# A more detailed view about the TGD counterpart of Langlands correspondence

April 19, 2025

Matti Pitkänen

**orcid**:0000-0002-8051-4364.
**email**: matpitka6@gmail.com,
**url**: http://tgdtheory.com/public_html/,
**address**: Rinnekatu 2-4 A 8, 03620, Karkkila, Finland.

**Abstract**

In TGD, geometric and number theoretic visions of physics are complementary. This complementarity is analogous to momentum position duality of quantum theory and implied by the replacement of a point-like particle with 3-surface, whose Bohr orbit defines space-time surface.

At a very abstract level this view is analogous to Langlands correspondence. The recent view of TGD involving an exact algebraic solution of field equations based on holography= holomorphy vision allows to formulate the analog Langlands correspondence in 4-D context rather precisely. This requires a generalization of the notion of Galois group from 2-D situation to 4-D situation: there are 2 generalizations and both are required.

1. The first generalization realizes Galois group elements, not as automorphisms of a number field, but as analytic flows in $H = M^4 \times CP_2$ permuting different regions of the space-time surface identified as roots for a pair $f = (f_1, f_2)$ of pairs $f = (f_1, f_2) : H \to C^2$, $i = 1, 2$. The functions $f_i$ are analytic functions of one hypercomplex and 3 complex coordinates of $H$.

2. Second realization is for the spectrum generating algebra defined by the functional compositions $g \circ f$, where $g : C^2 \to C^2$ is analytic function of 2 complex variables. The interpretation is as a cognitive hierarchy of function of functions of .... and the pairs $(f_1, f_2)$ which do not allow a composition of form $f = g \circ h$ correspond to elementary function and to the lowest levels of this hierarchy, kind of elementary particles of cognition. Also the pairs $g$ can be expressed as composites of elementary functions.

   If $g_1$ and $g_2$ are polynomials with coefficients in field $E$ identified as an extension of rationals, one can assign to $g \circ f$ root a set of pairs $(r_1, r_2)$ as roots $f_1, f_2) = (r_1, r_2)$ and $r_i$ are algebraic numbers defining disjoint space-time surfaces. One can assign to the set of root pairs the analog of the Galois group as automorphisms of the algebraic extension of the field $E$ appearing as the coefficient field of $(f_1, f_2)$ and $(g_1, g_2)$. This hierarchy leads to the idea that physics could be seen as an analog of a formal system appearing in Gödel's theorems and that the hierarchy of functional composites could correspond to a hierarchy of meta levels in mathematical cognition.

3. The quantum generalization of integers, rationals and algebraic numbers to their functional counterparts is possible for maps $g : C^2 \to C^2$. The counterpart of the ordinary product is functional composition $\circ$ for maps $g$. Degree is multiplicative in $\circ$. In sum, call it $+_e$, the degree should be additive, which leads to the identification of the sum $+_e$ as an element-wise product. The neutral element $1_\circ$ of $\circ$ is $1_\circ = Id$ and the neutral element $0_e$ of $+_e$ is the ordinary unit $0_e = 1$.

   The inverse corresponds to $g^{-1}$ for $\circ$, which in general is a many-valued algebraic function and to $1/g$ for *times*. The maps $g$, which do not allow decomposition $g = h \circ i$, can be

identified as functional primes and have prime degree. $f : H \to C^2$ is prime if it does not allow composition $f = g \circ h$. Functional integers are products of functional primes $g_p$.
The non-commutativity of $\circ$ could be seen as a problem. The fact that the maps $g$ act like operators suggest that the functional primes $g_p$ in the product commute. Functional integers/rationals can be mapped to ordinary by a morphism mapping their degree to integer/rational.

4. One can define functional polynomials $P(X)$, quantum polynomials, using these operations. In $P(X)$, the terms $p_n \circ X^n$, $p_n$ and $X$ should commute. The sum $\sum_e p_n X^n$ corresponds to $+_e$. The zeros of functional polynomials satisfy the condition $P(X) = 0_e = 1$ and give as solutions roots $X_k$ as functional algebraic numbers. The fundamental theorem of algebra generalizes at least formally if $X_k$ and $X$ commute. The roots have representation as a space-time surface. One can also define functional discriminant $D$ as the $\circ$ product of root differences $X_k -_e X_l$, with $-_e$ identified as element-wise division and the functional primes dividing it have space-time surface as a representation.

What about functional p-adics?

1. The functional powers $g_p^{\circ k}$ of primes $g_p$ define analogs of powers of p-adic primes and one can define a functional generalization of p-adic numbers as quantum p-adics. The coefficients $X_k$ $X_k \circ g_p^k$ are polynomials with degree smaller than $p$. The sum $+_e$ so that the roots are disjoint unions of the roots of $X_k \circ g_p^k$.
2. Large powers of prime appearing in p-adic numbers must approach $0_e$ with respect to the p-adic norm so that $g_P^n$ must effectively approach $Id$ with respect to $\circ$. Intuitively, a large $n$ in $g_P^n$ corresponds to a long p-adic length scale. For large $n$, $g_P^n$ cannot be realized as a space-time surface in a fixed CD. This would prevent their representation and they would correspond to $0_e$ and Id. During the sequence of SSFRs the size of CD increases and for some critical SSFRs a new power can emerge to the quantum p-adic.
3. Universal Witt polynomials $W_n$ define an alternative representation of p-adic numbers reducing the multiplication of p-adic numbers to elementwise product for the coefficients of the Witt polynomial. The roots for the coefficients of $W_n$ define space-time surfaces: they should be the same as those defined by the coefficients of functional p-adics.

There are many open questions.

1. The question whether the hierarchy of infinite primes has relevance to TGD has remained open. It turns out that the 4 lowest levels of the hierarchy can be assigned to the rational functions $f_i : H \to C^2$, $i = 1, 2$ and the generalization of the hierarchy can be assigned to the composition hierarchy of prime maps $g_p$.
2. Could the transitions $f \to g \circ f$ correspond to the classical non-determinism in which one root of $g$ is selected? If so, the p-adic non-determinism would correspond to classical non-determinism. Quantum superposition of the roots would make it possible to realize the quantum notion of concept.
3. What is the interpretation of the maps $g^{-1}$ which in general are many-valued algebraic functions if $g$ is rational function? $g$ increases the complexity but $g^{-1}$ preserves or even reduces it so that its action is entropic. Could selection between $g$ and $g_{-1}$ relate to a conscious choice between good and evil?
4. Could one understand the p-adic length scale hypothesis in terms of functional primes. The counter for functional Mersenne prime would be $g_2^n/g_1$, where division is with respect to elementwise product defining $+_e$? For $g_2$ and $g_3$ and also their iterates the roots allow analytic expression. Could primes near powers of $g_2$ and $g_3$ be cognitively very special?

# Contents

# 1 Introduction

The Quanta Magazine article (see this) related to Langlands correspondence and involving concepts like elliptic curves, modular functions, and Galois groups served as an inspiration for these comments. Andrew Wiles in his proof of Fermat's Last Theorem used a relationship between elliptic curves and modular forms. Wiles proved that certain kinds of elliptic curves are modular in the sense that they correspond to a unique modular form. Later it was proved that this is true for all elliptic surfaces. Later the result was generalized to real quadratic extensions of rationals by 3 mathematicians involving Samir Siksek and now by Caraiani and Newton for the imaginary quadratic extensions.

Could this correspondence be proved for all algebraic extensions of rationals? And what about higher order polynomials of two variables? Complex elliptic curves, defined as roots of third order polynomials of two complex variables, are defined in 2-D space with two complex dimensions have the special feature that they allow a 2-D discrete translations as symmetries: in other words, they are periodic for a suitable chosen complex coordinate. I have talked about this from TGD point of view in [L12]. Is the 1-1 correspondence with modular forms possible only for elliptic curves having these symmetries?

How are the Galois groups related to this? Indian mathematical genius Ramanujan realized that modular forms seem to be associated with so-called Galois representations. The Galois group would be the so- called absolute Galois group of the number field involved with the representation. Very roughly, they could be seen as representations of a Lie group which extends the Galois group. Also elliptic curves are associated with Galois representations. This suggests that the Galois representations connect elliptic curves, objects of algebraic geometry and modular forms, which correspond to group representations. These observations led to Langlands program which roughly states a correspondence between geometry and number theory.

The Galois group is indeed involved with Langlands duality. If the Lie group $G$ is defined over field $k$ (in the recent case extension of rationals), the Langlands dual ${}^LG$ of $G$ is an extension of the absolute Galois group of $k$ by a complex Lie group (see this. The representation of the absolute Galois group is finite-dimensional, which suggests that it reduces to a Galois group for a finite-dimensional extension of rationals. Therefore the effective Galois group used can be larger than the Galois group of extension of rationals. ${}^LG$ has the same Lie algebra as $G$.

In the following, I will consider the situation from a highly speculative view point provided by TGD. In TGD, geometric and number theoretic visions of physics are complementary: $M^8-H$ duality in which $M^8$ is analogous to 8-D momentum space associated with 8-D $H=M^4\times CP_2$ is a formulation for this duality and makes Galois groups and their generalizations dynamic symmetries in the TGD framework [L9]. This complementarity is analogous to momentum position duality of quantum theory and implied by the replacement of a point-like particle with 3-surface, whose Bohr orbit defines space-time surface.

At a very abstract level this view is analogous to Langlands correspondence [L10]. The recent view of TGD involving an exact algebraic solution of field equations based on holography= holomorphy vision allows to formulate the analog Langlands correspondence in 4-D context rather precisely. This requires a generalization of the notion of Galois group from 2-D situation to 4-D situation: there are 2 generalizations and both are required.

1. The first generalization realizes Galois group elements, not as automorphisms of a number field, but as analytic flows in $H=M^4\times CP_2$ permuting different regions of the space-time surface identified as roots for a pair $f=(f_1,f_2)$ of pairs $f=(f_1,f_2):H\rightarrow C^2$, $i=1,2$. The functions $f_i$ are analytic functions of one hypercomplex and 3 complex coordinates of $H$.

2. Second realization is for the spectrum generating algebra defined by the functional compositions $g\circ f$, where $g:C^2\rightarrow C^2$ is analytic function of 2 complex variables. The interpretation is as a cognitive hierarchy of function of functions of .... and the pairs $(f_1,f_2)$ which do not allow a composition of form $f=g\circ h$ correspond to elementary function and to the lowest levels of this hierarchy, kind of elementary particles of cognition. Also the pairs $g$ can be expressed as composites of elementary functions.

   If $g_1$ and $g_2$ are polynomials with coefficients in field $E$ identified as an extension of rationals, one can assign to $g\circ f$ root a set of pairs $(r_1,r_2)$ as roots $f_1,f_2)=(r_1,r_2)$ and $r_i$ are algebraic numbers defining disjoint space-time surfaces. One can assign to the set of root pairs the analog of the Galois group as automorphisms of the algebraic extension of the field $E$ appearing as the coefficient field of $(f_1,f_2)$ and $(g_1,g_2)$. This hierarchy leads to the idea that physics could be seen as analog of formal system appearing in Gödel's theorems and that the hierarchy of functional composites could correspond to a hierarchy of meta levels in mathematical cognition [L11].

Do the notions of integers, rationals and algebraic numbers generalize so that one could speak of their functional or quantum counterparts?

1. For maps $g:C^2\rightarrow C^2$, the counterpart of the ordinary product is functional composition $\circ$. Degree is multiplicative in $\circ$. In sum, call it $+_e$, the degree should be additive, which leads to the identification of the sum $\sum_e$ as an elementwise product $\times_e$. One can identify neutral element $1_\circ$ of $\circ$ as $1_\circ=Id$ and the neutral element $0_e$ of $+_e$ as ordinary unit $0_e=1$.

   The inverse corresponds to $g^{-1}$ for $\circ$, which is a many-valued algebraic function and to $1/g$ for $times$. The maps $g$, which do not allow decomposition $g=h\circ i$, can be identified as functional primes and have prime degree. $f$ is a functional prime if it does allow the decomposition $f=g\circ h$. One can construct functional integers as composites of functional primes $g_p$.

   The non-commutativity of $\circ$ could be seen as a problem. The fact that the maps $g$ act like operators suggest that for the quantum versions of functional primes $g_p$ the primes in the product commute. Functional integers/rationals can be mapped to integers by a morphism mapping their degree to integer/rational.

2. One can also define functional polynomials $P(X)$, quantum polynomials, using these operations. In the terms $p_n\circ X^n$ $p_n$ and $g$ should commute and the sum $\sum_e p_nX^n$ corresponds to $+_e$. The zeros of functional polynomials satisfy the condition $P(X)=0_e=1$ and give as solutions roots $X_k$ as functional algebraic numbers. The fundamental theorem of algebra generalizes at least formally if $X_k$ and $X$ commute. The roots have representations as

space-time surfaces. One can also define functional discriminant $D$ as the $\circ$ product of root differences $X_k -_e X_l$, with $-_e$ identified as element-wise division. The prime factors are representable as space-time surfaces.

What about functional p-adics?

1. The functional powers $g_p^{\circ k}$ of primes $g_p$ define analogs of powers of p-adic primes and one can define a functional generalization of p-adic numbers as quantum p-adics. The coefficients $X_k$ $X_k \circ g_p^k$ are polynomials with degree smaller than $p$. The sum $+_e$ so that the roots are disjoint unions of the roots of $X_k \circ g_p^k$.

2. Large powers of prime appearing in p-adic numbers must approach $0_e$ with respect to the p-adic norm so that $g_P^n$ must effectively approach $Id$ with respect to $\circ$. Intuitively, a large $n$ in $g_P^n$ corresponds to a long p-adic length scale. For large $n$, $g_P^n$ cannot be realized as a space-time surface in a fixed CD. This would prevent their representation and they would correspond to $0_e$ and Id. During the sequence of SSFRs the size of CD increases and for some critical SSFRs a new power can emerge to the quantum p-adic.

3. Universal Witt polynomials $W_n$ define an alternative representation of p-adic numbers reducing the multiplication of p-adic numbers to elementwise product for the coefficients of the Witt polynomial. The roots for the coefficients of $W_n$ define space-time surfaces: they should be the same as those defined by the coefficients of functional p-adics.

There are many open questions.

1. The question whether the hierarchy of infinite primes has relevance to TGD has remained open. It turns out that the 4 lowest levels of the hierarchy can be assigned to the rational functions $f_i : H \to C^2$, $i = 1, 2$ and the generalization of the hierarchy can be assigned to the composition hierarchy of prime maps $g_p$.

2. Could the transitions $f \to g \circ f$ correspond to the classical non-determinism in which one root of $g$ is selected? If so, the p-adic non-determinism would correspond to classical non-determinism. Quantum superposition of the roots would make it possible to realize the quantum notion of concept.

3. What is the interpretation of the maps $g^{-1}$ which in general are many-valued algebraic functions if $g$ is rational function? $g$ increases the complexity but $g^{-1}$ preserves or even reduces it so that its action is entropic. Could selection between $g$ and $g_{-1}$ relate to a conscious choice between good and evil?

4. Could one understand the p-adic length scale hypothesis in terms of functional primes. The counter for functional Mersenne prime would be $g_2^n/g_1$, where division is with respect to elementwise product defining $+_e$? For $g_2$ and $g_3$ and also their iterates the roots allow analytic expression. Could primes near powers of $g_2$ and $g_3$ be cognitively very special?

# 2 Two Galois groups

In TGD it is possible to define two generalizations of the Galois group: I call them internal and external Galois groups. Both notions of the Galois group are needed.

## 2.1 Internal Galois group

The 4-D Galois group, the internal Galois group, is assumed to permute the regions of a single connected component of the space-time surface realized as roots of the pair $(f_1, f_2)$ defining the space-time surface. The internal Galois group would act as analytic flows of $H$ transforming the regions as roots to ech other so that the action is analogous to that of a braid group.

1. It is easy to see that the space-time surface in general consists of several disjoint regions if $(f_1, f_2)$ is expressible as the composite $(f_1, f_2) = (g_1(h_1, h_2), g_2(h_1, h_2))$. In this case the space-time surface is union of disjoint surfaces $h_i = r_i$, where $r_i$ correspond to the roots of $g_i$. The permutations of the roots for a connected component of the space-time surface would realized as analogs of braidings.

2. The space-time regions identified as roots of $(f_1, f_2)$ for a single connected component would have string world sheets as interfaces having hypercomplex time coordinates $u, v$. Suppose that there are $n$ string world sheets. The number of string world sheets/folds can be larger than $n$. If folds are between any pair $i, j$ are present then the number of folds cannot be larger than $(n-1)n$: in this case all pairs $i, j$ would have two folds. Circle is a simple example: it has 2 sheets and 2-folds: 1,2 and 2,1.

   Since the $M^4$ complex coordinates $w$ and roots as its values labelling the string world sheets are in general complex, one can say that the fold is complexified. For a cusp (see this) the two folds can be ordered. Fold would now involve a string world sheet and cusp would combine two folds. At the vertex of the cusp where 3 roots co-incide, two folds would disappear. This suggests that the string world sheets connect at their ends associated with the disappearing folds and form a single string world sheet.

3. Catastrophe theory suggests that all catastrophes and hence also the space-time surfaces can be constructed from complexified cusps. The folds, which appear on a cloth, can be ordered. If so, folds between roots $i, i+1$ and $i-1, i$ are possible and would come from a single cusp but folds with $|i - j \geq 1$ would not be possible. This could give rise to the ordering of the roots $W_i$. Does this mean that the Galois group is cyclic?

4. This brings in mind twistor amplitudes and planar diagrams, which correspond to Feynman diagrams with no crossing lines and therefore embeddable in plane. Non-planar Feynman diagrams are a problem of the twistor Grasmannian approach [B2, B1] since they have no twistorial representation. The Feynman diagrams with crossing lines can be embedded in the plane with holes, whose boundaries are connected by cylinders as kinds of wormholes. In string models, the corresponding diagrams involve this kind of wormholes. This suggests that if the 2-D projection of the space-time sheets with constant values of hypercomplex coordinates has a topology with a genus $g$ larger than 0, the space-time surface contains wormholes connecting roots with $|i - j \geq 1$. In this case also the generalized Galois group is non-Abelian. Wormhole contacts defining Euclidean regions ($CP_2$ type extremals) could be such connections.

   To include wormhole contacts as connectors of the Minkowskian space-time sheets, one should allow besides the Minkowskian folds also the presence of the Euclidean $CP_2$ type extremals with a light-like $M^4$ curve, possibly geodesic, as $M^4$ projection. For these Euclidean regions the string world sheet would reduce to this curve since the second hypercomplex coordinate would be constant.

The internal Galois group could relate to the TGD view of topological qubits [L13].

1. The quantum-mechanical transfer of fermions between regions corresponding to roots of $(f_1, f_2)$ does not require a continuous path. Classical transfer requires a path going through a fold at which the two roots as space-time regions meet. Fold corresponds to a boundary of a string world sheet identified as fermion line. Folds are labelled by the values of the complex coordinate $w$ having interpretation as roots.

2. There is a direct analogy to the case of condensed matter majorana fermions suggested to define topological qubits. For a Majorana fermion two branches of the Fermi surface touch each other at point and the energy difference for the branches is zero at this point. Majorana fermions are assigned with these points and they would be located at the ends of a wire [L13]. In the TGD framework the folds would correspond to the seats of topological qubits.

## 2.2 Outer Galois group

The element-wise multiplication of the function for pairs $(f_1, f_2)$ is essential for the identification of the outer Galois group and gives an algebra, which is enough for identifying the Galois group as group of automorphisms for the algebraic extension of rationals involved. Outer Galois group permutes the roots of $g$, which are algebraic numbers in the extension of $E$ and label the disjoint components of the spacetime surfaces. These two Galois groups commute and the outer Galois group relates to the internal Galois group in the same way as the Galois group of an extension of rationals to the Galois group of complex rations generated by complex conjugation.

The outer Galois group is natural for the TGD realization of the Langlands duality, discussed from the TGD point of view in [L10].

1. A simpler version of the outer Galois group is associated with dynamical complex analytic symmetries $g : C \to C$: $(f_1, f_2) \to (g_1 \circ f_1, f_2)$. Here $g_1$ does not have a parametric dependence on $f_2$. The outer Galois group relates to each other *disjoint* space-time surfaces. When $g$ reduces to map $g : C \to C$, one can assign to it an ordinary Galois group relating to each other the disjoint roots of $g \circ f$ realized as disjoint 4-surfaces $(f_1, f_2) = (r_1, 0)$.

2. The notion of outer Galois group generalizes to the general situation $g = (g_1, g_2)$. Also now the roots of $g \circ f$ are disjoint space-time surfaces representing roots as pairs of algebraic numbers $(f_1, f_2) = (r_{i,1}, r_{i,2})$. Is it possible to assign to the roots the analog of the Galois group?

   This group should act as a group of automorphisms of some algebraic structure. This structure cannot be a field but algebra structure is enough. These arithmetic operations would be component-wise sum $(a, b) + (c + d) = (a + c, b + d)$ and componentwise multiplication $(a, b) * (c, d) = (ac, bd)$. The basic algebra would correspond to the points of $(x, y) \in E^2$ or rationals and the extension would be generated by the pairs $(f_1, f_2) = (r_{i,1}, r_{i,2})$. This structure has an automorphism group and would serve as a Galois group. The dimension of the extension of $E^2$ could define the value of the effective Planck constant.

   Also the notion of discriminant can be generalized to a pair $(D_1, D_2)$ of discriminants using the component-wise product for the differences of root pairs. Could $D_i$ be decomposed to a product of powers of algebraic primes of the extension of $E^2$?

3. In [L10] the idea that the space-time surfaces can be regarded as numbers was discussed. For a given $g$, one can indeed construct polynomials having any for algebraic numbers in the extension $F$ of $E$ defined by $g$. $g$ itself can be represented in terms of its $n$ roots $r_i = (r_{i,1}, r_{i,2})$, $i = 1, n$ represented as space-time surfaces as a product $\prod_i (f_1 - r_{i,1}, f_2 - r_{i,2})$ of pairs of monomials. One can generalize this construction by replacing the pairs $(r_{i,1}, r_{i,2})$ with any pair of algebraic numbers in $F$. Therefore all algebraic numbers in $F$ can be represented as space-time surfaces. Also the sets formed by numbers in $F$ can be represented as unions of the corresponding space-time surfaces.

# 3 Symmetries and dynamical symmetries

## 3.1 Maps $g : C^2 \to C^2$ as dynamical symmetries

### 3.1.1 The action of $g : C^2 \to C^2$ on $f : H \to C^2$

It is good to look first at the action of $g$ on $f$ in more detail.

1. In the simplest situation $f_i$ and $g_i$ are polynomials with coefficients in $E$. The functional composition $f \to g \circ f$ increases algebraic complexity if the degree of $g$ is higher than 1. The interpretation would be as a spectrum generating or dynamical symmetry.

   If $g = (a, b; c, d)$, $ad - bc = 1$ as an element of $SL(2, C)$ acts on $(f_1, f_2)$ linearly, the degree $g$ is 1 and the complexity does not increase. The conditions $(f_1, f_2) = (0, 0)$ imply $(af_1 + bf_2, cf_1 + df_2) = (0, 0)$ so that the original space-time surface $f_1, f_2) = (0, 0)$ is a solution. The interpretation would be as a gauge symmetry. However, also the space-time surfaces

$(af_1 + bf_2, cf_1 + df_2) = (0,0)$ with $(f_i, f_j) \neq (0,0)$ could be considered and one would obtain an entire $SL(2,C)$ orbit. The interpretation would be as a dynamical symmetry.

2. $f_i$ and $g_i$ could also be rational functions but this is not necessary. Since the roots of $f_i$ correspond to the roots of the polynomial $P$ defining the numerator of $R = P/Q$, $Q$ does not affect the roots as space-time surfaces. If one requires that inversion acts as $SL(2,C)$ symmetry for $f_i$, $Q$ must be polynomial so that it can have zeros. Inversion would map the sectors $P_i = 0$ and $Q_i = 0$ to each other.

### 3.1.2 Prime polynomials $g_p : C^2 \to C^2$ and complexity hierarchy

The polynomials $(P_1, P_2)$ and also the rational functions $(g_1 = P_1/Q_1, g_2 = P_2/Q_2)$ form a well-defined complexity hierarchy.

1. In the general case, the space-time surfaces $(f_1, f_2) = (0,0)$: $H \to C^2$ can have several disjoint components. It is also possible to have quantum superposition of these surfaces and also many-particle states consisting of a subset of the root. There are several disjoint components, when $(f_1, f_2)$ is a composite function of form $f = g \circ f$. In other words, one has $(f_1, f_2) = (g_1(h_1, h_2), g_2(h_1, h_2))$. The space-time surfaces correspond to roots $h_i = r_i$, which are disjoint.

2. $SL(2,C)$ transformations however act linearly in $C^2$ and the original space-time surface $(f_1, f_2) = (0,0)$ solves the conditions. Note that also the surfaces $(af_1 + bf_2, cf_1 + df_2) = (0,0)$ with $(f_1, f_2) \neq (0,0)$ define solutions. Their action is therefore equivalent with the action of a multiplicative unit. Hence $SL(2,C)$ acts either as a gauge symmetry or as a dynamical symmetry.

3. To avoid disjoint union of space-time surfaces $(f_1, f_2)$ must be a prime polynomial in the sense that it does not allow the functional composition $f = g \circ h$ with $g$ having degree higher than 1. For the polynomials of a single variable, this is the case if the degree of the polynomial is prime but this is not a necessary condition for primeness. As already found, this condition generalizes to the polynomials of 3 complex variables considered in the recent case.

   Space-time surfaces of these kinds are natural candidates for fundamental objects and the polynomial in question would have prime degree with respect to each of the 3 complex coordinates of $H$: this would make 3, presumably small primes. The composites formed of maps $g$ and of these fundamental function pairs $f$ would define cognitive representations of the surface defined by $f$ as kind of statements about statements. An interesting question is whether these surfaces could correspond to elementary particles.

Consider first the primality for pairs $(g_1, g_2)$.

1. For the polynomials of a single variable, this is the case if the degree of the polynomial is prime but this is not a necessary condition for primeness. In the recent case this means that by a suitable choices of gauge using $SL(2,C)$ transformation one could choose that for prime pairs $(g_1(f_1, f_2), g_2(f_1, f_2))$ $g_1$ has highest power of $f_1$ equal to prime. Single prime labels the prime polymial pair $(g_1, g_2)$.

2. There is also a natural measure of complexity as the number of maps $g$, which correspond to prime polynomial pairs $(g_1 = P_1, g_2 = P_2)$ appearing in the functional composite with a pair of prime polynomials $(f_1, f_2)$. Here the prime polynomials $P_i$ must have degree higher than 1 in order to increase the complexity and affect the space-time surface at all.

What about primality in the case of $(f_1, f_2) : H \to C^2$, which are polynomials of 3 complex coordinates of $H$. The situation reduces to that for $f_1$ by the above argument but does this mean that for prime polynomial pairs $(f_1, f_2)$, $f_i$ are characterized by 3 prime degrees?

1. In this case one would have reducible polynomials $f_1 = \xi_1^{p_1} \xi_2^{p_2} w^{p_3}$ as prime polynomials. The conditions $(f_1, f_2) = 0$ would define 3 disjoint 4-surfaces which might have common points. This situation is prevented if the the polynomials are required to be irreducible but one

would have polynomials $\xi_1^{p_1}$, $\xi_2^{p_2}$, and $w^{p_3}$ as polynomial primes for $f_1$ and $f_2$. The problem is that these polynomials involve only a single power and their roots are zero irrespective of the value of $p_i$ so that all these powers would correspond to the same space-time surface. One can say that all roots for these powers are degenerate and equal to zero. The physical interpretation would be in terms of maximal criticality.

2. The coefficients of powers of $p$ for p-adic numbers can be regarded as numbers in a finite field $F_p$. Now the numbers of $F_p$ would correspond to polynomials $P(z)$ of degree lower than $p$. Could the analogs of the number of $F_p$ correspond to the sums of the products of powers of $w$ *resp.* $\xi_1$ *resp.* $\xi_2$ with maximal exponents smaller $p_1$ *resp.* $p_2$ *resp.* $p_3$. If this picture is correct, the counterparts of prime powers would be $\xi_1^{n_1p_1}\xi_2^{n_2p_2}w^{n_3p_3}$. As if one had 3 p-adicities simultaneously. If irreducibility is required only $\xi_1^{n_1p_1}$, $\xi_2^{n_2p_2}$, $w^{n_3p_3}$ are possible.

   For the functional analogs of p-adic numbers as sums of polynomials of $\xi_1, \xi_2$, and $w$ expanded with respect to powers of powers of $\xi_1^{n_1p_1}, \xi_2^{n_2p_2}$ and $w^{n_3p_3}$ with coefficients as polynomials of single variable of lower degree, this problem is not encountered.

3. Space-time surfaces corresponding to prime pairs $(f_1, f_2)$ are candidates for fundamental objects and the polynomial in question would have prime degree with respect to each of the 3 complex coordinates of $H$: this would make 3, presumably small primes. The composites formed of maps $g$ and of these fundamental function pairs $f$ would define cognitive representations of the surface defined by $f$ as kind of statements about statements. An interesting question is whether these surfaces could correspond to elementary particles.

A highly interesting observation is that the numbers allowing expansions in powers of an integer $n$ having powers of primes belonging to some set can be regarded as p-adic integers for all these primes. One might say that these numbers belong to an intersection of these number fields. This could allow gluing of p-adic factors of adeles to single continuous structure. This suggests the possibility of multi-p p-adicity. The discriminant D of a polynomial defined as root differences can be expressed as a product of powers of so called ramified primes and the question is which of them is physically selected and why. Could multi-p p-adicity prevail that the expansions of physical quantities are in powers of D. I have also proposed that D, or its suitable power, is the number theoretical counterpart for the exponent of Kähler function as vacuum functional.

### 3.1.3 Ramified primes for the iterates $g_p^{\circ n}$

One can ask whether the analogs of ramified primes for the Witt polynomials assignable abstraction hierarchies $g \circ g \circ ... \circ f$ and powers $g^n = (g_1^n, g_2^n)$ for which the degree of the polynomials is $n \times p$, $p$ the primes assignable to $g$.

1. The ramified primes for the Witt polynomials for $g \circ g \circ ...g \circ f$ and $g^n$ defining analogs of powers $p^n$ of p-adic numbers. Note that the roots of $g \circ g \circ ...g \circ f$ are a property of $g \circ g \circ ... \circ g$ and do not depend on $f$ in case that they exist as surfaces inside the CD.

2. The interesting question is whether and how the ramified primes could relate to the ramified primes assignable to a generalized Witt polynomial $W_n$. The iterated action of prime $g$ giving $g \circ g...g \circ f$ is the best candidate.

   There is hope that even the p-adic length scale hypothesis could be understood as a ramified primes assignable to some functional prime. The large values of p-adic primes require that very large ramified primes for the functional primes $(f_1, f_2)$. This would suggest that the iterate $g \circ ..... \circ g \circ f$ acting on prime $f$ is involved. For $p \simeq q^k$, $k^{th}$ power of $g$ characterized by prime $g$ is the first guess.

Generalized p-adic numbers as such are a very large structure and the systems satisfying the p-adic length scale hypothesis should be physically and mathematically special. Consider the following assumptions.

1. Consider generalized p-adic primes associated restricted to the case when $f_2$ is not affected in the iteration so that one has $g = (g_1, Id)$ and $g_1 = g_1(f_1)$ is true. This would conform

with the hypothesis that $f_2$ defines the analog of a slowly varying cosmological constant. If one assumes that the small prime corresponds to $q = 2$, the iteration reduces to the iteration appearing in the construction of Mandelbrot fractals and Julia sets. If one assumes $g_1 = g_1(f_1, f_2)$, $f_2$ defines the analog of the complex parameter appearing in the definition of Mandelbrot fractals. The values of $f_2$ for which the iteration converges to zero would correspond to the Mandelbrot set having a boundary, which is fractal.

2. For the generalized p-adic numbers one can restrict the consideration to mere powers $g_1^n$ as analogs of powers $p^n$. This would be a sequence of iterates as analogs of abstractions. This would suggest $g_1(0) = 0$.

3. The physically interesting polynomials $g_1$ should have special properties. One possibility is that for $q = 2$ the coefficients of the simplest polynomials make sense in finite field $F_2$ so that the polynomials are $P_2(z \equiv f_1, \epsilon) = z^2 + \epsilon z = z(z + \epsilon)$, $\epsilon = \pm 1$ are of special interest. For $q > 2$ the coefficients could be analogous to the elements of the finite field $F_q$ represented as phases $exp(i2\pi k/3)$.

Consider now what these premises imply.

1. Quite generally, the roots of $P^{\circ n}(g_1)$ are given $R(n) = P^{\circ -n}(0)$. $P(0) = 0$ implies that the set $R_n$ of roots at the level $n$ are obtained as $R_n = R_n(new) \cup R_{n-1}$, where $R_n(new)$ consist of $q$ new roots emerging at level $n$. Each step gives $q^{n-1}$ roots at the previous level and $q^{n-1}$ new roots.

2. It is possible to analytically solve the roots for the iterates of polynomials with degree 2 or 3. Hence for $q = 2$ and 3 (there is evidence for the 3-adic length scale hypothesis) the inverse of $g_1$ can be solved analytically. The roots at level $n$ are obtained by solving the equation $P(r_n) = r_{n-1,k}$ for all roots $r_{n-1,k}$ at level $n - 1$. The roots in $R_{n-1}(new)$ give $q^{n-1}$ new roots in $R_n(new)$.

3. For $q = 2$, the iteration would proceed as follows:

$$0 \rightarrow \{0, r_1\} \rightarrow \{0, r_1\} \cup \{r_{21}, r_{22}\} \rightarrow \{0, r_1\} \cup \{r_{21}, r_{22}\} \cup \{r_{121}, r_{221}, r_{122}, r_{222}\} \rightarrow ... \ .$$

4. The expression for the discriminant $D$ of $g_1^{\circ n}$ can be deduced from the structure of the root set. $D$ satisfies the recursion formula $D(n) = D(n, new) \times D(n-1) \times D(n, new; n-1)$. Here $D(n, new)$ is the product

$$\prod_{r_i, r_j \in \in D(n,new)} (r_i - r_j)^2$$

and

$D(n, new; n - 1)$ is the product

$$\prod_{r_i \in D(n,new), r_j \in D(n-1)} (r_i - r_j)^2 \ .$$

5. At the limit $n \rightarrow \infty$, the set $R_n(new)$ approaches the boundary of the Fatou set defining the Julia set.

As an example one can look at the iteration of $g_1(z) = z(z - \epsilon)$.

1. The roots of $z(z - \epsilon) = 0$ are $\{0, r_1\} = \{0, \epsilon\}$. At second level, the new roots satisfy $z(z - \epsilon) = r_1 = \epsilon$ given by $\{(\epsilon/2)(1 \pm \sqrt{1 + 4r_1}\}$. At the third level the new roots satisfy $z(z - \epsilon) = r_2$ and given by $\{(\epsilon/2)(1 \pm \sqrt{1 + 4r_2}\}$.

2. The points $z = 0$ and $z = \epsilon$ are fixed points. Assume $\epsilon = 1$ for definiteness. The image points $w(z) = z(z - \epsilon)$ satisfy the condition $|w(z)/z| = |z - 1|$. For the disk $D(1,1) : |z - 1| \leq 1$ the image points therefore satisfy $|w| \leq |z| \leq 2$ and belong to the disk $D(0,2) : |z| \leq 2$.

   For the points in $D(0,2)\backslash D(1,1)$ the image point satisfies $|w| = |z - 1||z|$ giving $|z| - 1| \leq |w| \leq |z| + 1$. Inside $D(0,2)\backslash D(1,1)$ this gives $0 \leq |w| \leq 3$. Therefore $w$ can be inside $D(2,0)$ including $D(1,1)$ also inside disk $D(0,3)$.

   For the points $z$ outside $D(2,0)$ $|w| = |z - 1||z| \geq 2$. So that the iteration leads to infinity here.

3. For the inverse of the iteration relevant for finding the roots of $f^{\circ(-n)}$ leads from the exterior of $D(2,0)$ to its interior but cannot lead from interior to the exterior since in this case $f$ would lead to exterior to interior. Hence the values of the roots $w_n$ in $\cup_n f^{\circ(-n)}(0)$ belong to the disc $D(2,0)$.

4. One can look at the asymptotic situation for very large values of $n$. At $n^{th}$ step $2^{n-1}$ new roots emerge by doubling and one has $r_{n+1,\pm} = (1/2)(1 \pm \sqrt{1 + 4r_{n,\pm}})$. For $r_{n,\pm} < -1/4$ the root pair becomes complex and could stay complex at the next steps. This happens already at the step from $r_2 = 1/2(1 \pm \sqrt{\sqrt{5}}) \rightarrow r_3$. If the iteration gives at some step a double real root, its further iterations could approach a fixed point at this limit. This root $r_n \rightarrow r$ would satisfy $r = (1/2)(1 \pm \sqrt{1 + 4r})$ giving $r^2 - 2r = 0$ with root $r_1 = 2$ and $r_1 = 0$ these are the intersections of the disk $D(0,2)$ with real axis. Note that $r_1 = 2$ is not a fixed point of $z(z - 1)$.

   There should exist a root $r_n$, which at the real axes in the range $(0,2)$. This would require that $1 + 4r_n = 0$ giving a double root $r_n = -1/4$. The next steps would give $r_{n+1} = +1/2 \pm \sqrt{3} \rightarrow r_{n+2} = 1/2(1 \pm \sqrt{2 \pm \sqrt{3}})$. Second root would be complex. The positive real roots are $r_{n+1,+} \simeq 1.366$ and $r_{n+2,+} = 1.7708$. This suggests that the convergence to $r = 2$ takes place for the positive roots. If this is the case the $D$ discriminant contains the product of the differences for these positive roots approaching zero. There is however no guarantee that the double root $r_n = 1/2$ emerges in the iteration.

   The prime decompositions of $D$ for $k = 1, 2, ..., 7$ are $\{1 : 1\}, \{5 : 1\}, \{5 : 3, 11 : 1\}, \{5 : 7, 11 : 3, 311 : 1\}, \{2 : 48, 3 : 3, 43 : 1, 7^3 : 1, 6577 : 1, 5521801 : 1, -1 : 1\}$, $\{2 : 209, 59 : 2, 3117269 : 1, 356831 : 1\}$, $\{2 : 596, 2358900226164371 : 1, -1 : 1\}$, where $p : m$ denotes the prime and its multiplicity. $-1 : 1$ tells that the discriminant is negative.

The conjecture deserving to be killed is that the discriminant $D$ for the iterate has Mersenne primes as factors for primes $n$ defining Mersenne primes $M_n = 2^n - 1$ and that also for other values of $n$ $D$ contains as a factor ramified primes near to $2^n$.

## 3.2 About the identification of the Lie groups appearing in Langlands duality?

Transformations $(g_1, g_2)$ acting as symmetries should not increase the complexity and therefore should preserve the degree of the numerator or perhaps decrease it. Several alternatives can be considered.

Transformations $(g_1, g_2)$ acting as symmetries should not increase the complexity and therefore should preserve the degree of the numerator or perhaps decrease it. Several alternatives can be considered.

1. If it is required that the polynomials $f_i$ remain polynomials, then $SL(2, C)$ that acts on $(f_1, f_2)$ like in spinors is a natural alternative. A possible interpretation is as a Lorentz group or alternatively as a group $SL(2, C)$ assignable to the Virasoro algebra.

2. The allowance of rational transformations $g$ and also rational functions $f_i$ would conform with the notion of modular group representations. If they are are allowed and if one requires that there is no mixing of $f_1$ and $f_2$ as mildly suggested by the element-wise product for $(f_1, f_2)$, the group reduces to $SL(2, C) \times SL(2, C)$. $SL(2, C)$ consists of Möbius transformations $z \rightarrow (az + b)/(cz + d)$ (see this). For polynomials $f_1$, this gives new solutions except in the

case of inversion $f_1 \rightarrow 1/f_1$. In this case one does not obtain a new solution unless one assumes that $f_1$ is a rational function $f_1 = P/Q$ such that $Q$ has zeros.

$SL(2,C)$ has a rich spectrum of subgroups and the modular representations are invariant under some discrete subgroup of $SL(2,C)$. The modular group corresponds to $SL(2,Z)$ which has various discrete subgroups leaving modular forms invariant. There is an entire hierarchy of subgroups associated with the algebraic extensions of $Z$ and in this case the matrix elements would be algebraic integers. Now the integers for subgroup $SL(2,Z)$ would be replaced with the algebraic integers for $E$ appearing as the coefficients of $f_i$ and $g_i$.

3. If one allows the mixing of $f_i$, Möbius group is replaced with group $SL(3,C)$. What is interesting is that $SL(3,C)$ contains $SU(3)$ as a subgroup acting as isometries of $CP_2$. A second interesting observation is that also $SL(3,C)$ allows McKay correspondence in which the finite subgroups of $SU(2)$ are replaced by finite subgroups of $SU(3)$ [L2]. This is highly desirable in the TGD framework since $SU(3)$ acts as isometries of $CP_2$. An interesting question is whether the McKay correspondence holds true for $SL(n,C)$, $n > 3$.

Where should the Lie group for the analogs of Möbius transformations act? It is not natural to require that a discrete subgroup would leave the space-time surface invariant. The most natural option is that the action takes place in the "world of classical worlds" (WCW) formed by the generalized Bohr orbits satisfying holography= holomorphy principle. The counterparts of modular forms could correspond to WCW spinor fields invariant under the appropriate discrete subgroup of the generalized Möbius group.

## 3.3 Physical interpretation of the generalized modular group and spectrum generating group

One can consider several physical interpretations for the generalized modular group and dynamical spectrum generating algebra formed by the maps $g : C^2 \rightarrow C^2$.

1. Is the interpretation of $SL(2,C)$ as a Lorentz group reasonable? The McKay correspondence would refer to finite subgroups of $SU(2)$. This interpretation is not necessary since the Lorentz group and Poincare group act in the moduli space of causal diamonds (CDs). The discrete subgroups of $SU(2)$ appearing in Mac-Kay correspondence act in $C$ as modular transformations.

2. Could $SL(3,C)$ refer to $SU(3)$. It is known that $SL(3,C)$ allows the generalization of Mac-Kay correspondence to the finite subgroups of $SU(3)$. $SU(2)$ can be identified as a rotation group and a subgroup of the color group.

   Does this pose an interpretational problem? I have encountered a similar problem earlier in the twistorialization [L9]: the twistor spheres of $M^4$ twistor space and $CP_2$ twistor space are identified and this strongly suggests a close correspondence with the representations of rotation group and weak gauge group, the holonomy group $U(2)$ of $CP_2$, which is identifiable as a subgroup of $SU(3)$. The quark and lepton doublets are indeed spin and isospin doublets and this would allow us to realize this kind of correlation. In the recent formulation of the twistorialization without explicit introduction of the twistor spaces of $M^4$ and $CP_2$, the twistor spheres appear also as spheres embedded to the spacetime surfaces in $H$. Could the identification of these two $SU(2)$ subgroups be a part of the same story?

3. $SL(2,C)$ could also correspond to the sub-algebra of the Virasoro algebra of the string models. $SL(3,C)$ would naturally generalize this algebra to a 4-D situation. A generalization of Super Virasoro algebra involving two variables occurs naturally in TGD. The gauge conditions satisfied for the Super Virasoro algebra and associated Kac Moody type algebras are essential in string models. A possible interpretation of the Super Virasoro algebra in terms of infinitesimal analytic transformations which have interpretation as general coordinate transformations so that although they do not respect the degree of the polynomial they do not change the physics.

   In the TGD framework, a breaking of superconformal invariance is assumed to occur. The half-algebras associated with these algebras allow an infinite fractal hierarchy of sub-algebras isomorphic to the entire algebra and super-conformal symmetry can break down to this kind of sub-algebra [L7]. Therefore algebra generators with finite conformal weight below some maximum value would not act anymore as gauge symmetries but tranform to dynamical symmetries. In the recent case, these generators could correspond to maps $g$, which correspond to polynomial or rational functions with degree below some maximum value.

4. $SL(3, C)$ would naturally generalize this algebra to a 4-D situation and define the extension of Virasoro algebra two the case of two complex variables. This would be natural because the string world sheets are replaced by spacetime surfaces.

Also the representations of the analogs of Super Virasoro and Super Kac-Moody algebras (in particular super-symplectic algebra) are essential in TGD [L7]. A natural expectation is that they are also generalized modular representations and therefore involve the outer Galois group associated with the space-time surfaces at the various levels of the hierarchy defined by the maps $g$. This would conform with the view that the outer Galois group acts as physical symmetry group in the TGD Universe. I have earlier developed this view in detail in the construction of quantum states. The original identification of the Galois group was not however quite correct.

## 3.4 Langlands duality for the representations of the Lorentz group

In TGD, the modular forms defined in the hyperbolic space $H^3$ are especially interesting. Lorentz group acts on both. The earlier proposal is that modular forms can be generalized to $H^3$ as an analog of mass shell or Lorentz invariant cosmic time=constant hyperboloid. The discrete subgroup of $SL(2, C)$ as a symmetry group would define tessellations of $H^3$: this is a rather strong assumption.

Lorentz group and its discrete subgroups act on $H^3$ or possibly on the light-cone boundary at which the holographic data resides. Generalized modular forms could be also assigned with WCW spinor fields. The counterpart of the Galois group would be the same as in the above proposal. This picture applies also to color symmetries. This would give rise to the analogs of lattice waves in $E^3$. The holographic data invariant under a discrete subgroup would define tessellations as analogs of the lattices in $E^3$ [L4]. One application is a proposal of a universal realization of genetic code based on completely exceptional tessellation of $H^3$ involves instead of single Platonic solid the three Platonic solids with triangular faces. Also applications in cosmological scales are possible and there is some empirical evidence that stars could be assigned to a tessellation of $H^3$ [L8].

# 4 Quantum arithmetics

The function pairs $f = (f_1, f_2) : H \rightarrow C^2$ define a function field with respect to element-wise sum and multiplication. This is also true for the function pairs $g = (g_1, g_2) : C^2 \rightarrow C^2$. Now functional composition $\circ$ is an additional operation. This raises the question whether ordinary arithmetics and p-adic arithmetics might have functional counterparts.

## 4.1 Functional (quantum) counterparts of integers, rational and algebraic numbers

Do the notions of integers, rationals and algebraic numbers generalize so that one could speak of their functional or quantum counterparts? Here the category theoretical approach suggesting that degree of the polynomial defines a morphism from quantum objects to ordinary objects leads to a unique identification of the quantum objects.

1. For maps $g : C^2 \rightarrow C^2$, both the ordinary element-wise product and functional composition $\circ$ define natural products. The element-wise product does not respect polynomial irreducibility as an analog of primeness for the product of polynomials. Degree is multiplicative in $\circ$. In the sum, call it $+_e$, the degree should be additive. This leads to the identification of $+_e$ as an elementwise product. One can identify neutral element $1_\circ$ of $\circ$ as $1_\circ = Id$ and the neutral element $0_e$ of $+_e$ as ordinary unit $0_e = 1$. This is a somewhat unexpected conclusion.

The inverse of $g$ with respect to $\circ$ corresponds to $g^{-1}$ for $\circ$, which is a many-valued algebraic function and to $1/g$ for $+_e$. The maps $g$, which do not allow decomposition $g = h \circ i$, can be identified as functional primes and have prime degree. If one restricts the product and sum to $g_1$ (say), the degree of a functional prime $g$ corresponds to an ordinary prime. These functional integers/rationals can be mapped to integers by a morphism mapping their degree to integer/rational. $f$ is a functional prime with respect to $\circ$ if it does not allow a decomposition $f = g \circ h$. One can construct integers as products of functional primes.

2. The non-commutativity of $\circ$ could be seen as a problem. The fact that the maps $g$ act like operators suggest that for the functional primes $g_p$ the primes in the product commute. Since $g$ is analogous to an operator, this can be interpreted as a generalization of commutativity as a condition for the simultaneous measurability of observables.

3. One can also define functional polynomials $P(X)$, quantum polynomials, using these operations. In the terms $p_n \circ X^n$ $p_n$ and $g$ should commute and the sum $\sum_e p_n X^n$ corresponds to $+_e$. The zeros of functional polynomials satisfy the condition $P(X) = 0_e = 1$ and give as solutions roots $X_k$ as functional algebraic numbers. The fundamental theorem of algebra generalizes at least formally if $X_k$ and $X$ commute. The roots have representations as space-time surfaces. One can also define functional discriminant $D$ as the $\circ$ product of root differences $X_k -_e X_l$, with $-_e$ identified as element-wise division.

## 4.2 About the notion of functional primeness

There are two cases to consider corresponding to $f$ and $g$. Consider first the pairs $(f_1, f_2)$: $H \to C^2$.

1. Primeness could mean that $f$ does not have a composition $f = g \circ h$. Second notion of primeness is based on irreducibility, which states that $f$ does not reduce to an elementwise product of $f = g \times h$. Concerning the definition of powers of functional primes in this case, a possible problem is that the power $(f_1^n, f_2^n)$ defines the same surface as $(f_1, f_2)$ as a root with n-fold degeneracy. Irreducibility eliminates this problem but does not allow defining the analog of p-adic numbers using $(f_1^n, f_2^n)$ as analog of $p^n$.

2. Since there are 3 complex coordinates of $H$, $f_i$ are labelled by 3 ordinary primes $p_r(f_i)$, $r = 1, 2, 3$, rather than single prime $p$. By the earlier physical argument related to cosmological constant one could assume $f_2$ fixed, and restrict the consideration to $f_1$. Every functional p-adic number, in particular functional prime, corresponds to its own ramified primes. The simplest functional would correspond to $(f_1, f_2) = (0, 0)$ (could this be interpreted as stating the analog of $mod\ p = 0$ condition).

3. The degrees for the product of polynomial pairs $(P_1, P_2)$ and $(Q_1, Q_2)$ are additive. In the sum, the degree of the sum is not larger than the larger degree and it can happen that the highest powers sum up to zero so that the degree is smaller. This reminds of the properties of non-Archimedean norm for the p-adic numbers. The zero element defines the entire $H$ as a root and the unit element does not define any space-time surface as a root.

Also the pairs $(g_1, g_2)$ can be functional primes, both with respect to powers defined by element-wise product and functional composition $\circ$.

1. The ordinary sum is the first guess for the sum operation in this case. Category theoretical thinking however suggests that the element-wise product corresponds to sum, call it $+_e$. In this operation degree is additive so that products and $+_e$ sums can be mapped to ordinary integers. The functional p-adic number in this case would correspond to an elementwise product $\prod X_n \circ P_p^n$, where $X_n$ is a polynomial with degree smaller than $p$ defining a reducible polynomial.

2. A natural additional assumption is that the coefficient polynomials $X_n$ commute with each other and $P_p$. This is natural since the $X_n$ and $P_p$ act like operators and in quantum theory a complete set of commuting observables is a natural notion. This motivates the term quantum p-adics. The space-time surface is a disjoint union of space-time surfaces assignable to the

factors $X_k \circ P_p^k \circ f$. In quantum theory, quantum superpositions of these surfaces are realized. If the surface associated with $X_k \circ P_p^k \circ f$ is so large that it cannot be realized inside the CD, it is effectively absent from the pinary expansion. Therefore the size of the CD defines a pinary cutoff.

## 4.3 The notion of functional p-adics

What about functional p-adics?

1. The functional powers $g_p^{\circ k}$ of prime polynomials $g_p$ define analogs of powers of p-adic primes and one can define a functional generalization of p-adic numbers as quantum p-adics. The coefficients $X_k$ in $X_k \circ g_p^k$ are polynomials with degree smaller than $p$. The first idea which pops up in mind is that ordinary sum of these powers is in question. What is however required is the sum $+_e$ so that the roots are disjoint unions of the roots of the $+_e$ summands $X_k \circ g_p^k$.

2. Large powers of prime appearing in p-adic numbers must approach $0_e$ with respect to the p-adic norm so that $g_P^n$ must effectively approach $Id$ with respect to $\circ$. Intuitively, a large $n$ in $g_P^n$ corresponds to a long p-adic length scale. For large $n$, $g_P^n$ cannot be realized as a space-time surface in a fixed CD. This would prevent their representation and they would correspond to $0_e$ and Id. During the sequence of SSFRs the size of CD increases and for some critical SSFRs a new power can emerge to the quantum p-adic.

The very inspiring discussions with Robert Paster, who advocates the importance of universal Witt Vectors (UWVs) and Witt polynomials (see this) in the modelling of the brain, forced me to consider Witt vectors as something more than a technical tool. As the special case Witt vectors code for p-adic number fields.

1. Both the product and sum of ordinary p-adic numbers require memory digits and are therefore technically problematic. This is the case also for the functional p-adics. Witten polynomials solve this problem by reducing the product and sum purely digit-wise operations.

2. Universal Witt vectors and polynomials can be assigned to any commutative ring $R$, not only p-adic integers. Witt vectors $X_n$ define sequences of elements of a ring $R$ and Universal Witt polynomials $W_n(X_1, X_2, ..., X_n)$ define a sequence of polynomials of order $n$. In the case of p-adic number field $X_n$ correspond to the pinary digit of power $p^n$ and can be regarded as elements of finite field $F_{p,n}$, which can be also mapped to phase factors $exp(ik2\pi/p)$. The motivation for Witt polynomials is that the multiplication and sum of p-adic numbers can be done in a component-wise manner for Witt polynomials whereas for pinary digits sum and product affect the higher pinary digits in the sum and product.

3. In the general case, the Witt polynomial as a polynomial of several variables can be written as $W_n(X_0, X_1, ...) = \sum_{d|n} dX_d^{n/d}$, where $d$ is a divisor of $n$, with 1 and $n$ included. For p-adic numbers $n$ is power of $p$ and the factors $d$ are powers of $p$. $X_d$ are analogous to elements of a finite field $G_{p,n}$ as coefficients of powers of $p$.

Witt polynomials are characterized by their roots, and the TGD view about space-time surfaces both as generalized numbers and representations of ordinary numbers, inspires the idea how the roots of for suitably identified Witt polynomials could be represented as space-time surfaces in the TGD framework. This would give a representation of generalized p-adic numbers as space-time surfaces making the arithmetics very simple. Whether this representation is equivalent with the direct representation of p-adic number as surfaces, is not clear.

Could the *prime* polynomial pairs $(g_1, g_2) : C^2 \to C^2$ and $(f_1, f_2) : H = M^4 \times CP_2 \to C^2$ (perhaps states of pure, non-reflective awareness) characterized by ordinary primes give rise to functional p-adic numbers represented in terms of space-time surfaces such that these primes could correspond to ordinary p-adic primes?

# 5 Infinite primes, the notion of rational prime, and holography= holomorphy principle

The notion of infinite prime [K12, K4, K6] emerged a repeated quantization of a supersymmetric arithmetic quantum field theory in which the many-fermion states and many-boson formed from the single particle states at a given level give rise to free many-particle states at the next level. Also bound states of these states are included at the new level. There is a correspondence with rational functions as ratios $R = P/Q$ of polynomials and infinite prime can be interpreted as prime rational function in the sense that $P$ and $Q$ have no common factors. The construction is possible for any coefficient field of polynomials identified as rationals or extension of rationals, call it $E$.

At a given level implest polynomials $P$ and $Q$ are products of monomials with roots in $E$, say rationals. Irreducible polynomials correspond to products of monomials with algebraic roots in the corresponding extension of rationals and define the counterparts of bound states so that the notion of bound state would be purely number theoretic. The level of the hierarchy would be characterized by the number of variables of the rational functions.

Holography= holomorphy principle suggests that the hierarchy of infinite primes could be used to construct the functions $f_1 : H \rightarrow C$ and $f_2 : H \rightarrow C$ defining space-time surfaces as roots $f = (f_1, f_2)$. There is one hypercomplex coordinate and 3 complex coordinates so that the hierarchy for $f_i$ would have 4 levels. The functions $g : C^2 \rightarrow C^2$ define a hierarchy of maps with respect to the functional composition $\circ$. One can identify the counterparts of primes with respect to $\circ$ and it turns out that the notion of infinite prime generalizes.

## 5.1 The construction of infinite primes

Consider first the construction of infinite primes.

1. Two integers with no common prime factors define a rational $r = m/n$ uniquely. Introduce the analog of Fermi sea as the product $X = \prod_p p$ of all rational primes. Infinite primes is obtain as $P = nX/r + mr$ such that $m = \prod p_k$ is a product for finite number of primes $p_k$, $n$ is not divisible by any $p_k$, and m has as factors powers of some of primes $p_k$. The finite and infinite parts of infinite prime correspond to the numerator and denominator of a rational $n/m$ so that rationals and infinite primes can be identified. One can say that the rational for which $n$ and $m$ have no common factors is prime in this sense.

   One can interpret the primes $p_k$ dividing $r$ as labels of fermions and $r$ as fermions kicked out from the Fermi sea defined by $X$. The integers $n$ and $m$ as analogs of many-boson states. This construction generalizes also to athe algebraic extensions $E$ of rationals.

2. One can generalize the construction to the second level of the hierarchy. At the second level one introduces fermionic vacuum $Y$ as a product of all finite and infinite primes at the first level. One can repeat the construction and now integers $r, m$ and $n$ are products of the monomials $P(m/n, X) = nX/r + mr$ represented as infinite integers and . The analog of $r$ from the new fermionic vacuum away some fermions represented by infinite primes $P(m/n, X) = nX/r + mr$ by kicking them out of the vacuum. The infinite integers at the second level are analogous to rational functions $P/Q$ with the polynomials $P$ and $Q$ defined as the products of ratio of the monomials $p(m/n, X) = X/r + mr$ taking the role of $n$ and $m$. These polynomials are not irreducible.

   One can however generalize and assume that they factor to monomials associated with the roots of some irreducible polynomial $P$ (no rational roots) in some extension $E$ of rationals. Hence also rational functions $R(X) = P(X)/Q(X)$ with no common monomial factors as analogs of primes defining the analogs of primes for rational functions emerge. The lowest level with rational roots would correspond to free many-fermion states and the irreducible polynomials to a hierarchy of fermionic bound states.

3. The construction can be continued and one obtains an infinite hierarchy of infinite primes represented as rational functions $R(X_1, X_2, ..X_n) = P(X_1, X_2, ..X_n)/Q(X_1, X_2, ..X_n)$ which have no common prime factors of level $n - 1$. At the second level the polynomials are $P(X, Y) = \sum_k P_{n_k}(X)Y^k$. The roots $Y_k$ of $P(X, Y)$ are obtained as ordinary roots of a

polynomials with coefficients $P_{n_k}(X)$ depending on $X$ and they define the factorization of $P$ to monomials. At the third level the coefficients are irreducible polynomials depending on $X$ and $Y$ and the roots of $Z$ are algebraic functions of $X$ and $Y$.

Physically this construction is analogous to a repeated second quantization of a number theoretic quantum field theory with bosons and fermions labelled/represented by primes. The simplest states at a given level of free many-particle states and bound states correspond to irreducible polynomials. The notion of free state depends on the extension $E$ of rationals used.

## 5.2 Infinite primes and holography= holomorphy principle

How does this relate to holography= holomorphy principle? One can consider two options for what the hierarchy of infinite prime could correspond to.

1. One considers functions $f = (f_1, f_2) : H \to C^2$, with $f_i$ expressed in terms of rational functions of 3 complex coordinates and one hyperbolic coordinate. The general hypothesis is that the function pairs $(f_1, f_2)$ defining the space-time surfaces as their roots $(f_1, f_2) = (0, 0)$ are analytic functions of generalized complex coordinates of $H$ with coefficients in some extension $E$ of rationals.

2. Now one has a pair of functions: $(f_1, f_2)$ or $(g_1, g_2)$ but infinite primes involve only a single function. One can solve the problem by using element-wise sum and product so that both factors would correspond to a hierarchy of infinite primes.

3. One can also assign space-time surfaces to polynomial pairs $(P_1, P_2)$ and also to pairs rational functions $(R_1, R_2)$. One can therefore restrict the consideration to $f_1 \equiv f$. $f_2$ can be treated in the same way but there are some physical motivations to ask whether $f_2$ could define the counterpart of cosmological constant and therefore could be more or less fixed in a given scale.

The allowance of rational functions forces raises the question whether zeros are enough or whether also poles needed?

1. Hitherto it has been assumed that only the roots $f = 0$ matter. If one allows rational functions $P/Q$ then also the poles, identifiable as roots of $Q$ are important. The Compactification of the complex plane to Riemann-sphere $CP_1$ is carried out in complex analysis so that the poles have a geometric interpretation: zeros correspond to say North Pole and poles to the South pole for the map of $C \to C$ interpreted as map $CP_1 \to CP_1$. Compactication would mean now to the compactification $C^2 \to CP_1^2$.

   For instance, the Riemann-Roch theorem (see ) is a statement about the properties of zeros and poles of meromorphic functions defined at Riemann surfaces. The so called divisor is a representation for the poles and zeros as a formal sum over them. For instance, for meromorphic functions at a sphere the numbers of zeros and poles, with multiplicity taken into account, are the same.

   The notion of the divisor would generalize to the level of space-time surfaces so that a divisor would be a union of space-time surfaces representing zero and poles of $P$ and $Q$? Note that the iversion $f_i \to 1/f_i$ maps zeros and poles to each other. It can be performed for $f_1$ and $f_2$ separately and the obvious question concerns the physical interpretation.

2. Infinite primes would thus correspond to rational functions $R = P/Q$ of several variables. In the recent case, one has one hypercomplex coordinate $u$, one complex coordinate $w$ of $M^4$, and 2 complex coordinates $\xi 1, \xi_2$ of $CP_2$. They would correspond to the coordinates $X_i$ and the hierarchy of infinite primes would have 4 levels. The order of the coordinates does not affect the rational function $R(u, w, \xi 1, \xi_2)$ but the hypercomplex coordinate is naturally the first one. It seems that the order of complex coordinates depends on the space-time region since not all complex coordinates can be solved in terms of the remaining coordinates. It can even happen that the coordinate does not appear in $P$ or $Q$.

   The hypercomplex coordinate $u$ is in a special position and one can ask whether rational functions for it are sensical. Trigonometric functions and Fourier analysis look more natural.

What could be the physical relationship between the space-time surfaces representing poles and zeros?

1. $P$ and $Q$ would have no common polynomial (prime) factors. The zeros *resp.* poles of $R = P/Q$ as zeros of $P$ *resp.* $Q$ are represented as space-time surfaces. Could the zeros and poles correspond to matter and antimatter. Could one assign baryon and lepton numbers to $f_1$ and $f_2$ and so that the total baryon and lepton numbers for $P_i$ and $Q_i$ would sum up to zero for meromorphic functions $f_i$.

   Note that besides the fermionic vacuum annihilated by annihilation operations there is also fermionic vacuum annihilated by the creation operators and these vacua correspond to opposite boundaries of CD in ZEO.

2. Could infinite primes have two representations. A four-levelled hierarchy represented as space-time surfaces in terms of holography= holomorphy principle and as fermion states represented as hierarchy of second quantizations for both quarks and leptons and corresponding bosonic states. What could these 4 quantizations mean physically?

3. Can the space-time surfaces defined by zeros and poles intersect each other? If BSFR permutes the two kinds of space-time surfaces, they should intersect at 3-surfaces defining holographic data. The failure of the exact classical determinism implies that the 4-surfaces are not identical.

Does the time reversal in BSFR have a geometric counterpart? Inversion and complex conjugation at the level of $C^2$ are the obvious candidates.

1. Could the time reversal occurring in "big" state function reduction (BSFR) change zero to poles and vice versa and correspond to the inversion $f_i \to 1/f_i$ inducing $P/Q \to Q/P$? The inversion $f_i \to 1/f_i$ mapping zeros to poles and vice versa can be carried independently for $f_i$. This does not support the assignment of inversion with the time reversal. This interpretation would also require that the 3-D regions at the boundary of CD defining holographic data are invariant under the inversion. This also forces us to ask whether both zeros and poles present for a given arrow of time or only for one arrow of time? Therefore the interpretation as analog of charge conjugation mapping fermions to antifermions looks more natural.

2. Complex conjugation replaces the Hamilton-Jacobi structure of $H$ with its conjugate. Complex conjugation makes sense also for $C^2$. Complex conjugation performed for both $H$ and $C^2$ does not affect the space-time surfaces. Holomorphic space-time surfaces and their antiholomorphic complex conjugates need not be disjoint. For instance, in $CP_2$ a homologically non-trivial geodesic sphere can be self-conjugate.

   If matter and antimatter were related by complex conjugation, holomorphy would require that matter *resp.* antimatter resides at holomorphic *resp.* space-time surfaces: could this relate to matter-antimatter asymmetry?

   Instead of inversion, complex conjugation in $C^2$ could be involved with the time reversal occurring in BSFR (it would not be the same as time reflection $T$). This would require that the 3-D regions defining holographic data (at the boundary of CD) are invariant under complex conjugation. The classical worlds with opposite arrows of geometric time would be related by complex conjugation.

## 5.3 Hierarchies of functional composites of $g : C^2 \to C^2$

One can consider also rational functions $g = (g_1, g_2)$ with $g_i = R = P_i/Q_i : C^2 \to C^2$ defining abstraction hierarchies. Also in this case elementwise product is possible but functional composition $\circ$ and the interpretation in terms of formation of abstractions looks more natural. Fractals are obtained as a special case. $\circ$ is not commutative and it is not clear whether the analogs of primes, prime decomposition, and the definition of rational functions exist.

1. Prime decompositions for $g$ with respect to $\circ$ make sense and can identify polynomials $f = (f_1, f_2)$ which are primes in the sense that they do not allow composition with $g$. These primal spacetime surfaces define the analogs of ground states.

2. The notion of generalized rational makes sense. For ordinary infinite primes represented as $P/Q$, the polynomials $P$ and $Q$ do not have common prime polynomial factors. Now / is replaced with a functional division $(f,g) \to f \circ g^{-1}$ instead of $(f,g) \to f/g$. In general, $g^{-1}$ is a many-valued algebraic function and the manivaluedness distinguishes between the analogs of polynomials and their inverses. The only exceptions are Möbius transformations forming a group. In the one-variable case for polynomials the inverse involves algebraic functions appearing in the expressions of the roots of the polynomial. This means a considerable generalization of the notion of infinite prime.

   What matters physically are the roots of $g \circ h^{-1}$. The condition $g \circ h^{-1} f = 0$ has as roots $h(r_n)$, where $r_n$ is the roots of $g \circ f = 0$. Therefore the situation is simple at the level of space-time surfaces. Could one think of generalizing the notion of group so that the counterparts of group operations would be many-valued?

3. One obtains the counterpart for the hierarchy of infinite primes. The analog for the product of infinite primes at a given level is the composite of prime $g$:s. The irreducible polynomials as realization of bound states for ordinary infinite primes replaces the coefficient field $E$ with its extension. The replacement of the rationals as a coefficient field with its extensions $E$ does the same for the composes of $g$:s. This gives a hierarchy similar to that of irreducible polynomials: now the hierarchy formed by rational functions with increasing number of variables corresponds to the hierarchy of extensions of rationals.

4. The conditions for zeros and poles are not affected since they reduce to corresponding conditions for $g \circ f$.

# 6 Some questions related to the maps $g$

The maps $g$ and possibly also their inverses which would be central in the realization of cognition and reflective hierarchies. These ideas are however far from their final form and in the following I try to imagine and exclude various alternatives.

## 6.1 What could happen in the transition $f \to g \circ f$?

The proposal is that in SSFR the transition $f \to g \circ f$ takes place. The number of roots becomes n-fold if $g$ is a rational function of form $P/Q$. What could this transition mean physically? One can consider two options.

### 6.1.1 The option allowing quantum realization of concept

The $nm$ roots (poles and zeros) for $g \circ f$, where $f$ as $m$ roots would be alternative outcomes of SSFR of which only a single outcome, or possible quantum superposition of the outcomes would be selected. What is so nice is that the classical non-determinism crucial for the TGD view of consciousness would follow automatically from the holography= holomorphy hypothesis without any additional assumptions.

Conservation laws conform with this view. All the alternative Bohr orbits would have the same classical conserved charges. The quantum superposition of the roots would represent a particular quantum realization of a concept and $f \to g \circ f$ would mean a refinement of the quantum concept defined by $f$.

The hypothesis that the classical non-determinism correspond to the p-adic non-determinism would transform to a statement that different Bohr orbits associated $g^{\circ k}$ define analogs for the sequences of $k$ pinary digits if there are $p$ outcomes for $g \circ f$. A possible interpretation would be in terms of a $k$-digit pinary digit sequence in powers of $p$. The largest integer would correspond to $n = 2^k$ for $g^{\circ k}$. The generalization of the notion of the notion of p-adic numbers for which $p$ is replaced by a functional prime $g$ and based on the generalization of Witt polynomials is suggestive. It remains unclear whether this could allow us to understand the generalization of the p-adic length scale hypothesis stating that a large prime $p \simeq p^k$ can be assigned to this set of Bohr orbits.

### 6.1.2 The option allowing a classical realization of concept

The union of $nm$ space-time surfaces, where $n$ is the degree of $g$ and $m$ is the number of roots of $f$, is generated in the step $f \to g \circ f$. The set of the $nm$ space-time surfaces would give a classical realization of a concept as a set. Does this make sense? The first grave objection is that there is no continuous time evolution between $f$ and $g \circ f$ multiplying the number of space-time surfaces by $n$. Second objection relates to the conservation laws which seem to be violated. The third objection is that classical non-determinism is lost. It seems that this objection cannot be circumvented. One can however consider the analogs of many-particle states in which some surfaces of this set carry fermionic zero energy states.

One can try to imagine ways to overcome the first two objections.

**Option I**: ZEO interpreted in the "eastern" sense in principle allows the creation of $n$ space-time surfaces from each of the $m$ space-time surfaces associated with $f$. This is because the total classical charges of the zero energy states as sums of those for states at the boundaries of CD vanish. Zero energy state would be analogous to a quantum fluctuation.

**Option II**: In standard ontology, the classical realization of the concept as union of space-time surfaces defining its instances is possible only in a situation in which space-time surfaces are vacua or nearly vacua. Could this kind of surface serve as a template for the non-vacuum physical systems?

Cell replication, which would correspond to $n = 2$ for $g$, was motivated by the consideration of both options, at least half-seriously. The instantaneous replication of the space-time surface representing the cell does not look sensible since the generation of biomatter requires a feed of metabolics and metabolic energy. Could a replicated field body serve as a kind of template for the formation of a final state involving two cells generated in $f \to g \circ f$? Could the replication occur at the level of the field body, proposed to control the biological body?

For **Option II**, conservation laws pose a problem for replication. In ZEO the classical charges of the space-time $nm$ surfaces should be those associated with the passive boundary of CD and therefore same as those for $f$.

1. Could the space-time surfaces be special in the sense that the classical charges vanish? The vanishing of classical conserved charges is not possible unless the classical action reduces to Kähler action allowing vacuum extremals. The finite size of CD indeed allows by Uncertainty Principle a slight violation of the classical conservation laws assignable to the Poincare invariance [L6]. This cannot be excluded and the original proposal [K3, K10] indeed was that Kähler action defines the classical action by its unique property of having huge classical non-determinism defining the 4-D analog of spin-glass degeneracy [K11] which could play a key role in biology.

   If one assigns to $M^4$ the analog of the Kähler structure [L5], this argument weakens since the induced $M^4$ and $CP_2$ Kähler forms must vanish for the vacuum extremals. However, for a given Hamilton-Jacobi structure defining the $M^4$ Kähler form, there exist space-time surfaces of this kind. They are Cartesian products of Lagrangian 2-manifolds of $M^4$ and $CP_2$ defining vacuum string world sheets.

   Holography= holomorphy principle, implying that Bohr orbits are minimal surfaces, seems to hold true for any classical action, which is general coordinate invariant and is determined by the induced geometry. For the Kähler action, the coefficient $\Lambda$ of the volume term, defining the analog of cosmological constant, would vanish. Holography= holomorphy principle does not allow Cartesian products of Lagrangian 2-manifolds of $M^4$ and $CP_2$. One could hope that their vacuum property could change the situation but this does not look an elegant option.

2. For the standard ontology, one can also consider another option. The classical action, and therefore the classical conserved charges, are for the twistor lift proportional to $1/\alpha_K$, where $\alpha_K$ is Kähler coupling strength. The conservation of charges would suggest $\alpha_K \to n\alpha_K$ requiring $h_{eff} \to h_{eff}/n$ in the n-fold multiplication. For $h_{eff} = h$ this would require $h \to h/n$. This looks strange.

   $h$ need not however be the minimal value of $h_{eff}$ and I have considered the possibility that one has $h = n_0 h_0$ [L1], where $n_0$ corresponds to the ratio $R^2(CP_2)/l_P^2$. $CP_2$ size scale would be given by Planck length $l_P$ size scale but for $h = n_0 h_0$ the size scale would be scaled up to $R^2 \sim n_0 l_P^2$, $n_0 \in [10^7, 10^8]$. The estimate for $n_0$ is given by $n_0 = (7!)^2$ having numbers $2, 3, 5, 6, 7$ (primes $2, 3, 5, 7$) as factors [L1]. $R(CP_2)$ would naturally correspond to the $M^4$ size of a wormhole throat. $h$ could be reduced by a factor appearing in $n_0$ and there is some evidence for the reduction of $h_eff$ by a small power of 2 [D1]. This mechanism could work for a functional prime $g$ characterized by prime $p \in \{2, 3, 5, 7\}$.

To classical realization of concept does not look realistic except possibly for **Option I**.

## 6.2 About the interpretation of the inverses of the maps $g$

What could be the interpretation of the inverse maps $g^{-1}$ for $g = P/Q$, assuming that they can occur? $g^{-1}$ is a multivalued algebraic function analogous to $z^{1/n}$. In $f \to g^{-1} \circ f$ the roots $r_n$ of $f$ are mapped to $g(r_n)$ so that their number does not increase. For the iterate of $g$, $g^{-1}$ means the reduction of the number of roots by $1/n$. The complexity does not increase and can even decrease.

This is just the opposite for what occurs in $f \to g \circ f$. The increase of complexity is assigned with number theoretic evolution and NMP. Suppose for a moment that the inverses $g^{-1}$ are allowed. What could be their interpretation?

1. The sequence of the inverses $g^{-1}$ does not correspond to non-determinism and does not give rise to a refinement of either classical or quantum concept. There is no increase of complexity and it can be reduced for iterates.

2. Could the reduction of the cell to stem cell level as a reverse of cell differentiation, which occurs by cell replications, correspond at the level of the field body to a sequence of $g^{-1}$:s reducing the complexity. Could cancer correspond to this kind of process? This would conform with the interpretation in terms of the reduction of negentropy.

3. The first option is that the maps of type $g^{-1}$ are possible for both arrows of the geometric time. For the iterates of $g$, $g^{-1}$ destroys complexity and information and reduces the level of cognition in this case. $g^{-1}$ would obey anti-NMP in this case. Both maps $g$ and $g^{-1}$ make possible a trial and error process. If an iterate of $g$ is not involved, the roots $r_n$ of $h \circ f$ are mapped by $g$ to roots $g(r_n)$ and the number of roots is preserved. It is not clear whether the algebraic complexity is increased or reduced.

   This suggests that NMP [K7] is not lost if both maps of type $g$ and $g^{-1}$ are allowed? Furthermore, there is a lower bound for algebraic complexity but no upper bound so that it seems that NMP remains true even if maps of type $g^{-1}$ are allowed.

   Any quantum theory of consciousness should be able to say something about the quantum correlates of ethics [K13]. In TGD, one can assign the notion of good to state function reductions (SFRs) inducing the increase of quantum coherence occurring in a statistical sense in SFRs. It would correspond to the increase of algebraic complexity and would be accompanied by the increase of $h_{eff}$ and the amount of potentially conscious information. Is evil something something analogous to a thermodynamic fluctuation reducing entropy or can one speak of an active evil? Could the notion of evil as something active be assigned with the occurrence of maps of type $g^{-1}$?

4. The maps of type $g$ and $g^{-1}$ are reversals of each other and differ unless they act as symmetries analogous to Möbius transformations. Could they be assigned with SSFRs with opposite arrows of geometric time? If so, negentropy would not increase for both arrows of the geometric time and there would be a universal arrow of time analogous to that assumed in standard thermodynamics and defined by negentropy increase. If a universal arrow of time exists, it should somehow relate to the violation of time reflection symmetry $T$. To me this option does not look plausible.

   If this is the case, the trial and error process allowed by ZEO and based on pairs of BSFRs would involve a map of type $g^{-1}$ induced by SSFRs whereas the second BSFR would correspond to a map of type $g$. The sequence of SSFRs after the first BSFR would preserve or even reduce complexity and would mean starting from a new state at the passive boundary (PB) of CD. If the first BSFR is followed by a sequence of SSFRs of type $g$, it in general leads to a more negentropic new initial state at PB.

## 6.3 Could one understand p-adic length scale hypothesis?

What could be the physical interpretation of the prime polynomials $(f_1, f_2)$ and $(g_1, g_2)$, in particular $(g_1, Id)$ and how it relates to the p-adic length scale hypothesis [L3]?

1. p-Adic length scale hypothesis states that the physically preferred p-adic primes correspond to powers $p \simeq 2^k$. Also powers $p \simeq q^k$ of other small primes $q$ can be considered [K9] and there is empirical evidence of time scales coming as powers of $q = 3$ [?, ?]. For Mersenne primes $M_n = 2^n - 1$, $n$ is prime and this inspires the question whether $k$ could be prime quite generally.

2. Probably the primes as orders of prime polynomials do not correspond to very large p-adic primes ($M_{127} = 2^{127} - 1$ for electron) assigned in p-adic mass calculations to elementary particles.

   The proposal has been that the $p$ and $k$ would correspond to a very large and small p-adic length scale. The short scale would be near the $CP_2$ length scale and large scale of order elementary particle Compton length.

Could small-p p-adicity make sense and could the p-adic length scale hypothesis relate small-p p-adicity and large-p p-acidity?

1. Could the p-adic length scale hypothesis in its basic form reflect 2-adicity at the fundamental level or could it reflect that $p = 2$ is the degree for the lowest prime polynomials, certainly the most primitive cognitive level. Or could it reflect both?

2. Could $p \simeq 2^k$ emerge when the action of a polynomial $g_1$ of degree 2 with respect to say the complex coordinate $w$ of $M^4$ on polynomial $Q$ is iterated functionally: $Q \rightarrow PcircQ \rightarrow$ $...P \circ ...P \circ Q$ and give $n = 2^k$ disjoint space-time surfaces as representations of the roots. For $p = 2$ the iteration is the procedure giving rise to Mandelbrot fractals and Julia sets. Electrons would correspond to objects with 127 iterations and cognitive hierarchy with 127 levels! Could $p = M_{127}$ be a ramified prime associated with $P \circ ... \circ P$.

   If this is the case, $p \simeq 2^k$ and $k$ would tell about cognitive abilities of an electron and not so much about the system characterized by the function pair $(f_1, f_2)$ at the bottom. Could the $2^k$ disjoint space-time surfaces correspond to a representation of $p \simeq 2^k$ binary numbers represented as disjoint space-time surfaces realizing binary mathematics at the level of space-time surfaces? This representation brings in mind the totally discontinuous compact-open p-adic topology. Cognition indeed decomposes the perceptive field into objects.

3. This generalizes to a prediction of hierarchies $p \simeq q^k$, where $q$ is a small prime as compared to $p$ and identifiable as the prime order of a prime polynomial with respect to, say, variable $w$.

I have considered several identification of the p-adic primes and arguments for why the p-adic length scale hypothesis should be true.

1. One can imagine I have tentatively identified p-adic primes as ramified primes [L3] appearing as divisors of the discriminant $D$of a polynomials define as the product of root differences, which could correspond to that for $g = (g_1, Id)$.

   Could the 3 primes characterizing the prime polynomials $f_i : H \rightarrow C^2$ correspond to the small primes $q$? Could the ramified primes $p \simeq 2^k$ as divisors of a discriminant $D$ defined by the product of non-vanishing root differences be assigned with the polynomials obtained to their functional composites with iterates of a suitable $g$?

   Similar hypotheses can be studied for the iterates of $g : C^2 \rightarrow C^2$ alone. The study of this hypothesis in a special case $g = P_2 = x(x - 1)$ described in an earlier section did not give encouraging results. Perhaps the identification of p-adic prime as ramified primes is ad hoc. There is also the problem that there are several ramified primes, which suggests multi-p-p-adicity. The conjecture also fails to specify how the ramified prime emerges from the iterate of $g$.

2. A new identification of p-adic primes suggested by quantum p-adics is that p-adic primes correspond to primes defining the degrees of prime polynomials $g$ and that the Mersenne primes $N_n = 2^n - 1$ correspond to rational functions $P_2^{\circ n}/P_1$, where / corresponds to element-wise-division and $P_2$ can be any polynomials of degree 2. This would mean category theoretic morphism of quantum p-adics to ordinary p-adics. A more general form of the conjecture is that the rational functions $P_p^{\circ n}/P_k$ correspond to preferred p-adic primes.

   The reason could be that for these quantum primes it is possible to solve the roots as zeros and poles analytically for $p < 5$. This might make them cognitively very special. The primes $p = 2$ and $p = 3$ would be in a unique role information theoretically. For these primes there is indeed evidence for the p-adic length scale hypothesis and these primes are also highly relevant for the notion of music harmony [K8, K5, K14, K2, L4].

**Acknowledgements:** I want to thank Tuomas Sorakivi for inspiring discussions and for a valuable help in the computation of the roots of the iterated polynomials.

# REFERENCES

## Theoretical Physics

[B1] Huang Y-T Elvang H. Scattering amplitudes, 2013. Available at: https://arxiv.org/pdf/1308.1697v1.pdf.

[B2] Arkani-Hamed N et al. The S-Matrix in Twistor Space, 2009. Available at:https://arxiv.org/abs/0903.2110.

## Condensed Matter Physics

[D1] Mills R et al. Spectroscopic and NMR identification of novel hybrid ions in fractional quantum energy states formed by an exothermic reaction of atomic hydrogen with certain catalysts, 2003. Available at: https://www.blacklightpower.com/techpapers.html.

## Biology

## Books related to TGD

[K1] Pitkänen M. An Overall View about Models of Genetic Code and Bio-harmony. In *Genes and Memes: Part II. https://tgdtheory.fi/tgdhtml/Bgenememe2.html*. Available at: https://tgdtheory.fi/pdfpool/tessellationH3.pdf, 2023.

[K2] Pitkänen M. Geometric Theory of Bio-Harmony. In *Genes and Memes: Part II. https://tgdtheory.fi/tgdhtml/Bgenememe2.html*. Available at: https://tgdtheory.fi/pdfpool/harmonytheory.pdf, 2023.

[K3] Pitkänen M. Identification of the WCW Kähler Function. In *Quantum Physics as Infinite-Dimensional Geometry. https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/kahler.pdf., 2023.

[K4] Pitkänen M. Infinite Primes and Consciousness. In *Mathematical Aspect of Consciousness. https://tgdtheory.fi/tgdhtml/mathconsc.html*. Available at: https://tgdtheory.fi/pdfpool/infpc.pdf, 2023.

[K5] Pitkänen M. More Precise TGD View about Quantum Biology and Prebiotic Evolution. In *Evolution in TGD Universe. https://tgdtheory.fi/tgdhtml/Btgdevolution.html*. Available at: https://tgdtheory.fi/pdfpool/geesink.pdf, 2023.

[K6] Pitkänen M. Motives and Infinite Primes. In *TGD as a Generalized Number Theory: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdnumber3.html*. Available at: https://tgdtheory.fi/pdfpool/infmotives.pdf, 2023.

[K7] Pitkänen M. Negentropy Maximization Principle. In *TGD Inspired Theory of Consciousness: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdconsc1.html*. Available at: https://tgdtheory.fi/pdfpool/nmpc.pdf, 2023.

[K8] Pitkänen M. p-Adic Physics as Physics of Cognition and Intention. In *TGD Inspired Theory of Consciousness: Part II.* *https://tgdtheory.fi/tgdhtml/Btgdconsc2.html*. Available at: https://tgdtheory.fi/pdfpool/cognic.pdf, 2023.

[K9] Pitkänen M. Possible Role of p-Adic Numbers in Bio-Systems. In *Bio-Systems as Self-Organizing Quantum Systems.* *https://tgdtheory.fi/tgdhtml/BbioSO.html*. Available at: https://tgdtheory.fi/pdfpool/biopadc.pdf, 2023.

[K10] Pitkänen M. Recent View about Kähler Geometry and Spin Structure of WCW . In *Quantum Physics as Infinite-Dimensional Geometry.* *https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/wcwnew.pdf, 2023.

[K11] Pitkänen M. Spin Glasses, Complexity, and TGD. In *TGD and Condensed Matter.* *https://tgdtheory.fi/tgdhtml/BTGDcondmat.html*. Available at: https://tgdtheory.fi/pdfpool/sg.pdf, 2023.

[K12] Pitkänen M. TGD as a Generalized Number Theory: Infinite Primes. In *TGD as a Generalized Number Theory: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/visionc.pdf, 2023.

[K13] Pitkänen M. Time and Consciousness. In *TGD Inspired Theory of Consciousness: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdconsc1.html*. Available at: https://tgdtheory.fi/pdfpool/timesc.pdf, 2023.

[K14] Pitkänen M. What Music Could Teach about Consciousness? In *TGD and EEG: Part II.* *https://tgdtheory.fi/tgdhtml/Btgdeeg2.html*. Available at: https://tgdtheory.fi/pdfpool/musicophilia.pdf, 2023.

## Articles about TGD

[L1] Pitkänen M. Questions about coupling constant evolution. https://tgdtheory.fi/public_html/articles/ccheff.pdf., 2021.

[L2] Pitkänen M. McKay Correspondence from Quantum Arithmetics Replacing Sum and Product with Direct Sum and Tensor Product? . https://tgdtheory.fi/public_html/articles/McKayGal.pdf., 2022.

[L3] Pitkänen M. Two objections against p-adic thermodynamics and their resolution. https://tgdtheory.fi/public_html/articles/padmass2022.pdf., 2022.

[L4] Pitkänen M. About tessellations in hyperbolic 3-space and their relation to the genetic code . https://tgdtheory.fi/public_html/articles/tessellationH3.pdf., 2023.

[L5] Pitkänen M. Holography and Hamilton-Jacobi Structure as 4-D generalization of 2-D complex structure. https://tgdtheory.fi/public_html/articles/HJ.pdf., 2023.

[L6] Pitkänen M. New result about causal diamonds from the TGD view point of view. https://tgdtheory.fi/public_html/articles/CDconformal.pdf., 2023.

[L7] Pitkänen M. Symmetries and Geometry of the "World of Classical Worlds" . https://tgdtheory.fi/public_html/articles/wcwsymm.pdf., 2023.

[L8] Pitkänen M. The TGD view of the recently discovered gravitational hum as gravitational diffraction. https://tgdtheory.fi/public_html/articles/gravhum.pdf., 2023.

[L9] Pitkänen M. A fresh look at $M^8 - H$ duality and Poincare invariance. https://tgdtheory.fi/public_html/articles/TGDcritics.pdf., 2024.

[L10] Pitkänen M. About Langlands correspondence in the TGD framework. https://tgdtheory.fi/public_html/articles/Frenkel.pdf., 2024.

[L11] Pitkänen M. Gödel,Lawvere and TGD. https://tgdtheory.fi/public_html/articles/Gtgd.pdf., 2025.

[L12] Pitkänen M. Holography = holomorphy vision and elliptic functions and curves in TGD framework. https://tgdtheory.fi/public_html/articles/Weierstrass.pdf., 2025.

[L13] Pitkänen M. The realization of topological qubits in many-sheeted space-time. https://tgdtheory.fi/public_html/articles/majorana.pdf., 2025.