Holography and Hamilton-Jacobi Structure as a 4-D generalization of 2-D complex structure

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Abstract

Hamilton-Jacobi structure is a key notion of Topological Geometrodynamics proposed to generalize 2-D complex structure to 4-D case. Space-time surfaces $X^4 \subset H = M^4 \times CP_2$, as analogs of Bohr orbits realizing almost deterministic holography, are simultaneous extremals of both volume action and Kähler action if X^4 is determined by the simultaneous vanishing of 2 complex valued functions of H coordinates. In the simplest situation, the functions depend on a light-like coordinate of $M^2 \subset M^4$, complex coordinate of E^2 orthogonal to M^2, and 2 complex coordinates of CP_2.

The complex structures of 2-D Riemann surface are parameterized by the moduli space (Teichmüller parameters). The same should be true in 4-D case. The proposal is that Hamilton-Jacobi structure, defining an integrable distribution of partonic 2-surfaces and string world sheets, slices X^4 by partonic 2-surfaces X^2 and string world sheets Y^2. The complex (hyper-complex) structure for X^2 would depend on point of Y^2 (X^2). Space-time would define a 4-D surface in the product of Teichmüller space for the partonic 2-surfaces and its analog for string world sheets.

Contents

1 Introduction

It is good to begin with a brief introduction to Topological Geometrodynamics (TGD) and Quantum TGD. Physics as geometry and physics as number theory are the two complementary views of physics provided by TGD. I have collected to the two subsections of the Appendix a vocabulary giving a brief summary of the basic notions of TGD as geometry and TGD number theory.

1.1 TGD as a unification of standard model interactions and gravitation

The energy problem of General Relativity Theory (GRT) was the starting point of TGD for about 46 years ago. For a curved space-time the isometries of empty Minkowski space M^4 are lost so that the conservation laws implied by Noether's theorem are also lost.

The solution of the problem is obtained by assuming that the space-times at the fundamental level are 4-surfaces in $H = M^4 \times CP_2$. CP_2 codes for color and electroweak symmetries and M^4 for Poincare symmetries, and one obtains a geometrization of the standard model and unification of electroweak and color interactions and gravitation.

M^4 and CP_2 the only spaces allowing twistor space with Kähler structure [A5]. This makes possible the twistor lift of TGD [K12, K3, K2] for which space-time surfaces are replaced with 6-D surfaces representing twistor spaces as sphere bundles in the product $T(M^4) \times T(CP_2)$ of the twistor spaces of M^4 and CP_2. The 6-D Kähler action dimensionally reduces to a sum of Kähler action and volume term for the space-time surface. The p-adic length scale dependent coefficient of the volume term has an interpretation as cosmological constant.

H is also number-theoretically unique [L3, L4, L8] and one ends up to number theoretic *resp.* geometric view of physics in terms of 4-surfaces in complexified M^8 *resp.* H. $M^8 - H$ duality as the analog of momentum-position duality maps 4-surfaces in octonionic M^8_c to space-time surfaces in H. $M^8 - H$ duality is also analogous with Langlands duality [L5].

1.2 What Quantum TGD is?

Quantum TGD can be seen as a generalization of the wave mechanics of a point-like particle replaced with a 3-surface.

1. One can end up to TGD also as a generalization of a point-like particle (string model) by replacing a point-like particle (string) with a 3-particle. Space-time as a 4-surface is the orbit of a 3-D surface.

2. The first guess is that the configuration space for the positions of the particle is replaced with the "world of classical worlds" (WCW) as the space of 3-D surfaces in H. 4-D general coordinate invariance however demands that a given 3-surface of a 4-D space-time surface is accompanied by a 4-surface at which the 4-D general coordinate transformations can act. This surface must be highly unique and is analogous to Bohr orbit. This means holography. The alternative would be the assignment of all possible space-time surfaces and path integral quantization: this however fails due to an infinite hierarchy of divergences with no hope of renormalization.

   Since the Bohr orbits are not expected to be completely unique, one must improve the first guess: WCW is replaced with the space of 4-D space-time surface as Bohr orbits. In biology this would correspond to the replacement of 3-D structure replaced with function. From the computer science perspective, space-time surfaces are analogous to almost deterministic computer programs.

   (a) Quantum-classical correspondence is the first implication. Quantum state, zero energy state, is a superposition of classical time evolutions. Zero energy state means that the 3-surfaces at the opposite boundaries of causal diamond (CD) containing the space-time surface have identical total quantum numbers. This conforms with the general convention used in quantum field theories.

   (b) The second implication is that there is no path integral troubled by mathematical ill-definedness and divergences. Quantum states correspond to wave functions in the space of Bohr orbits defining WCW. The description of fermions forces to replace wave functions with WCW spinor fields. The construction of WCW spinor structure reduces to second quantization of free spinor fields of H.

   (c) The third implication is zero energy ontology (ZEO). Quantum jumps/state function reductions occur between the superpositions of the space-time surface as Bohr orbits and one gets rid of the basic problem of quantum measurement theory. The non-determinism of state function reduction implies no violation of classical determinism of the field equations defining the space-time surface almost uniquely.

3. The classical geometrization of gravitation and standard model gauge fields in terms of induced geometry extends a geometrization of the entire quantum physics in terms of the Kähler geometry and spinor structure of WCW. Kähler geometry is needed to geometrize hermitian conjugation. In the case of loop spaces the geometry is unique from its mere existence and the isometry group must be maximal [A3]. Same is expected in the 3-D case.

   The Kähler geometry of WCW is determined by Kähler function, which can be identified as the classical action for the Bohr orbit-like 4-surface. The classical action gives a direct connection with classical physics and Kähler action with volume term is forced by the twistor lift of TGD [K12, K10].

   The intuitive guess is that WCW is a union of infinite-D analogs of symmetric spaces labelled by zero modes, which do not contribute to the line element but appear as parameters of the Kähler metric of WCW. The isometries of WCW are induced from the symplectic transformations of $\delta M^4 \times CP_2$ by holography. Supersymplectic algebra corresponds to the isometries and the generalized Kac-Moody type algebras associated with the light-like partonic orbits correspond to holonomies acting as spectrum generating symmetries.

   The 2-D conformal invariance of $S^2 \subset M^4_+$ extends to of light-cone boundary δM^4_+ and at 3-D light-like orbits of partonic 2-surfaces extend their conformal symmetries to analogs of local gauge transformations depending on the light-like coordinate. Holography allows to extend these 3-D symmetries to 4-D symmetries.

4. Momentum position duality is central in wave mechanics. Which could be the analog of momentum space for the particles as an orbit of 3-surface? Momentum space is flat and the space of 4-surfaces as Bohr orbits in M^8 would naturally serve as analog of Bohr orbits in momentum space. Polynomials of a real variable define the 3-D data for number theoretic holography, which determines 4-surfaces in complexified M^8 as analogs of orbits in momentum space. The 3-D holographic data correspond to 3-surfaces at the mass shells

of $H^3 = M^4 \subset M_c^8$ determined by the roots of the polynomials. Associativity of the normal space of the 4-surface in M_c^8 defines the number theoretic holography. $M^8 - H$ duality maps these 4-surfaces of M^8 to space-time surfaces in H. CP_2 and its isometries and holonomies symmetries emerge in this way naturally. Color group corresponds to a subgroup of octonionic automorphisms at the level of M^8.

1.3 Could one realize the Bohr orbit property exactly in terms of 4-D generalization of complex structure?

Holography requires Bohr orbit property. What could this mean? String models serve as a guideline. String world sheets are extremals of area action of a very special kind. The induced metric defines a generalization of complex structure. The stringy coordinates consisting of 2 light-like coordinates (u, v), are hypercomplex analogs of complex coordinates. In these coordinates, the metric has no diagonal components. Hyper complex analyticity requires that the target space coordinates depend on the light-like coordinate u or v but not both. Field equations reduce to purely algebraic conditions and are solved by this ansatz.

Can one generalize this picture to the case of 4-D space-time surface or 4-D Minkowski space or possibly both as suggested by the general vision? For a general 4-D space-time surface, one cannot expect a generalization of the 2-D complex structure induced by the metric alone. One can however have such a structure as an induced structure implied by the proposed solution ansatz.

Holography requires Bohr orbit property. What does it mean?

1. The general ansatz for the preferred extremals involves only generalized complex structure for M^4 and complex structure for CP_2. One avoids the possible existence problems. The roots of two analytic functions of CP_2 complex coordinates and of generalized complex coordinates of M^4, defining what I have called Hamilton-Jacobi structure, would define the solution ansatz and realize Bohr orbit property. Hamilton-Jacobi structure would characterize M^4 as a dynamically generated structure rather than a structure defined at the fundamental level. The situation would be analogous to that for the spinor structure and twistor structure, whose non-existence in the general case produces problems in General Relativity.

2. What is especially nice is that the field equations would reduce to purely algebraic conditions for any general coordinate invariant action constructible in terms of the induced geometry. To see this consider first the 2-D case. Field equations involve only contractions of tensors having index pairs $(z, \overline{z})$ and $(\overline{z}, z)$ with tensors having only index pairs (z, z) and $(\overline{z}, \overline{z})$: they vanish identically. This generalizes to the 4-D situation.

   This means universality and realizes the quantum criticality of the TGD Universe. The differences between various actions is related to the values of conserved Noether charges, the value of the action and the boundary conditions at singularities, which include light-like boundaries or light-like interfaces between Minkowskian and Euclidian space-time regions.

2 About the definition of Hamilton-Jacobi structure

The notion of Hamilton-Jacobi structure is one of the key notions of TGD and is proposed to generalize the 2-D complex structure at space-time level. The physical vision is that space-time surfaces, as analogs of Bohr orbits realizing almost deterministic holography, are preferred extremals of some action defining classical TGD. These extremals are 4-D minimal surfaces in $H = M^4 \times CP_2$, which are simultaneous extremals of both volume action and Kähler action apart from lower-dimensional singularities.

As a matter of fact, and in accordance with the universality of quantum criticality, these surfaces would be preferred extremals for any general coordinate invariant action defined in terms of the induced geometry. In this framework, quantum TGD reduces to an analog of wave mechanics for point-like particles generalized to 2-D surfaces, or rather, their quite non-unique 4-D Bohr orbits.

This is true if the space-time surface is determined by the simultaneous vanishing of 2 complex valued functions of H coordinates giving 4 conditions and thus defining the space-time as a 4-surface. In the simplest situation, the functions are generalizations of functions of several complex variables and are functions of a light-like coordinate of $M^2 \subset M^4$ as an analog of complex coordinate, a complex coordinate of E^2 orthogonal to M^2, and 2 complex coordinates of CP_2.

2-D Riemann surfaces allow several complex structures parameterized by the moduli space (Teichmüller parameters). The same is expected to be true in the 4-D case, and the natural expectation that Hamilton-Jacobi structure, defining an integrable distribution of partonic 2-surfaces and string world sheets, defines a slicing of the space-time surface by partonic 2-surfaces and string world sheets such that the complex structure for the partonic 2-surface depends on space-time point as also its possibly existing analog for the string world sheet. Space-time could be seen as a 4-D surface in the product of Teichmüller space for the partonic 2-surfaces and its analog for string world sheets. Generalized conformal invariance would state that only these modular degrees of freedom are physical, which would mean huge reduction of degrees of freedom in accordance with the holography implying that space-time surfaces are analogous to Bohr orbits.

The notion of Hamilton-Jacobi structure is not new [K1] but its reduction to an integrable distribution of products of 2-D complex and hypercomplex structures is. In the sequel this notion will be discussed in detail.

2.1 Hermitian and Hyper-Hermitian structures

The starting point is the observation that besides the complex numbers forming a number field there are hyper-complex numbers. Imaginary unit i is replaced with e satisfying $e^2 = 1$. One obtains an algebra but not a number field since the norm is Minkowskian norm $x^2 - y^2$, which vanishes at light-cone $x = y$ so that light-like hypercomplex numbers $x \pm e)$ do not have inverse. One has "almost" number field.

Hyper-complex numbers appear naturally in 2-D Minkowski space since the solutions of a massless field equation can be written as $f = g(u = t - ex) + h(v = t + ex)$ whith $e^2 = 1$ realized by putting $e = 1$. Therefore Wick rotation relates sums of holomorphic and antiholomorphic functions to sums of hyper-holomorphic and anti-hyper-holomorphic functions. Note that u and v are hyper-complex conjugates of each other.

Complex n-dimensional spaces allow Hermitian structure. This means that the metric has in complex coordinates $(z_1, ..., z_n)$ the form in which the matrix elements of metric are non-vanishing only between z_i and complex conjugate of z_j. In 2-D case one obtains just $ds^2 = g_{z\overline{z}}dzd\overline{z}$. Note that in this case metric is conformally flat since line element is proportional to the line element $ds^2 = dzd\overline{z}$ of plane. This form is always possible locally. For complex n-D case one obtains $ds^2 = g_{i\overline{j}}dz^id\overline{z}^j$. $g_{i\overline{j}} = \overline{g_{j\overline{i}}}$ guaranteeing the reality of ds^2. In 2-D case this condition gives $g_{z\overline{z}} = \overline{g_{z\overline{z}}}$.

How could one generalize this line element to hyper-complex n-dimensional case. In 2-D case Minkowski space M^2 one has $ds^2 = g_{uv}dudv$, $g_{uv} = 1$. The obvious generalization would be the replacement $ds^2 = g_{u_iv_j}du^idv^j$. Also now the analogs of reality conditions must hold with respect to $u_i \leftrightarrow v_i$.

2.2 How to end up with Hamilton-Jacobi Structure

Consider next the path leading to the notion of Hamilton-Jacobi structure. 4-D Minkowski space $M^4 = M^2 \times E^2$ is Cartesian product of hyper-complex M^2 with complex plane E^2, and one has $ds^2 = dudv + dzd\overline{z}$ in standard Minkowski coordinates.

One can also consider more general integrable decompositions of M^4 for which the tangent space $TM^4 = M^4$ at each point is decomposed to $M^2(x) \times E^2(x)$. The physical analogy would be a position dependent decomposition of the degrees of freedom of massless particle to longitudinal ones ($M^2(x)$): light-like momentum is in this plane) and transversal ones ($E^2(x)$) polarization vector is in this plane). Cylindrical and spherical variants of Minkowski coordinates define two examples of this kind of coordinates (it is perhaps a good exercise to think what kind of decomposition of tangent space is in question in these examples). An interesting mathematical problem highly relevant for TGD is to identify all possible decompositions of this kind for empty Minkowski space.

The integrability of the decomposition means that the planes $M^2(x)$ are tangent planes for 2-D surfaces of M^4 analogous to Euclidian string world sheet. This gives slicing of M^4 to Minkowskian string world sheets parametrized by euclidian string world sheets. The question is whether the sheets are stringy in a strong sense: that is minimal surfaces. This is not the case: for spherical coordinates the Euclidian string world sheets would be spheres which are not minimal surfaces. For cylindrical and spherical coordinates howeverr $M^2(x)$ integrate to plane M^2, which is minimal surface.

Integrability means in the case of $M^2(x)$ the existence of light-like vector field J whose flow lines define a global coordinate. Its existence implies also the existence of its conjugate and together these vector fields give rise to $M^2(x)$ at each point. This means that one has $J = \Psi\nabla\Phi$: Φ indeed defines the global coordinate along flow lines. In the case of M^2 either the coordinate u or v would be the coordinate in question. This kind of flows are called Beltrami flows. Obviously the same holds for the transversal planes E^2.

Is it possible to generalize this metric to the case of general 4-D space with Minkowski signature of metric? At least the elements g_{uv} and $g_{z\overline{z}}$ are non-vanishing and can depend on both u, v and $z, \overline{z}$. They must satisfy the reality conditions $g_{z\overline{z}} = \overline{g_{\overline{z}z}}$ and $g_{uv} = \overline{g_{vu}}$ where complex conjugation in the argument involves also $u \leftrightarrow v$ besides $z \leftrightarrow \overline{z}$.

The question is whether the components g_{uz}, g_{vz}, and their complex conjugates are non-vanishing if they satisfy some conditions. They can. The direct generalization from complex 2-D space would be that one treats u and v as complex conjugates and therefore requires a direct generalization of the hermiticity condition

$$g_{uz} = \overline{g_{v\overline{z}}} \quad , \quad g_{vz} = \overline{g_{u\overline{z}}} \quad .$$

This would give complete symmetry with the complex 2-D (4-D in real sense) spaces. This would allow the algebraic continuation of hermitian structures to Hamilton-Jacobi structures by just replacing i with e for some complex coordinates.

2.3 A proposal for an explicit realization of the Hamilton-Jacobi structures

There are two options corresponding to the Hamilton-Jacobi structure in M^4 and for a general space-time surface.

1. The first option corresponds to the TGD view realizing holography as generalized holomorphy: the generalization of complex structure of space-time surface is induced from that of H. This is analogous to the induction of the complex structure of CP_2 to the complex 2-surfaces of CP_2 representable as roots of complex functions of complex coordinates of CP_2. For an arbitrary 2-surface of CP_2 the complex structure determined by the induced metric is not consistent with this complex structure.

2. The second option corresponds to a generalization of the stringy option for which the 2-dimensionality of the string world sheet or its Euclidian analog implies the existence of complex structure. This is not of course for space-time surfaces in the general case but would be realized for the first option and maybe even more generally for the preferred extremals.

For both options, the Hamilton-Jacobi structure should define a slicing of the space-time surface, not necessarily orthogonal, partonic 2-surfaces and 2-D string worlds sheets. The Hamilton-Jacobi structure could be seen as a 4-D generalization of 2-D complex structure.

This raises the question whether Hamilton-Jacobi structures could be seen as convolutions of 2-D complex structures with their Minkowskian analogs such that the conformal moduli of the partonic 2-surfaces and strings world sheets (possibly making sense) depend on the point space-time surface?

1. The conformal structures of partonic 2-surfaces are classified by their conformal moduli expressible in terms of Teichmüller parameters [K5]. Could one consider some kind of analytical continuation of the moduli spaces of the partonic 2-surfaces with different topologies to moduli spaces of time-like string world sheets?

This would give a direct connection with p-adic mass calculations in which the (p-adic counterparts of) these moduli spaces are central. In p-adic mass calculations [K5, K8], one assumes only partonic 2-surfaces. However, the recent proposal for the construction of preferred extremals of action as minimal surfaces, realizing holography in terms of a 4-D generalization of the holomorphy of string world sheets and partonic 2-surfaces, assumes a 4-D generalization of 2-D complex structure and this generalization could be just Hamilton-Jacobi structure.

2. The boundaries of a string world sheet can also have space-like portions and they would be analogous to the punctures of 2-D Euclidian strings (now partonic 2-surfaces are closed).

3. Can Minkowskian string world sheets have handles? What would the handle of a string world sheet look like? String world sheets have ends at the boundaries of the causal diamond (CD) playing a central role in zero energy ontology (ZEO).

   If the 1-D throat of the handle is a smooth curve, it must have portions with both space-like and time-like normal: this is possible in the induced metric but the portion with a time-like normal should carry vanishing conserved currents. For area action this is not possible.

   Intuitively, the throat of the handle would look physically to a splitting of a planar string to two pieces such that the conserved currents go to the handle. If this is the case, the 1-D throat would have two corners at which two time-like halves of the throat meet.

   The handle would be obtained by moving the throat along a space-like curve such that its Minkowskian signature and singularity are preserved. CP_2 contribution to the induced metric might make this possible.

2.3.1 Hamilton-Jacobi structure as a convolution of Minkowskian and Euclidian 2-D conformal structures

The proposal would mean the following.

1. The basic property of the slicing implies that to each point of the space-time surface X^4 can assign a unique closed 2-D partonic surface Y^2 and a unique 2-D string world sheet X^2. If the above conjecture is true one can assign to partonic 2-surface genus g and unique conformal moduli represented by a point of Teichmüller space [A1, A4, A2] assignable to the genus [K5]. Same is true for string world sheets.

2. Neither the partonic nor stringy genus need not be the same at all space-time points. Therefore one must allow the union of the Teichmüller spaces for various values of the genus glued together along partonic 2-surfaces/string world sheets topologically intermediate between two different genera. A torus which is pinching to a sphere is a good illustration of this. One might call this union a universal Teichmüller space. This notion is needed in the description of CKM mizing in terms of topological mixing for partonic 2-surfaces [K5].

3. Hamilton-Jacobi structure can be regarded as an embedding/immersion of the space-time surface to a product of the universal Teichmüller spaces for partonic 2-surfaces and string world sheets.

This interpretation allows the Hamilton-Jacobi structure to have singularities at which the genus of the partonic 2-surface of string world sheets changes. For instance, topological mixing proposed to be behind CKM mixing could correspond to a light-like orbit of partonic 2-surfaces as the boundary between Minkowskian region and Euclidian wormhole contact for which the genus of the partonic 2-surface changes.

It is convenient to speak of generating partonic 2-surfaces. They would correspond to partonic 2-surfaces assignable with the boundaries between space-time regions with Minkowskian *resp.* Euclidian signature of the induced metric. The Euclidean regions would correspond to wormhole contacts connecting two space-time sheets with Minkowskian signature and they would define building bricks of elementary particles.

The genus changing singularities would in general be 3-D surfaces since regions of a fixed genus are 4-D. The singularities are unavoidable if the space-time surface contains the orbits of wormhole contacts for which partonic 2-surfaces have different genera. Partonic 2-surfaces belong for a fixed value of time coordinate to a slicing for which the partonic 2-surfaces have the same genus and at some 3-D surfaces the partonic 2-surfaces of slicings have topologies intermediate between two different genera.

2.3.2 How to define Teichmüller parameters in Minkowskian case?

Could the space of Teichmüller parameters parameterizing the moduli space of Riemann surfaces genelize to the Minkowskian signature?

The definition and the basic facts about Teichmüller parameters are discussed in [K5]. Essential notions are as follows. Canonical homology basis (a_i, b_j) is defined by $2g$ homology generators such that the pair (a_i, b_i) corresponds intuitively to the two homologically non-trivial circles for the i:th handle. g holomorphic 1-forms ω_i and their complex conjugates define a basis for cohomology.

One can define Teichmüller parameters Ω_{ij} as values of ω_i for the g homology equivalence classes a_j of Riemann surface. The normalization condition is $\omega_i(a_j) \equiv \oint_{a_j} \omega_i dz = \delta_{i,j}$. Teichmüller parameters are defined as the values of ω_i for the generators b_j: $\Omega_{ij} = \omega_i(b_j) \equiv \oint_{b_j} \omega_i dz$. The complex dimension of the space of conformal equivalence classes is $d = 3(g-1)$ and for $g \geq 2$ smaller than the dimension of the $D = g(g+1)/2$ space of the Teichmüller parameters.

For $g \leq 2$ one has $d(g) = D(g)$, which presumably relates to the fact that all Riemann surfaces with $g \leq 2$ allow Z_2 as conformal symmetry. Physically, genera $g \leq 2$ are special and this provides an explanation for why there are only 3 fermion families. Since the metric of the partonic 2-surface is an induced metric in the TGD framework, this does not produce problems and for $g \leq 2$ the problems are absent in any case. The imaginary parts of Ω_{ij} are positive. The space of Teichmüller parameters can be regarded as a coset space $Sp(2g, R)/U(g)$ [A2], where $Sp(2g, R)$ is symplectic group for a symplectic space with g momentum-like and g position-like coordinates and has interpretation as quaternionic unitary group.

How could the Minkowskian counterparts of the Teichmüller parameters be defined?

1. What does one mean with hypercomplex numbers? Should one introduce the hypercomplex counterpart e of imaginary unit i satisfying $e^2 = 1$ so that hypercomplex M^2 coordinate would be $u = z + et$. One can solve e as $e = \pm 1$. This would correspond to the physical situation in which holomorphic functions correspond to functions of $t + z$ or $t - z$.

2. Suppose that it is possible to speak of homology of the string world sheets and assign homology generators with the handles. The complex coordinates $(z, \overline{z})$ is replaced in the case M^2 as analog of complex plane with $(u = t + z, v = t - z)$ and complex analytic functions correspond to hypercomplex analytic functions $u \to f(u)$. The physical intuition suggests that the counterpart of the upper half of the complex plane corresponds to $t > 0$. In the general case u and v would correspond to light-like coordinates for a string world sheet and in these coordinates the non-vanishing components of the induced metric would be $g_{uv} = g_{v,u}$. The induced metric would define these coordinates. These kinds of conditions are also applied in string models.

3. The analogs of holomorphic one-forms ω_i would be of form $\omega_i(u)du$ and also now the normalization condition would hold true. Now however the values of the forms would be real unless one introduces the hypercomplex counterpart e of the imaginary unit satisfying $e^2 = 1$. The Minkowskian counterparts of Teichmüller parameters would be real and the dimension of Teichmüller space would be only 1/2 of that for the Euclidian counterpart. Physically this would correspond to the fact that time direction does not correspond to a dynamical degree of freedom.

4. The naive intuitive guess is that the complex dimension $D = g(g+1)/2$ of the Euclidian Teichmüller space is replaced with a real dimension $D = g(g+1)/2$ and that the similar interpretation applies to the complex conformal moduli. Could complex continuation as an analog of Wick rotation work? This would mean the replacement of complex coordinates $(\Omega_{ij}, \overline{\Omega}_{ij})$ for Teichmüller space with light-like coordinates $(t_{ij} + z_{ij}, t_{ij} - z_{ij})$ combined with light-likeness condition $t_{ij} = z_{ij}$ eliminating the second degree of freedom work. This would imply the intuitively plausible reduction of dimension by 1/2.

This operation could have an interpretation in terms of a Wick rotation of the coset space $Sp(2g,R)/U(g)$, which has a complex structure. The complex coordinates $(\Omega_{ij},\overline{\Omega}_{ij})$ would be replaced with light-like coordinates and the second light-like coordinate would vanish. One might also think of replacing the Euclidian metric of the Teichmüller space of complex dimension D with Minkowskian metric of with signature $(d,-d)$ and identify the Minkowskian Teichmüller space with a light cone for which d coordinates are light-like.

Preferred extremal property, e. g. Bohr orbit of space-time surface, property is expected to pose quantization conditions, and an interesting question is whether both the Euclidian Teichmüller parameters and their Minkowskian counterparts satisfy quantization conditions. The integrals defining the values of Ω_{ij} are analogous to the integrals appearing in Bohr quantization conditions so that integer valuedness is what comes first in mind.

In the TGD framework, number theoretical quantization conditions are suggestive. Cognitive representations mean a restriction to an extension of rationals, in the case of momenta the components of momenta are algebraic integers associated with the extension. For physical states the total momenta would have integer value components. Could the values of both kinds of Teichmüller parameters be algebraic integers of the extension? In the Minkowskian case the algebraic integers would be real. A straightforward guess would be that the discretization means the replacement of real numbers R appearing in $Sp(2g,R)/U(g,R)$ with an extension of rationals in question so that the quantization would follow automatically.

2.3.3 WCW geometry and Hamilton-Jacobi structure

The generating partonic 2-surfaces are identified as geometric building bricks of elementary particles. Their 3-D light-like orbits are also a key piece of the description. They would carry fermionic lines as the light-like boundaries of string world sheets interpreted as counterparts of fermions in the quantum field theoretic picture. The justifications for this assumption were already given.

The Hamilton-Jacobi structure however inspires the question whether also a description in terms of the orbits of other than these generating partonic 2-surfaces could be possible and whether this condition could bring to the theory an additional gauge symmetry. I have indeed proposed that this could be the case.

1. If a gauge symmetry is in question, the effect of the symmetries relating pairs of string-world sheet and partonic 2-surfaces at different points of the space-time surface should leave the Kähler metric of the WCW unaffected.

   This is true if the Kähler function K_{WCW} is transformed by an addition of a real part of holomorphic function of WCW coordinates Z^k: $K\rightarrow K+f(Z^k)+\overline{f(\overline{Z^k})}$. This transformation can be regarded as a $U(1)$ gauge transformation.

2. A good candidate for these gauge transformations is provided by a 4-D generalization of 2-D holomorphic transformations leaving by definition the Hamilton-Jacobi structure unaffected. The generalization of holomorphic transformations restricted to the light-cone boundary $S^2\times R^+$ (ends of CD) would correspond to conformal transformation of S^2 which are localized also with respect to the light-like coordinate r of R^+.

   The generalization of the holomorphic transformation acting along the 3-D light-like orbit of the partonic 2-surface would have a similar structure. Now the holomorphic transformations would act on partonic 2-surfaces with a complex structure determined by the induced metric and would be localized with respect to the light-like time coordinate (in the induced metric) along the orbit of the partonic 2-surface.

   One might hope that the combination of these transformations could allow the construction of general 4-D analogs of 2-D holomorphic transformation leaving the Hamilton-Jacobi structure unaffected.

3. The counterparts for the Kac-Moody algebras of conformal theories would be supersymplectic transformations of S^2 and CP_2 localized with respect to the light-like radial coordinate of R^+ and the Kac-Moody algebras of H isometries localized with respect to the complex coordinate and light-like coordinate of the partonic orbit.

2.3.4 What it means that WCW Kähler form defines a homological analog of $U(1)$ magnetic charge?

Ordinary $U(1)$ gauge symmetry allows a $U(1)$ gauge charge or magnetic charge with a physical meaning. Could $U(1)$ magnetic charge have a counterpart at the level of WCW naturally assignable to the WCW Kähler form as analog of $U(1)$ gauge field in WCW? Genuine magnetic monopoles break gauge invariance. Could the homological monopole character of this Kähler field lead to an effective breaking of gauge invariance which would be realized in terms of generalized conformal symmetries. How could this happen?

1. What could be the interpretation of the monopole charge at the level of WCW. $\delta M^4_\pm \times CP_2$ represent the center of mass degrees of freedom of 3-D surface at the boundaries of CD.

   The symplectic isometries of WCW are assumed to reduce to the symplectic symmetries of $\delta M^4_\pm \times CP_2$ acting at the boundaries of CD. Does the homological non-triviality of WCW reduce to the non-triviality of CP_2 homology and possibly also $\delta M^4_\pm$ homology? If the light-cone boundary $\delta M^4_\pm$ does not actually contain its tip and CD does actually contain the line connecting its tips, $\delta M^4_\pm$ *resp.* CD has a point-like *resp.* line-like hole. This would give rise to homological S^2 monopole?

2. TGD strongly suggests the breaking of generalized conformal symmetries based on the fractal structure of super symplectic and generalized Super-Kac Moody half-algebras having conformal weights coming as non-negative integer multiples of the basic conformal weights. They possess a fractal hierarchy of isomorphic sub-algebras A_n with conformal weights coming as a positive integer $n > 0$-multiples of the entire algebra. The proposal is A_n and the commutator $[A_n, A]$ annihilate the physical states for a given integer $n > 0$ [K13].

   One obtains a hierarchy of the breakings of the generalized conformal symmetry in the sense that the generators with $h < n$ define genuine physical states and conformal symmetry is transformed for them to a genuine physical symmetry having an interpretation as finite-dimensional dynamical Lie group symmetry. The symmetry breaking for the generalized conformal symmetries would generate a dynamical symmetry.

   There are many options to consider. One option is that the super-symplectic algebra as gauge algebra transforms to a dynamical sub-algebra isomorphic to a super Kac-Moody algebra assignable to the light-like orbits of partonic 2-surfaces. This would realize a strong form of holography in the sense that partonic 2-surfaces at the boundaries of CD would code for the 4-D physics apart from these dynamical degrees of freedom.

   These groups might correspond to simply-laced Lie algebras or Kac-Moody algebras assignable to the hierarchy of inclusions of hyperfinite factors of type II_1 [K14, K7] and having root diagrams coinciding with the McKay graphs for finite discrete subgroups of $SU(2)$ [K13]. Physically this could correspond to a finite measurement resolution in which the excitations with $h \geq n$ would appear as gauge degrees of freedom.

3. The possibly existing extension of these 3-D light-like super-conformal algebras would allow hierarchical breaking of generalized super conformal symmetries. One would have a conformal symmetry breaking without conformal symmetry breaking as Wheeler might have put it. This would conform with the fact that the monopoles are homological rather than real.

4. What could this mean from the point of view of the Hamilton Jacobi structure? Does it mean that WCW spinor fields correspond to non-constant wave functions in the product of the universal Teichmüller spaces associated with partonic 2-surfaces and string world sheets in the non-gauge degrees of freedom? Could these wave functions would generalize the modular invariant elementary particle vacuum functionals constructed in the universal Teichmüller space for partonic 2-surfaces [K5].

3 About the field equations and boundary conditions

It is appropriate to start with some general comments.

1. The action principle for interior can be deduced from the twistor lift of TGD. Twistor lift geometrizes twistor fields in the same way as the fields of the standard model are geometrized. One can say that twistor structure is induced. Embedding space $M^4 \times CP_2$ is replaced with the product $T(M^4) \times T(CP_2)$ of the corresponding twistor spaces. This solves the problem due to the fact that the ordinary twistor space does not exist for a typical space-time geometry. Twistor space as 6-surface would be an S^2 bundle and dimensional reduction for the action involved would guarantee the bundle structure with the space-time surface as base-space.

   The action would be the 6-D counterpart of Kähler action for the 6-surface. This is possible only if both $T(M^4)$ and $T(CP_2)$ have Kähler structure. Intriguingly, M^4 and CP_2 are the only 4-D spaces having twistor space with Kähler structure. Also M^4 must have an analog of Kähler structure. Dimensional reduction reduces this action to 4-D Kähler action and volume action, which has interpretation in terms of cosmological constant, which depends on p-adic length scale and approaches zero in long length scales.

2. Field equations in the interior in the case of volume action are 4-D minimal surface equations expressing the vanishing of the trace of the second fundamental form. They express the fact that the sum of the external curvatures vanishes so that the minimal surface looks like a saddle.

3. Quite generally, the field equations are hydrodynamic in the sense that for any general coordinate invariant actions constructible in terms of the induced geometry they express conservation of Nother currents for the isometries of H. The generalized holomorphy implies that minimal surface equations are true for any action constructible in this way and are thus minimal surface equations. Differences between action appear only at the level of boundary conditions at the light-like boundaries and interfaces. Physically this property corresponds to the universality of the dynamics at quantum criticality.

   Minimal surface equations have two interpretations bringing in mind field-particle duality. Minimal surface equations correspond to a geometrized, non-linear variant of massless field equations. On the other hand, the particle-like minimal surface is analogous to the orbit of a massless particle as a generalization of a light-like geodesic, which is also a minimal surface.

3.1 Choice of the action

Concerning the choice of the interior part of the action, the basic question is whether the induced action is the sum of the Kähler actions for M^4 and CP_2 or whether the sum of the induced Kähler forms should appear in the action. One must consider both options. Same question is encountered in the case of Chern-Simons action at the light-like boundaries and interfaces.

The induced metric for the light-like 3-surface is degenerate and its inverse does not exist. Therefore the only possible action for the light-like boundaries or interfaces between Euclidian and Minkowskian space-time regions is Kähler-Chern-Simons action (K-C-S) action.

The C-S action is not manifestly gauge invariant under the $U(1)$ gauge transformations induced by the symplectic transformations of CP_2 and those of light-cone boundary in the case of M^4 C-S action. However, symplectic transformations do *not* leave invariant the induced metric and are not gauge symmetries of the theory at the classical level although they are assumed to act as isometries of WCW. This actually does allow them to be Noether symmetries of the action. The induced Kähler form of CP_2 is interpreted as a part of classical Kähler field, which suggests that Maxwellian gauge invariance is slightly broken by gravitation for the induced gauge fields.

3.1.1 First option: Induced Kähler form is the sum of the M^4 and CP_2 contributions

Consider first the option assuming that the induced Kähler form and gauge potential are sums of the induced Kähler forms and gauge potentials for M^4 and CP_2.

1. The field equations express the vanishing of the divergences of the canonical momentum densities $\Pi^k_\alpha = \partial L/\partial\partial_\alpha h^k$. Their contractions with isometry generators give rise to conserved Noether currents. The K-C-S action density is given by

$$L_{K-C-S} = \epsilon^{\alpha\beta\gamma} A_\alpha J_{\beta\gamma} \quad .$$

Here A_α and $J_{\beta\gamma}$ are sums of the Kähler gauge potentials and -forms of M^4 and CP_2. The canonical momentum currents are given by

$$\begin{aligned}\Pi_k^\alpha(CP_2) &= \epsilon^{\alpha\beta\gamma}A_k(CP_2)J_{\beta\gamma} + \epsilon^{\alpha\beta\gamma}A_\beta J_{kl}(CP_2)\partial_\gamma s^l \ , \\ \Pi_k^\alpha(M^4) &= \epsilon^{\alpha\beta\gamma}A_k(M^4)J_{\beta\gamma} + \epsilon^{\alpha\beta\gamma}A_\beta J_{kl}(M^4)\partial_\gamma s^l \ .\end{aligned}$$

The canonical momentum currents are not $U(1)$ gauge invariant since they depend on the total induced Kähler gauge potential and the components of the M^4 and CP_2 Kähler gauge potential. Hence symplectic transformations of $\delta M^4_\pm$ can affect the canonical momentum currents for the holographic data at light-like 3-surface and also the Noether charges.

2. If $J_{\alpha\beta}$ vanishes, the currents reduce to

$$\begin{aligned}\Pi_k^\alpha(CP_2) &= \epsilon^{\alpha\beta\gamma}A_\beta J_{kl}(CP_2)\partial_\gamma s^l \ , \\ \Pi_k^\alpha(M^4) &= \epsilon^{\alpha\beta\gamma}A_\beta J_{kl}(M^4)\partial_\gamma s^l \ .\end{aligned}$$

Now A_α is gradient and if gauge transformations correspond to Hamiltonian transformations, it is a sum of gradients of symplectic transformation of $\delta M^4_\pm \times CP_2$ at the boundary and one could interpret it as a conserved Noether charge associated with this symplectic transformation.

3. The vanishing of the divergences of the canonical momentum currents reduces to the conditions

$$\begin{aligned}&B^\alpha J^k_{\ l}(M^4)\partial_\alpha m^k = 0 \ , \\ &B^\alpha J^k_{\ l}(CP_2)\partial_\alpha s^k = 0 \ , \\ &B^\alpha = \epsilon^{\alpha\beta\gamma}(J_{\beta\gamma}(M^4) + J_{\beta\gamma}(CP_2)) \ .\end{aligned}$$

In the case of CP_2 (M^4) h^k corresponds to CP_2 (M^4) coordinates s^k (m^k). B^α is the analog of a magnetic field.

By contracting with $J^r_{\ k}$, and assuming $J_{kl}(M^4)$ is invertible, one obtains from these conditions $B^\alpha\partial_\alpha h^r = 0$. This seems to imply $B^\alpha = 0$ since the gradients of H coordinates define a tangent space basis for the light-like 3-surface. If B is light-like, then only the partial derivatives with respect to this light-like coordinate are involved. This derivative cannot vanish. To see this, suppose that the light-like 3-surface has light-like geodesic $U = m^0 - m^3 = constant$ as M^4 projection and the light-like coordinate $V = m^0 + m^3$ varies along the light-like 3-surface and defines its light-like coordinate v by the condition $V = v$. This implies $B^\alpha = 0$.

Can one overcome the problem? Does the non-vanishing of $B^\alpha(CP_2) = -B^\alpha(M^4)$ have physical implications?

1. Should one drop M^4 Kähler form or assume that it has only transversal part? One can argue that this is in accordance with the physical intuition that only transversal degrees of freedom are present and also with the corresponding conditions in string models?

2. Note that the vanishing of B^α does not imply the vanishing of the canonical momentum densities $\Pi_k^\alpha = \partial L/\partial(\partial_\alpha h^k)$, whose contractions with the imbedding space gamma matries define modified gamma matrices appearing in the modified Dirac equation. Only the canonical momentum currents associated with M^4 (CP_2) Chern-Simons term contract with M^4 (CP_2) gamma matrices so that the $A_\alpha(M^4)$ and $A_\alpha(CP_2)$ are visible via the M^4 and CP_2 parts of the modified gamma matrices at the level of the modified Dirac equation $\Gamma^\alpha D_\alpha\Psi = 0$. The $U(1)$ part of the induced spinor connection is proportional to $A_\alpha = A_\alpha(M^4) + A_\alpha(CP_2)$ which must be gradient but need not vanish.

3.1.2 Second option: Induced Kähler action is the sum of the M^4 and CP_2 contributions

For the second option, the action is sum of the M^4 and CP_2 Kähler actions and the vanishing of the divergences of the canonical momentum currents gives the conditions

$$\begin{array}{ll} B^{\alpha}(M^4)J_l^k(M^4)\partial_{\alpha}m^k = 0 \ , & B^{\alpha}(M^4) = \epsilon^{\alpha\beta\gamma}J_{\beta\gamma}(M^4) \ , \\ B^{\alpha}(CP_2)J_l^k(CP_2)\partial_{\alpha}s^k = 0 \ , & B^{\alpha}(CP_2) = \epsilon^{\alpha\beta\gamma}J_{\beta\gamma}(CP_2) \ . \end{array}$$

In this case the field equations in CP_2 degrees of freedom can be solved if the CP_2 coordinates do not depend on the light-like longitudinal coordinate so that $B^{\alpha}(CP_2)$ is light-like.

In M^4 degrees of freedom one encounters the same problem as for the first option and the solution seems to be that only $B^{\alpha}(M^4)$ vanishes. This is true if the light-like 3-surface is a Lagrangian manifold for $J_{\alpha\beta}(M^4)$. Note however that $A_{\alpha}(M^4)$ need not vanish so that the conserved Noether currents for momentum and angular momentum can be non-vanishing and there is a gamma matrix term in the modified Dirac equation.

3.1.3 The coupling between the interior and boundary/interface degrees of freedom

There is natural coupling of the boundary degrees of freedom with the interior degrees of freedom Volume action plus Kähler action would be a natural choice for the interior action. The coupling expresses the conservation of the Noether charges. The normal component of the conserved Noether current is equal to the divergence of the corresponding Noether current at the light-like 3-surface. In this case it is in principle possible to have a non-vanishing $B^{\alpha}(M^4)$ since in linear Minkowski coordinates $B^{\alpha}(M^4)J_{kl}(M^4)\partial_{\alpha}m^k$ equals to the normal component of the momentum current and need not vanish.

4 Appendix: Main concepts of TGD

There are two complementary visions of TGD: geometric and number theoretic one, and the basic principles and ideas of both visions are summarized in the sequel. Also zero energy ontology (ZEO) central in TGD inspired quantum measurement theory is discussed. I want to thank Marko Manninen for the help in listing the basic concepts.

4.1 Physics as geometry

Consider first the basic notions associated with the vision of physics as geometry.

1. Embedding space: This is the 8-dimensional space $H = M^4 \times CP_2$, where M^4 is the 4-dimensional Minkowski space of special relativity and CP_2 is the 4-dimensional complex projective space [L9]. The embedding space has a rich geometric structure that allows for the unification of gravity, electromagnetism, and the weak and strong interactions. For instance, the isometries and holonomies of CP_2 correspond to color interactions and weak interactions, which implies a profound connection between these two interactions. In the standard model they are completely independent.

2. Hyperbolic 3-space H^3: H^3 is a 3-dimensional space that has a constant negative curvature. The mass shell of momentum space is a physical realization of H^3 as is also the light-cone proper time $a = constant$ slice of M^4. In cosmology, the cosmic time $a = constant$ section corresponds to H^3. H^3 can be modeled by a Poincare ball model generalizing the Poincare disk familiar from Escher's paintings. H^3 has highly interesting properties, such as the notion of hyperbolic angles and angle deficit for a geodesic triangle, infinite volume, and exponential growth of area of a sphere with its radius. Hyperbolic space also allows an infinite number of tessellations analogous to lattices in Euclidian 3-space. These tessellations could appear in all scales, even cosmological.

3. Space-time surface: This is a 4-dimensional surface in H that represents a possible classical time evolution. The basic objects of TGD are 4-dimensional surfaces in H, which can be seen as generalizations of the space-time of general relativity. The space-time surface has

induced metric and gauge potentials that are induced from the geometry of the embedding space. In TGD, the geometrization eliminates gauge fields and metric as primary dynamical variables and one can say that all classical fields emerge. The space-time surface can have different topologies and forms a many-sheeted structure.

4. Twistor space: Twistor space is a 6-dimensional space having 4-D space-time as base space and sphere S^2 as fiber having interpretation as a space of light-like rays emanating from a given point of M^4. Locally the twistor space looks like $M^4 \times S^2$ but since the spheres S^2 at points connected by a light-like ray intersect at a point, the bundle structure is non-trivial.

   In the case of M^4, twistor space can be identified as CP_3. In the case of CP_2, the twistor space is $SU(3)/U(1) \times U(1)$ and represents the space of choices of quantization axes for color isospin and hypercharge. The twistor space encodes the information about the second quantized spinor fields on H and corresponding induced spinor fields on the space-time surface and provides an alternative description of spin in terms of wave functions in S^2.

5. World of Classical Worlds (WCW): In the original formulation, WCW was identified as the set of all possible 3-dimensional surfaces or 'worlds' embedded within the 8-dimensional embedding space $H = M^4 \times CP_2$. Holography, which is forced by 4-dimensional(!) general coordinate invariance, forces a modification: a given 3-surface identified as a particle is replaced with an almost unique 4-D surface identifiable as an orbit of the 3-surface. This 4-surface, the 'world', is analogous to Bohr orbit. These 'worlds' are solutions of the field equations that define these Bohr orbits as preferred extremals of the action principle.

4.1.1 Twistor lift of TGD

The existence of a twistor lift of TGD is what makes TGD unique.

1. Twistorialization: The twistorialization of gauge theories has been rather successful although it works best for $N = 4$ SUSY. The problem of twistorialization in GRT is that the twistor space exists mathematically only for conformally flat space-time geometries. The second problem is that the twistor approach works only for massless particles. The absence of scale is a further problem and is reflected as infrared divergences.

2. Twistor lift of TGD: The successes of twistorialization in quantum field theories motivated the twistor lift of TGD [K12, K10], which geometrizes also the notion of twistor field by a generalization of the induction procedure. The idea is to geometrize the twistor fields in the same way as one geometrizes standard model gauge fields. The proposal is to represent twistor space as a 6-D surface in the product $T(M^4) \times T(CP_2)$ of twistor spaces of M^4 and CP_2.

   Intriguingly, only M^4 (also E^4) and CP_2 allow twistor space with Kähler structure [A5] so that TGD is unique from this condition. The dynamics of the 6-surfaces is naturally based on 6-D Kähler action requiring the Kähler structure. The condition that 6-surface is an S^2 bundle requires dimensional reduction and the action reduces to 4-D Kähler action plus volume term. The coefficient of the volume term has an interpretation as cosmological constant depending on p-adic length scale and approaching zero in long scales.

4.1.2 Space-time as 4-surface

In TGD, space-times are identified as 4-D surfaces in $H = M^4 \times CP_2$. Space-time surface can be seen as a generalization of the space-time of classical general relativity, where there is only one 4-surface that corresponds to the entire universe [K4, K1]. Space-time surface can be topologically non-trivial in all scales. The M^4 projection of the space-time surface has a finite size and can be very small. The space-time of GRT corresponds to space-time surfaces with 4-D M^4 projection.

1. Many-sheeted space-time: Many-sheeted space-time corresponds to the 'Einsteinian' space-time surfaces with a 4-dimensional M^4 projection, which makes them analogous to the space-time of GRT as a small deformation of M^4. Space-time sheets however have a finite size so that their M^4 projection has a boundary. Many-sheetedness means that several points of the

space-time surface are projected to the same point of M^4. The hemispheres of the sphere represent the analog of 2-sheeted space-time. Since the 4-surface has finite size the sheets must meet along some 3-D surface defining the boundary of M^4 projection just like the hemispheres of the sphere meet at the 1-D equator.

The many-sheeted space-time allows for a hierarchical organization of physical systems, such as elementary particles, atoms, molecules, cells, organs, organisms, etc., each having their own space-time sheets with different sizes and properties. Also this distinguishes TGD from GRT.

One can also consider many-sheetedness with respect to CP_2 taking the role of space-time and M^4 the role of field space. This kind of many-sheeted structures are proposed to define quantum coherent structures with very large values of effective Planck constant.

2. Cosmic string: In GUTs, cosmic strings are primordial defects in the fabric of space-time that may have been formed during the early stages of the universe. They are not strings in the sense of string theory, but rather, one-dimensional objects with very high energy density and tension. Cosmic strings can have various effects on the surrounding matter and radiation, such as gravitational lensing, gravitational waves, and cosmic microwave background anisotropies.

   In TGD, cosmic strings are not strings in the sense of GUTs or string theory but 4-D space-time surfaces with a 2-D string world sheet as M^4 projection [K6, K11, K9]. Flux tubes are deformations of cosmic strings and have a very thin M^4 projection and look like string world sheets.

3. Monopole flux tube: Monopole flux tubes distinguish between TGD and GRT. They are magnetic flux tubes that carry a net magnetic flux [L6, L7]. If they had ends, there would be opposite magnetic charges at them. Monopole flux tubes i.e. monopole fluxes without monopole charges reflect the non-trivial homology of CP_2 and are not possible in gauge theories. Monopole flux tubes have a 2-D closed surface as a cross section and are not possible in space-times with trivial 2-homology.

   The magnetic fields associated with the monopole flux tubes require no currents ('inductance coils') to maintain them. Therefore these magnetic fields are possible even in cosmological scales and solve the mystery of the magnetic fields appearing in cosmic scales and also explain [L1] why the Earth's magnetic field has not disappeared long time ago by the dissipation of currents needed to create them in the Maxwellian world.

   Monopole flux tubes can get knotted and linked and also interact by reconnection. The reconnection of U-shaped monopole flux tubes acting as tentacles defines a universal interaction mechanism possible in all scales. In biocatalysis, reconnection makes it possible for the molecules to find each other in the molecular crowd. Monopole flux tubes are generated as deformations of cosmic strings and connect particle-like 3-surfaces to networks. They play a vital role in various astrophysical and biological phenomena.

4. Magnetic body: This is a term used to describe a hierarchy of magnetic structures consisting of flux tubes and sheets that carry dark matter and energy. Magnetic flux tubes are 3-dimensional (but effectively one-dimensional) objects that have a magnetic field along their length and can act as quantum wires or springs. In TGD, the explanation for the failure to find dark matter is that it resides at the magnetic body as phases with non-standard value of Planck constant and remains undetected as long as one does not take seriously the existence of the magnetic body.

   Magnetic body allows for the communication and control between various levels of the hierarchy formed by the layers of the magnetic body and the space-time sheets carrying ordinary matter, and it plays a crucial role in TGD inspired theory of consciousness and biology.

4.1.3 The dynamics of the space-time surface

The dynamics of the space-time surface is determined by an action principle but holography means dramatic deviation from the standard field theory.

1. Action principle: A general way of describing the dynamics of physical systems by using an integral called the action. The action is a function of the system s configuration and its history, and it depends on the choice of a Lagrangian function that characterizes the system. The action principle states that the actual path followed by the system is such that the action is stationary, meaning that it does not change under small variations of the path. This condition leads to the equations of motion for the system, which can be derived using variational calculus. The action principle can be applied to different kinds of systems, such as classical mechanics, electromagnetism, relativity, and quantum mechanics.

   In TGD, the action principle is stronger since holography is required and the space-time surface is analogous to Bohr orbit.

2. Preferred extremal: These are special solutions of the field equations that define the dynamics of the space-time surfaces. Twistor lift of TGD implies that the action is the sum of volume action and Kähler action, which is analogous to the Maxwell action for electromagnetism. Preferred extremals are analogs of Bohr orbits for particles identified as 3-surfaces. They have many remarkable properties, such as generalized holomorphy, minimal surface property, and number theoretic symmetries. Furthermore, the preferred extremals are minimal surfaces for *any* general coordinate invariant action constructed in terms of the induced geometry. This universality reflects the universality of quantum criticality. Only the boundary conditions and values of conserved Noether charges distinguish between different actions.

3. Bohr orbit: This is a term used to describe the orbits of electrons around atomic nuclei in the Bohr model of the atom. Bohr orbit is a solution of Newton's equations but satisfies additional semiclassical quantization conditions. As a consequence, Bohr orbit satisfies almost exact holography: when the initial position of the particle is known, the orbit of the particle is almost uniquely determined.

   The Bohr model implies that electrons can only have certain discrete energy levels and angular momenta, and that they emit or absorb photons when they jump between orbits. The Bohr model can explain the hydrogen spectrum, but it fails for more complex atoms. In TGD many-electron atoms require the presence of monopole flux tubes connecting electrons to compensate for the repulsive interaction between electrons overcoming the repulsive interactions with the nucleus in the classical sense. In standard atomic physics this problem is not considered.

4.1.4 Basic principles of TGD as geometrization of physics

The following basic principles dictate the vision of physics as geometry to a high degree.

1. 4-dimensional General Coordinate Invariance: 4-D general coordinate transformations must act as gauge symmetries of the theory meaning that they do not affect the physical properties of the state.

2. Holography: If 3-dimensional surfaces are the basic objects, there must exist a principle assigning to them a 4-D space-time surface at which the 4-D general coordinate transformations can act. Path integral formalism would allow all 4-surfaces but is plagued by divergences. Second option is that there exist only a small, possibly finite number of these 4-surfaces. These highly unique space-time surfaces are analogous to Bohr orbits of a particle identified as a 3-surface. This 4-D surface is a preferred extremal of some general coordinate invariant action principle constructible in terms of the induced geometry.

   If holography were unique, 3-surfaces correspond to Bohr orbits in 1-1 manner. This is however not the case so one must take the 4-D Bohr orbits as the basic entities instead of 3-surfaces. Therefore the infinite-dimensional configuration space, WCW, consists of Bohr-orbit-like 4-surfaces rather than 3-surfaces.

   This leads to zero energy ontology (ZEO) with profound implications for both quantum measurement theory and for the view about time. The principle states that the data on the 3-surface almost completely determines the 4-D space-time surface. A 3-surface can be a boundary of a 4-surface in H, which can be either at the opposite ends of a causal diamond (CD), which is a region of space-time defined by the intersection of future and past light-cones, or it can be a light-like interface between regions of space-time surface with Euclidian and Minkowskian signatures (light-like partonic orbit). These 3-surfaces determine the holographic data, which includes the induced metric and other fields that are inherited from H. They also determine the holographic data about fermionic dynamics.

The dynamics of the 4-surface is governed by the extremization of action. Twistor lift of TGD predicts that the action is the sum of the Kähler action and volume term, and the quantum states and observables, which are defined by the symmetries of the 'world of classical worlds' (WCW). Holography implies that quantum theory can be formulated as an analog of wave mechanics in the space of Bohr orbits.

3. Zero Energy Ontology (ZEO): In ZEO, fermionic states can be regarded as pairs of fermionic states associated with 3-surfaces located at the opposite boundaries of a causal diamond CD= $cd \times CP_2$, where cd is a region of M^4 defined by the intersection of future and past light-cones). In the geometric degrees of freedom, physical states are superpositions of space-time surfaces identified as preferred extremals (Bohr orbits) connecting the 3-surfaces at the boundaries of CD.

   Conservation laws imply that total quantum numbers of these 3-surfaces are identical (in the case of translational symmetries, Uncertainty Principle implies that this is true only for very large CDs). It is convenient to use quantum field theoretic conventions and say that the total quantum numbers of boundaries are of opposite sign. This motivates the term "zero energy state". This does not mean that the 3-D states of the state pairs would have zero energy.

   ZEO based quantum measurement theory implies that zero energy states have a well-defined arrow of time. This means that only the state at the "passive" boundary of CD remains invariant under repeated measurements of the same observables (Zeno effect) whereas the active boundary of CD is shifted in these the repeated measurements and 3-D states at them change. This sequence of repeated measurements corresponds in the TGD inspired theory of consciousness theory to a flow of consciousness of a conscious entity, self. In ordinary quantum measurements the arrow of time changes.

4.1.5 Symmetries of TGD as geometrization of physics

Symmetries play a crucial role in TGD.

1. Isometry: This is a transformation that preserves the distance between any two points in a Riemann space. For example, a rotation or a translation is an isometry of Minkowski space. Isometries can be used to define the notion of congruence and symmetry in geometry.

2. Holonomy: This is a measure of how much a vector changes its direction when it is parallel transported along a closed loop in a curved space. For example, if a vector is parallel transported along the equator of a sphere and then back to the north pole, it will point in a different direction than before. The holonomy depends on the curvature of the space and the shape of the loop. Holonomy can be used to classify the possible symmetries and structures of Riemannian manifolds.

3. Symmetries of the embedding space: The starting point of TGD was the energy problem of GRT, i.e. the loss of classical conservation laws due to the loss of the Poincare invariance of Special Relativity due to the fact that space-time becomes curved. The assumption that space-time is 4-D surface in $M^4 \times CP_2$ saves Poincare symmetries and allows us to understand the color and electroweak symmetries of the standard model in terms of isometries and holonomies of CP_2.

4. WCW symmetries: These are the symmetries of the world of classical worlds (WCW), which is the configuration space of all possible 3-dimensional space-time surfaces in TGD. The basic result for loop spaces is that the geometry exists only if it has maximal isometries. This result is expected to be true also for WCW.

(a) Supersymplectic transformations: WCW isometries include symplectic transformations of $\delta M^4_+ \times CP_2$ and their super-symmetric variants. Supersymplectic symmetries are analogs of local gauge transformations with the light-like radial coordinate of δM^4_+ appearing as a parameter.

(b) Superconformal transformations: In TGD, the super-conformal symmetries extend the ordinary super-conformal symmetries. Conformal transformations of $\delta M^4_+ = S^2 \times R_+$ correspond to conformal transformations of S^2, which depend on the radial light-like coordinate r of light-cone boundary delta M^4_+. The dependence of r makes these transformations local conformal transformations of S^2.

(c) Extended Super Kac-Moody symmetries correspond to WCW holonomies. In the same way as at the level of H, these isometry and holonomy algebras can be used to define the quantum states and dynamics of TGD.

The special feature is that there is the following. All these symmetry algebras are fractals and include the hierarchy of subalgebras A_n isomorphic to the entire algebra A. The isomorphic sub-algebra A_n and the commutator $[A_n, A]$ can act as a gauge algebra annihilating the physical states. The generators not belonging to A_n act as genuine physical symmetries. One has a hierarchy of symmetry breakings transforming gauge symmetries to symmetries. Symmetry breaking without symmetry breaking!

5. Super-symplectic algebra: This is an infinite-dimensional Lie algebra associated with the isometry group of WCW. This algebra consists of generators of symplectic transformations of $\delta M^4 \times CP_2$ that preserve the symplectic form of $\delta M^4 \times CP_2$, as well as fermionic super generators that correspond to super-symplectic transformations. Super-symplectic algebra can be used to construct representations of physical states and operators in TGD.

6. (Extended) Super Kac-Moody algebra: This is an infinite-dimensional Lie superalgebra associated with the holomy group of WCW. This algebra extends the super-Kac Moody algebras of string models and conformal field theories in the sense that besides complex coordinate z there is a light-like coordinate u of which these analogs of local transformations depend. It consists of Kac-Moody generators that correspond to gauge transformations deforming the light-like partonic orbits, as well as fermionic generators that correspond to super-Kac Moody transformations. Super Kac-Moody algebra as analog of holonomy algebra can be used to construct representations of physical states and operators in TGD.

7. Generalized holomorphy: Generalized holomorphy is a proposal for a 4-D generalization of the ordinary 2-D holomorphy for the representations of string world sheets and their Euclidian counterparts. Generalized holomorphy would define 4-D space-time surfaces as preferred extremals satisfying holography making space-time surfaces analogous to Bohr orbits. The generalized holomorphy leads to a proposal for an explicit construction of preferred extremals: the space-time surface would be determined by the vanishing of two complex valued functions depending on a light-like coordinate and complex coordinate of M^4 and two complex coordinates of CP_2. This reduces the field equations to purely algebraic conditions satisfied identically if generalized holomorphy makes sense. TGD would be an exactly solvable theory.

4.2 Physics as number theory

Physics as number theory is a vision complementary to physics as geometry vision.

4.2.1 Basic notions and ideas of physics as number theory

Physics as number theory, or adelic physics [L2], both real numbers and various p-adic number fields are in a democratic position. Reals would describe the physics of sensory perception and various p-adic number fields the physics of cognition.

1. Number theoretic universality: The requirement of number theoretic universality leads to the notion of cognitive representation as a discrete set of points of space-time surface for which the coordinates of points belong to an algebraic extension of rationals defining also extension of p-adic numbers. This discretization is unique for a given extension and for fixed and

highly unique coordinates of octonionic M^8. The hierarchy of the extensions corresponds to an evolutionary hierarchy with an increasing resolution and approaching algebraic numbers dense in the set of reals. The cognitive representation can be interpreted in all extensions of p-adic number fields induced by this extension so that the discretized dynamics is theoretically universal.

2. Measurement resolution: The notion of measurement resolution is the ugly duckling of theoretical physics and p-adic numbers provide a universal number theoretic characterization of finite measurement resolution. p-Adic length scale hypothesis stating that physical preferred primes are near to some powers of two has very strong predictive power and suggests a fractal hierarchy of scaled variants of physics. At this moment, perhaps the most important applications of the p-adic length scale hypothesis are in elementary particle physics, hadron physics, nuclear physics, and atomic physics.

3. M^8-H-duality: This is a duality between two different descriptions of TGD - one based on $H=M^4\times CP_2$ and one based on the ocotions O, which with respect to number theoretic norm $Re(o^2)$, which is an 8-dimensional Minkowski space with octonionic structure [L10]. The original belief [L3, L4, L8] that complexification of O is needed led to interpretational problems but turned out to be wrong. The duality maps spacetime surfaces in H to those in $O\equiv M^8$ and vice versa.

   M^8-H duality has a physical interpretation. The replacement of the point-like particle with 3-surface implies that quantum TGD is essentially wave mechanics for these 3-D particles. Momentum-position duality of the ordinary wave mechanics generalizes to M^8-H duality and 4-surfaces in M^8 as counterparts of orbits in momentum space correspond to 4-D spacetime surfaces in H as analogs of Bohr orbits. The number theoretical M^8 description and the geometric H description are the TGD counterparts for momentum space and space-time descriptions of physics.

4.2.2 Symmetries of physics as number theory

In physics as number theory isometries as basic symmetries of the geometric vision are replaced with Galois groups.

1. Galois group: This is a symmetry group for algebraic extensions, which are fields of numbers that contain some roots of polynomials with rational coefficients. The Galois group leaves rational numbers invariant but permutes the roots of a polynomial defining the extension of rationals. For example, the Galois group for the extension Q($\sqrt{2}$) over Q is Z_2, which consists of two elements: the identity and the automorphism that maps $\sqrt{2}$ to -$\sqrt{2}$. Galois groups can be used to study the properties and solvability of polynomial equations and their roots.

2. Galois confinement: This is a general number theoretic mechanism for the formation of bound states. Bound states are Galois singlets in the same way as hadrons are color singlets in QCD. This means that the components of the momenta are integers for a suitable momentum unit although they are algebraic integers for the fermion forming the state. Galois confinement can be also used to explain some features of TGD inspired quantum biology, such as genetic code and related bio-harmony.

   A stronger, highly speculative form of Galois confinement requires that the extensions, whose Galois groups act as discrete subgroups of the isometry group of H at space-time surfaces, are physically favored. In the case of the 3-D rotation group, this has very powerful implications and would leave only Galois groups identifiable as finite discrete subgroups of the rotation group: they correspond to the symmetry groups of Platonic solids and planar polygons.

3. Adelic physics: An approach to unify real and p-adic physics by using adeles, which are tuples consisting of real and of various p-adic numbers, p=2,3,..... Adelic physics allows for a description of both matter and cognition in terms of number theory and algebraic geometry.

4. Algebraic extension of rationals: This is a field of numbers that contains all the rational numbers and also some algebraic numbers, which are roots of polynomials with rational coefficients. For example, the field $Q(\sqrt{2})$ contains all the rational numbers and also the square root of 2 and its multiples. Algebraic extensions of rationals are important for studying the properties of algebraic numbers and their symmetries. The roots of irreducible polynomials define algebraic extensions of rationals.

4.2.3 Hierarchy of Planck constants and dark matter

According to TGD, the Planck constant is not a universal constant, but instead, its values are integer multiples of the minimal value. The proposal is that there are different 'layers' of dark matter, each associated with a different value of the Planck constant. This allows for the prediction of macroscopic quantum effects, where higher layers of the hierarchy correspond to larger quantum coherence lengths, typically proportional to the value of Planck constant.

The hypothesis was inspired by several anomalies (such as quantal effects of a radiation at EEG frequencies on mammalian brain physiology and behavior), whose quantum mechanical explanation required a very large value of Planck constant. In the number theoretic vision of TGD, the hierarchy of Planck constants follows as a prediction and effective Planck constant corresponds to the dimension of algebraic extension of rationals assignable to a given space-time region if $M^8 - H$ duality is true [L10].

4.2.4 p-Adic physics

1. p-Adic physics: This is the study of physical phenomena using p-adic numbers, which are numbers that have a different notion of distance and size than the usual real numbers. The proposal is that p-adic numbers serve as correlates of cognition.

2. p-Adic number field: This is a field of numbers that is obtained by completing an algebraic extension of rationals with respect to the p-adic norm, which measures how divisible a number is by a given prime number p. For example, the field Qp is the completion of Q with respect to the p-adic norm. p-adic number fields have many applications in number theory, such as solving Diophantine equations and proving theorems about primes.

3. Extension of p-adic number field: This is a field of numbers that contains a given p-adic number field as a subfield. For example, $Q(\sqrt{2})$ is an extension of Q that contains the square root of 2 as an element. Extensions of p-adic number fields can be classified by their degree, ramification, and Galois group.

4. p-adic topology: This is a topology on a set of numbers that is induced by the p-adic norm or distance function. In this topology, two numbers are close if they have many common digits in their p-adic expansion, which starts from the right and goes to the left. For example, in the 3-adic topology, 1 and 10 are close because they have the same last digit (1), but 1 and 2 are far apart because they have no common digits. p-adic topology gives rise to many interesting properties and structures, such as ultrametric spaces and profinite groups.

5. Ultrametricity: This is a property of a metric space that satisfies a stronger version of the triangle inequality: $d(x, z) \leq max(d(x, y), d(y, z))$ for any points x, y, z in the space. In other words, the distance between two points is determined by the most distant pair among them. Ultrametric spaces have many peculiar features, such as having only one open ball around each point and being totally disconnected. Ultrametric spaces arise naturally in p-adic analysis, hierarchical clustering, and spin glass theory.

6. Ramified primes: In TGD the so called ramified primes appearing as divisors of the determinant define by the rational polynomial are expected to define very special p-adic lengths scales the proposal is that the the preferred p-adic topologies emerging from p-adic mass calculations correspond to the ramified primes.

7. p-Adic thermodynamics: This is a generalization of thermodynamics that uses p-adic numbers instead of real numbers to describe the thermal mass squared (instead of energy) and

entropy of physical systems. The condition that p-adic valued counterparts of Boltzmann weights exists mathematically is very powerful and in TGD it is satisfied if the system has a stringy mass squared spectrum coming as integers. Also p-adic temperature is quantized. p-adic thermodynamics allows for a discrete and hierarchical description of complex systems, such as living organisms. The interpretation of p-adic thermodynamics could be as a cognitive representation provided by Nature itself and coding only for the most important bits. p-adic thermodynamics also provides a way to calculate the masses of elementary particles using quantized p-adic temperature and conformal symmetry.

8. p-Adic mass calculations: This is a method of computing the masses of elementary particles using p-adic thermodynamics and super-conformal symmetry. The idea is to assume that particles are in thermal equilibrium at some p-adic temperature, which is inverse integer and equal to its maximal value 1 for fundamental fermions, and then use the partition function to obtain their p-adic masss squared spectrum mapped to real numbers by canonical identification. p-adic mass calculations can reproduce the observed masses of quarks, leptons, and bosons with remarkable accuracy.

4.3 ZEO and quantum measurement theory

There are two types of state function reductions (SFRs) in TGD. "Small" SFRs (SSFRs) and "big" SFRs (BSFRs).

1. SSFR: In standard quantum mechanics SSFR would correspond to a repeated measurement of the same observables and would not change the state of the system (Zeno effect). In quantum optics this idealization is too strong and the notion of weak measurement is introduced. SSFR is the TGD counterpart of the weak measurement of quantum optics.

   Also in SSFR, the observables O, whose eigenstates the states at the passive boundary of CD are, are repeatedly measured. Besides this some additional observables commuting with O can be measured in SSFR or only a subset of O can be measured. New observables can emerge due to the failure of the complete determinism of classical time evolution (holography is not quite unique). Therefore SSFR can change the state of the system but does not change the arrow of the geometric time. The Zeno effect is replaced with the sequence of SSFRs giving rise to a flow of consciousness. SSFRs define the flow of consciousness and one can assign free will to them.

2. BSFR: BSFR is the counterpart of the ordinary state function reduction. BSFR occurs when the system is perturbed so the observables measured in SSFR do not commute with O and the state at the passive boundary must change. System can also intentionally induce BSFR.

   BSFR changes the state of the system and also the arrow of the geometric time. Passive and active boundaries of CD change their roles. One can assign to BSFR terms like universal death , "sleep state", and reincarnation with an opposite arrow of time. The motivation for the identification is that in the time-reversed state (say sleep) all classical signals propagate to the geometric past and after the wake-up it is not possible to remember anything of this period. Dreams would correspond to partial awakenings of the brain. Since the state at the passive boundary is not changed in SSFRs, the changes of the space-time surface far from the active boundary are small so that the system looks "dead".

   Pairs of BSFRs returning the original arrow of time make possible a trial and error process and together with NMP it makes possible a goal directed behavior. One can say that the system returns temporarily to the geometric past and starts again.

3. Time reversal: A concept that refers to the possibility of reversing the direction of time in physical processes. In classical physics, time reversal is equivalent to changing the sign of all velocities and angular momenta in a system, which preserves the laws of motion and conservation. In quantum physics, time reversal is more subtle, because it also involves changing the sign of some quantum numbers, such as spin and magnetic moment, which may not be conserved.

In TGD, one must distinguish between two variants of time reversal. Time reflection T is an isometry but T as symmetry, as also CP, is slightly violated in both TGD and standard model. Time reversal can be also interpreted as a reversal of the arrow of time assignable to the zero energy states. Time reversal in this sense is neither geometric transformation nor symmetry. This time reversal corresponds to thermodynamic time reversal, which is ot allowed in standard thermodynamics but is possible in TGD.

In contrast to subjective time as a sequence of "small" state function reductions (SSFRs), geometric time corresponds to a space-time coordinate. In ZEO, the arrow of time is determined by the asymmetry between passive and active boundaries of CD. At the passive boundary of CD, the states are not affected in the repeated measurements of the same observables (Zeno effect). At the active boundary of CD, the states change and the size of the CD increases in statistical sense. Therefore the geometric time identified as the distance between the tips of CD increases in a definite direction.

The classical signals propagate to the direction of the active boundary, the future. The arrow of geometric time can be also defined classically by the direction of the energy flow from the passive (past) boundary to the active (future) boundary of a causal diamond (CD).

4. Negentropy Maximization Principle (NMP): NMP is mathematically analogous to the second law of thermodynamics and states that the amount of conscious information, which apparently looks just the opposite of the second law.

   Contrary to the original proposal, the recent view is that this is not a variational principle of consciousness but follows automatically from the number theoretic evolution implied by the hierarchy of extensions of rationals. Number theoretic complexity unavoidably increases during the sequence of SFRs and implies the increase of the negentropic resources. Universe becomes more complex and can understand more about itself.

   The p-adic negentropy measuring the information content of quantum entanglement is closely related to ordinary entanglement entropy which measures the lack of information of the state of either entangled system. Therefore NMP is consistent with the second law. NMP relates to conscious (subjective) information of the system about itself and second law to conscious (objective) information about the external world: this allows us to avoid the paradox.

5. Scattering amplitudes: In TGD, the scattering amplitudes are determined by zero energy states and that code for the scattering rates of particles in quantum field theory as well as in TGD. Scattering amplitudes depend on the kinematics and dynamics of the particles involved and can be calculated in quantum field theories using various methods such as Feynman diagrams or twistor techniques. In TGD, their calculation requires a generalization of Feynman and twistor diagrams to take into account that the point-like particles are replaced with 3-surfaces.

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