ of the CP_2 radius. The proposal discussed in the model of the Weinberg rotation is that CP_2 radius is "dark" CP_2 is related by a scaling to fundamental CP_2 scale, which is Planck length l_P: one has $R^2(CP_2) = (\hbar_{eff}/\hbar_{min}) \times l_P^2$ with $\hbar/\hbar_{min} = (7!)^2$.

2. There is an entire hierarchy of values of $h_{eff}/\hbar_{min}$. Could the values of $\hbar_{eff}/\hbar$ for Z^0 and W differ by the factor $(m_Z/m_W)^2 = 1/(1-p)$? This would predict that $1-p$ and also p is

rational. This gives a strong condition of the values of p. For Pythagorean triangles $sin(\theta)$ and $cos(\theta)$ are rational. Now they would be square roots of rationals.

The prime factors of $N = (7!)^2$ are 2, 3, 5 and 7, so that the number of possible ratios $1/(1-p) = n_W/n_Z$, with n_Z and n_W obtained by dropping a finite number of factors from N, is finite. One would have $\hbar_{eff}(W)/\hbar_{eff}(Z) = 1/(1-p)$. This is true for $c^2 = 1 - s^2 = cos^2(\phi_W) = 0$. For $c^2 \geq 0$ the condition reads as $1/(1-ps^2) = n_W/n_Z$. Also the parameter c^2 appearing in the ratio $m^2_{G_s}/(m^2_W) = 1/c^2$ is rational. This will be discussed in the sequel.

3. The formula for the "dark" radius of CP_2 appears in the proposed formula for the spectrum of Weinberg angle as $p = 1/(2Z(q,k))$, where (q,k) labels monopole harmonics. The motivating observation was that the monopole charge q defines a winding and thus a covering of δM^4_+ or CP_2 so that the 7! factors of $\hbar_{eff}/h_0$ define coverings of M^4 and CP_2.

Are these two identifications of the Weinberg angle mutually consistent? This is not plausible if these two mechanisms fixing p are realized in the same sector (M^4 or CP_2) associated with $\hbar_{eff}/h_0 = (7!)^2$. There is however no obvious objection against their realization in separate sectors. This will be discussed in detail in the sequel.

There should also be a connection to warping as a representation of massivation at the level of fermion lines as light-like geodesics of H.

1. The reduction of M^4 light velocity to $g_{tt} = c^2_{\#} = 1 - \omega^2 R^2$ gives rise to the generation of M^4 mass. The geometric formula does not involve effective Planck constant: instead of masses only frequency appears in the formula. This conforms with the fact that only frequencies appear in space-time description involving waves and energies and momenta requiring Planck constant emerge at quantum level description involving particles.

2. The representation $m^2(B) = \sum(m^2(f_i) + m^2(f_j))Q_iQ_j$ with $m^2(f_i)$" = "$A^2\omega^2(f_i)$ weighted by the products Q_iQ_j of charges for the couplings of B defines the space-time correlate for the quantum superposition of fermion pairs and in this way would realize the quantum classical correspondence. As suggested, the values of $R^2(CP_2) = (\hbar_{eff}/\hbar_0)l^2_P$ defining the mass scales for gauge bosons and for fermion pairs defining them are slightly different and for Z and W. This translates to different frequency scales in the warping formula $f^2_i(Z)/f^2_i(W) = \hbar_{eff}(W)/\hbar_{eff}(Z)$ for the mass scales. The mass squared ratio involves also the ratio $A^2(Z)/A^2(W)$.

   To avoid confusion, it should be noticed that I have used in my articles h_{eff} with two meanings. h_{eff} appears in $R^2(CP_2, h_{eff}) \propto h_{eff}$. h_{eff} has referred also to bilocal electric or gravitational Planck constant h_{gr} or h_{em} and its increase increases quantum coherence scale. h_{int} would be a better notation. Mass scale of m depends on h_{eff} and frequency scales down like $1/h_{int}$ but mass does not depend on h_{int}.

   The fermions would be naturally located to the wormhole throats of closed monopole flux tube representing particle as a 3-D string like object. The fermion and anfermion can reside at the opposite throats of a single wormhole contact and also at throats of opposite wormhole contacts.

The challenge is to really understand the mathematics behind the proposed mass squared formula. p-Adic thermodynamics is used to calculate quark and lepton mass [L15, L21]. The additional complication is that p-adic thermodynamics is involved for fermions and perhaps also for the fermions composing the bosons. If real fermion masses appear in the mass formulas for bosons, calculation involves calculation mass squared values for several p-adic primes simultaneously and the numbers should make sense in several p-adic number fields simultaneously. Adelic physics [L3] involves all p-adic primes and inspiring the notion of multi-p p-adicity [L14, L15] could help here. If the p-adic integers are involved and if they are small (below the smallest p-adic prime involved) there are no problems. Multi-p p-adicity is possible even in a more general case if the p-adic numbers have expansion powers of integers divided by all p-adic primes involved.

2.1.2 General structure of boson states

With these assumptions it is possible to estimate the gauge boson masses by using simple additive mass squared formula for the fermion pairs and test whether the estimates are realistic. Consider first the general structure of the gauge boson states. The wave mechanics inspired guess is that the gauge boson states are normalized.

1. The mass squared is expectation value of mass squared in gauge boson state $|B\rangle$ coupling to fermion pair by charge $Q_{ij}(B)$ given by

$$|B\rangle = \sum_{ij} Q(B)_{ij}|\overline{f}_i\rangle|f_j\rangle \ . \tag{2.1}$$

   This expression can be decomposed to leptonic and quark parts and there is sum over the fermion families and sum over 3 color degrees of freedom: this assuming that in final states quark/lepton color partial waves combine to form color triplets/singlets.

2. For Z^0 and $W^\pm$ the charge matrix is

$$\begin{aligned} Q(Z)_{ij} &= [I_L^3(i) - pQ_{em}(i)]\delta_{i,j} \ , \\ Q(W^\pm)_{ij} &= I^\pm \delta_{i+1,j} \end{aligned} \tag{2.2}$$

   Concerning the presence of pQ_{em} term in Z^0 two options corresponding to pQ_{em} and peQ_{em} were considered. Z^0 decays however select the pQ_{em} option so that the peQ_{em} option will not be considered. This gives

$$\begin{aligned} Q_{i,j}g_Z &\to g_Z[I_L^3(i) - pQ_{em}(i)]\delta_{i,j} \ , \\ Q(W^\pm)_{ij} &\to = gI^\pm \delta_{i+1,j} \ . \end{aligned} \tag{2.3}$$

2.1.3 Mass squared formulas

Consider next the mass squared formulas for weak bosons. The matrix element $m^2(W)$ given by

$$m^2(W) = \frac{g^2}{4}\sum_i m^2(L_i) + m^2(\nu_i) + 3\sum_i [m^2(U_i) + m^2(D_i)] \ . \tag{2.4}$$

Note that only charged leptons contribute and the sum is over 3 generations.

1. The matrix element $m^2(W)$ without normalization factor given by

$$N_W m^2(W) = \frac{g^2}{4}[\sum_i m^2(L_i) + 3\sum_i [m^2(U_i) + m^2(D_i)]] \ . \tag{2.5}$$

   Note that only charged leptons contribute and the sum is over 3 generations. g_W^2 disappears in the mass formula.

2. Assuming that the normalization factors are not included, one obtains the expression for the matrix element $\langle Z|M^2|Z\rangle$:

$$\begin{aligned} N(Z)m^2(Z) &= \frac{g_Z^2}{4}[m_1^2(Z) + m_2^2(Z) + m_3^2(Z)] \ , \\ m_1^2(Z) &= \sum_i [m^2(L_i) + m^2(\nu_i) + 3m^2(U_i) + 3m^2(D_i)] \ , \\ m_2^2(Z) &= 4p^2\sum_i Q_{em}^2(i)m_i^2 = 4p^2\sum_i [m^2(L_i) + (4/3)m^2(U_i) + (1/3)m^2(D_i)] \ , \\ m_3^2(Z) &= -4p\sum_i [m^2(L_i) + 2m^2(U_i) + m^2(D_i)] \ . \end{aligned}$$

(2.6)

2.1.4 Calculation of Z/W mass ratio

For the ratio $m^2(Z)/m^2(W)$ the model gives the estimate

$$\frac{m^2(Z)}{m^2(W)} = \frac{1}{1-p}\frac{N_W}{N_Z}(1 + m_2^2/m_1^2 + m_3^2/m_1^2) \ . \tag{2.7}$$

The factor $1/(1-p)$ is empirically motivated and the most plausible proposal is that it corresponds to different mass scales appearing as the inverse of dark CP_2 length scale in p-adic mass calculation. The dark CP_2 length scale satisfying $R^2(CP_2, h_{eff})/l_P^2 = h_{eff}/h_{min}$ would differ for the fermion lines associated with W and Z. In the lowest order in $sin(\theta_W)$ one obtains $m^2(Z)/m^2(W) = h_{eff}(W)/h_{eff}(Z) = n_W/n_Z = 1/(1-p)$. n_W and n_Z would be numbers dividing the number $N = (7!)^2$. The correction terms depend on the squares of the masses of quarks and leptons so that the expression is analogous to that obtained as radiative corrections in the standard model.

The explicit form of X depends on what option is assumed. There are 2 options to consider.

1. In the TGD framework, the Aleph anomaly inspires the proposal [L25] that the official top with mass about 172 GeV belongs to M_{89} hadron physics and the real top has much smaller mass about 56 GeV. If the official top quark corresponds to the u quark of M_{89} hadron physics, it has a d partner, which, on base of experience with standard model physics, is expected to have the same mass scale. Here one must be however cautious: current quark masses for ordinary u and d differ are estimated to be 2.2 MeV and 4.7 MeV, whereas constituent quark masses are about 300 MeV both. Since low energy M_{89} hadron physics is in question, the interpretation of u and d as constituent quarks looks reasonable.

2. It was not quite clear to me whether one should assume the em coupling of Z to be Q_{em} or eQ_{em}. It however turned out that Z^0 decays allow to select the Q_{em} option. There are also two options depending on whether the heaviest quarks correspond to a single top quark of M_{107} hadron physics or u and d quarks of M_{89} hadron physics.

This gives rise to 2 cases corresponding to M_{107} and M_{89} options.

1. The expression for $X \equiv 1 + m_2^2/m_1^2 + m_3^2/m_1^2$ for the Q_{em} option can be written as

$$\begin{aligned} &X = 1 + \ (4x^2C - 4xA)/S \ , \qquad x = p \ , \\ &S = 3 \times m^2(U) + 3 \times m^2(D) \ , \quad A = 2 \times m^2(U) + m^2(D) \ , \quad C = (4/3) \times m^2(U) + (1/3) \times m^2(D) \ . \end{aligned} \tag{2.8}$$

   For a single top quark and b left out, one has $C/S = 4/9$ and $A/S = 2/3$. For u and d M_{89} quarks one has $C/S = 5/18$ and $A/S = 1/2$.

2. More concrete formulas read as

$$\begin{aligned} X &= 1 + \ p^2 \times 16/9 - p \times 8/3 \ , \quad M_{107} \text{ option} \ , \\ X &= 1 + p^2 \times 10/9 - \ 2p \ , \qquad\quad M_{89} \text{ u-d pair alone} \ , \end{aligned} \tag{2.9}$$

3. Since the normalization factor $1/N_Z$ appears in the ratio defining the radiative corrections to mass squared, radiation corrections vanish for $X = X_N \equiv N_Z/N_W$, the analog of fixed point of RG flow. M_{89} u-d pair alone gives $X - X_N = -(8/45)x^2$. The M_{107} option, the top alone over the 24 states of the three generations, gives $X - X_N = (4/9)p^2 - (2/3)p$. In neither case, $X - X_N$ vanishes for $0 < p < 1$.

4. An interesting question is whether the radiative corrections could be interpreted as a discrete coupling constant normalization so that one would have iterations defining a discrete variant of renormalization group.

If the mass squared values and normalization factors are defined by assuming the Q_{em} option, forced by Z^0 decays and implying $x = p$, one obtains

$$N(W)/N(Z) = 3/(3 - 6p + 4p^2) \quad , \quad M_{107} \text{ option} \quad , \tag{2.10}$$

$$\tag{2.11}$$

$$N(W)/N(Z) = 45/(45 - 90p + 58p^2) \quad , \quad M_{89} \text{ option} \quad , \tag{2.12}$$

$$\tag{2.13}$$

This gives the analog of radiative correction to the mass ratio m_Z/m_W. With the masses of the W test below one obtains, in the M_{89} option, with the real top t_{107} at 56 GeV and its b, $X = .5906$ and $N(W)/N(Z) = 1.6500$ for $x = p = .2315$, so that $\sqrt{XN(W)/N(Z)} = .9872$: $m(Z)/m(W)$ is reduced below $1/cos(\theta_W)$ by 1.3 percent, the M_{89} u-d pair alone giving .79 (1.2 or 1.4 percent in all with $m(t_{107}) = 50$ or 62 GeV). This should be compared to the .52 percent measured; $m(d_{89})$ 8 percent above $m(u_{89})$ gives .52 percent. For the M_{107} option the reduction is 11 percent.

What about the contribution of the proposed $t(56)$ quark with mass 56 GeV. This contribution vanishes in the lowest order p. It is essential that the mass ratio is in lowest order $1/(1-p)$ but this assumption lacks a convincing justification.

The change of the W/Z mass ratio due to the $t(56)$ quark mass is

$$\begin{aligned} &[(dm^2(Z)/dm^2(56))1/m^2(W) - m^2(Z)(dm^2(W)/dm^2(56))]3m^2(56) \\ &= [\tfrac{1}{N(Z)(1-p)} - \tfrac{m^2(Z)}{m^2(W)}\tfrac{1}{N(W)}\tfrac{3m^2(56)}{m^2(W)} \quad . \end{aligned} \tag{2.14}$$

Apart from terms proportional to $x = p$ and its square x^2 appearing in $m^2(Z)$ and $N(Z)$, $N(Z) = N(W)$ is true so that the correction vanishes in this order. One has $x = 0.2315$ and $x^2 = 0.054$ so that the term proportional x is dangerous. These terms contribute to the mass squared of both Z and $N(Z)$ a contribution of the same form. For the M_{89} pair these terms cancel in the lowest order; for the t_{107}, whose partner b is light, they do not.

2.1.5 Calculation of W mass as a test

It is possible to test the proposal by calculating the prediction for the W boson mass. Using GeV as a unit, the relevant masses $m(u_{89}) = m(d_{89}) = 172.52$; $m(t_{107}) = 56; m(b) = 4.18; m(W) = 80.36$. The relevant coupling parameters are $\alpha_{em} = 1/128$, $p = .2315$, and $g^2 = e^2/p$. In the formula for m_W^2 the weight of both $m(U_i)^2$ and $m^2(D_i)$ is $N = 3$. The relevant mass squared ratios are

$$r_1 = m^2(u_{89})/m^2(W) = m^2(d_{89})/m^2(W) \quad , \quad r_2 = m^2(t_{107})/m^2(W) \quad , \quad r_3 = m^2(b)/m^2(W) \quad . \tag{2.15}$$

The ratio of the predicted W mass squared to the real one is, for normalized states, with $N_W = 15g^2/4$ in the M_{89} option and $N_W = 3g^2$ in the M_{107} option, with $r_t = m^2(t)/m^2(W)$ if its single top is the official one,

$$\begin{aligned} &r \equiv m^2(W, pred)/m^2(W) = \tfrac{g^2}{4N_W}(6r_1 + 3r_2 + 3r_3) = \tfrac{6r_1+3r_2+3r_3}{15} \quad , \quad M_{89} \text{ option} \quad , \\ &r = \tfrac{g^2}{4N_W}(3r_t + 3r_3) = \tfrac{r_t+r_3}{4} \quad , \qquad\qquad\qquad\qquad M_{107} \text{ option} \quad . \end{aligned} \tag{2.16}$$

For normalized states g^2 cancels. In the M_{89} option the 15 doublets of the sum, the 12 of the three generations and the 3 color doublets of the M_{89} pair, give $N_W = 15g^2/4$, $r = (6r_1 + 3r_2 + 3r_3)/15 = 1.94$ and $m(W) = 112$ GeV, 39 percent above the measured value; the real top contributes 3 GeV of it.

In the M_{107} option, if the single top is the official one, $m(t) = 172.52$ GeV, with its b, the 12 doublets give $N_W = 3g^2$, $r = (m^2(t) + m^2(b))/(4m^2(W)) = 1.15$ and $m(W) = 86$ GeV, 7 percent above.

$N_W = 3$ without g^2, with the masses of the M_{89} option, gives $r = 1.029$; then $m^2(W) \simeq g^2 m^2(u_{89})/2$, the standard model relation $m(W) = gv/2$ ($v = 246$ GeV) with the Yukawa coupling $\sqrt{2}m(u_{89})/v = .99$ of the official top put equal to one. If the p-adic primes of the bound state fermions are near 2^{k+1} instead of the 2^k for free fermions, their masses and $m(W)$ scale down by $1/\sqrt{2}$, in the M_{89} option to 79.2 GeV, 1.5 percent below the measured value; a twice longer p-adic length scale with $k \to k+2$ would halve them. For M_{107} option the scaling by $1/\sqrt{2}$ would give $m_W = 61.0$ GeV.

Could the crucial scaling up of the p-adic length scale from k to $k+1$ be assigned to the periodicity of the dynamics of the conformal scaling generator L_0 with respect to the Lorentz invariant light-cone proper time, whose eigenvalues correspond to the mass squared operator?

1. p-Adic mass calculations assume that p-adic primes near powers of two are favored. As described earlier, Mersenne primes are theoretically special, being extremely near to powers of two. Period doubling, appearing universally in chaotic systems, in particular those involving iteration, means that closed orbits cease to close in a single full turn but do so after two full turns. In many-sheeted space-time, this could come from many-sheetedness causing formerly closed orbits to make 2 full turns before closing.

2. If holography-holomorphy correspondence holds [L14] true, the scalings by 2^k would naturally correspond to a dynamical symmetry as iteration sequences $f \mapsto g^{\circ k} \circ f$ with the function pair $f = (f_1, f_2) : H \to C^2$ determining the space-time surface as its root. Here g would be a polynomial of degree 2.

   The increase of the p-adic length scale could be interpreted in terms of "bound state mass squared" or "bound state conformal weight". $k \to k+1$ means a minimal increase of the p-adic length scale and minimal increase of complexity.

2.1.6 The values of W, Z and G masses for scaled variants of M_{89} hadron physics

The values of W, Z and G masses for scaled variants of M_{89} hadron physics provide a test for the hierarchy of hadron (standard model) physics predicted by TGD. The p-adic primes involved correspond to 3 Mersenne primes M_k, $k = 89, 107, 127$ characterized by scaling factors $2^{(89-k)/2} \in \{1, 2^{-9}, 2^{-19}\}$ for mass and Gaussian Mersenne prime $M_{G,113}$ with scaling factor $2^{(89-113)/2} = 2^{-12}$.

This allows us to estimate the masses for the possibly existing scaled variants of W, Z and G_s. For $m_W = 80.36$ GeV the masses of the scaled variants of W are $m_{W(k)} = cos(\theta_W) m_{Z(k)}$ are $[80.36\ GeV, 157.0\ MeV, 19.6\ MeV, .15\ MeV]$. The estimate for the mass of the mediator of fifth force is 17 MeV [L1, C1] [L1]. The current quark mass for s quark is estimated to be 95 MeV and 486-500 MeV. TGD predicts leptopions with the mass slightly below the mass $2m_L$ [K10].

For G_s the masses are obtained by multiplying with the factor $\sqrt{\alpha_{G_s}/\alpha_W} = 1/\sqrt{\alpha_s \alpha_W} = \sqrt{p}/\sqrt{\alpha_s \alpha_{em}}$. This gives $m(G_s(89)) = 1273.4$ GeV. The masses of the scaled variants $G_s(k)$ are $[1.27\ TeV, 2.49\ GeV, 311\ MeV, 2.43\ MeV]$.

2.1.7 Are the two mechanisms proposed to determine Weinberg angle mutually consistent?

As already explained, TGD proposes two identifications of Weinberg angle. The first proposal, explained in previous section, is $2p = 1/Z(q,k)$ and the second proposal is $1/(1-p) = R^2(CP_2, W)/R^2(CP_2, Z)$. Both proposals involve "dark" CP_2 radius $R(CP_2)$ given by the formula $R^2 = (h_{eff}/h_{min}) l_P^2 = n_1 n_2 l_P^2$. The condition $h/h_{min} = (7!)^2$ fixes the minimal value of h_{eff}. n_1 and n_2 can be identified as the numbers of sheets for the space-time surface as coverings of M^4 and CP_2.

The consistency of these models is not plausible unless they are realized in different sectors (M^4 or CP_2) associated with $\hbar_{eff}/h_{min} = (7!)^2$. There is however no obvious objection against their realization in separate sectors.

Consider first the realization of $h_{eff}/h_{min} = n_1 n_2$ hierarchy at space-time levels in terms of space-time sheets defining n_1. *resp.* n_2-fold coverings of M^4 and CP_2.

1. The hierarchy of effective Planck constants gives rise to a 7-layered fractal hierarchy labelled by $k = 1, 2..., 7$ (there are also higher levels not relevant for what follows). At the k:th level of the hierarchy, the values of effective Planck constant are $h_{eff}/h_{min} = k!/n_k$, where n_k

divides k but is not divisible by k. The k:th level is assumed to consist of k-sheeted surfaces with $h_{eff}/h_{min} = (k-1)!/n_{k-1}$. In accordance with the basic idea of fractality, the k sheets decompose to $k-1$-sheetes which in turn decompose to $k-2$ sheets. This hierarchy would define a fractal hierarchy of standard model physics and rather detailed identifications are suggestive.

2. The "dark" sheets at level $n-1$ correspond to space-time surfaces of level n with winding number determined by the ratio $h_{eff}(n)/h_{eff}(n-1)$ of effective Planck constant equal to $nm(n-1)/m(n)$. For dark excitations at the level $n-1$, angular momentum fractionation takes place with unit $1/n$. The unit of angular momentum at level $n-1$ is $[(n-1)!/m(n-1)] \times h_{min}$. Many-sheetedness implies fractionization with a reduced unit $[(n-1)!/m(n-1)]/n!(m(n)) = [m(n)/m(n-1)]/n$.

   By eliminating the largest common factor of $[m(n)$ and $m(n-1)$, one can write $[m(n)/m(n-1)] = m_R(n)/m_R(n-1)/$. The rotation by $m_R(n-1) \times n$ 2π at the level of M^4 corresponds to 2π rotation at the level of the partonic 2-surface (space-time surface). One can say that $m_R(n-1) \times n$ defines the winding number having interpretations in terms of homology of δM^4_+ and corresponding monopole charge.

The following argument suggests that these two models for p must be realized in different factors associated with $h_{eff}/h_{min} = n_1 n_2$.

1. In the model for the mass ratio $m_Z^2/m_W^2 = 1/(1-p) = R^2(CP_2, W)/R^2(CP_2, Z)$, the ratio for the squares of dark CP_2 length scales is $R^2(CP_2, W)/R^2(CP_2, Z) = q(Z)n_W/(n_Z q_W) = 1/(1-p)$, which suggests $qh_{min}(Z) = qh_{min}(W) = 7$ giving $1 - p = n_Z/n_W$. The more general options can be easily tested.

2. The proposal for the model of Weinberg angle discussed in the earlier section is that the value of $2p = 1/Z(q,k)$ can be identified by considering states of $k = 7$ level as dark states identifiable $h_{eff}/h_{min} = 7!$ as excitations of states for which $qh_{min} = q = 1$. The value of k was fixed from the condition that the Weinberg angle and the parameter $cos^2(\phi_W)$ are reproduced as well as possible.

   The additional parameter is the integer k analogous to the projection of the angular momentum in the direction of the quantization axis and a given value of $2p = 1/Z(q,k)$ assigned with a monopole harmonic corresponds to $q = 7!$. A rather restricted set of values of k is consistent with the empirical constraints coming from the Weinberg angle and strong coupling strength.

   $qh_{min} = 1$ is not the only option although it gives the widest spectrum of (q,k) values. One can consider all values of q at various levels of the hierarchy labelled by $qh_{min} \in [1,7]$. For instance, the transition $q = 7!/n_1 \to 7!/n_2$ would correspond to the winding number $q = n_1/n_2$ where n_i are factors of $7!$ not divisible by 7. This is not a realistic option.

It is not enough that the two models are assigned with different sectors. They must also reproduce the same value for p. This might serve as a killer test for the proposal.

2.2 Some questions and remarks

2.2.1 Could the option $L(KCS, H)$ involving the sum of M^4 and CP_2 Kähler forms be realistic?

As explained in the previous section, one can consider two options for $L(KCS)$. For option **a)** $L(KCS, H = M^4 \times CP_2)$ is defined for the sum $J_K(M^4) + J_K(CP_2)$ of the induced Kähler forms as the dynamical variable. For option **b)** $L(KCS)$ is the sum $L(KCS, M^4) + L(KCS, CP_2)$ for $J_K(M^4)$ and $J_K(CP_2)$: this option has been considered hitherto.

Whether the option **a)** makes sense physically, can be tested. For this option, the KCS action $L(KCS, H)$ is proportional to the sum $B_K(M^4)_u + B_K(CP_2)_u$ of M^4 Kähler gauge potential $B_K(M^4)$ and of the induced Kähler gauge potential $B_K(CP_2)$. The coordinate u associated with the Hamilton-Jacobi structure.

There are good arguments suggesting $B_K(M^4)_u = 0$ or very small (it could correspond to the Newtonian gravitational potential). If $B_K(M^4)_u$ is not small $B_K(CP_2)_u$ and warping modifying the Hamilton Jacobi structure gives rise to the massivation, the modified condition determining p depends on $\omega R = mR/\hbar_{eff}$ so that Z and p would depend on fermion mass. The $B_K(CP_2)$ contributes the factor

$$B_u = \omega R \times (F-1)/F = \omega R \times r^2/(1+r^2)$$

in the $L(KCS, M^4)$. If this factor dominates, it modifies dramatically the condition determining the value of p. The cautious conclusion is that option **a)**, which assumes $J_K(M^4) + J_K(CP_2)$, is not physically favored.

It is interesting to see how much one can conclude from the values of Z for option **a)** involving $B_u(CP_2) \propto (F-1)/F)$ as an additional weight factor by using the facts about ScMHs.

1. The independence of the parameter Z determining the value of p same for all holomorphic ScMHs makes very power predictions for the expectation values of $X_2(n) = \langle (F-1)/F \rangle_n$ and $X_1(n) = \langle 1/F \rangle_n$. $X_2(n)$ can be written as

$$X_2(n) \equiv \langle Id \rangle_n - \langle 1/F \rangle_n = 1 - X_1(n) \ . \tag{2.17}$$

   This gives $X_1(n) + X_2(n) = 1$ and from $2Z(n) = 1$ one can conclude $X_1(n) = X_2(n) = 1/2$.

2. The expectations $X_2(n)$ and $X_1(n)$ are modified by the presence of $B_K(CP_2)_u \propto (F-1)/F$:

$$\begin{aligned} X_2(n) = \langle (F-1)/F \rangle_n \to \tilde{X}_2 = \langle [(F-1)/F]^2 \rangle_n \\ = \langle (F-1)/F)(1-1/F) \rangle_n = X_2(n) - \langle (F-1)/F^2 \rangle_n \ . \end{aligned} \tag{2.18}$$

   If the ScMHs Ψ_{n+1} and Ψ_n are related by (this is crucial!)

$$|\Psi_{n+1}|^2 = [r^2/(1+r^2)^2]|\Psi_n|^2 = [(F-1)/F^2]|\Psi_n|^2 \ .$$

   The latter expectation is apart from a normalization factor the expectation for $\langle Id \rangle_{n+1}$ and can be written as a ratio of normalization factors:

$$\langle (F-1)/F^2 \rangle = N(n+1)/N(n) \ .$$

   This gives

$$X_2(n) \to \tilde{X}_2 = X_2(n) - N(n+1)/N(n) = 1/2 - N(n+1)/N(n) \ . \tag{2.19}$$

3. For X_1 one obtains

$$X_1(n) = \langle 1/F \rangle_n \to \tilde{X}_1(n) = \langle (F-1)/F^2 \rangle_n = N(n+1)/N(n) \ . \tag{2.20}$$

4. From these two equations one obtains the equations

$$\begin{aligned} &\tilde{X}_1(n) + \tilde{X}_2(n) = 1/2 \ , \\ &\tilde{X}_1(n)/\tilde{X}_2(n) = \tilde{X}_1(n)/(1/2 - \tilde{X}_1(n)) \ , \\ &\tilde{X}_1(n) = N(n+1)/N(n) \ . \end{aligned} \tag{2.21}$$

   This is a parameter free prediction and can be made explicit and tested once the ratios $N(n+1)/N(n)$ are known.

5. Martin Ebner has calculated the value of

$$R(n) = N(n+1)/N(n) = (n+1)/(2(2n+3)) \ . \tag{2.22}$$

for ScBH:s and obtained the series $R(n) = 1/6, 1/5, 3/14, 2/9, ..$ rising to $1/4$. Expressing Z in terms of $R(n) = N(n+1)/N(n)$ gives $\tilde{Z}(n) = R(n)/(1-2R(n))$ and $R(n) = Z(n)/(1+2Z(n))$. This gives the series $\tilde{Z}(n) = 1/4, 1/3, 3/8, ...$ with asymptotic value $\tilde{Z}(\infty) = 1/2$. The universality for the holomorphic situation is lost.

2.2.2 How to predict the overall strength of the weak and strong interactions?

The proposed model for the Weinberg rotation involves only the ratios of coupling constants and says nothing about the overall scale of electroweak and strong coupling strengths fixed by the fine structure constant. The hypothesis that the values of α_K are fixed by quantum criticality as analogs of critical temperature in principle fixes the scale. The needed constraint comes from the value of Fermi coupling constant $G_F/\sqrt{2} = g^2/8m_W^2$ fixing the value of g^2. This is a prediction of a full quantum theory. The condition $(e/g)^2 = psin^2(\phi_W)$ implies that $sin^2(\phi_W)$ must be near unity.

In TGD, p-adic mass calculations performed at the level of embedding space and WCW, predict the masses of fundamental fermions using Ramond representations. p-Adic thermodynamics can predict also boson masses. The simplest option is that fermionic p-adic mass squared given by Ramond representation is additive: this is in accordance with the interpretation of mass squared as conformal weight. I have also considered N-S representation as an attractive but not so attractive, option. As noticed, it might make sense at the space-time level for an Principal Series representations of the Lorentz group with conformal weight with fractional conformal weights for which inner product is defined by a bilocal kernel.

The fact that warping [L23] allows to code the mass of a fundamental fermion to the reduced M^4 light velocity $c_\#$ characterizing light-like geodesic of H raises the hope that warping could code also gauge boson masses. The proposed additivity of mass squared for fundamental fermions would suggest that the squares of warping parameters are additive for bound states so that the component of the induced metric would be $g_{tt} = 1 - c_{\#,1}^2 - c_{\#,2}^2$ for the light-like geodesic of H associated with the bosonic space-time sheet. The contributions of the warpings on metric would be additive just like they are in Newtonian theory of gravitation.

2.2.3 Some remarks related to strong interactions

The predictions for both the Weinberg angle and for the strong coupling constant strength $\alpha_s(H)$ in the old-fashioned description of strong interactions in terms of the field $g_s G_2$. Also the possible duality between old-fashioned and QCD type description implying $\alpha_s(QCD) = 1/\alpha_S(H)$ is suggested by electric-magnetic duality [K1] and discussed in more detail in the previous section. The view emerging from the attempt to understand Weinberg angle concretizes the general vision based on super-symplectic invariance [K3, K9] suggesting that the scattering amplitudes reduce to correlations function of super-symplectic generators.

1. The above considerations suggest a generalization and reinterpretation of the Weinberg mixing at classical level: otherwise unitary is lost. The generalized Weinberg mixing could occur between Kähler gauge potential, vector counterpart of V_{03} proportional to Z^0 boson and a vector boson V_{12} coupling to I_V^3.

   The electroweak vector isospin and hyper charge are realized in the holonomy algebra $U(2)_{ew}$ algebra realized in the tangent space of CP_2 as holonomies acting on fermions. Electroweak algebra is embeddable as a subalgebra to $SU(3)$ but color and electroweak charges are independent.

   The question arises whether color isospin and hypercharge could be represented at space-time level as anomalous isospin and magnetic/homology charge labelling the monopole harmonics. After many unsuccessful trials it became clear that there is no need for this. Electric-magnetic duality provides the description of strong interactions at the space-time level. There is no

need to describe quark color and gluons at the space-time level since color appears at the level of embedding space and WCW description. The large value of $\alpha_s(U) = 1/\alpha_s(QCD)$ is not a problem since holography implies that there is no path integral.

The emerging picture would give a more precise content for the earlier crazy sounding idea forced by the TGD based view of color stating that strong interactions in some sense reduce to electroweak interactions or vice versa. It would also conform with the old-fashioned view of strong interactions.

2. The close relationship between strong and weak interactions is also an empirical fact and old-fashioned hadronic physics formulated this in terms of CVC and PCAC hypothesis. In QCD strong and weak interactions decouple and it seems to me that this relationship was regarded as a trivial consequence of weak interactions. For instance, vector boson dominance, involving a transformation of photon to vector boson, is one of the key notions of old-fashioned hadron physics. In TGD, the Weinberg mixing of the photon with the gluons of the color Cartan algebra would allow the formulation of this notion at the fundamental level.

3. M^4 Kähler action gives rise to a new $U(1)$ interaction via $B_K(M^4)$. The interpretation in terms of gravitational interaction looks natural. For the large values of the reduction of light velocity, the modification of the induced metric in warping would correspond to a strong gravitational potential with constant value. Could both these interactions be involved with strong interactions as hadronic string model in which transversal M^4 deformations of the string are in central role, would suggest? Note also that warping would provide a space-time description of massivation. This interaction could relate to the view of strong interactions as strong gravitation becoming important in hadron physics.

4. In TGD, also leptons would have strong interactions. If this prediction is a catastrophe, it would not be the first one. TGD predicts an entire hierarchy of color partial waves and also the counterpart of color confinement for both quarks and leptons [L18, L16, L19]. There is evidence for bound states of color octet leptons [K10]. There are good aesthetic reasons for why the couplings to the M^4 Kähler charge should obey the same formulas as in the case of CP_2. Could the striking difference between leptons and quarks be due to the fact that color singlet property for leptons does not require several leptonic strings forming hadron-like states.

2.3 Summary and conclusions

In the following I try to explain the big picture as clearly as possible.

2.3.1 Some dimensional analysis

The TGD based notion of gauge potential differs from that of the standard model and the notion of Planck constant generalizes.

1. The relationship of the induced spinor connection A to the standard model gauge potential is given by $A = gA_{st}$. The dimension of the induced gauge potential gA_{st} is $[gA_{st}] = [\hbar/L]$ in the Standard Model. This brings new elements even at the level of dimensional analysis.

   In the Standard Model, the dimension $[hbar/L]$ for A_{st} follows from a definition relying on the linear coordinates of the Minkowski space so that this notion is not a generally coordinate-invariant notion. In TGD, the radius $R(CP_2)$ of CP_2 provides the scale for the induced gauge potentials of CP_2: CP_2 gauge potential A is proportional to $1/R$. The scale of the causal diamond cd provides the overall scale for the M^4 Kähler gauge potential as a natural length unit based on the properties of M^4.

2. What does the notion h_{eff} mean? There are two basic variants of h_{eff} and they are not equivalent.

   (a) Gravitational *resp.* electric Planck constants $\hbar_{gr}$ *resp.* $\hbar_{em}$ characterize a particle pair rather than a single particle as the ordinary Planck constant. Particle masses do not

depend on h_{gr} and h_{em}, whose values define hierarchies of quantum coherence scales for a particle with a given mass. The frequency and wavelength depend on h_{gr} or h_{em}.

The proposed interpretation of $\hbar_{gr}$ and $\hbar_{em}$ relies on multilocal Yangian symmetries [L2] involving generalization of ordinary geometric symmetries to multilocal symmetries. The very fact that gravitational and electromagnetic interaction energies involve description in terms of bi-local potential energies reflects this.

(b) Second hierarchy of Planck constants is associated with the postulated "dark" hierarchy CP_2 size scales $R^2(CP_2, dark) = h_{eff}/h_{min} l_P^2$. h_{eff} is a single-particle parameter and $h_{eff} = h$ corresponds to $R^2(CP_2, dark) = (7!)^2 l_P^2$. This hypothesis realizes the idea that Planck length scale l_P is the only fundamental scale. $R^2(CP_2, n_1 n2) = n_1 n_2 l_P^2$ defines the mass scale of the particle in p-adic mass calculations. This assumption allows us to understand why the mass ratios Z/W mass ratio is different from unity.

2.3.2 The two ways to understand particle masses

There are two ways to understand particle massivation.

1. Warping makes possible the TGD counterpart of the Higgs mechanism. The basic formula emerges by looking at the coupling of induced spinor fields to the induced electroweak gauge potential A having various components. This term defines in the standard model the analog of mass term, denoted somewhat loosely by $m^2 = A^2\omega^2 R^2$, with the dimension $[\hbar^2\omega^2] = [m^2]$ as required. The crucial point is that A^2 is constant at the fermion lines, whose projections are geodesic lines of CP_2 the value is different for homologically trivial *resp.* non-trivial geodesic spheres assignable to W and Z^0.

   The original naive proposal was that a quantity with dimensions of mass squared is obtained as $\hbar_{eff}^2\omega^2$ where $\hbar_{eff}$ was identified as $\hbar_{gr}$ or $\hbar_{em}$ and scaling of $\hbar_{eff}$ was compensated by opposite scaling of ω.

2. p-Adic mass calculations predict the masses satisfy the following proportionality $m^2 \propto \hbar^2/R(CP_2)^2$ with "dark" CP_2 length scale $R^2(CP_2, dark) = \hbar_{eff}/\hbar_{min} l_P^2$, where one has $h_{eff}/h_{min} = n_1 n_2$ corresponding to winding numbers with respect to M^4 and CP_2, so that $m^2 \propto \hbar^2 \hbar_{min}/(\hbar_{eff} l_P^2)$ is true. This allows us to understand the Z/W mass ratio in terms of slightly different "dark" CP_2 length scales.

3. For warping *resp.* Higgs type massivation the mass squared formulas are given by $m^2 \propto \hbar^2/R^2(CP_2, dark)$ *resp.* $m^2 = A^2\omega^2 R^2(CP_2, dark) = A^2\omega^2(\hbar_{eff}/\hbar_{min}) l_P^2$ and are in an apparent conflict. The condition that the two formulas give the same result implies $A^2\omega^2 \propto \hbar^2/R^4(CP_2, dark)$ having as a solution

$$\omega \propto \hbar/(\hbar_{eff} l_P) \quad , \quad A \propto \hbar_{min}/l_P \quad . \tag{2.23}$$

   The varying mass scale comes from $R(CP_2, dark)^2 \propto n_1 n_2$.

4. The prediction for G_s goes as follows. One has $\alpha_Z = \alpha_W/(1-p)$ with $\alpha_W = g^2/(4\pi\hbar)$. The model for Weinberg rotation, assuming electric-magnetic duality, gives $\alpha_{G_s} = \alpha_{em}/pc^2 = 1/\alpha_s$, implying $m^2(G_s)/m^2(W) = 1/(\alpha_s \times \alpha_W)$. The argument can be however questioned. Can the mass scale really increase? One must however keep in mind that TGD predicts an entire hierarchy of standard model physics labelled by Mersenne primes and their Gaussian counterparts.

2.3.3 Are there alternative options?

The model for generalized Weinberg rotation predicts $psin^2(\phi_W)$, when combined with the assumption that bosons provide representations for their decay amplitudes, predicts that W/Z mass ratio is unity. The model based on the variable dark CP_2 scale solves the problem and also predicts the value of p. Both models involve dark CP_2 scale and h_{eff}. The models can in principle

be consistent if the different sectors (M^4 anad CP_2) associated with h_{eff} are involved with their predictions.

The condition $\langle\gamma\rangle = 0$ and monopole harmonics lead to the constraint $p(2-3c^2) = 1/Z(q,k)$ and the latter model leads to the condition $1 - ps^2 = m_W^2/m_Z^2 = R^2(CP_2,Z)/R^2(CP_2,W)$. $m_{G_s}^2/m_W^2$ is also predicted and the condition that the two predictions are identical gives hopes for predicting the entire spectrum for (p,c^2).

In the prediction of $m_{G_s}^2/m_W^2$ magnetic-electric duality is used but one can ask whether there is a more direct alternative. This picture looks rather nice: there are two sectors corresponding to $\hbar_{eff}/\hbar_{min} = n_1 n_2$ and two parameters p and c^2 characterizing the Weinberg rotation. The entire spectrum of coupling constants and the evolution of coupling constants would follow as predictions, including also masses.

One can also consider weaker variants of this model. For instance, $\langle\gamma\rangle = 0$ condition could be dropped but the notion of Weinberg rotation implying the notions of p and c^2 could be kept. Also in this case the value of h_{eff} would provide a weaker constraint on the values of p and c^2.

2.3.4 Critical summary

The model enables a transition from TGD to the description provided by field theory. The aim is to predict both coupling constant and masses. Skeptic's critique is easy to guess. There is too much new physics and untested assumptions. There are two kinds of reasons for criticism.

1. The general assumptions go beyond the standard model. The fundamental assumption that space-times are 4-surfaces in H; the surfaces predicted by holography = holomorphy hypothesis and minimal Lagrangian surfaces; the notion of WCW and the absence of boson fields as primary fields; the predicted hierarchy of Standard Model physics; the predicted tight relationship between strong and weak interactions implied by CP_2 geometry; p-adic thermodynamics and warping as a mechanism of massivation reflecting masslessness at the level of H; the notions of string world sheets, partonic 2-surface, partonic orbit, and of light-like fermion line in H; the h_{eff} hierarchy predicting "dark" CP_2 scales as well as the notion of h_{gr} and h_{em}, which are however not used in the model.

2. There are also model-specific assumptions. The $\langle\gamma\rangle = 0$ hypothesis as a generalization of the statement that the emission of the wrong helicity does not occur; the notion of 3-D Weinberg rotation to include dual Weinberg rotation for charges; electric-magnetic duality; monopole harmonics as a model for parton-level physics.

Even if this general view were correct, the model is bound to contain errors at the level of detail.

To sum up, for normalized boson states, in which g^2 cancels, the W test gives $m(W) = 112$ GeV for M_{89} option and, if the single top is the official one, 86 GeV in the M_{107} option: the error is 7 percent.

With p-adic primes of bound state fermions near 2^{k+1} instead of 2^k for free fermions, the M_{89} option gives 79.2 GeV, 1.5 percent too small. M_{107} option would give $m_W = 61.0$ GeV. With the choice $x = p$ that the decays of Z fix, left-handed pairs and $m(d_{89}) = m(u_{89})$, $m(Z)/m(W)$ lies 1.3 percent below $1/cos(\theta_W)$ in the M_{89} option, where .52 percent is measured; the size rests on $m(d_{89})$, and 8 percent more gives .52 percent. In the M_{107} option it lies 11 percent below. Therefore M_{89} option looks more plausible than M_{107} option. As explained, p-adic thermodynamics and holography = holomorphy hypothesis suggest a justification for the increase $k \to k+1$ for bound state fermions.

3 Appendix: Conformal involution,quantum homology, Lagrangian minimal surfaces and unitary representations of Lorentz group

3.1 The role of conformal involution

$g \leq 2$ partonic 2-surfaces are important in TGD and one reason would be that they are always hyperelliptic and allow an involution as a Z_2 conformal symmetry. This involution is intuited to be highly relevant for conformal field theory defined on the partonic 2-surface (X^2) and could define the counterpart of $z \to 1/z$ in CFTs.

The condition that Z_2 is a global conformal symmetry, predicts 3 fermion generations corresponding to the genera $g = 0, 1, 2$. For $g > 2$, only a small fraction of 2-D metrics are hyperelliptic. If the handles behave like free particles for non-hyperelliptic $g > 2$ surfaces, the mass spectrum is continuum. For $g = 2$ hyper-ellipticity would make a bound state of handles possible and the mass spectrum would be discrete.

Vesa Ala-Mattila has gathered basic information about conformal involutions of 2-surfaces using Claude (see this). Vesa gathered also information about conformal surfaces in E^3 (see this) and conformal symmetries of surfaces (see (this). Vesa gathered also information about the connection between homology and genus for curves (2-D complex surfaces) of $CP_1 \times CP_2$ (this).

1. Canonical involution is degree two map existing always for Riemann surfaces with genus $g \leq 2$. For higher genera it exists only for hyper-elliptic surfaces having Z_2 as a subgroup of conformal symmetries.

2. In the case of CP_1, the map $z \to 1/z$ defines the involution. In the case of torus, the lattice representation of torus in terms of a non-orthogonal lattice spanned by complex numbers ω_1 and ω shows that the involution is $z \to -z$ and leaves the points of the half-lattice invariant. The points $\omega_1/2$, $\omega_2/2$ and $(\omega_1+\omega_2)/2$ are fixed points defining branch points. For $g = 2$ the 6 fixed points are known as Weierstrass points and depend only on the metric of the surface. Involution is in the center of the group of conformal automorphisms and thus commutes with it.

   Torus is special because of the underlying lattice structure. $g = 2$ is special because of its canonical map, depending only on the conformal equivalence class of the surface and mapping the point p of the surface to $[\omega_1(p) + \omega_2(p)] \in CP_1$ is of degree 2 and therefore involution (here ω_1 and ω_2 are the holomorphic 1-forms defining homology basis and characterizing the complex structure). For $g > 2$ the degree of the canonical map is $2(g-1) \geq 2$ so that involution is not in question.

   The locus of hyperelliptic surfaces a has complex dimension $2g-1$ whereas the complex dimension of conformal equivalence classes is $3g-3$. The co-dimension is $g-2$ so that for $g > 2$ the generic 2-surface does not allow a canonical involution.

3. One can understand the situation also geometrically. A two-surface with genus g has $2(g+1)$ coordinate charts: 2 for sphere, 4 for torus and 6 for $g = 2$ as is rather easy to see in the case of sphere and torus. More generally, each handle corresponds to a torus having 4 charts. The covering projection maps these chart pairs to the chart pair for CP_1. The Z^2 involution has $2g+2$ branch points as fixed points of the involution, one pair for each handle.

   One can visualize 2-surface as a union of $g+1$ disjoint copies of CP_1, which touch each other at branch points so that the topology becomes multiply connected. In the topology change $g \to g-1$ in which a handle splits, the two fixed points fuse and disappear.

4. In conformal field theories (CFTs) hyper-elliptic surfaces are of special interest since the existence of the involution means that the CFT can be formulated as a lift of a theory defined in CP_1 to a theory at the sheets of the covering of CP_1. Therefore the generic non-hyper-ellipticity of $g > 2$ poses a difficult technical problem for CFTs and in the calculation of loop corrections in string theories.

5. In the TGD framework, the existence of the involution is also highly desirable since for H-H hypothesis the complex partonic 2-surfaces could be projected to a homologically non-trivial geodesic sphere of CP_2. The calculation of $\langle\gamma\rangle$ and of the expectation values $\langle r\rangle$ and $\langle 1/r\rangle$ discussed in the previous sections could be reduced to corresponding calculations for the sheets of the covering defined by the partonic 2-surface with $g \leq 2$. It would not be surprising if the situation remains the same in the case of super minimal surfaces with a homologically trivial geodesic sphere as a CP_2 projection.

   Involution makes it possible to understand why Z_2 symmetry implies that handles form a bound state. The involution projects all handles to CP_1 and all these handles are inverse images of the same CP_1: this implies strong correlations giving rise to a bound state of handles.

In the TGD framework the embeddings of the partonic 2-surfaces as complex submanifolds of CP_2, $E^2 \times CP_2 = C \times CP_2$ ($M^4 = M^2 \times E^2$), and of $\delta M^4_{\pm} \times CP_2$ are of special interest. $\delta M^4_{\pm}$ is metrically CP_1 as a sphere with a constant value of the light-like radial coordinate so that also the complex embeddings to $CP_1 \times CP_2$ can be considered. The basic question is whether these holomorphic embeddings exist. As noted, also superminimal surfaces can be considered. Google Gemini provides the following information.

1. Are the holomorphic embeddings of $g \geq 2$ surfaces as hyperelliptic surfaces into CP_2 possible? Surprisingly, the $g = 2$ surfaces do not allow an embedding as complex, hyperelliptic or any other kind of surface to CP_2. Even more surprisingly: $g > 2$ surfaces allow hyperelliptic embeddings to CP_2 but they represent a minority!

2. What about a holomorphic embedding into $CP_2 \times C$, where $C = E^2$ would correspond to 2-D transverse degrees of freedom in $M^4 = M^2 \times E^2$. These embeddings do not exist although there is more room available!

3. A holomorphic embedding to $CP_2 \times CP_1$ is possible. The situation actually reduces to the existence of the embedding to $CP_1 \times CP_1$, where the second CP_1 has an interpretation as a homologically non-trivial geodesic sphere of CP_2. The embeddings are defined by a curve as a root of a polynomial of bidegree as (a, b). The smoothness condition relates the bidegree (a, b) to the genus via $g = (a-1)(b-1)$ (adjunction formula). For $g = 2$ this allows only $(a, b) = (2, 3)$ or $(3, 2)$.

   The adjunction formula provides surprisingly detailed information about partonic 2-surfaces. For the spherical topology either $a = 1$ or $b = 1$ is satisfied and $a = b = 1$ is required by the condition that there is only a single copy of CP_1. For torus topology $(a, b) \in \{(2, 2)\}$ is satisfied.

Holomorphic embeddings could be realized in the zero energy ontology (ZEO) of TGD.

1. Causal diamonds (CDs) are the key geometric aspect of ZEO: they form a scale hierarchy and all systems, including elementary particles, are accompanied by a CD. A CD is analogous to a big bang followed by a big crunch and the tip of the CD is analogous to the first moment of the big bang.

2. The 2 halves of the causal diamond (CD) can be thought of as stacks of the 3-D light-like surfaces $\delta M^4_{\pm}$ identifiable as boundaries of CD ($t^2 - r^2 = 0$). $\delta M^4_{\pm}$ has a slicing by spheres CP_1 for which the light-like radial coordinate is constant. The generalized conformal transformations of $\delta M^4_{\pm}$ affecting the radial coordinate by conformal scaling give new slicings of this kind and as a special case one obtains an infinite-D group of isometries.

3. The singular line at which CP_1 contracts to a point connecting the vertices of the CD must be removed from the CD to make the slicing and the associated Hamilton-Jacobi structure [L6] non-singular.

   A Hamilton-Jacobi structure of M^4 allows CP_1 slicing for the time translations of $\delta M^4_{\pm}$, when the time-like line connecting the vertices of the causal diamond CD is removed. Therefore a holomorphic partonic 2-surface with $g = 2$ topology is possible.

4. The partonic surface would be extended in the direction of the transversal M^4 degrees of freedom and the CP_2 projection could be CP_1. The extension would be a double covering having involution as a projection map.

   The covering involution would project the 2-surface in $\delta M^4_{\pm} \times CP_1$ onto $CP_1 \subset CP_2$ holomorphically onto CP_1, which in fact must be a homologically non-trivial geodesic sphere. The covering would be $4 \to 1$ for the torus and $6 \to 1$ for $g = 2$ and have 2 *resp.* 4 branch points. For these coverings, one can even find general analytical expressions as roots of polynomials. By generalized conformal invariance, generalized conformal transformations acting as isometries of δM^4_+ define an infinite number of physically equivalent coverings in the TGD framework.

Is it possible to understand the difference between genera $g \in \{0, 1, 2\}$ and higher genera topologically? Why would the existence of involution as conformal Z_2 symmetry make bound states formed by handles possible?

1. One can start from an algebraic representation of the partonic 2-surface as cover CP_1, which corresponds to either $CP_1 \subset CP_2$ or $CP_1 \subset \delta M^4_+$. The partonic 2-surface is determined as a root of a polynomial $P(z_1, z_2)$ defined in $CP_1 \times CP_1$ and is characterized by bidegree (n_1, n_2).

2. The basic formula of algebraic geometry to be used is $g = (n_1 - 1)(n_2 - 1)$. Sphere correspond to either $n_1 = 1$ or $n_2 = 1$, torus to $(n_1, n_2) = (2, 2)$, $g = 2$ to $(n_1, n_2) = (3, 2)$ or $(n_1, n_2) = (2, 3)$ and $g = 3$ to $(n_1, n_2) = (4, 2)$ or $(n_1, n_2) = (2, 4)$.

   If Z_2 is possible as a symmetry then there is a 2-sheeted cover and the fixed points of the involution are touching points for the sheets of the same sheet pair. This should make it possible to form bound states between handles.

   The number of fixed points is $2g + 2$ and therefore $n = 2$ for a sphere, $n = 4$ for torus and $n = 6$ for $g = 2$. The handles due to the fixed points are constrictions of a cylinder splitting into two disks. This might be special feature of algebraic surfaces.

3. What happens for $g > 2$? Consider as an example $g = 3$ surface determined by a polynomial $P(z_1, z_2)$ with bidegree $(n_1, n_2) = (4, 2)$. The 2-surface is union of the roots of $P(z_1, z_2)$. The 4 roots for z_1 can be solved and they correspond to 4 regions of the 2-surface. The touching points of the regions are also roots of $P(z_1, z_2)$ with respect to z_2. There are 2 of them.

   The touching points are associated with the leaf pairs 1-2, 1-3, 1-4, 2-3, 2-4, 3-4: 6 in total corresponding to 3 handles. Only 1 touching point is associated with the same sheet pair in absence of Z_2 symmetry. If the bound state conjecture holds true, a bound state of 3 handles is possible only when the handles correspond to touching point pairs between the same pair of leaves. In the case of $g > 2$, this would require hyper-ellipticity, i.e. the existence of Z_2 symmetry.

3.2 A connection between TGD and quantum homology

Vesa Ala-Mattila has gathered highly relevant information about the connection between homology and genus for curves (2-D complex surfaces) of $CP_1 \times CP_2$ (this). Quantum cohomology is a modern generalization of cohomology theory applicable also to singular situations, such as complex surfaces with pinches and in the following some results relevant to TGD will be summarized.

Let us summarize first the basic ideas of the TGD view.

1. Both holography = holomorphy (H-H) principle and minimal Lagrangians for action consisting of Kähler action (assuming that Kähler current is light-like [L22]) and volume term give rise to a holographic dynamics in which holomorphy associated with inherent 2-D complex structure and the induced complex structure appear. Hypercomplex holomorphy appears in the 2 longitudinal degrees of freedom of M^4 and relates to the H-J structure [L6]. What might be called standard H-J structure involves slicing of M^4 by light-cone boundaries $\delta M^4_{\pm}$. Space-time surfaces can be regarded as Bohr orbits.

2. Interactions are identified as contact interactions at 2-D intersections and self-intersections giving rise to string world sheets. String world sheets intersect partonic orbits along fermion lines which in turn intersect partonic 2-surfaces at points defining the vertices. Classical non-determinism implies that fermion lines give rise to orbits of 8-D Brownian motion.

3. Another view is that interactions are mediated by monopole flux tubes connecting 3-surfaces. The monopole flux tubes have M^4 monopole/homology charge in CD associated with the fermion and CP_2 monopole/homology charge in CP_2. Therefore the description involves homology theory. The 4-D Bohr orbits of monopole flux tubes are analogs of lines of exchanged particles in Feynman diagrams.

4. δM^4_+ is metrically CP_1 since the radial light-like coordinate r is non-dynamical. This gives rise to a generalized conformal invariance. Could the fluctuations of the local value of radial coordinate of r at a partonic 2-surface generate tentacles and the touching of these tentacles essential for interactions. This picture would conform with the TGD view of the role of monopole flux tubes and with the view of their role in quantum gravitational coherence. cd:s associated with fermions are analogous to expanding spherical light fronts. The touching of the tentacles becomes possible when these spherical light fronts meet.

5. In standard algebraic geometry one studies intersections that are 2-D complex curves with various objects, points, complex curves,.... H-H is true, points as intersections of complex curves with geometric objects are replaced by string world sheets as 2-D (self-)intersections of space-time surfaces.

3.2.1 Challenges

The challenges are as follows.

1. The partonic 2-surfaces $Y^2_i \subset CP_1 \times CP_2$ can be characterized homologically. The values of M^4_+ and CP_2 homological charges/mopole fluxes as analogs of winding numbers provide this classification. A rather general representation of these surfaces as roots of pairs (P_1, P_2) of bihomogeneous polynomials in the $2 + 3$ homogeneous coordinates of $CP_1 \times CP_2$ with bidegree (a_1, b_1) and (a_2, b_2). The bi-degrees (a_1, b_1) and (a_2, b_2) define the numbers of sheets for covering over CP_1 and shadowing of CP_2.

2. One must also characterize the genera g of these surfaces. There are surprisingly strong correlations between the bi-degrees, homological invariants, and genera: Google Gemini provides information of these correlations. The 2-surfaces (complex curves) in $CP_1 \times CP_1$ are of special interest now. For a smooth curve in $CP_1 \times CP_2$ cut out by two polynomials of bi-degrees (a_1, b_1) and (a_2, b_2), the two homology charges are $b_1 b_2$, the number of sheets over CP_1, and $a_1 b_2 + a_2 b_1$, the degree in CP_2 or the number of sheets shadowing CP_2.

   The genus is given by $2g-2 = (a_1+a_2-2)b_1b_2+(b_1+b_2-3)(a_1b_2+a_2b_1)$; the pinches lower the geometric genus. With $(a_1, b_1) = (0, 1)$ in the $CP_1 \times CP_1$ case, this gives $g = (a_2-1)(b_2-1)$; with $(1, 0)$ it gives $g = (b_2 - 1)(b_2 - 2)/2$; bidegrees $(a_1, b_1) = (a_2, b_2) = (1, 1)$ give a rational curve ($g = 0$) with one sheet over CP_1 and degree 2 in CP_2.

   The book "Algebraic Geometry" by Robin Hartshorne published in Spring seriese Graduate Texts in Mathematics [A2] explains the basic formulas: (Ch. I § 7 (Bezout in the plane), Ch. V § 1 (intersection numbers on a surface, the quadric $CP_1 \times CP_1$ among its examples) and Prop. V.1.5, the adjunction formula $2g - 2 = C \cdot (C + K)$ for a smooth curve C on a surface, which on the surface $P_1 = 0$ gives your genus formula.

3. The enumerative challenge is as follows. Assume that a collection of vertices is fixed. Consider collections of partonic surfaces X^2_i and allow their homology charges to vary. How many such collections intersect partonic surfaces with fixed genera. Since the vertices appear as end points of fermion lines, this number is highly relevant for the calculation of scattering matrix elements.

4. The intersection form is the mathematical notion allowing to deduce this number for given homological invariants of X^2 as M^4_+ and CP_2 homological charges. The situation resembles

that in the topological string model pioneered by Witten. Now this model would have $CP_1 \times CP_2$ as the ambient space. The calculation of intersection numbers leads to the notions of quantum homology and homology and to Gromov–Witten invariants [A3] (see this, this and this) classifying the partonic 2-surfaces in $CP_1 \times CP_2$ and their intersections. In the M-theory framework and for topological string models of type A and B the counterparts of the partonic 2-surfaces would be as branes of various dimensions needed to satisfy Dirichlet boundary conditions for strings.

The motivation of the pioneers for identifying these invariants was the need to take into account also various limiting cases such as the development of pinches as contacts between sheets of the covering of CP_1 which change the topological genus. Their inclusion compactifies the moduli space of the curves in question. A partonic 2-surface intermediate between sphere and torus is the basic example in the TGD framework [K2].

The highly unprofessional TGD based view of Gromov–Witten invariants is discussed in [K6, K4, K5, K8]. The connection with topological string models and branes is as follows. Topological string models are reduced string models without propagating metric degrees. In the TGD framework, partonic 2-surfaces take the role of branes of string theory. The counterparts of topological strings emerge when hypercomplex holomorphy is satisfied. The second hyper-complex coordinate v identifiable as the radial coordinate r of δM^4_+ slice of cd is dynamically passive. Putting $u = constant$ effectively gives partonic 2-surfaces in $CP_1 \times CP_2$ associated with strings as coordinate lines for v. Partonic 2-surfaces with the strings associated with fermions at them are 3-dimensional. When the cd:s associated with 2 partonic 2-surfaces intersect, these 3-surfaces intersect at discrete points defining interaction vertices for fermions. One can say that partonic 2-surfaces intersect with the mediary of strings acting as tentacles.

3.2.2 What does "intersection" mean?

"What does the notion of intersection mean?" sounds like a trivial question. This is not the case in quantum homology nor in TGD if hypercomplex holomorphy is assumed.

1. Interactions are identified as contact interactions. In $D = 8$ it would consist of a discrete set point in the generic case. The intersection of 4-surfaces in H or CD define interaction regions. By H-H principle the points are however replaced by string world sheets. If the H-J structures of two surfaces are identical, they have identical hypercomplex longitudinal spaces coordinated by the dynamically active hyper complex coordinate u. The dynamically passive coordinate v does not show itself in the dynamics.

   The radial coordinate of δM^4_+ must be also passive. Otherwise there would be violation of hypercomplex conformal invariance. Therefore for H-H the intersections and also self-intersections are 2-D string world sheets. This is true also for the Lagrangian minimal surface since they satisfy hypercomplex holomorphy.

2. What does the intersection for the partonic 2-D surfaces $X^2_i \subset X^4$ mean? Usually it would consist of points in the generic case. Hypercomplex holomorphy implies that points are effectively replaced with strings defined by the coordinate lines of v. This implies a non-locality as a correlate for quantum non-locality.

   Two partonic 2-surfaces intersect if they are connected by a string. Two partonic orbits intersect if they are connected by a string world sheet. This view is adopted also in quantum cohomology.

3. Consider partonic 2-surfaces $X^2_i \subset X^4$ and string worlds sheets $Y^2_j \subset X^4$. A family of homologically non-trivial 2-surfaces X^2 labelled by the radial light-cone coordinate r of δM^4_+ is determined as roots of pairs of homogeneous polynomials f_1 and f_2 of complex coordinates variables of $CP_1 \times CP_2$, which do not depend on the radial coordinate r at all. The active coordinate u appears as a parameter.

   One can solve the partonic 2-surfaces in $CP_1 \times CP_2$ as roots of analytic functions f_i (homogeneous polynomials algebraic geometry) and obtain partonic 2-surfaces as complex curves.

Since r is a passive coordinate, these surfaces are the same for all values of r and their unions define 3-D surfaces Y_i^3.

Partonic 2-surfaces X_i^2 do not in general intersect as 2-surfaces of $CP_1 \times CP_2$. The associated 3-surfaces Y_i^3 however can intersect and this occurs if the light fronts δM_+^4, that is CDs associated with them, meet. The intersections consist of points. The notion of intersection is thus non-local and intersections correspond to strings connecting the partonic 2-surfaces inside the same CD or associated with different CDs.

4. Also 2 3-D partonic orbits intersect in the intersection of corresponding cd:s. The 4-D orbits of the M^4 monopole flux tubes have a part of the string world sheet as intersection. These connecting space-time regions would be orbits of M^4 monopole flux tubes. This conforms with the vision that formation of flux tube connections is essential for the interactions utilized also in TGD inspired quantum biology.

5. Fermion lines as intersections of string world sheets and partonic orbits and orbits of the intersection points are proposed to be light-like geodesics of H giving rise to warping and reduced light-velocity $c_\#$ at the level. The boundaries of modified cd are light-like geodesic of H and cd as the light-front expanding with velocity $c_\# < c$ rotates along a geodesics of CP_2. Also fermion lines having separate CDs can be said to intersect if they are connected by a string. This would correspond to classical description of interactions as exchange of particles with light-like H-momenta.

In quantum homology the intersection form coding the numbers of homologically non-trivial surfaces intersecting give surfaces at given points codes. Since the intersection points correspond to vertices serving as endpoints of fermion propagators, the computation of the intersection form is the first step in the computation of scattering amplitudes.

3.2.3 Challenge: find the space-time surfaces as Bohr orbits containing given set of interaction vertices

The calculation of scattering amplitudes requires finding the algebraic surfaces for various values of u and if H-H holds true, the problem reduces to a problem of algebraic geometry.

1. Assume 2-surface X^2 and partonic 2-surfaces $Y_i^2 \subset \delta M_+^4 \times CP_2$. Partonic 2-surfaces contain vertices at which fermion lines have an edge. Fermions can also turn back in time direction [L11, L4, L13]: this occurs in pair creation or annihilation.

2. Assume that the intersection points $x_{i,j}$ of X^2 with partonic 2-surfaces Y_i^2 are known. Also Y_i^2 can have intersection points. These points define the possible end points of propagator lines of fermion.

3. Assume that these 2-surfaces are given as roots of two homogeneous polynomials in $CP_1 \subset \delta M_+^4$ and in CP_2. For $CP_1 \subset \delta M_+^4$ local radial deformations of light-like coordinates act as gauge symmetries and define Virasoro algebra. This makes possible the radial extrusions/tentacles making possible interactions, in particular gravitational interactions. Conformal invariance implies that the wave function is scale invariance as a function of the light-like coordinate r of δM_+^4.

3.2.4 How do topological string models generalize in the TGD framework?

1. One can regard string world sheets as analogs of the intersection points and as a purely algebraic phenomenon happening automatically in H-H picture once the functions $f_1(w, \xi^1, \xi^2, u)$ and $f_2(w, \xi^1, \xi^2, u)$ are known. u is the light-like coordinate of H varying along a partonic orbit. For a fixed u the situation would reduce to algebraic geometry with u appearing as a parameter in the polynomials involved.

2. There are however differences. In TGD, one has 4-D surfaces and hypercomplex holomorphy so that the functions $f_1(w, \xi^1, \xi^2, u)$ and $f_2(w, \xi^1, \xi^2, u)$ determining space-time surface as

their roots. The second light-like coordinate v interpreted as the hypercomplex conjugate of u does not appear as argument at all and is dynamically passive. For a fixed value u, the dynamics reduces effectively to that in the space with complex coordinates w, ξ^1, ξ^2 and for "standard" H-J structure w is the complex coordinate of $CP_1 \subset \delta M^4_+$.

3. The presence δM^4_+ might make the generalization non-trivial. Is w just the spherical complex coordinate without any scale or is it proportional to the distance r from the tip of the CD of fermion defining the radius of CP_1?

   One can argue that the radial light-like coordinate r is analogous to the coordinate passive coordinate v. r is also a dimensional parameter: its presence would mean the breaking of conformal invariance. The metric of the light-cone boundary is effectively 2-D implying that radial conformal transformations act as gauge symmetries. Can one fix the value as function of S^2 complex coordinate as a gauge choice or do the Virasoro conditions in radial direction imply scaling invariance in radial direction and fix the wave function for r to a constant? The latter option looks the physical one. If this is the case, the reduction to standard algebraic geometry would take place for a given value of u. The coordinate u appears as a parameter on which the polynomials defining the partonic 2-surfaces depend on and the algebraic geometry becomes dynamical. The situations in which CDs intersect could involve classical non-determinism.

3.2.5 The dynamical aspect of algebraic geometry

What could the dynamical aspect of algebraic geometry mean classically and quantally?

1. It would be nice to have a connection with the description of interactions in terms of classical signals/particles propagating with finite velocity $v < c$, say with light-velocity c or $c_{\#} < c$. The first idea is that the tentacles of partonic 2-surfaces develop in the dynamics given by the dependence on u. This cannot be the case if r corresponds to the passive coordinate v.

2. There is however an inherent dynamics associated with cd:s as analogous to expanding spherical light fronts. Contact interactions between the partonic 2-surface appear as these light fronts associated with the fermions intersect. For certain critical values of u cd:s would intersect and partonic 2-surfaces in the superposition would fuse together. The amplitude for this would be determined by the scaling invariant wave functions in the radial coordinate r.

3. The term "tentacle" and biological interpretation induce the association that free will is involved with state function reductions (SFRs), which come in two varieties "small" SFRs (SSFRs) and "big" SFRs (BSFR). Could the extrusion emerge instantaneously in SSFR or BSFR instead of a classical dynamical process? ZEO is forced by the classical non-determinism having as loci the partonic 2-surfaces identified also as generalized vertices. SSFRs would be made possible by classical non-determinism. It seems that the SSR would naturally occur when the cds associated with two partonic 2-surfaces (fermions) as analogous to classical light fronts intersect.

3.3 Are the minimal Lagrangian surfaces needed in TGD?

In the section "Should also Lagrangian mimimal surfaces be allowed?" it was found that are arguments supporting the view that H-H vision [L10, L14] is not enough. Also Lagrangian minimal surfaces are possible as extremals of action which sum of Kähler action and volume term and there are arguments suggesting that they are needed: I have discussed this possibility in [L12]. For instance, Maxwellian magnetic fields could be associated with the Maxwellian flux tubes without monopole flux. They would have vanishing induced Kähler form but non-vanishing CP_2 curvature R_{12} giving rise to non-vanishing electromagnetic field when the Weinberg angle is non-vanishing. One can aqlso considerthe possibility that Maxwellian flux tubes have boundaries. A coil would represent the physical situation in which this is the case. This is so because the boundary conditions requiring vanishing magnetic flux through the boundary can be satisfied.

As was found, a quite general feature of complex sub-manifolds produced by H-H vision is that the induced W gauge potentials and fields vanish (as already found in the main tex) : this is due to the functional dependence of the complex coordinates ξ^1 and ξ^2 of CP_2. For the Lagrangian minimal surface as a partonic 2-surface defining the vertex however, W gauge potentials can be non-vanishing and would make possible non-trivial W propagators.

A very large set of extremals of Kähler action plus volume term predicted by the twistor lift would be of form $X^4 = X^2 \times Y^2$, where X^2 is string world sheet in M^4 consistent with H-J structure [L6] and degenerate transversal Kähler form and Y^2 is Lagrangian minimal surface in CP_2. The second kind of Lagrangian surface would be of the form $X^4 = X^1 \times Y^3$, where Y^3 is a Lagrangian minimal surface in $W^2 \times CP_2$, where W^2 is the transversal space associated with the H-J structure.

For these reasons it is interesting to look what is known about the minimal Lagrangian surfaces of CP_2: I have considered this problem already earlier [L12] but thanks to Google Gemini it was easy to gain new important information summarized in the sequel. An especially interesting question is whether the H-H principle of generalizes for Lagrangian minimal surfaces. The construction method as a solution of initial value problem suggests that this is the case.

3.3.1 Some examples of superminimal surfaces

Consider first some examples of superminimal surfaces.

1. Clifford torus is the standard embedded Lagrangian torus in $\mathbb{C}P^2$, which is Hamiltonian stable and totally real. Clifford torus corresponds to the surface $|\xi_1|^2 = |\xi_2|^2 = 1$. In Eguchi-Hanson coordinates $(\xi^1, \xi^2) = (rcos(\Theta/2)exp(i(\Psi + \Phi)), rsin(\Theta/2)exp(i(\Psi - \Phi)))$ [L8] this condition gives $\Theta = \pi/2$ and $r = \sqrt{2}$.

2. Real projective plane ($\mathbb{R}P^2$): The totally geodesic Lagrangian embedding inside $\mathbb{C}P^2$. It is obtained by assuming that ξ^1 and ξ^2 are real.

3. Equivariant surfaces have one parameter symmetry group and are divided into translationally and rotationally equivariant families, which serve as the foundational building blocks for more complex configurations.

3.3.2 Construction methods for Lagrangian minimal 2-surfaces in CP_2

There are several construction methods for Lagrangian minimal 2-surfaces in CP_2. I am not a specialists and the following summarizes what one can learn from Google Gemini.

1. Lagrangian minimal surfaces in $\mathbb{C}P^2$ lift via the Hopf fibration $S^5 \to \mathbb{C}P^2$ to 2-D Legendrian surfaces in S^5.

2. Loop group methods involve complexified loop group $SU(3)$. In TGD framework the interpretation is in terms of color symmetries. This loop group appears also in the construction 4-surfaces at the level of M^8 mapped by $M^8 - H$ duality to space-time surfaces in H [L7, L24, L17, L20]. The complexification is with respect to the inherent complex structure of the 2-D surface determined by its induced metric rather than due to inherited complex structure implied by complex manifold property as in the case of H-H vision. Integrable systems techniques (such as loop groups like $\Lambda SU(3)_\sigma$) are heavily utilized to construct and parameterize general minimal Lagrangian cylinders and higher-genus surfaces from spectral data.

   In the TGD framework Lagrangian cylinder would naturally correspond to the 3-D partonic orbit and the physically motivated identification is an orbit of partonic 2-surface along light-like geodesic of H.

3. The Gauss–Codazzi equations for these surfaces often reduce to elliptic sinh-Gordon or Toda-type differential equations known as Tzitzeica/Toda Equations.

The loop group construction method (also known as the DPW method or integrable systems approach) integrates loop group theory with complex geometry. It constructs minimal Lagrangian

surfaces in $\mathbb{C}P^2$ by solving a pair of linear differential equations. This transforms a complex nonlinear PDE (the Gauss–Codazzi equations) into a linear system.

1. The underlying Lax Pair is a central notion. It is possible to assign a Lax pair to some highly non-linear differential equations.

   (a) The Lax pair consists of operators A and B analogous to t- and y- components of a gauge potential. For integral systems, the original non-linear equations reduce to differential equations $D_t\Psi = (\partial_t + A)\Psi = 0$ and $D_x\Psi = (\partial_x + B)\Psi = 0$.

   (b) The integrability condition $\partial^2/\partial_t\partial_x\Psi = \partial^2/\partial_x\partial_t$ is equivalent to the vanishing of $[D_t, D_x]\Psi$ implying that the analog of gauge field defined by $F = \partial_t B - \partial_x A + [A, B]$ vanishes. The interpretation is as vanishing of the curvature associated with the gauge potential. One can also consider eigenstates for which $\partial_x\Psi = -B\Psi = \lambda\Psi$ so that one has $(\partial_t + A)\Psi = 0$. It is of crucial importance that the time evolution for Ψ preserves the spectrum.

   (c) For an integrable system, one can derive the scattering potential V at the initial moment from the scattering data $u(0, x)$ given by transmission and reflection coefficients. From this data one can solve the initial value problem by deducing the time development of the scattering data and after that perform inverse scattering method (Gelfand–Levitan–Marchenko integral equation).

2. A surface in $\mathbb{C}P^2$ is framed by moving frames in the special unitary group $SU(3)$. The minimality and Lagrangian conditions require the frame map $F : U \to SU(3)$ to satisfy a zero-curvature condition (Lax pair):$\mathbf{d}F = F\alpha(\lambda)$.

   Here, $\lambda \in S^1 \subset \mathbb{C}$ is a spectral parameter. The $\mathfrak{su}(3)$-valued 1-form $\alpha(\lambda)$ is a loop of $SU(3)$ connections decomposed by the $\mathbb{Z}_3$-automorphism of $\mathfrak{su}(3)$:$\alpha(\lambda) = \lambda^{-1}\Phi + \alpha_0 + \lambda\Phi^*\Phi$ is a holomorphic $\mathfrak{sl}(3, \mathbb{C})$-valued (1,0)-form called the holomorphic cubic differential. α_0 represents the Levi-Civita connection of the surface. Note that the holomorphicity means that $SU(3)$ complexified.

   In the TGD framework, an attractive physical interpretation is in terms of $SU(3)$ color gauge potentials which do not couple to spinor fields in TGD. The connection is defined in complexified group $SU(3, C)$ and the integrablity states that it is flat. This does not however imply the vanishing of the real part of the connection so that physical color gauge potentials would be well-defined.

   In TGD, the expectation is that λ corresponds to the coordinate of geodesic circle S^1 in CP_2. The construction would give the partonic orbit as a motion of the Lagrangian surface along the light-like geodesic of H.

3. The Dorfmeister–Pedit–Wu (DPW) method generates these surfaces systematically from basic holomorphic data through four primary steps:

   (a) Select a holomorphic form: Choose a domain Σ and a holomorphic 1-form $\eta(z, \lambda) = \lambda^{-1}\Phi(z)dz$ taking values in the loop algebra. This is the holomorphic data. Here holomorphy is with respect to the complex structure defined by the induced metric. For H-H one has complex manifold and the holomorphy has different interpretation: complex coordinate for X^2 is a complex coordinate of H.

   (b) Solve the complex ODE: Solve the linear ordinary differential equation $\mathbf{d}\phi = \phi\eta(z, \lambda)$ for a holomorphic loop group frame $\phi(z, \lambda)$.

   (c) Perform the loop group factorization (Iwasawa decomposition): The complex loop group element $\phi(z, \lambda)$ is factored into a unitary part F and a solvable/positive part B:$\phi(z, \lambda) = F(z, \overline{z}, \lambda) \cdot B(z, \overline{z}, \lambda)$, where one has $F \in \Lambda(SU(3))$ and B extends holomorphically to the interior of the unit disk.

   (d) Project via Hopf fibration: Evaluate the unitary frame F at a specific value (typically $\lambda = 1$). The projection of the first column of $F(z, \overline{z}, 1)$ from $S^5 \to \mathbb{C}P^2$ yields the coordinate vectors of the minimal Lagrangian surface.

   In TGD, the union over 2-surfaces with different values of λ should give the light-like orbit of the partonic 2-surface along H geodesic.

4. There is a profound connection with soliton theory. When applying this construction, the underlying geometry forces the metrics to satisfy integrable scalar equations. The compatibility condition of the Lax pair reduces to the affine Toda field equations for $\mathfrak{sl}(3,\mathbb{C})$. For specific geometric symmetries, this simplifies further into the Tzitzeica equation ($u_{z,\overline{z}} = e^u - e^{-2u}$), which natively governs minimal Lagrangian surfaces with specialized transformation groups.

The Iwasawa factorization (or loop group Birkhoff/Iwasawa decomposition) is the crucial non-linear step in the DPW method. It splits a complex, purely holomorphic loop into a unitary loop (which yields the geometry) and a positive/solvable loop (which contains the metric information).This process directly solves the non-linear Gauss–Codazzi equations through a algebraic splitting mechanism.

1. To construct a minimal Lagrangian surface, we define specific loop groups based on the $\mathbb{Z}_3$-symmetry of $\mathfrak{su}(3)$. Let $\sigma : \mathrm{SL}(3,\mathbb{C}) \to \mathrm{SL}(3,\mathbb{C})$ be the automorphism of order 3.
   (a) $\Lambda\mathrm{SL}(3,\mathbb{C})_\sigma$: The complex loop group of maps $\gamma : S^1 \to \mathrm{SL}(3,\mathbb{C})$ satisfying the twisting condition $\sigma(\gamma(\lambda)) = \gamma(e^{2\pi i/3}\lambda)$.
   (b) $\Lambda\mathrm{SU}(3)_\sigma$: The unitary subgroup consisting of loops where $\gamma(\lambda) \in \mathrm{SU}(3)$ for all $\lambda \in S^1$.
   (c) $\Lambda^+\mathrm{SL}(3,\mathbb{C})_\sigma$: The subgroup of twisted loops that extend holomorphically to the interior of the unit disk ($|\lambda| < 1$), with the identity map or a specific upper-triangular structure at $\lambda = 0$.

2. The factorization theorem

   Let $\phi(z,\lambda)$ be the holomorphic loop frame obtained by integrating the holomorphic data η. For each point z in the domain, $\phi(z,\lambda)$ is an element of $\Lambda\mathrm{SL}(3,\mathbb{C})_\sigma$.

   The loop Iwasawa decomposition states that the complex loop group factorizes uniquely as a product: $\phi(z,\lambda) = F(z,\overline{z},\lambda) \cdot B(z,\overline{z},\lambda)$ the components belong to the following spaces: $F(z,\overline{z},\lambda) \in \Lambda\mathrm{SU}(3)_\sigma$ (the real, unitary loop frame) and $B(z,\overline{z},\lambda) \in \Lambda^+\mathrm{SL}(3,\mathbb{C})_\sigma$ (the complex, positive/solvable loop). Because ϕ depends holomorphically on z, but the factorization separates unitary and non-unitary parts, the resulting factors F and B depend smoothly on both z and $\overline{z}$.

3. Extraction of the metric and surface geometry

   Once the factorization is performed, the properties of the minimal Lagrangian surface are fully unlocked by looking at how F and B evolve.

   The metric momponent (B)at $\lambda = 0$, the positive loop $B(z,\overline{z},\overline{z},0)$ takes values in a solvable subgroup (Borel subgroup) of $\mathrm{SL}(3,\mathbb{C})$. For minimal Lagrangian surfaces, it is expressed as a diagonal matrix containing the conformal factor of the metric: $B(z,\overline{z},0) = \begin{pmatrix} e^{u_1} & 0 & 0 \\ 0 & e^{u_2} & 0 \\ 0 & 0 & e^{u_3} \end{pmatrix}$ where $u_1 + u_2 + u_3 = 0$. The functions $u_i(z,\overline{z})$ are exactly the smooth scalar fields that solve the affine Toda equations.

   The unitary loop $F(z,\overline{z},\lambda)$ acts as the generalized moving frame. To project this down to $\mathbb{C}P^2$ evaluate the frame at $\lambda = 1$, giving a smooth map $F(z,\overline{z},1) \in \mathrm{SU}(3)$. Take the first column vector of this matrix, $f_1(z,\overline{z}) \in S^5 \subset \mathbb{C}^3$.Project via the Hopf fibration $\pi : S^5 \to \mathbb{C}P^2$. The immersion $[f_1(z,\overline{z})]$ is guaranteed to be a minimal Lagrangian surface.

4. While solving the initial ODE for ϕ is standard linear complex analysis, the Iwasawa factorization is highly non-linear. It cannot be written explicitly in closed form for general holomorphic data. In practice, computing F and B explicitly relies on solving a Riemann–Hilbert problem or using numerical methods. For specialized setups —such as homogeneous surfaces or surfaces of constant curvature — the factorization reduces to algebraic operations, making explicit global constructions possible.

Can one assign holography to minimal Lagrangian surfaces? It seems that the loop space method assigns a holography to the 3-surface of CP_2 defined by the motion of partonic 2-surface along the light-like geodesic of H. The initial data correspond to the partonic surface with $\lambda = 1$ and the slices $\lambda \geq 1$ define the orbit.

3.3.3 Equivariant Lagrangian minimal surfaces in CP_2

Equivariant Lagrangian minimal surfaces in $\mathbb{CP}^2$ are Lagrangian minimal surfaces, which possess continuous symmetry group actions. In the complex projective plane $\mathbb{CP}^2$ these surfaces are deeply connected to integrable systems, specifically through the DPW (Dorfmeister–Pedit–Wu) loop group method already discussed.

1. Every local Lagrangian surface in $\mathbb{CP}^2$ with a conformal metric $g = 2e^u dzd\bar{z}$ and a cubic differential $\Psi = \psi dz^3$ is governed by structural equations. When the Lagrangian surface is minimal, the cubic differential Ψ is strictly holomorphic ($\partial_{\bar{z}}\psi = 0$). Therefore the holomorphy is present also now albeit in different way as in the case of H-H principle.

   The metric profile u satisfies the generalized Toda lattice equation (also structurally related to the $\mathbb{Z}_3$-elliptic Tzitzeica/sinh-Gordon equation):$u_{z\bar{z}} = e^u - e^{-2u}$.

   For equivariant surfaces, the symmetry constraints dramatically reduce this complex 2D PDE into standard ordinary differential equations (ODEs), allowing explicit classification and integration.

2. By examining the 1-parameter subgroups of PU(3) acting on $\mathbb{CP}^2$, equivariant minimal Lagrangian surfaces are classified into two distinct geometric categories:

   (a) Translationally equivariant surfaces are invariant under a 1-parameter group acting by translation-like transformations. These surface provide the foundational building blocks for more complex topologies. The partial differential equation collapses down directly to an ODE in terms of a single real parameter. The geometric properties and invariants of these surfaces can be explicitly parameterized using Weierstrass elliptic functions.

   (b) Rotationally equivariant surfaces are surfaces invariant under a 1-parameter group of rotational symmetries. A core classification theorem states that any rotationally equivariant minimal Lagrangian surface is either a standard canonical model (like the Clifford torus) or arises via a 1-to-1 correspondence with periodic translationally equivariant surfaces (specifically, 2π-periodic immersions).

3. Global geometric bounds strictly dictate how these surfaces exist globally in $\mathbb{CP}^2$ based on their topology:

   (a) Any minimal Lagrangian surface of genus zero in $\mathbb{CP}^2$ is strictly totally geodesic. It is unique up to congruence and corresponds to the standard embedding of $\mathbb{RP}^2$ or a totally geodesic $\mathbb{S}^2$.

   (b) The Clifford torus is the unique flat, minimal Lagrangian torus that exhibits full $S^1 \times S^1$ invariance. By perturbing the loops using the DPW method, mathematicians construct Delaunay-type minimal Lagrangian cylinders that model periodic structures.

   (c) While simple equivariance under a 1-parameter group yields tori or open cylinders, modern research utilizes discrete group representations (such as surface group representations into $\mathrm{SU}(2,1)$ or $\mathrm{SL}(3,\mathbb{C})$) to generate equivariant minimal Lagrangian surfaces of high genus.

3.3.4 Inverse scattering transform

The inverse scattering transform (IST) is a powerful mathematical method used to solve specific nonlinear partial differential equations (PDEs), much like how the Fourier Transform is used to solve linear PDEs. It allows mathematicians and physicists to solve notoriously difficult nonlinear wave equations—such as the Korteweg–de Vries (KdV) equation or the Nonlinear Schrödinger equation—by transforming them into simpler, linear problems.

(a) To understand the IST, it is helpful to look at how we solve a linear wave equation using a standard Fourier Transform:

   i. Take a complicated initial wave profile in physical space and break it down into its individual frequencies (Fourier components).

ii. Let those frequencies evolve over time. In a linear system, each frequency evolves independently and trivially.

iii. Use the Inverse Fourier Transform to stitch those evolved frequencies back together into the final physical wave profile.

The Inverse Scattering Transform follows this exact three-step architecture, but it adapts the process specifically for nonlinear waves.

(b) How the IST Works (Step-by-Step)

Instead of frequencies, the IST views the initial nonlinear wave profile as a scattering potential in a quantum mechanics-like problem.

i. The Direct Scattering Problem (Time = 0)What you have: An initial nonlinear wave profile, $u(x, 0)$. This wave profile is treated as a localized physical potential barrier or well. An imaginary quantum particle (or wave) is projected at it.

ii. You calculate how much of that particle's wave function gets reflected, how much gets transmitted, and whether any part of it gets trapped as a "bound state" (an electron caught in a well). This data — bound states, reflection coefficients, and transmission coefficients — is called the scattering data.

(c) While the actual physical wave $u(x, t)$ is morphing in a highly complicated, nonlinear way, the scattering data evolves linearly and independently for integrable systems. Here the Lax pair or zero-curvature condition comes in. Because the system is isospectral, the bound state energies do not change at all over time, and the reflection/transmission coefficients evolve via simple linear differential equations. Calculating the scattering data at a later time t becomes trivial.

(d) The linear time evolution gives scattering data at time t. The question is "What shape must a potential barrier equivalent with the solution of the non-linear problem could produce this scattering data?"

Solving this inverse problem requires solving a linear integral equation (known as the Marchenko or Gelfand–Levitan–Marchenko (GLM) equation). Solving it directly yields the exact shape of the nonlinear wave $u(x, t)$ at time t.

4. The Inverse Scattering Transform explains one of the most famous phenomena in modern physics: solitons (stable, localized, non-dispersive wave packets that behave like particles).When you break down an initial wave using the IST, the scattering data splits into two distinct mathematical parts:

(a) The continuous spectrum isrepresents the background radiation or ripples. Over time, these linear-like waves disperse, fade away, and separate from each other.

(b) The discrete spectrum (bound states) represents the solitons. Because the bound state energies are invariant under the Lax equation, these wave packets retain their energy, amplitude, and shape forever. If two solitons collide, they pass right through each other, suffering only a minor shift in position (phase shift), mimicking the behavior of colliding physical particles.

3.3.5 Cubic differential

Also the notion of cubic differential is involved. A cubic differential is a symmetric tensor or a holomorphic section of the third power of the canonical bundle on a Riemann surface, locally written as $q(z)dz^3$. It appears in advanced differential geometry, Teichmüller theory, and the study of affine spheres and projective structures.

1. A holomorphic cubic differential on a Riemann surface S locally takes the form $q(z)dz^3$, where $q(z)$ is a holomorphic function, transforming under coordinate changes $z \to w(z)$ with a $(dw/dz)^3$ factor.

Cubic differential arises naturally as the Pick form or components of the third fundamental form in affine differential geometry and minimal Lagrangian surfaces. Cubic differential is

used to parameterize geometric structures, such as convex projective manifolds and certain polygon configurations in the projective plane.

Cubic differential differs from cubic differential equations (polynomial ODEs with degree-3 non-linearities), though both deal with cubic scaling behavior.

Cubic differential relates to the geometry of the surfaces via so called third fundamental form which measures the rate for the change of the second fundamental form defining a normal of the surface whose trace vanishes for minimal surfaces.

If S is a regular surface in $\mathbb{R}^3$ and N is its unit normal vector field, the third fundamental form evaluated on two tangent vectors u and v is the dot product of their differentials with respect to the normal map: $\mathrm{III}(u,v) = \langle dN(u), dN(v)\rangle$. In terms of the shape operator (or Weingarten map) $A_p(u) = -dN(u)$, it can be written as:$\mathrm{III}(u,v) = \langle A(u), A(v)\rangle$.

2. The third fundamental form is completely determined by the First Fundamental Form (I, which measures metric/distance) and the Second Fundamental Form (II, which measures curvature). They are bound together by a linear identity:$\mathrm{III} - 2H\mathrm{II} + K\mathrm{I} = 0$. H is the Mean Curvature of the surface an vanishes for minimal surfaces.K is the Gaussian Curvature of the surface.

   If I, II, and III are represented by matrices G, L, and M respectively, the relationship can be expressed using the shape operator matrix $A = G^{-1}L$:$M = LG^{-1}L$.

3. For minimal surfaces ($H = 0$), the identity simplifies to $\mathrm{III} = -K\mathrm{I}$. This means the third fundamental form is a conformally scaled version of the metric.

3.3.6 Could the inverse scattering method serve as a paradigm in the TGD framework?

The inverse scattering method applies to completely integrable systems and there are excellent reasons to believe that TGD is such a system. The H-H principle reduces the field equations to local algebraic equations. There is however a problem. How to relate these nice algebraic surfaces to concrete physics in a more explicit way. In the case of super-minimal surfaces, one can use scattering data. Could this idea generalize?

The space-time edges at which the reduced value of light velocity $c_\#$ for particle mass changes, partonic orbits representing particle dynamics, partonic 2-surfaces, fermion lines as intersections of string world sheets as self-intersections of space-time surfaces for space-time surfaces satisfying H-H principle provide the needed data geometric classical data. In particular, reduced light velocity $c_\#$ is associated with warping distinguishing between GRT and TGD [L23, L19] and it is possible to speak about reflection and transmission of fermions from the edges and partonic orbits and partonic 2-surfaces.

Scattering provides the information, which should fix the holographic data. Scattering in the simplest situation involves reflection and transmission. Could the inverse scattering method generalize to a kind of paradigm behind the conjecture of complete integrability of TGD? The intersections of the space-time edges and partonic orbits define partonic 2-surfaces. Spinor modes reflect and are transmitted at the edges. Could this information provide the scattering data connection TGD to experimental reality at a general level.

3.4 About exceptional unitary representations of $SL(2,C)$

Principal series of unitary representations (PSUR) and exceptional series of unitary representations (ESUR) [A1] are reasonals candidates concerning the identification of wave functions for the position of fermion vertex at partonic 2-surface appearing in the condition $\langle\gamma\rangle = 0$ proposed to determine the value of the Weinberg angle. Google Gemini gives a nice summary of these representations and the following adds to this summary the TGD point of view.

3.4.1 The two Casimir operators

The Lie algebra $\mathfrak{so}(1,3)$ consists of three rotation generators J_i and three boost generators K_i. There are two independent Casimir invariants. $C_1 = \mathbf{J}^2 - \mathbf{K}^2$ is a generalization of the angular

momentum squared by including also the infinitesimal generators of hyperbolic rotations (Lorentz boosts). $C_2 = \mathbf{J}\cdot\mathbf{K}$ is a helicity-like invariant and measures the projection of the boost generators along the axis of rotation.

1. Unitary irreducible representations are traditionally labeled by two complex numbers (j_0, c) or (ν, ρ), where j_0 is an integer or half-integer representing the lowest spin in the representation, and c (or ρ) is a continuous complex parameter. When acting on a specific irreducible representation space, the Casimir operators evaluate to the following scalar eigenvalues:$C_1 = j_0^2 + c^2 - 1$ and $C_2 = -j_0 c$.

2. For a representation to be unitary, the eigenvalues of the Casimir operators must restrict to specific real domains analyzed by Bargmann:

   (a) Principal Series representations (PSUR) are labeled by $j_0 \in \mathbb{Z}/2$ and $c = i\rho$, $\rho \in \mathbb{R}$.

$$C_1 = j_0^2 - \rho^2 - 1 \ , \quad C_2 = -ij_0\rho \ . \tag{3.1}$$

   (b) Supplementary (Exceptional) Series representations (ESUR) are labeled by $j_0 = 0$ and $c \in \mathbb{R}$ within the strict open interval $0 < |c| < 1$.$C_1 = c^2 - 1$ (where $-1 < C_1 < 0$), $C_2 = 0$

Both these representation series are possible but ESUR is of special interest.

3.4.2 Inner product is bilocal for the ESUR

For ESUR the local inner product is not unitary. To see this, consider the representation space realized as functions $f(\xi)$ on the cone or the celestial sphere CP_1 (where ξ parameterizes the directions of light-like vectors).

Under a Lorentz transformation Λ, a wave function transforms as:$[U(\Lambda)f](\xi) = |\Lambda^{-1}\xi|^{-(1+c)} f(\Lambda^{-1}\xi)$If we try to define the standard local L^2 inner product:$\langle f_1, f_2\rangle_{\text{local}} = \int \overline{f_1(\xi)} f_2(\xi)\, d\mu(\xi)$

Checking for invariance under a boost reveals that the Jacobian does not cancel out because c is real ($c \in \mathbb{R}$). To force the representation to be unitary, the transformation must preserve the inner product, requiring a radical structural shift in how distances between wavefunctions are calculated.

To construct a truly invariant, positive-definite inner product for the exceptional series, one must introduce a continuous, non-local kernel $K_c(\xi, \eta)$ to define inner product. Instead of multiplying the fields at the exact same point ξ (local interaction), the metric cross-correlates the value of the wavefunction at point ξ with its value at a completely different point η. The physical meaning of the bilocality is that states in the exceptional representation cannot be interpreted as tightly localized, point-like physical particles.

For the Lorentz group $SO(1,3)$, this kernel is inversely proportional to the geometric distance between the two points on the projective space CP_1: $K_c(\xi,\eta) = \frac{1}{|\xi-\eta|^{2(1-c)}}$. The kernel $K_c(\xi,\eta)$ functions as a Fourier-like intertwining operator. When transformed into momentum space or harmonic modes, its eigenvalues are proportional to gamma functions. For $0 < c < 1$, all the eigenvalues of this bilocal kernel are strictly positive, defining a valid Hilbert space.

3.4.3 How does the parameter c relate to the central extension of conformal algebra?

The global $SL(2,\mathbb{C})$ subalgebra can be identified as the global $\{L_{-1}, L_0, L_1\}$ sub-algebra of conformal (Virasoro) algebra so that there is a natural connection to the conformal symmetries. The central extension term vanishes identically ($m^3 - m = 0$ for $m = -1, 0, 1$). Thus, the Lorentz parameter c acts on a subspace where the central extension is hidden, but c determines how states are "raised and lowered" into the broader Virasoro tower where c_{Vir} becomes visible.

There indeed is a connection to conformal field theories (CFT)s: The exceptional representations are identical to the state spaces of so called generalized free fields, which have conformal central extension parameter $c_{Vir} > 1$ whereas for $c_{Vir} < 1$ one has series of minimal CFTs with finite number of primary fields. For $c_{Vir} > 1$ the number of primary fields is infinite.

The bilocal kernel $K_c(\xi,\eta)$ is exactly the two-point correlation function $\langle \mathcal{O}(\xi)\mathcal{O}(\eta)\rangle$ of a conformal primary operator of anomalous dimension $\Delta = 1-c$.

The Lorentz parameter c dictates the conformal dimension (weight) of an operator, while the central extension $c_{\rm Vir}$ dictated by the algebra can be said to count the total number vacuum degrees of freedom of the quantum system.

1. In the exceptional representation of $SL(2,\mathbb{C})$, the states are characterized by a single non-vanishing Casimir invariant $C_1 = c^2 - 1$ (with $0 < c < 1$). In CFT, the eigenvalue of the dilatation generator $D = (L_0 + \bar{L}_0)$ is the conformal dimension Δ. For a primary state created by an operator $\mathcal{O}(\xi)$, the $SL(2,\mathbb{C})$ quadratic Casimir maps and the conformal weight are related by the equations

$$C_1 = \Delta(\Delta-2) \ , \quad c^2 - 1 = \Delta(\Delta-2) \ , \quad \Delta = 1 \pm c \ . \tag{3.2}$$

   c is restricted to the exceptional window $0 < c < 1$ and the conformal dimension is restricted by the conditions $0 < \Delta < 2$ and $\Delta \neq 1$. Therefore the exceptional representation to so called anomalous or generalized free fields in the underlying strongly coupled CFT.

2. Both c $c_{\rm Vir}$ act as gatekeepers of unitarity through two different constraints.

   (a) Lorentz Parameter $c0 < c < 1$ eliminates negative-norm states within the global $SL(2,\mathbb{C})$ primary space.

   (b) Central Extension $c_{\rm Vir}c_{\rm Vir} \geq 1$ (or discrete minimal models) eliminates negative-norm "descendant" states ($L_{-2}|\Delta\rangle$, etc.) created by the Virasoro algebra. If you try to embed the exceptional representation into a full Virasoro algebra, the bilocal metric fixes the primary state's norm, but the central extension $c_{\rm Vir}$ dictates whether the rest of the infinite tower has positive norms.

3. In a fully interacting quantum conformal field theory (like Liouville field theory), the central extension $c_{\rm Vir}$ and the exceptional conformal dimensions $\Delta = 1 \pm c$ are unified via a background charge parameter Q:$c_{\rm Vir} = 1 + 6Q^2$ where $Q = b + \frac{1}{b}$.

4. $c_{Vir} < 1$ corresponds to discrete "minimal models" (like the Ising model at $c_{Vir} = 1/2$). For $c_{Vir} > 1$, the theory exits this discrete classification, typically representing continuous families of theories like multiple free bosons (e.g., N compactified free scalar fields give $c_{Vir} > 1 = N > 1$).

   For $c_{Vir} \geq 1$, consistent unitary representations of the Virasoro algebra exist for all conformal weights $h \geq 0$, meaning there are no negative-norm ghost states.

   Unlike $c_{Vir} < 1$ models, $c_{Vir} \geq 1$, theories often contain marginal operators that allow continuous deformations, meaning physical exponents can change continuously rather than being fixed to rigid rational values.

3.4.4 The origin of unitary irreps of $SO(1,3)$ in TGD

In TGD, the appearance of the unitary representations of $SL(2,C)$ has two alternative interpretations.

1. $SO(1,3)$ acts as isometries for the hyperbolic 3-space H^3 realized as light-cone proper time = constant hyperboloids inside light-cone and CD and the distance from either tip of the CD defines Lorentz invariant distance. For the quantum states defined at H^3, the natural wave functions belong naturally to the unitary irreps of Lorentz group taking the role of unitary representations of Poincare group.

   The irreps of Poincare group, in particular those of translation group, are realized in the moduli space of CDs in which the Poincare group acts. One can say that CD replaces point of M^4 and the internal structure of particle corresponds to the zero energy states in WCW consisting of analogs of Bohr orbits for a particle identified ass 3-surface.

2. Conformal symmetries are basic symmetries of quantum TGD. The p-adic mass calculations rely on conformal invariance realized at the partonic 2-surfaces [L5, L15, L21]. The genus-generation correspondence [K7, K2] relies on heuristic arguments leading to a formula for mass squared of the fermion in terms of the genus of the partonic 2-surface associated with it.

   Therefore the association of a CFT with the partonic 2-surfaces looks natural. The unitary irreps of Lorentz group could be assigned either to the internal degrees of freedom of a particle in WCW or to the conformal modes of induced spinor fields inside partonic 2-surfaces.

   Note that non-unitary finite-dimensional representations can be fused with the representations of translation group to unitary representations of Poincare group.

3. One can imagine two realizations of conformal invariance. If H-H principle is true, the conformal invariance is related to the complex sub-manifold property.

   Also Lagrangian minimal surfaces are allowed by the action as a combination of Kähler action and volume term, which is predicted by the twistor lift of TGD. For Lagrangian option, the conformal invariance would be due to the 2-dimensionality of partonic 2-surface.

4. How do the M^4 and CP_2 projections relate to each other conformally? First of all M^4 projection could be a point: this is the case for string-like objects $X^2 \times Y^2$, $X^2 \subset M^4$ string world sheet and $Y^2 \subset CP_2$ complex manifold or minimal Lagrangian surface.

   For H-H option, the complex transversal complex coordinate w for the M^4 projection associated with the H-J structure is a holomorphic function of the coordinate complex coordinate of the CP_2 projection. If the CP_2 and M^4 projections are spheres, this map is characterized by a winding number and by its generalization to higher genus in the general case. For H-H option this map would be a conformal map between the M^4 and CP_2 projecions of the partonic surface. If the M^4 projection is not homotopically equivalent to a point the M^4 and CP_2 projections of the partonic surface would be conformally equivalent.

   For a spherical topology of parton surface, the moduli space of conformal structures is trivial and the conformal structures defined by the induce metric induced and complex sub-manifold property would be conformally related when M^4 projection cannot be reduced to a point by a homotopy. For higher genera the moduli spaces are finite-dimensional [K2] and the two conformal structures are in general not equivalent anymore for Lagrangian minimal surfaces.

   Could one speak of a kind of bulk-boundary duality between the projection of the partonic 2-surface to M^4 (the bulk) and CP_2 (the boundary)? Same kind of duality might apply in the case of superminimal surfaces.

3.4.5 How could the bilocality emerge in the TGD framework?

Could the emergence of bilocality be understood in the TGD framework? Suppose that the inner product is proportional to the Feynman propagator of the second quantized H spinor field between modes of a fixed color multiplet for fermions.

What matters is the behavior of the propagator as a function of M^4 distance at the M^4 projection of the partonic 2-surface. Microcausality fails slightly for the M^4 Feynman propagator at space-like distances. The propagator decays exponentially like $exp(-mr/\hbar_{eff})$ for space-like distances. The mass of the fermion would determine the length sale of below which the correlations are non-vanishing and this length scale should correspond to the size of the partonic 2-surface. Could this give rise to an inner product as a propagator increasing like $1/|(z-w)|^{2(1-c)}$? Also the divergence of the correlation for small distances could be understood.

Note that also the approximate masslessnes for fermions of H which can have very large mass can be understood. The restriction to light-like geodesics of M^4 implies that the propagator behaves in good approximation as a massless propagator. For the light-like geodesics of H, the fermion behaves as almost massless particle and the M^4 mass is expected to correlate with the frequency parameter ω characterizing the geodesic defining mass as $m \propto \hbar_{eff}\omega$.

It is important to notice that $\hbar_{eff}$ does not refer here to $\hbar_{eff}$ appearing in $R^2(CP_2, \hbar_{eff})$ but to bilocal gravitational (electric) Planck constant $\hbar_{gr}$ ($\hbar_{em}$) making possible long quantum coherence scales and affects frequency but not the mass m.

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