$M^8 - H$ duality reduces to local G_2 invariance

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Abstract

The idea of $M^8 - H$ duality has progressed through frustratingly many several twists and turns. I have discussed several variants of $M^8 - H$. Basic technical problem relates to the lack of the concrete realization of the $M^8 - H$ duality since explicit realization of the parametrization of the quaternionic normal space has been missing. The second open question is whether $M^8 - H$ duality is between 4-D surfaces in M^8 and space-time surfaces in H or is it enough that only the 3-D holographic data in H are fixed by $M^8 - H$ duality. There are indeed some intuitive arguments supporting the latter view. Could it be that M^-H duality is not an alternative view to see the dynamics but a way to determine holographic data in H so that a consistency with holography = holomorphy hypothesis is obtained.

It turns out that a modification of the original form of the $M^8 - H$ duality formulated in terms of a real analytic function of octonions leads to a possible solution of these problems. The original form of the $M^8 - H$ duality based on real analytic octonion valued functions of octonions $f(o)$ representables as power series works. The conditions $f(o) = 0$ and $f(o) = 1$ are invariant under local G_2 and the local G_2 acts as a dynamical spectrum generating symmetry group since $f \circ g_2 = g_2 \circ f$ holds true. The roots of $f(o) = 0/1$ for the simplest situation in which the quaternionic normal space is fixed M^4, are 4-spheres S^6.

The 4-surface Y^4 for the simplest solutions is identifiable as the intersection $Y^4 = E^4 \cap S^6$, where E^4 is the normal space of a fixed quaternionic $M^4 \subset M^8$. This guarantees that the normal space of Y^4 at each point is quaternionic M^4. The assumption that M^4 contains commutative subspace M^2 guarantees that the normal space correspond to a point of CP_2 so that the Y^4 can be mapped to $X^4 \subset H = M^4 \times CP_2$. Local G_2 transformations give more general surfaces Y^4. One can choose the function $f(o)$ to be an analytic function of a hypercomplex coordinate of M^4 and 3 complex coordinates of M^8. One can hope that the image X^4 of Y^4 satisfies the holography = holomorphy hypothesis.

1 Introduction

The idea of $M^8 - H$ duality has progressed through frustratingly many several twists and turns. Consider first the development of the key ideas.

1. The first key idea was that one can interpret octonions O as M^8 by using the number theoretic inner product defined by the real part of the octonion product. Later I gave up this assumption and considered complexified octonions, which do not form a number field, but finally found that the original option is the only sensible option.

2. The second key idea was that if either the tangent or normal space of the surface $Y^4 \subset M^8$ is quaternionic and therefore associative and if it also contains a commutative subspace, it can be parameterized by a point of CP_2 and mapped to $H = M^4 \times CP_2$. This would be the first half or $M^8 - H$ duality. How to map the $M^4 \subset M^8$ projection to $M^4 \times CP_2$? This question did not have an obvious answer. The simplest map is direct identification whereas inversion is strongly suggested by Uncertainty Principle (UP) and the interpretation of M^8 coordinates as components of 8-momentum. Note that one can considerably generalize the simplest view by replacing the fixed commutative subspace of quaternion space M^4 with an integrable distribution of them in M^8.

3. I considered first the option in which tangent space was assumed to be associative. The cold shower was that this option allows only trivial solutions [L1, L2]. Quaternionic normal space however works: any integrable distribution of quaternionic normal spaces defines an associative surface Y^4.

4. If M^8 is not complexified the surfaces Y^4 in M^8 are necessarily Euclidean. [L8]. This is in sharp conflict with the original intuitive idea that they have a number theoretic Minkowski signature. It is the normal space, which must have a Minkowskian signature and this forces us to rethink how the $M^8 - H$ duality is realized in Minkowskian degrees of freedom.

5. I have considered also the minimal option in which $M^8 - H$ duality determines only the 3-D holographic data as 3-surfaces $Y^3 \subset M^8$ mapped by $M^8 - H$ duality to H. The images of Y^3 could define holographic data consistent with the holography = holomorphy (H-H) vision. Both M^8 and H sides of the duality would be necessary.

   The physical interpretation for the space of 4-surfaces Y^4 in M^8 is as the analog of momentum space for particles identified as 3-D surfaces. In this interpretation the Y^4 would be analog of time evolution with time replaced with energy. This is in conflict with the physical intuition and suggests that maybe the minimal option is correct and indeed consistent with the fact that for point-like particles the momenta are at 3-D mass shells. One must be however extremely cautious here.

6. A further criticism against the $M^8 - H$ duality is that its explicit realization is missing. The problem is the identification of the CP_2 coordinates for the normal space of Y^4 at a given point.

In the following a formulation of $M^8 - H$ duality possibly solving these problems in terms of local G_2 invariance is proposed.

2 Understanding $M^8 - H$ duality in terms of local G_2 invariance

The motivation for reconsidering the $M^8 - H$ duality came from the fact that the H-H hypothesis works extremely nicely for the space-time surfacex $X^4 \subset H$. The roots of two generalized analytic functions f_1, f_2 of hypercomplex coordinate and 3 complex coordinates of H give as their roots space-time surfaces as minimal surfaces and the ansatz works for any action, which is general coordinate invariant and expressible in terms of the induced geometry. One would expect that H-H hypothesis appears also at the level of M^8: How?

One can also argue that the might be problems with the 3-D holographic data. How to fix them in such a way that they are consistent with functions f_1 and f_2 as analytic functions of H coordinates involving hypercomplex coordinate and 3 complex coordinates?

These issues led back to the original idea that the associative 4-surfaces $Y^4 \subset M^8$ might be definable in terms of real analytic functions $f(o)$ of octonions as an octonionic generalization of the notion of holomorphy. The 3 alternative conditions $f(o) = 0$, $f(o) = 1$ and the reality of $f(o)$ are promising since they are invariant under octonionic automorphism group G_2. The argument goes as follows.

1. Since G_2 acts as automorphisms one has $f(g_2(o)) = g_2(f(o))$, where $g_2(0)$ is any *local* G_2 automorphism. If $f(o) = 0/1$ is true then also $f(g_2(o))$ is true for any $g_2 \subset G_2$. This is true also for the roots of $Im(f(o)) = 0$, where Im refers to the octonionic imaginary part. Since G_2 maps the decomposition of octonion to quaternion and to a part orthogonal to it, also the conditions $RE(f(o) = 0$ and $IM(f) = 0$, where RE refers to the quaternionic part of the octonion is invariant under local G_2.

   One could have a huge dynamical spectrum generating symmetry analogous to the holomorphic symmetries of H-H vision. It would map the quaternionic normal spaces to quaternionic normal spaces and complex subspaces to complex subpaces.

2. Consider first the condition $f(o) = 0/1$. The Taylor (or even Laurent -) expansion in powers of o gives only two terms. The first term is proportional to the octonionic real unit 1 of o and the second term to the octonionic imaginary part of $Im(o) = o_7$ of o.

   For o^2 one obtains $o^2 = o_0^2 - o_7 \cdot o_7 + 2o_0o_7$. The coefficients of these parts depend on the real part o_0 of o and the length r_7 of the imaginary $Im(o)$. The higher powers of o involve products of two octonions of form $o_1 = \alpha_1 + \beta_1 o_7$ and $o_2 = \alpha_2 + \beta_2 0_7$ and the product is of form $o_1o_2 = (\alpha_1\alpha_2 - \beta_1\beta_2) + (\alpha_1\beta_2 + \alpha_3\beta_1)o_7$. By induction one finds that the coefficients for any power depend only on o_0 and the radius r_7 of 6-sphere only. In particular, the function $f(o)$ is expressible has the general form

   $$f(o) = f_1(o_0, r_7) + f_2(o_0, r_7) \quad . \tag{2.1}$$

   The detailed forms of these functions have been discussed in the earlier articles [L1, L2, L6] but are not relevant for what follows.

3. The condition $Im(f(o)) = 0$ fixes the relationship between 0_0 and r_7 and gives a time evolution of the radius r_7 of a 6-sphere as function of time parameter o_0. The condition $RE(f(o)) = 0$ requiring the vanishing of the quaternionic part implies the vanishing of both $Re(f)$ and $Im(f)$ and reduces to the condition $f(o) = 0$. The condition $IM(f(o)) = 0$ implies the vanishing of $Im(f(o))$.

Consider now various G_2 invariant options.

1. The condition $f(o) = 0$ or $f(o) = 1$ gives the roots of f_1 and f_2 as $o_0 = h_1(r_7)$ and $o_0 = h_2(r_7)$. Together these conditions give a discrete set of roots $(o_0, r_7)_n$.

   These roots define a discrete set of 6-spheres S^6 with o_0 constant and $r_7 = constant$. Can one assign an associative 4-surface $Y^4 \subset M^8$ to a given S^6? The condition that the normal space is quaternionic is satisfied if one fixes complement E^4 of quaternionic sub-space M^4 and restricts the points of S^6 to the intersection $E^4(o_0) \cap S^6(o_0)$. The normal space M^4 of E^4 would define the quaternionic subspace M^4, and it should be the same for all points of Y^4. What is the 4-dimension of $E^4(o_0) \cap S^6$. It deserves to be noticed that S^6 can be represented as a coset space $G_2/SU(3)$. S^6 has an almost complex structure induced by the octonionic cross product, which makes it nearly Kähler manifold.

   Since both surfaces in the intersection are contained in the hyper-plane $E^7(o_0)$, the dimension of $E^4(o_0) \cap S^6(o_0)$ is from the basic rule $6 + 6 - 7 = 3$. Clearly, the intersection is identifiable as 3-sphere $S^3(o_0)$. It is difficult to understand how this 3-sphere could serve as a holographic data allowing us to construct 4-D $Y^4 \subset M^8$. Since 0_0 corresponds to M^4 time coordinate, $M^8 - H$ duality could cannot map this surface to a time-like or light-like 3-D parton orbit contributing to the holographic data in holography= holomorphy vision.

2. The third option for which $Im(f(o)) = 0$ invariant under local G_2 would give a 4-D union of 3-spheres $Y^4 = S^3(o_0) = E^4 \cap S^6(o_0)$ and could work. If this proposal indeed works, one

can construct more general solutions for Y^4 by applying local G_2 automorphisms to these basic solutions for which the CP_2 coordinate for $X^4 \subset H$ is constant. The CP_2 points for the image of Y^4 in H would not be constant anymore.

3. The condition $IM(f(o)) = 0$ is equivalent with the condition $Im(f(o)) = 0$. The condition $RE(f(o)) = 0$ gives a discrete set of 3-spheres as roots is equivalent with the condition $f(o) = 0$. An interesting question is whether these roots correspond to "very special moments of time", which emerged in the original form of $M^8 - H$ duality. Could these spheres correspond at the level of H singularities at which the local G_2 element becomes multiple-valued so that the quaternionic normal space is not unique. The situation is analogous to a singularity of a vector field. This would give rise to cosmic strings and CP_2 type extremals. These singularities could be also associated with the classical non-determinism of holography = holomorphy principle.

Some remarks are in order.

1. $M^8 - H$ duality requires that M^4 contains $M^2 \subset M^4$ defining a commutative sub-space. Since $U(2) \subset SU(3)$ respects this choice, the normal spaces satisfying this condition are parameterized by $CP_2 = SU(3)/U(2)$ and $M^8 - H$ duality allows to assign to a given point of Y^4 a point of CP_2.

   An integrable distribution of these subspaces is possible. The local elements of G_2 map these distributions to each other. The subgroup leaving the distribution invariant corresponds to local $SU(3)$, which at the H side has interpretation as color group whereas $U(2)$ leaving the normal space invariant corresponds to the electroweak gauge group.

   The distribution of these choices together with generalized complex coordinates for M^8 defines the analog of Hamilton-Jacobi structure (H-J) [L3] in $M^4 \subset M^8$ mapped to its counterpart in H and playing a key role in H-H vision [L8]. Rather remarkably, the local $G_2/U(2)$ can therefore be identified as the moduli space of H-J structures [L3]. The division by $U(2)$ is because the quaternionic normal space with complex subspace is invariant under $U(2) \subset G_2$. Note that $G_2/U(2)$ is 10-D (here Google AI claims that the dimension is 7).

2. What about $M^8 - H$ duality for M^4 coordinates? Could the $M^4 \subset H$ point correspond to the projection of the $Y^4 = E^4 \times S^6$ point to $M^4 \subset M^8$ as such or is an inversion suggested by Uncertainty Principle and the interpretation of M^8 as 8-D momentum space? This question remains open.

3. What can one say about the elements $g_2(o)$ of the local G_2? The action of G_2 on octonions allows a matrix representation but the matrix elements are octonions so that the rules of multiplication are not standard and non-associative. Associativity is obtained if one considers only elements of G_2 belonging to a local $SU(3)$ subgroup having physical interpretation as a color group.

   Holomorphy= holography vision [L7, L9, L10] inspires the question whether $g_2(o)$ can be regarded as a real part of an analytic function of the generalized complex coordinates of M^8 for the Hamilton-Jacobi structure in question. Could this guarantee that the image of Y^4 in H is consistent with the holomorphy in H?

4. The real analytic functions $f(o)$ and $g(o)$ can be multiplied and summed so that the analog of a function field is in question. Also iterations of $f(o)$ are possible. The roots $Im(g) = 0$ of $g = f \circ f ... \circ f$ contain the roots of f plus roots of higher iterates. A complexity hierarchy analogous to that appearing for function pairs (f_1, f_2) at H sides emerges and the interpretation in terms of cognitive hierarchies is suggestive An interesting question is whether there is a simple relationship between functions $f(o)$ and function pairs (f_1, f_2).

5. The ramified primes of a polynomial of a single variable are expected to play an important role in the number theoretic view of TGD. If f assigned to $M^8 - H$ duality [L11] is a polynomial P with rational coefficients, the ramified primes would be assigned with the discriminant of P. The conjecture has been that the classical action defining the space-time surface is expressible as a power of discriminant of some polynomial P defined by the differences

of ramified primes [L4]. This would be a central aspect of the 4-D version of Langlands duality [L7, L9].

This would imply a huge degeneracy since all space-time surfaces related by local G_2 transformations as analogs of conformal transformations would have the same classical action defining the Kähler metric of WCW and give excellent hopes that also the functional integral over the 4-D "Bohr orbits" is calculable [L4]. Local G_2 would define zero modes for the WCW metric and symplectic degrees of freedom would correspond to non-zero modes as also conjectured [L5].

6. Besides space-time surfaces X^4 representable as graphs of maps $M^4 \rightarrow CP_2$ also surfaces for which M^4 projection has dimension smaller than 4, are possible. These could correspond to the singularities of the map g_2 such that the quaternionic normal space M^4 labelled by a CP_2 point depends on the direction in which one approaches a lower-dimensional surface X of M^8. This would give rise to CP_2 type extremals with 1-D X and cosmic strings with 2-D X.

The above mentioned conjecture that the classical action equals some kind of discriminant and is thus a number theoretic invariant, can be sharpened in the recent picture.

1. The condition $Im(f) = 0$ ($Re(f) = 0$) has a discrete set of roots $Y^4(n) \subset M^8$ as time evolutions $r_7 = h_n(o_0)$ of S^6, in turn giving rise to 4-surfaces $Y^4(n)$ as time evolutions $S^3(o_0) = E^4(o_0) \cap S^6(o_0)$ with respect to time coordinate o_0 mapped. Different roots $Y^4(i)$ as 4-surfaces can be interpreted as free particles, mapped to space-time surfaces $X^4(i)$ in H by $M^8 - H$ duality.

2. For each orbit $Y^4(i)$ of S^3, the condition $f(o) = 0$ defines a discrete set of "very special moments of time" $o_0(n,i)$ as its roots. The roots can be also complex but for real polynomials appear as complex conjugate pairs. One can define discriminant D as the product of differences of squares of roots in the usual manner [L7, L9]. This is true also when f is analytic function rather than only polynomial.

   One can assign a discriminant $D(i)$ to each $Y^4(i)$. The product $\prod_{i \in U} D(i)$ is well defined for the system of all $Y^4(i)$ or a subset U of them. These discriminants would define exponents of "free" actions for each $Y^4(i)$. Interactions are not taken into account yet.

3. How to assign "interaction action" to this system as a discriminant. The proposal is that the interactions between particles are at the level of H are contact interactions made possible by the intersection of space-time surfaces. For identical H-J structures the intersection $X_1^4 \cap X_2^4$ consists of 2-D string world sheets rather than a discrete set of points. Identical H-J structures would mean that they correspond to same element of local $G_2/U(2)$ since $U(2)$ leaves the quaternionic normal space containing a preferred commutative plane invariant.

4. As found, the H-J structures of M^8 and H naturally correspond to each other. If so then also the intersection $Y_1^4 \cap Y_2^4$ consists of string world sheets. One should be able to assign to the intersection an "interaction action". The conditions $o_0(1) = o_0(2)$ and $r_7(1) = r_7(2)$ (radii of S^6) must be satisfied. This gives a set $\{o_0(n)\}$ of roots. These 6-spheres now define "very special moments" for the interaction. The 3-spheres $S^3(o_0, i) = E^4(o_0, i) \cap S^6(o_0)$, $i = 1, 2$ must intersect in $Y_1^4 \cap Y_2^4$. The intersection of two 3-spheres should consist of 2-D string world sheets for the same H-J structures. This looks sensical since the hypercomplex M^4 coordinates appearing in the functions f_1 and f_2 are the same and one condition is eliminated. Also self-intersections for Y^4 are possible and would contribute to the action terms having an interpretation in terms of self-interactions of Y^4.

   One can assign discriminant D_{12} to the intersection $Y_1^4 \cap Y_2^4$ as a product of squares of root differences in $\{o_0(n)\}$. This would define an additional multiplicative contribution to the action exponential.

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