How infinite primes relate to other views about mathematical infinity?

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Contents

Abstract

In the following I will discuss the Cantorian view about infinite ordinals and cardinals and summarize the basic problems of the approach which relate to the axiom of choice, continuum hypothesis, and Russell's antinomy. After that I will compare this purely set theoretic approach to the notion of number with that provided by infinite primes. The approaches are not equivalent. For instance, sum and product for Cantorian ordinals are not commutative unlike for infinite integers defined in terms of infinite primes. I will also discuss what the foundations of physical mathematics might be if one accepts the TGD inspired view about infinity, about the notion of number, and the restrictions on the notion of set suggested by classical TGD.

1 Cantorian view about infinity

The question which I have but repeatedly under the rug during the last fifteen years concerns the relationship of infinite primes to the notion of infinity as Cantor and his followers have understood it. I must be honest: I have been too lazy to even explain to myself what Cantor really said. Therefore the reading of the New Scientist article "The Ultimate logic: to infinity and beyond" [1] was a pleasant surprise since it gave a bird's eye of view about how the ideas about infinity have evolved after Cantor as a response to severe difficulties in the set theoretic formulation for the foundations of Mathematics.

1.1 Cantor's paradize

I try to summarize Cantor's view about infinity first. Cantor was the pioneer of set theory, in particular the theory of infinite sets. Cantor started his work around 1870. His goal was to formulate all notions of mathematics in terms of sets, in particular natural numbers. Cardinals and ordinals define two kind of infinite numbers in Cantor's approach.

1. Cantor realized that real numbers are "more numerous" than natural numbers and understood the importance of one-to-one correspondence (bijection) in set theory. One can say that two sets related by bijection have same cardinality. This led to the notion of cardinal number. Cardinals are represented as sets and two cardinals are same if a bijection exists between the corresponding sets. For instance, the infinite cardinals assignable to natural numbers and reals are different since no bijection between them exists.

2. The definition of ordinal relies on successor axiom of natural numbers generalized to allow infinitely large ordinals. Given prdinal can be identified as the union of all ordinals strictly smaller than it. Well ordering is a closely related notion and states that every subset of ordinals has smallest element. One can classify ordinals to three types: 0, elements with predecessor, and elements without predecessor such as ω, which corresponds to the ordinal defined as the union of all natural numbers.

   The number of ordinals much larger than the number of cardinals. This is clear since the notion of ordinal involves additional structure coming from their ordering. A given cardinal corresponds to infinitely many ordinals and one can identify the cardinal as the smallest ordinal of this kind. For instance, ω and $\omega+n$ correspond to same cardinal α_0 (countable infinity) for all finite values of n.

3. Cantor introduced the notion of power set as the set of all subsets of the set and proved that the cardinality of the power set is larger than that of set. Cantor introduced also the continuum hypothesis stating that there are no cardinals between the cardinal $\aleph_0$ *resp.* $\aleph_1$ assignable to natural numbers *resp.* reals. Hilbert represented continuum hypothesis as one of his 23 problems in his talk at the 1900 International Congress of Mathematicians in Paris. Hilbert was also a defender of Cantor and introduced the term Cantor's paradize.

4. Cantor developed the arithmetics of ordinals based on sum, product, and power: each of these operations is expressible in terms of set theoretic concepts. For infinite ordinals multiplication and sum are not commutative anymore. This looks highly counter intuitive and requires detailed definition of the sum and product. Sum means just writing the ordered sequences representing ordinals in succession. To see the non-commutativity of sum it is enough to notice that the number of elements having predecessor is not the same for $\omega + n$ and $n + \omega$.

   To see the non-commutativity of product it is enough to notice that the product is define as cartesian product $S \times T$ of the ordered sets representing the ordinals. This means that every element of T is replaced with S. It is easy to see that $n \times \omega$ and $\omega \times n$ are different.

   One can define also the powers (exponentials) in the arithemetics of ordinals: exponent must reduce to the notion of power set X^Y, which can be realized as the set of maps $Y \rightarrow X$ and has formally $\#X^{\#Y}$ elements.

It is pity that the we physicists have so pragmatic attitude to mathematics that we do not have time to realize the beauty of the idea about reduction of all mathematics to set theory. This is even more regrettable since it might well be that the manner to make progress in physics might require replacing the mathematics with a mathematics which does not rely on set theory alone.

1.2 Snakes in Cantor's paradize

Cantor's paradize is extremely beautiful place but there are snakes there. Continuum hypothesis looked to Cantor intuitively obvious but the attempts to prove it failed. Betrand Russel showed in 1901 that the logical basis of Cantor's set theory was flawed. This manifested itself via a simple paradox. Assume that it makes sense to speak about the set of all ordinals. This is by definition ordinal itself since ordinal is a set consisting of all ordinals strictly smaller than it. But this would mean that the set of all ordinals is its own member! The famous barber's paradox is a more concrete manner to express Russel's antinomy. One cannot speak of the set of ordinals and must introduce the notion of class. Russell introduced also the notion of types and type theory.

At 1920 Ernst Zermelo and Abraham Fraenkel devised a series of rules for manipulating sets but these rules did not allow to resolve the status of the continuum hypothesis. The stumbling block was the rule known as "axiom of choice" stating that if you have a collection of sets you can form a new set by picking one element from each of them. At first this sounds rather obvious but in the case when there is no obvious rule telling how to do it, situation becomes non-trivial. Then Polish mathematicians Stefan Banach and Alfred Tarski managed to show how the axiom would allow the division of a spherical ball to six subsets which can then be arranged to two balls with the same size as the original ball using only rotations and translations. These six sets are non-measurable in terms of Lebesque measure. The non-intuitive outcome must relate to the definition of the volume of the ball that is integration or measure theory: the axioms of measure theory should bring in constraints preventing construction of the six sets.

Around 1931 Kurt Gödel proved the incompleteness theorem that it is not possible to axiomatize arithmetics using any axiom system. There always remain unprovable propositions, which are true and cannot be proved to be true. This kind of statement is analogous to "I am a statement which cannot be proved to be true". If this statement could be proved to be true it would not be true.

1.3 Constructing logical universes

The attempts to expel the snakes from Cantor's paradize led to the idea that by posing some constraints it might be possible to construct logically consistent set theory obeying Zermelo-Fraenkel axioms such that continuum hypothesis and the axiom of choice would hold true and which would be free of paradoxes such as Banach-Tarski paradox.

Around 1938 Gödel introduced what he called "constructible universe" or L world satisfying these constraints. The structure of L world is hierarchical and one can say that the successor idea manifests itself directly in the construction. The levels are labeled by ordinals and one can always add a new level. The introduction of a new level to the hierarchy means that new axioms are introduced to the system bringing in meta level to the mathematical structure. The axiom system can be extended indefinitely. Gödel's theorem holds true at given level of hierarchy but by adding new levels non-probable truths can be made provable.

1963 Paul Cohen however demonstrated that there is infinite number of this kind of L worlds. In some of them continuum hypothesis holds true, in some of them the number of cardinals between $\aleph_0$ andα_1 can be arbitrary large - even infinite. This initiated a boom of constructions brings in mind the inflation of GUTs in particle physics and tge endless variety of brane constructions and the landscape misery of M-theory. From the point of view of physicist the non-uniqueness in foundations of mathematics does not seem to matter much since the everyday mathematics would remain the familiar one. One can of course ask what about quantum theory: should quantum physics replace classical physics in the formulation of fundamental fo mathematics.

For instance, von Neuman proposed one particular L world. In von Neumann unverse one starts from natural numbers and constructs its power set and at each step in the construction one consideres power set assigned to the sete obtained at the previous level. It is clear that one imagine several options. One could consider all subsets, only finite subsets, or only subsets which have cardinality smaller than the set itself. Power sets identified as the set of all finite subsets would give minimal option. Power set identified as the set of all subsets would give the maximal option.

The work of Hugh Woodin represented in 2010 International Congress of Mathematicians in Hyderabad, India represents the last twist in the story. Woodin argues that one must step outside the system that is conventional mathematical world to solve the problem. Woodin has introduced so called Woodin cardinals whose existence implies that all "projective" subsets of reals have a measurable size: it is not an accident that the word "measure" appears here when one recalls what Banach-Tarski paradox states. Woodin was motivated by the problems of set theory. He expresses this by saying "Set theory is riddled with unsolvability. Almost any problem of set theory is unsolvable".

Woodin proposed his own constructive universe which he calls ultimate L. It has all the desired properties: in particular, continuum hypothesis holds true. Physicists reader need not get frustrated if he fails to intuit why this is the case: for a decade ago Wooding himself did not believe in this. Also this L world is infinite tower to which one can add new levels.

2 The notion of infinity in TGD Universe

The construction of infinite primes, integers, and rationals brings strongly in mind the L worlds of Gödel and followers and this inspires the idea about concrete comparison of these approaches to see the differences.

2.1 Rule of thumb

It is good to start with a rule of thumb allowing to make strong conclusions about the cardinalities of infinite primes. If one considers the set formed by all finite subsets of a countable set you get a countable set because these subsets can be expressed as bit sequences with finite number if non-vanishing binary digits telling whether given element of set belongs to the subset or not: this bit sequence corresponds to a unique integer. If *all* subsets (also infinite) are allowed the set is not countably finite. If continuum hypothesis holds true it has at least as many elements as real line.

2-adic integers are good example. Consider first all 2-adic numbers with a *finite* number of non-vanishing bits (finite as real numbers). You get a countably infinite set since you can map these bit sequences to natural numbers in an obvious manner.

Consider next all possible bit sequences: most of them have infinite number bits. These numbers form naturally 2-adic continuum with 2-adic topology and differentiability. 2-adics can be mapped to real continuum in simple manner: canonical identification allows to do this continuously. The cardinality of these bit sequences is same as for reals as the rule of thumb would predict.

The hierarchy of infinite integers is based on number theoretical view about infinity and it would seem that these infinities are between the countable infinity and infinity defining the number of points of real axis. This reflects the fact that number theoretic infinity is much more refined notion than the infinities associated with cardinals and even ordinals. For instance, one can divide these infinities whereas Cantorian arithmetics contains only sum, product and power.

2.2 How Cantor's ordinals relate to the construction of infinite primes?

The fascinating question is whether the comparison of the construction of infinite primes, integers and rationals could relate to the work of Cantor and Gödel and his followers could provide new insights about infinite primes themselves.

1. What is intriguing that L-worlds are defined as infinite hierarchies just as the hierarchy of infinite primes and associated hierarchies. The great idea is that these constructions are essentially set theoretic in accordance with the vision that mathematics should reduce to set theory. As already noticed, naive set theory however leads to paradoxes which motivates the work of Gödel and followers. The basic physical philosophy is the identification of physical state as a set: this is essentially a notion belonging to classical physics.

2. TGD approach is algebraic rather than set theoretic. The construction is based on explicit formulas assuming the existence of weird quantities defined as product of all primes at previous level. These quantities can be taken as purely algebraic notions without any attempt to find a set theoretic definition.

   The possibility to interpret the construction as a repeated second quantization of a supersymmetric arithmetic quantum field theory with bosons and fermions labeled by ordinary primes at the loweset level of hierarchy replaces the set theoretic picture. These weird products of all primes represent Dirac sea at a given level of hierarchy and the many particles states of previous level become elementary particles at the new level of hierarchy. This construction is proposed to have a direct physical realization in terms of many-sheeted space-time and generalized to the level of octonionic primes is suggested to allow number theoretic interpretation of standard model quantum numbers.

   Perhaps it is not mere arrogance of quantum physics to argue that the classical set theoretic view about physical state is replaced with quantum view about it. Algebra replaces set theory and real and p-adic topologies are essential: for instance, infinite primes are infinite only in real topology.

One can raise many interesting questions. Although the underlying philosophies are very different, one can ask whether it might be possible to reduce TGD inspired construction to set theory playing key role in the construction of ordinals?

1. Can one assign to a given infinite integer a set in a natural manner? At the lowest level of hierarchy infinite prime can be mapped to a rational. Could one assign to this rational a set in cartesian product $N \times N$? Does this argument generalize to higher levels? Could the construction discussed in [1] allow to realize the set theoretic representation?

2. The notion of divisibility and explicit formulas for infinite integers obviously imply that the number of infinite numbers is much larger than cardinals of Cantor. This is true also for the ordinals of Cantor. How infinite integers relate to the ordinals of Cantor for which successor axiom is true? Also now it makes sense to form successors and in general they correspond to products of infinite primes which can be mapped to polynomials of several variables. For infinite integers however also the predecessor always exists. For instance $X \pm 1$ are infinite primes, where X represents the product of primes at previous level. Only zero fails to have predecessor for infinite natural numbers.

3. In TGD framework one loses the very essential notion of well-orderedness stating that every ordinal corresponds to a set with smallest element: that is element without predecessor. For instance, the infinite numbers known as limits and by definition are infinite and have no predecessor, the simplest example about limit is ω, which corresponds to the union of all natural numbers. The study of predecessors allowed to conclude that the sum and product are non-commutative for ordinals. Since the notion of well-ordered set does not make sense for infinite integers, one cannot identify infinite integers as ordinals.

   One must however remember that just the well-orderedness hypothesis together with successor axiom allows to express ordinal as a union of strictly smaller ordinals. This in turn leads to the conclusion that ordinals cannot form a set and to Russel's antinomy and are responsible for the many problems of set theory forcing Wooding to sigh "Set theory is riddled with unsolvability. Almost any problem of set theory is unsolvable". Maybe the well-orderedness axiom is simply too strong for infinite ordinals.

4. Sum, product, and power are the basic operations in the arithmetics of ordinals. All they reduce to set theoretic constructions. One can however define these operations purely algebraically. The algebraic definition of sum and product makes sense since one can map the infinite integers to polynomials of several variables. The possibly existing set theoretic definition of infinite integers using infinite sets cannot be consistent with the commutativity of sum and product defined algebraically. Either algebra or set theory but not both!

5. Also the notion of power makes sense for ordinals and relies on the notion of power set. Could the algebraic definition of exponential make sense? If the exponent N of M^N is finite integer, then the exponent makes sense for infinite M. If N is infinite integer it does not. Hence it seems that the analogs of numbers like ω^ω do not exist in TGD inspired L universe.

6. The failure of set theoretic reductionism brings in mind the motivic approach to integration as purely algebraic approach applied to the symbol defining the integral instead of a number approach based on set theoretic notions. The motivation of the motivic approach in p-adic context is that p-adic numbers are not well-ordered so that one loses the notion of boundary and orientation as topological concepts although they can make sense algebraically.

For the hierarchy infinite integers the notion of infinity relies on real norm, which is essentially length rather than on the cardinality of a set. This infinity is essentially non-Cantorian and it is perhaps useless to try to relate it to that of ordinal or cardinal. There is just an infinite hierarchy of infinities which replaces the hiererachy of ordinals and for which the real norm of ratio makes possible partial ordering. Clearly the notion of infinity is extremely slippery and one must carefully specify what one means with infinite.

2.3 Cardinals in TGD Universe

What about cardinals in TGD framework? There seems to be no reason for giving them up and the first guess is that TGD replaces cardinals and ordinals of Cantor with cardinals and the hierarchy of infinite primes, integers, and rationals.

1. The first question is what is the cardinal assignable to infinite primes at the first level of hierarchy. For the set of finite primes the cardinal is $\aleph_0$. For the first level of infinite primes the situation is not so simple. The simple infinite primes correspond to free Fock states constructed from fermions and bosons labelled by primes. They are in one-one correspondence with rationals. There is however infinite number of many particle bound states representable as products of irreducible polynomials of one variable with integer coefficients and having finite number of roots which are algebraic numbers. The set of algebraic numbers is countable. This suggests that the cardinality of set of infinite primes at the first level of hierarchy corresponds to $\aleph_0$. This if course assuming that infinite integers and rationals for a set although they themselves cannot be described as sets.

   If one allows Fock states containing infinite number of particles and having thus infinite energy one obtains formally polynomials of infinite degree identifiable as Taylor expansions. In this case the roots can be transcendental numbers and one expects that a cardinal larger than $\aleph_0$, say $\aleph_1$ emerges. In von Neumann's Universe one indeed allows all subsets and $\aleph_1$ appears already at the first level. The higher cardinals appear at higher levels.

   One cannot exclude the Fock states containing infinite number of quanta if one accepts the idea that infinite prime representing quantum state characterizing entire Universe make sense. Does this mean that $\aleph_1$ has meaning only for entire universe and for states carrying infinite energy (in ZEO the positive energy part of zero energy state would carry the infinite energy)?

2. What happens at the next levels of the hierarchy? One possibility is that infinite primes at each level define a countable set. The point is that in polynomials representation one considers only finite degree polynomials depending on finite number of variables, having rational coefficients. Therefore everything at the level of definitions is countable and finite and the product X of primes of previous level is just an algebraic symbol identifiable as a variable of polynomial.

3. In an alternative construction of infinite integers suggested in [**?**]ne considers the first level of the hierarchy the set of finite subsets of algebraic numbers and the set of finite subsets of this set at the next level and so on. All these sets are countable which suggests that the number of infinite primes at each level of the hierarchy is countable and that only the completion of algebraic number to reals or p-adic can give rise to $\aleph_1$. This would conform with the fact that quantum physics is basically based on counting of quanta and that finite measurement resolution is an essential restriction on what we can know.

2.4 What about the axiom of choice?

Axiom of choice has several variants. One variant is axiom of countable choice. Second variant is generalized continuum hypothesis states that the cardinality of an infinite set is between that of infinite set S and its power set: in other words there is no cardinal satisfying $\aleph_\alpha < \lambda < 2^{\aleph_\alpha}$ or equivalently: $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$. For a finite collection of sets it can be proved but already when on has a countable collection of nonempty set and in the case that one cannot uniquely specify some preferred element of each set, axiom of choice must be postulated. For instance, each subset of natural numbers has smallest element so that there is no need to postulate axiom of choice separately. Also closed intervals of real axis have smallest element.

What happens to the axiom of choice in TGD Universe. TGD is a physical theory and this means that the laws of classical physics strong considerations on the allowed sets. Classical physics is in TGD framework the dictated by the Kähler action and by a principle selecting its preferred extremals. Although several almost formulations for this principle exist, it is far from being well-understood and it is not clear whether one can give explicit formula for preferred extremals. One formulation is as quaternionic sub-manifolds of 8-D imbedding space allowing octonionic structure in its tangent space and defined by octonionic representation of the gamma matrices defining the Clifford algebra.

1. The world of classical worlds can be regarded as the space of preferred extremals of Kähler action identifiable as certain 4-surfaces in $M^4 \times CP_2$. The mere extremal property implies also smoothness of the partonic 2-surfaces so that very powerful constraints are involved: therefore situation is very far from the extreme generality of set theory where one does assumes neither continuity nor smoothness. Zero energy ontology means that this space effectively reduces to a

collection of spaces assignable to causal diamonds. Strong form of holography reduces this space effectively to the space consisting of collectings of partonic 2-surfaces at the light-like boundaries of CD plus 4-D tangent space data at them which very probably cannot be chosen freely.

2. In this kind of situation it might well happen that all collections of sets, say are finite or in the case that that they are countable they allow a unique choice of preferred point. Axiom of choice would not be needed. The specification of a preferred point of every 4-surface in the collection does not look a problem for a pragmatic physicist, since one can restrict the consideration to the boundaries of causal diamonds and consider for instance minimum of light-like radial coordinate. In fact, finite measurement resolution leads to the effective replacement of partonic 2-surfaces with the collection of ends of braid strands and the ends of braid strands define the preferred points. One might say, that physics with finite measurement resolution performs the choice automatically. A stronger form of this choice is that the points in question are rational points for a natural choice of the imbedding space coordinates.

2.5 Generalization of real numbers inspired by infinite integers

Surreal numbers define a generalization of reals obtained by introducing a hierarchy of infinite reals and infinitesimals as their inverses. Infinite integers and rationals in TGD sense could give rise to a similar generalization so that one would have an infinite hierarchy of 8-D imbedding space such that at given level previous level would represent infinitesimals.

TGD suggests another generalization of reals. One can construct from infinite integers rationals with unit norm. A possible interpretation would be as zero energy states with denominator and numerator represention positive and negative energy parts of the zero energy state and vanishing of total quantum numbers represented by real unit property. These numbers would have arbitrarily complex number theoretical anatomy however.

This structure has enormous representative power and one could dream that the world of classical worlds and spinor fields in this space could allow representation in terms of these real units. Brahman Atman Identity would be realized: the structure of single space-time point invisible to ordinary physics would represent the world of classical worlds! Single space-time point would be the Platonia!

Could one say that the space of all infinite rationals which are real units is countable? If previous arguments are correct this would seem to be true. If this is true, then TGD inspired notion of infinity would be extremely conservative as compared to the view proposed by Cantor and his followers using the Cantorian criteria. Just $\aleph_n$, $n = 0, 1$ and hierarchy of infinite integers which are countable sets. One can of course, ask how many surfaces WCW contains, what $\aleph$ is in question. This depends on the properties of preferred extremals. If partonic 2-surfaces can be choosen freely at the boundaries of CDs the restrictions come only from smoothness of the imbedding of the partonic 2-surfaces and tangent space data. The space of all functions from reals to reals has cardinality $2^{\aleph_1}$ which suggests that the cardinality is not larger than this, perhaps smaller since continuity and smoothness poses strong conditions. The natural guess is that the tangent space of WCW can be modelled as and infinite-dimensional separable Hilbert space which has cardinality $\aleph_1$.

TGD leads also a second generalization of the number concept motivated by number theoretical universality inspiring the attempt to glue different number fields (reals and various p-adics) together among common numbers -rationals in particular- to form a larger structure [2].

To sum up, the distinctions between Cantorian and TGD inspired approaches are clear. Cantorian approach relies on set theory and TGD on number theory. What is common is the hierarchy of infinities.

3 What could be the foundations of physical mathematics?

Theoretical physicists do not spend normally their time for questioning the foundations of mathematics. They calculate. There are exceptions: Von Neuman was both a theoretical physicist developing mathematical foundations of quantum theory and mathematician buildingthe mathematics of quantum theory and also proposing his own L world for foundations of mathematics.

A physicist posing the question "What should be done for the foundations of mathematics?" sounds blasphemous and the physicist should add the attribute "physical" to "mathematics" to avoid irritation. In any case, the fact is that the problems plaguing set theory and therefore the foundations of mathematics had been discovered roughly century ago and no commonly accepted solution to these problems have been found. The foundations of mathematics rely on classical physics and quantum view about existence suggests that the foundations of mathematics might need a revision.

Again the work of von Neuman comes readily into mind. The goal of von Neuman was to build a non-commutative measure theory: the outcome was the three algebras bearing his name and defining the mathematical backbone of three kinds of quantum theories. Factors of type I are natural for wave mechanism with finite number of degreees of freedom. In QFT hyperfinite factors of type III appear. In TGD framework hyperfinite factors of type II (and possibly of type III) are natural.

Connes who has studied von Neumann algebras highly relevant to quantum physics proposed the notion of non-commutative geometry in terms of a spectral triplet defined by C^* algebra A, Hilbert space H, and Dirac operator D with some additional properties. As a special case one re-discovers Riemannian manifolds using commutative function algebra, the Hilbert space of continuous functions, and certain kind of Dirac operator.

Physicists are usually mathematical opportunists and do not want to use time to ponder the foundations of mathematics My belief is that physicists should get rid of this attitude and make fool of themselves by posing the childish questions of physicist in the hope that some real mathematician might get interested. In order to not irritate mathematicians too much I will talk about physical mathematics instead of mathematics in the sequel.

3.1 Does it make sense to speak about physical set theory?

For the physicist set theory looks quite too general. In the recent day physical theories sets are typically manifolds, submanifolds, or orbifolds. Feynman diagrams represent example of 1-D singular manifolds and in TGD generalized Feynman diagrams of TGD fail to be 3-manifolds only at the vertices represented as 2-D partonic surfaces. In string theories and in twistor approach to gauge theories algebraic geometry is important. Branes are typically algebraic surfaces. The spaces are endowed with various structures: besides metric induced topology one differential structure, differential forms, metric, spinor structure, complex and Kähler structure, etc...

1. In algebraic geometry sets are replaced with varieties and basic set theoretic operations such as intersection and union are algebraized. Physicists should not fail to realize how profound this algebraization of the set theory is. The price that must be paid is that varieties are manifolds only locally. What limitations does this mean for set theory? Is it enough to formulate set theory algebraically? In TGD framework this could be possible in the intersection of real and p-adic worlds (WCWs) since set theoretic operations would have algebraic representation. For instance, $A \subset B$ would be formulated by adding additional functions for which the intersection of zero locus with B defines A.

   The algebraic notion of set as a variety is extremely restrictive: maybe the problems of set theory are partially due to the neglect of the fact that allowed sets must have a physical realization. Every physicists of course has her own pet theory, which he regards as the real physics, and one naturalcondition on any accceptable physics is that it can emulate sufficiently general spaces - to act as a kind of mathematical Turing machine. At least real and complex manifolds with arbitrary dimension should have some kind of physical representation. One can imagine this kind of representation in terms of unions of partonic 2-surfaces since union can be regarded also as a Cartesian product as long as the surfaces do not intersect.

2. The introduction of topology is the first step in bringing structure to the set theoretic primordial chaos. Metric topology is standard in physics at space-time level. More refined topologies can be certainly found in highly technical mathematical physics articles. In algebraic geomery Zariski topology is important but has its problems realized by Groethendienck in his attempts to build a universal cohomology theory working in all number fields. The closed sets of Zariski topology are varieties. Their complements would be open sets open also in norm based topology. Zariski topology is obviously much rougher than the metric opology. Zariski topology makes sense also for p-adic number fields. This kind of topology might make sense in TGD framework if one restricts the consideration to the intersection of real and p-adic worlds identified at the level of WCW as the space of algebraic surfaces defined using polynomials with rational coefficients and having finite degree.

3. In TGD framework preferred extremals of Kähler action define space-time surfaces and strong form of holography makes the situation effectively 2-dimensional. The conjecture is that preferred extremals correspond to quaternionic surfaces of octonionic 8-space. Octonionic structure is associated with the octonionic representation of the imbedding space gamma matrices (not actually matrices any more!) defining the Clifford algebra. Associativity would be the basic dynamical principle. Does this mean that number theory- in particular classical number fields- should appear in the formulation of the foundations ofphysical mathematics? This idea is attractive even when one does not assume that TGD Universe is the Universe.

What is beautiful that algebraic geometry brings in also number theory. One might hope that the foundations of physical mathematics could be based on the fusion of set theory, geometry, algebra, and number theory.

3.2 The restrictions of mathematical cognition as a guideline?

With the birth of quantum theory physicists ceased to be outsiders since it was impossible to consider quantum measurement as something not affecting the measured system in any way. With the advent of consciousness theory physicists have been forced to give up the idea about uni-directional action with with reality and have become a part of quantum Universe - self. This also requires dramatic modification of the basic ontology forcing to give up the physicalistic dogmas. Consciousness involves free will manifested in ability to select and create something completely new in each quantum jump. Physical Universe is not given but is re-created again and again and evolves.

In standard mathematics mathematician is still a complete outsider, and the possible limitations of mathematical cognition are not considered seriously in the attempts to formulate the foundations of mathematics. Mathematicians still choose effortlessly one element from each set of infinite collection of sets. We know that in numerics one is always bound to introduce cutoff on the number of bits and use finite subset of rational numbers but also this has not been taken into account in the formulation of foundations as far as I know. If one takes consciousness theory seriously one is led to wonder what are the physical restrictions on mathematical cognition and therefore on physical mathematics. What looks obvious that the idea about mathematics based on fixed axiomatics must be given up. The evolution of the physical universe and of consciousness means also the evolution of (at least physical) mathematics. The paradox of self reference plaguing conventional view about consciousness and leading to infinite regress disappears when this regress is replaced with evolution.

Suppose that life resides and cognitive representations are realized in the intersection of real and p-adic worlds reducing to intersections of real and p-adic variants of partonic 2-surfaces at space-time level. At the level of WCW the intersection of real and p-adic worlds could correspond to the space of partonic 2-surfaces defined by rational functions constructed using polynomials of finite degree with rational coefficients.

What kind of restrictions of this picture poses set theory, topology, and logic? The reader can of course imagine restrictions on some other fields of mathematics involved. The question in the case of the set theory and topology has been already touched. In the case of logic the key question seems to concern the operational meaning of $\forall$ and $\exists$, when the finite resolution of measurements and cognitive representation are taken into account. What these universal quantors really mean: what is their domain of definition?

Consider first the domain of definition at space-time level.

1. Should all theorems be formulated using $\forall$ and $\exists$ restricted to the dense subset rationals of 8-D imbedding space. Since continuous function is fixed from its values in a dense subset, this assumption is not so strong unless there are other restrictions.

2. At space-time surface and partonic 2-surfaces the situation is different. The assumption that only the common rational points of real and p-adic surfaces define cognitive representations poses a strong limitation since typically the number of rational points of 2-surface is expected to be finite. Algebraic extensions of p-adic numbers extend the number of common points and one can imagine an evolutionary hierarchy of mathematics realized in terms of geometry of partonic 2-surfaces reflecting itself as the geometry of space-time surfaces by strong form of holography.

3. The orbits of the rational points selected at the ends of partonic 2-surfaces are braids along light-like 3-surfaces. At space-time level one has world sheets or strings which form in general case 2-braids. This picture leads to a what I have used to call almost topological QFT.

What about the domain of definition of existence quantors at the level of WCW? The natural conjecture is that the surfaces in the intersection of real and p-adic worlds form a dense set of full WCW so that everything holding true in the intersection would hold true generally and one coul dhope that systems which are living in the proposed sense are able to discover interesting mathematics.

Suppose that the partonic 2-surfaces decompose into patches such that in each patch the surface is a zero locus of polynomials with rational coefficients. Since polynomials can be seen as Taylor series with cutoff one can hope that they form a dense subset. Since rationals are dense subset of reals, one can hope that also the restriction to rational coefficients preserves the dense subset property and living subsystems are able to represent all that is needed and completion takes care of the rest as it does for rationals. The notion of completion leading from rationals to various algebraic numbers fields and also to reals and complex numbers would become the fundamental principle leading from number theory to metric topology.

Physicist reader has certainly noticed that "rational point" does not represent a general coordinate invariant notion.

1. The coordinates of point are rational in preferred coordinates and the symmetries of the 8-D imbedding space suggest families of preferred coordinates. The moduli space for CDs would be characterized by the choice of these preferred coordinates dictating also the choice of quantization axes so that quantum measurement theory would be realized as a decomposition of WCW to a union corresponding to different choices. State function reduction would involve also a localization determining quantization axes.

2. There are many possible choices of quantization axes/preferred coordinates and this means a restriction of general coordinate invariance from group of all coordinate transformations to a discrete subgroup of isometries which is not unique. Cognition would break the general coordinate invariance. The world in which the mathematician thinks using spherical coordinates differs in some subtle manner from the world in which she thinks using Cartesian coordinates. Mathematician does not remain outside Platonia anymore just as quantum physicists is not outside the physical Universe!

Axiom of choice relates to selection, which can be regarded as a cognitive act. The question whether axiom of choice is needed at all has been already discussed but a couple of clarifying comments are in order.

1. At quantum level selection would be naturally assigned with state function reduction, also the state function reduction selecting quantization axes. The cascade of state function reductions - starting from the scale of CD and proceeding fractally downwards sub-CD by sub-CD and stopping when only negentropic entanglement stabilized by NMP remains - could be how Nature performs the choice. State function reduction would involve also the choice of quantization axes dictating possible subsequence choices. Note that non-deterministic element would be involved with the quantum choice.

2. If life and cognitive representations are at the intersection of real and p-adic worlds, it would seem that rational points are chosen at space-time level and algebraic 2-surfaces at WCW level. As explained, it is easy to imagine the collection of sets from which one selects points is always finite or that there is a natural explicit criterion allowing to select preferred point from each set. Finite measurement resolution implying braids and string world sheets could provide this criterion. If so, the axiom of choice would be un-necessary in physical mathematics. Finite measurement resolution suggests that for partonic 2-surfaces the ends of braid strands define preferred points.

Platonia is a strange place about which many mathematicians claim to visit regularly. I already proposed that the generalization of space-time point by bringing in the infinite number theoretical anatomy of real (and octonionic) units might allow to realize number theoretical Brahman=Atman identity by representing WCW in terms of the number theoretic anatomy of space-time points. This kind of representation would certainly be the most audacious idea that physical mathematician could dare to think of.

Books related to TGD

[1] M. Pitkänen. Non-standard Numbers and TGD. In *TGD as a Generalized Number Theory*. Onlinebook. http://tgdtheory.com/public_html/tgdnumber/tgdnumber.html#infsur, 2006.

[2] M. Pitkänen. TGD as a Generalized Number Theory: p-Adicization Program. In *TGD as a Generalized Number Theory*. Onlinebook. http://tgdtheory.com/public_html/tgdnumber/tgdnumber.html#visiona, 2006.

Mathematics

[1] R. Elwes. The Ultimate logic: to infinity and beyond. http://www.newscientist.com/article/mg21128231.400-ultimate-logic-to-infinity-and-beyond.html, 2011.