Symmetries and Geometry of the "World of Classical Worlds"

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January 13, 2025
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Abstract

The view of the symmetries of the TGD Universe has remained unclear for decades. The notion of "World of Classical Worlds" (WCW) emerged around 1985 but found its basic form around 1990. Holography forced by the realization of General Coordinate Invariance forced/allowed to give up the attempts to make sense of the path integral.

A more concrete way to express this view is that WCW does not consist of 3-surfaces as particle-like entities but almost deterministic Bohr orbits assignable to them as preferred extremals of Kähler action so that quantum TGD becomes wave mechanics in WCW combined with Bohr orbitology. This view has profound implications, which can be formulated in terms of zero energy ontology (ZEO), solving among other things the basic paradox of quantum measurement theory. ZEO forms also the backbone of TGD inspired theory of consciousness and quantum biology.

After the developments towards the end of 2023 leading to a discovery of explicit solution of field equations based on the 4-D geneneralization of holomorphy realizing holography, it seems that the extension of conformal and Kac-Moody symmetries of string models to the TGD framework is understood. What about symplectic symmetries, which were originally proposed as isometries of WCW? In this article this question is discussed in detail and it will be found that these symmetries act naturally on 3-D holographic data and one can identify conserved charges. By holography this is in principle enough and might imply that the actions of holomorphic and symplectic symmetry algebras are dual. Holography=holomorphy principle generalizes also to the construction of the solutions of the modified Dirac action.

1 Introduction

The view of the symmetries of the TGD Universe has remained unclear for decades. The notion of "World of Classical Worlds" (WCW) emerged around 1985 but found its basic form around 1990. Holography forced by the realization of General Coordinate Invariance forced/allowed to give up the attempts to make sense of the path integral.

A more concrete way to express this view is that WCW does not consist of 3-surfaces as particle-like entities but almost deterministic Bohr orbits assignable to them as preferred extremals of Kähler action so that quantum TGD becomes wave mechanics in WCW combined with Bohr orbitology. This view has profound implications, which can be formulated in terms of zero energy ontology (ZEO), solving among other things the basic paradox of quantum measurement theory. ZEO forms also the backbone of TGD inspired theory of consciousness and quantum biology.

WCW geometry exists only if it has maximal isometries: this statement is a generalization of the discovery of Freed for loop space geometries [A2]. I have proposed [K2, K1, K5, K3] that WCW could be regarded as a union of generalized symmetric spaces labelled by zero modes which do not contribute to the metric. The induced Kähler field is invariant under symplectic transformations of CP_2 and would therefore define zero mode degrees of freedom if one assumes that WCW metric has symplectic transformations as isometries. In particular, Kähler magnetic fluxes would define zero modes and are quantized closed 2-surfaces. The induced metric appearing in Kähler action is however not zero mode degree of freedom. If the action contains volume term, the assumption about union of symmetric spaces is not well-motivated.

Symplectic transformations are not the only candidates for the isometries of WCW. The basic picture about what these maximal isometries could be, is partially inspired by string models.

1. A weaker proposal is that the symplectomorphisms of H define only symplectomorphisms of WCW. Extended conformal symmetries define also a candidate for isometry group. Remarkably, light-like boundary has an infinite-dimensional group of isometries which are in 1-1 correspondence with conformal symmetries of $S^2 \subset S^2 \times R_+ = \delta M^4_+$.

2. Extended Kac Moody symmetries induced by isometries of δM^4_+ are also natural candidates for isometries. The motivation for the proposal comes from physical intuition deriving from string models. Note they do not include Poincare symmetries, which act naturally as isometries in the moduli space of causal diamonds (CDs) forming the "spine" of WCW.

3. The light-like orbits of partonic 2-surfaces might allow separate symmetry algebras. One must however notice that there is exchange of charges between interior degrees of freedom

and partonic 2-surfaces. The essential point is that one can assign to these surface conserved charges when the dual light-like coordinate defines time coordinate. This picture also assumes a slicing of space-time surface by by the partonic orbits for which partonic orbits associated with wormrhole throats and boundaries of the space-time surface would be special. This slicing would correspond to Hamilton-Jacobi structure.

4. Fractal hierarchy of symmetry algebras with conformal weights, which are non-negative integer multiples of fundamental conformal weights, is essential and distinguishes TGD from string models. Gauge conditions are true only the isomorphic subalgebra and its commutator with the entire algebra and the maximal gauge symmetry to a dynamical symmetry with generators having conformal weights below maximal value. This view also conforms with p-adic mass calculations.

5. The realization of the symmetries for 3-surfaces at the boundaries of CD and for light-like orbits of partonic 2-surfaces is known. The problem is how to extend the symmetries to the interior of the space-time surface. It is natural to expect that the symmetries at partonic orbits and light-cone boundary extend to the same symmetries.

After the developments towards the end of 2023, it seems that the extension of conformal and Kac-Moody symmetries of string models to the TGD framework is understood. What about symplectic symmetries, which were originally proposed as isometries of WCW? In this article this question is discussed in detail and it will be found that these symmetries act naturally on 3-D holographic data and one can identify conserved charges. By holography this is in principle enough and might imply that the actions of holomorphic and symplectic symmetry algebras are dual. Holography=holomorphy hypothesis is discussed also in the case of the modified Dirac equation.

2 The reduction of holography to a generalized holomorphy

The reduction of holography to generalized holomorphy reduced field equations to a ridiculously simple form. Field equations are satisfied because contractions of holomorphic tensors of type (1,1) with tensors of type (2,0)+(0,2) are identically vanishing. This ansatz works already for string sheets as minimal surfaces.

Preferred extremals as analogs of Bohr orbits are minimal surfaces irrespective of the action as long as it is a general coordinate invariant constructed using induced geometry and the minimal surface property fails only at lower-dimensional singularities analogous to the frames of a soap film.

At singularities the other parts of the action become visible by boundary conditions guaranteeing that conservation laws expressed by field equations are not violated. The other parts of action are visible only via the classical conservation laws and at interaction vertices [L4].

Twistor lift fixes the 4-D action to a sum of Kähler action and volume term emerging as a dimensional reduction of 6-surface in the Cartesian product of twistor spaces of M^4 and CP_2 to 6-D twistor space to twistor space as S^2 bundle over space-time surface. Only M^4 and CP_2 allow twistor space with Kähler structure so that TGD is unique from its mathematical existence [A3].

2.1 The conserved charges associated with holomorphies

Generalized holomorphy not only solves explicitly the equations of motion but, as found quite recently, also gives corresponding conserved Noether currents and charges.

1. Generalized holomorphy algebra generalizes the Super-Virasoro algebra and the Super-Kac-Moody algebra related to the conformal invariance of the string model. The corresponding Noether charges are conserved. Modified Dirac action allows to construct the supercharges having interpretation as WCW gamma matrices. This suggests an answer to a longstanding question related to the isometries of the "world of the classical worlds" (WCW).

2. Either the generalized holomorphies or the symplectic symmetries of $H = M^4 \times CP_2$ or both together define WCW isometries and corresponding super algebra. It would seem that symplectic symmetries induced from H are not necessarily needed and might correspond to

symplectic symmetries of WCW. One would obtain a close similarity with the string model, except that one has half-algebra for which conformal weights are proportional to non-negative integers and gauge conditions only apply to an isomorphic subalgebra. These are labeled by positive integers and one obtains a hierarchy.

3. By their light-likeness, the light cone boundary and orbits of partonic 2-surfaces allow an infinite-dimensional isometry group. This is possible only in dimension four. Its transformations are generalized conformal transformations of 2-sphere (partonic 2-surface) depending on light-like radial coordinate such that the radial scaling compensates for the usual conformal scaling of the metric. The WCW isometries would thus correspond to the isometries of the parton orbit and of the boundary of the light cone! These two representations could provide alternative representations for the charges if the strong form of holography holds true and would realize a strong form of holography. Perhaps these realizations deserve to be called inertial and gravitational charges.

   Can these transformations leave the action invariant? For the light-cone boundary, this looks obvious if the light-cone is sliced by a surface parallel to the light-cone boundary. Note however that the tip of this surface might produce problems. A slicing defined by the Hamilton-Jacobi structure would be naturally associated with partonic orbits.

4. What about Poincare symmetries? They would act on the center of mass coordinates of causal diamonds (CDs) as found already earlier [L7]. CDs form the "spine" of WCW, which can be regarded as fiber space with fiber for a given CD containing as a fiber the space-time surfaces inside it.

The super-symmetric counterparts of holomorphic charges for the modified Dirac action and bilinear in fermionic oscillator operators associated with the second quantization of free spinor fields in H, define gamma matrices of WCW. Their anticommutators define the Kähler metric of WCW. There is no need to calculate either the action defining the classical Kähler action defining the Kähler function or its derivatives with respect to WCW complex coordinates and their conjugates. What is important is that this makes it possible to speak about WCW metric also for number theoretical discretization of WCW with space-time surfaces replaced with their number theoretic discretizations.

2.2 Could generalized holomorphy allow to sharpen the existing views?

This picture is rather speculative, allows several variants, and is not proven. There is now however a rather convincing ansatz for the general form of preferred extremals. Could it help to make the picture more precise?

1. As explained, the explicit solution of field equations in terms of the generalized holomorphy is now known. The solution ansatz is independent of action as long it is general coordinate invariance depending only on the induced geometric structures.

   Space-time surfaces would be minimal surfaces apart from lower-dimensional singular surfaces at which the field equations involve the entire action. Only the singularities, classical charges and positions of topological interaction vertices depend on the choice of the action [L4]. Kähler action plus volume term is the choice of action forced by twistor lift making the choice of H unique.

2. The universality has a very intriguing implication. One can assign to any action of this kind conserved Noether currents and their fermionic counterparts (also super counterparts). One would have a huge algebra of conserved currents characterizing the space-time geometry. The corresponding charges can be made conserved by suitably modifying the form of holomorphic functions of the ansatz and therefore the time derivatives $\partial_t h^k$ at the 3-D end of space-time surface at the boundary CD. This need not be the case for all deformations of partonic orbits. In any case, the 3-D holographic data seem to be dual as the strong form of holography suggests. The discussion of the symplectic symmetries leads to the conclusion that they give rise to conserved charges at the partonic 3-surfaces obeying Chern-Simons-Kähler dynamics, which is non-deterministic.

3. Hamilton-Jacobi structures emerge naturally as generalized conformal structures of space-time surfaces and M^4 [L6]. This inspires a proposal for a generalization of modular invariance and of moduli spaces as subspaces of Teichmüller spaces.

4. One can assign to holomorphy conserved Noether charges. The conservation reduces to the algebraic conditions satisfied for the same reason as field equations, i.e. the conservation conditions involving contractions of complex tensors of type (1,1) with tensors of type (2,0) and (0,2). The charges have the same form as Noether charges but it is not completely clear whether the action remains invariant under these transformations. This point is non-trivial since Noether theorem says that invariance of the action implies the existence of conserved charges but not vice versa. Could TGD represent a situation in which the equivalence between symmetries of action and conservation laws fails?

   Also string models have conformal symmetries but in this case 2-D area form suffers conformal scaling. Also the fact that holomorphic ansatz is satisfied for such a large class of actions apart from singularities suggests that the action is not invariant.

5. The action should define Kähler function for WCW identified as the space of Bohr orbits. WCW Kähler metric is defined in terms of the second derivatives of the Kähler action of type (1,1) with respect to complex coordinates of WCW. Does the invariance of the action under holomorphies imply a trivial Kähler metric and constant Kähler function?

   Here one must be very cautious since by holography the variations of the space-time surface are induced by those of 3-surface defining holographic data so that the entire space-time surface is modified and the action can change. The presence of singularities, analogous to poles and cuts of an analytic function and representing particles, suggests that the action represents the interactions of particles and must change. Therefore the action might not be invariant under holomorphies. The parameters characterizing the singularities should affect the value of the action just as the positions of these singularities in 2-D electrostatistics affect the Coulomb energy.

   Generalized conformal charges and supercharges define a generalization of Super Virasoro algebra of string models. Also Kac-Moody algebra assignable to the isometries of $\delta M^4_+ \times CP_2$ and light H generalizes trivially.

6. An absolutely essential point is that generalized holomorphisms are *not* symmetries of Kähler function since otherwise Kähler metric involving second derivatives of type (1,1) with respect to complex coordinates of WCW is non-trivial if defined by these symmetry generators as differential operators. If Kähler function is equal to Kähler action, as it seems, Kähler action cannot be invariant under generalized holomorphies.

   Noether's theorem states that the invariance of the action under a symmetry implies the conservation of corresponding charge but does *not* claim that the existence of conserved Noether currents implies invariance of the action. Since Noether currents are conserved now, one would have a concrete example about the situation in which the inverse of Noether's theorem does not hold true. In a string model based on area action, conformal transformations of complex string coordinates give rise to conserved Noether currents as one easily checks. The area element defined by the induced metric suffers a conformal scaling so that the action is not invariant in this case.

There are several questions to be answered. Could also the symplectic symmetries act as isometries of WCW geometry? Could symplectic transformations act on 3-D holographic data without any continuation to the space-time interior and allow to assign conserved quantum charges with the 3-D data? Holographic generators act on 4-D space-time surfaces and can be associated with the boundary data at the space-like 3-surfaces at the boundaries of CD (at least). Could symplectomorphisms and generalized holomorphisms define algebras, which by holography are dual in some sense? This is possible since the quantum realizations of both algebras rely on second quantized free Dirac fields in H.

3 The twistor space of $H = M^4 \times CP_2$ allows Lagrangian 6-surfaces: what does this mean physically?

I received from Tuomas Sorakivi a link to the article "A note on Lagrangian submanifolds of twistor spaces and their relation to superminimal surfaces" [L13] (see this). The author of the article is Reinier Storm from Belgium.

The abstract of the article tells roughly what it is about.

In this paper a bijective correspondence between superminimal surfaces of an oriented Riemannian 4-manifold and particular Lagrangian submanifolds of the twistor space over the 4-manifold is proven. More explicitly, for every superminimal surface a submanifold of the twistor space is constructed which is Lagrangian for all the natural almost Hermitian structures on the twistor space. The twistor fibration restricted to the constructed Lagrangian gives a circle bundle over the superminimal surface. Conversely, if a submanifold of the twistor space is Lagrangian for all the natural almost Hermitian structures, then the Lagrangian projects to a superminimal surface and is contained in the Lagrangian constructed from this surface. In particular this produces many Lagrangian submanifolds of the twistor spaces and with respect to both the Kähler structure as well as the nearly Kähler structure. Moreover, it is shown that these Lagrangian submanifolds are minimal submanifolds.

The article examines 2-D minimal surfaces X^2 in the 4-D space X^4 assumed to have twistor space. From superminimality which looks somewhat peculiar assumption, it follows that in the twistor space of X^4 (assuming that it exists) there is a Lagrangian surface, which is also a minimal surface. Superminimality means that the normal spaces of the 2-surface form a 1-D curve in the space of all normal spaces, which for the Euclidian signature is the 4-D Grassmannian $SO(4)/SO(2) \times SO(2) = S^2 \times S^2$ ($SO(1,3)/SO(1,1) \times SO(2)$ for M^4). Superminimal surface is therefore highly flattened. Of course, already the minimal surface property favours flatness. It is interesting to examine the generalization of the result to TGD because the interpretation for Lagrange manifolds, which are vacuum extremals for the Kähler action with a vanishing induced symplectic form, has remained open. Certainly, they do not fulfill the holomorphy=holography assumption, i.e. they are not surfaces for which the generalized complex structure in H induces a corresponding structure at 4-surface.

Superminimal surfaces look like the opposite of holomorphic minimal surfaces (this turned out to be an illusion!). In TGD, they give a huge vacuum degeneracy and non-determinism for the pure Kähler action, which has turned out to be mathematically undesirable. The cosmological constant Λ, which follows from twistoralization, was thought to correct the situation.

I had not however notice that the Kähler action, whose existence for $T(H) = T(M^4) \times T(CP_2)$ fixes the choice of H, gives a huge number of 6-D Lagrangian manifolds! Are they consistent with dimensional reduction, so that they could be interpreted as induced twistor structures? Can a complex structure be attached to them? Certainly not as an induced complex structure. Does the Lagrangian problem of Kähler action make a comeback? Furthermore, should one extend the very promising looking holography=holomorphy picture by allowing also Lagrangian 6-surfaces $T(H)$?

Do the Lagrangian surfaces of $T(H)$ have a physical interpretation, most naturally as vacuums? The volume term of the 4-D action characterized by the cosmological constant Λ does not allow vacuum extremals unless Λ vanishes. For the twistor lift Λ is however dynamic and can vanish! Do Lagrangian 6-surfaces in $T(H)$ correspond to 4-D minimal surfaces in H, which are vacuums and have a vanishing $\Lambda = 0$? Would even the original formulation of TGD be an exact part of the theory and not just a long-length-scale limit? And does one really avoid the original problem due to the huge non-determinism spoiling holography!

The question is whether the result presented in the article could generalize to the TGD framework even though the super-minimality assumption does not seem physically natural at first?

3.1 Lagrangian surfaces in the twistor space of $H = M^4 \times CP_2$

Let us consider the 12-D twistor space $T(H) = T(M^4) \times T(CP_2)$ and its 6-D Lagrangian surfaces having a local decomposition $X^6 = X^4 \times S^2$. Assume a twistor lift with Kähler action on $T(H)$. It exists only for $H = M^4 \times CP_2$ [L1, L2].

Let us first forget the requirement that these Lagrangian surfaces correspond to minimal surfaces in H. Consider the situation in which there is no generalized Kähler and symplectic structure in M^4.

One can actually identify Lagrangian surfaces in 12-D twistor space $T(H)$.

1. Since $X^6 = X^4 \times S^2$ is Lagrangian, the symplectic form for it must vanish. This is also true in S^2. Fibers S^2 together with $T(M^4)$ and $T(CP_2)$ are identified by an orientation-changing isometry. The induced Kähler form S^2 in the subset $X^6 = X^4 \times S^2$ is zero as the *sum* of these two contributions of different signs. If this sum appears in the 6-D Kähler action, its contribution to the 6-D Kähler action vanishes. Λ vanishes because the S^2 contribution to the 4-D action vanishes.

2. The 6-D Kähler action reduces in X^4 to the 4-D Kähler action plus, which was the original guess for the 4-D action. The problem is that in its original form, involving only CP_2 Kähler form, it involves a huge vacuum degeneracy. The CP_2 projection is a Lagrangian surface or its subset but the dynamics of M^4 projection is essentially arbitrary, in particular with respect to time. One obtains a huge number of different vacuum extremals. Since the time evolution is non-deterministic, the holography, and of course holography=holomorphy principle, is lost. This option is not physically acceptable.

How the situation changes when also M^4 has a generalized Kähler form that the twistor space picture strongly suggests, and actually requires.

1. Now the Lagrangian surfaces would be products $X^2 \times Y^2$, where X^2 and Y^2 are the Lagrangian surfaces of M^4 and CP_2. The M^4 projections of these objects look like string world sheets and in their basic state are vacuums.

   Furthermore, the situation is deterministic! The point is that X^2 is Lagrangian and highly fixed as such. In the previous case much more general surface M^4 projection, even 4-D, was Lagrangian. There is no loss of holography! Neither is the holography=holomorphy principle lost: by their 2-D character X^2 and Y^2 have a holomorphic structure.

   What is important is that these Lagrangian 4-surfaces of H are obtained also when Λ is non-vanishing. In this case they must be minimal surfaces. Physically this option means that one has Lagrangian strings.

2. For $\Lambda = 0$, the symplectic transformations of H produce new vacuum surfaces. If they are allowed, one might talk of symplectic phase. $J = 0$ phase gives rise to both classical and fermionic vacuum since the modified gamma matries vanish since they are propertional to vanishing canonical momentum currents. So that Lagrangian phase does not contribute to physics for $\Lambda = 0$. There are however non-vacuum extremals for which the induced Kähler field is non-vanishing (having induced complex structure).

   For $\Lambda \neq 0$ Lagrangian surfaces which are non-vacuum extermals and only isometries are allowed as symmetries. One can say that symplectic symmetr breaks down to isometries. Irrespective of the value of Λ, the second phase with a induced complex structure would be present and give rise to color interactions and hadrons and probably also elementary particles. The interpretation of Lagrangian surfaces, which are string like entities, remains open.

3. In the Lagrangian phase induced Kähler form J and the induced color gauge fields vanish and it does not involve monopole fluxes. This phase might be called Maxwell phase. For $\Lambda \neq 0$ one would have two kinds of non-vacuum string like objects with string tension to which Λ contributes.

   Could the Lagrangian phase for $\Lambda \neq 0$ correspond to the Coulomb phase as the perturbative phase of the gauge theories, while the monopole flux tubes (large h_{eff} and dark matter) would correspond to the non-perturbative phase in which magnetic monopole fluxes are present? If so, there would be an analogy with the electric-magnetic duality of gauge theories although the two phases does not look like two equivalent descriptions of one and the same thing unless one restricts the consideration to fermions.

3.1.1 Can Lagrangian 4-surfaces be minimal surfaces?

I have not yet considered the question whether the Lagrangian surfaces can be minimal surfaces. For non-vanishing Λ they must be such but for $\Lambda = 0$ this need not be the case. One can of course ask whether this does matter at all for $\Lambda = 0$. In this case, one has only vacuum extremals and the modified gamma matrices are proportional to the canonical momentum currents, which vanish. Both bosonic and fermionic dynamics are trivial for $\Lambda = 0$. Therefore $\Lambda = 0$ does not give any physics.

In the theorem the minimal Lagrangian surfaces were superminimal surfaces. For super-minimal surfaces, a unit vector in the normal direction defines a very specific curve in normal space.

For a non-vanishing cosmological constant, the field equations for the Kähler action do not force the Lagrangian surfaces to be minimal surfaces. For $\Lambda \neq 0$ there exists a lot of minimal Lagrangian surfaces.

3.1.2 Lagrangian minimal surfaces in CP_2

Consider first the Lagrangian minimal surfaces in CP_2

1. In CP_2, a homologically trivial geodesic sphere is a minimal surface. Note that the geodesic spheres obtained by isometries are regarded here as equivalent. Also a $g = 1$ minimal Lagrangian surface (Clifford torus) in CP_2 is known.

2. There are many other minimal Lagrangian surfaces and second order partial differential equations for both Lagrangian and minimal Lagrangian surfaces are known (see this). In the article "A new look at equivariant minimal Lagrangian surfaces in CP_2 by Dorfmeister and Ma [A1] Lagrangian minimal surfaces in CP_2 are discussed and general partial differential equations for them are deduced.

   (a) An essential role is played by the used of complex coordinates in which the induced metric of X^2 is of form $ds^2 = e^u dz d\bar{z}$ and X^2 corresponds to immersion f.

   (b) The Lagrangian property makes it possible the lift of f and to an immersion defined to unit sphere $S^5 \subset C^3$ and therefore of X^2 to a surface in $S^5 \subset C^3$ defined by a complex triplet F. This allows to combine F, F_z and $F_{\bar{z}}$ to an orthgonal Hermitian tripet which can be can be replaced with a orthonormalized triplet $\mathcal{F} = (F, e^{-u/2}F_z, e^{-u/2}F_{\bar{z}})$.

   (c) At the next step minimal surface property is introduced. It translation to statement that

   $$\mathcal{F}_z = \mathcal{F}\mathcal{U} \quad , \quad \mathcal{F}_{\bar{z}} = \mathcal{F}\mathcal{N} \quad .$$

   Here one has

   $$\mathcal{U} = \begin{pmatrix} u_z/2 & 0 & e^u \\ e^{-u}\psi & -u_z/2 & 0 \\ 0 & -e^u/2 & 0 \end{pmatrix}$$
   $$\mathcal{N} = \mathcal{U}^\dagger$$

   Here ψdz^3 is so called Hopf differential with ψ given by

   $$\psi = F_{zz}\overline{F_{\bar{z}}} \quad .$$

   Clearly, $\mathcal{U}$ is the negative of the hermitian conjugate of $\mathcal{N}$. One can say that complex differentiation corresponds to the action of $SU(3)$ Lie algebra generator so that $\mathcal{F}$ defines an element of $SU(3)$ loop group at X^2.

   (d) The condition of integrability $(\mathcal{F}_z)_{\bar{z}} = (\mathcal{F}_{\bar{z}})_z$ gives

   $$\mathcal{U}_{\bar{z}} = -\mathcal{N}_z \quad .$$

   and the final equations

   $$u_{z\bar{z}} = e^{-2u}|\psi|^2 - e^u \quad , \quad \psi_{\bar{z}} = 0 \quad .$$

   The Hopf differential is therefore a holomorphic function.

Since any stable stable minimal submanifold in CP_n is a complex submanifold, the Lagrangian minimal surfaces cannot be stable under general variations.

3.1.3 Lagrangian minimal surfaces in M^4

Consider next the situation in M^4.

1. In M^4, the plane M^2 is an example of a minimal surface, which is a Lagrangian surface. Are there others? Could Hamilton-Jacobi structures [L6] that also involve the symplectic form and generalized Kähler structure (more precisely, their generalizations) define Lagrangian surfaces in M^4?

2. The Lagrangian surfaces, and as a special case Lagrangian minimal surfaces in R^4 are discussed in [A4]. The result of the article can be phrased as follows.

   Let L be a simply connected domain in C. Then for any smooth conformal Lagrangian immersion $f : L \to R^4$, there exist smooth functions $\beta : L \to R/2\pi Z$, which is the Lagrangian angle, and $s_1, s_2 : L \to C$, not simultaneously vanishing, that satisfy the Dirac-type equation

$$\begin{pmatrix} 0 & \partial_z \\ -\partial_{\overline{z}} & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ \overline{s}_2 \end{pmatrix} = \begin{pmatrix} \overline{U} & 0 \\ 0 & -U \end{pmatrix} \begin{pmatrix} s_1 \\ \overline{s}_2 \end{pmatrix} \ .$$

   with complex potential $U = \partial_z\beta/2$. Conversely, given β and any solution (s_1, s_2) to the Dirac equation satisfying $(|s_1|^2 + |s_2|^2 \geq 0)$ gives rise to a conformal Lagrangian immersion given by

$$f(z) = Re\left[\int^z exp(\beta J/2) \begin{pmatrix} s_1 \\ s_2 \\ -is_1 \\ is_2 \end{pmatrix}\right] \quad , \quad J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \ .$$

   Here the 4×4 matrix J defines the standard symplectic structure.

3. When the Lagrange angle is constant, one obtains minimal Lagrangian immersion. Note that this in this case one has free massless Dirac equation.This suggests quantum classical correspondence in which the solutions of massless Dirac equation in M^4 correspond to Lagrangian minimal surfaces.

4. This solution is defined for Euclidian E^4 rather than M^4 but the analytic continuation to M^4 case should be straightforward. This requires an appropriate modification of J. In TGD one must consider the possibility, that Hamilton-Jacobi structures defines large number of non-quivalent Kähler- and symplectic structures for M^4. The naive guess is that J in the exponential is replaced with the matrix $J_{kl}\sigma^{kl}$ in order to obtain a more general solution.

In the case considered now, the Lagrangian surfaces in H would be products $X^2 \times Y^2$. Interestingly, in the 2-D case the induced metric always defines a holomorphic structure. Now, however, this holomorphic structure would not be the same as the one related to the holomorphic ansatz: it is induced from H.

3.1.4 So What?

These findings raise several questions related to the detailed understanding of TGD. Should one allow only non-vanishing values of Λ? This would allow minimal Langrangian surfaces $X^2 \times Y^2$ besides the holomorphic ansatz. The holomorphic structure due to the 2-dimensionality of X^2 and Y^2 means that holography=holomorphy principle generalizes.

If one allows $\Lambda = 0$, all Lagrangian surfaces $X^2 \times Y^2$ are allowed but also would have a holomorphic structure due to the 2-dimensionality of X^2 and Y^2 so that holography=holomorphy principle would generalize also now! Minimal surface property is obtained as a special case. Classically the extremals correspond to a vacuum sector and also in the fermionic sector modified Dirac equation is trivial. Therefore there is no physics involved.

Minimal Lagrangian surfaces are favored by the physical interpretation in terms of a geometric analog of the field particle duality. The orbit of a particle as a geodesic line (minimal 1-surface) generalizes to a minimal 4-surface and the field equations inside this surface generalizes massless field equations.

4 Modified Dirac equation and the holography=holomorphy hypothesis

The understanding of the modified equation as a generalization of the massless Dirac equation for the induced spinors of the space-time surface X^4 [K5, ?] is far from complete. It is however clear that the modified Dirac equation is necessary [L4] and its failure at singularities, analogous to the failure of minimal surface property at them, leads to an identification of fundamental interaction vertices as 2-vertices for the creation of fermion pair in the induced classical electroweak gauge fields.

These singularities are lower-dimensional surfaces are related to the 4-D exotic diffeomorphic structures [A6, A7] and are discussed from the point of view of TGD in [L3]. They can be interpreted as defects of the standard diffeomorphic structure and mean that in the TGD framework particle creation is possible only in dimension $D = 4$.

A fermion-antifermion pair as a topological object can be said to be created at these singularities. The creation of particles, in the sense that the fermion and antifermion numbers (boson are identified as fermion-antifermion bound states in TGD) are not preserved separately, is only possible in dimension 4, where exotic differentiable structures are possible.

Two problems should be solved.

1. It is necessary to find out whether the modified Dirac equation follows from the generalized holomorphy alone. The dynamics of the space-time surface is trivialized into the dynamics of the minimal surface thanks to the generalized holomorphy and is universal in the sense that the details of the action are only visible at singularities which define the topological particle vertices. Could holomorphy solve also the modified Dirac equation? The modified gamma matrices depend on the action: could the modified Dirac equation fix the modified gamma matrices and thus also the action or does not universality hold true also for the modified Dirac action?

   (a) Let us consider Dirac's equation in M^2 as a simplified example. Denote the light like coordinates (u, v) by $(z, \overline{z})$. The massless Dirac equation reduces to an algebraic condition if the modes are proportional to z^n or $\overline{z}^n$. $\gamma^z\partial_z$ *resp.* $\gamma^{\overline{z}}\partial_{\overline{z}}$ annihilates such a mode if γ^z *resp.* $\gamma^{\overline{z}}$ annihilates the mode.

   (b) These conditions must be generalized to the case of a 4-D space-time surface X^4. Now the complex and Kähler structure are 4-dimensional and holomorphy generalizes. γ^z is generalized to modified gammas Γ^{z_i}, determined by the action principle, which is general coordinate invariant and constructible in terms of the induced geometry. Modified gamma matrices $\Gamma^\alpha = \gamma^k T^\alpha_k$, $T^\alpha_k = \partial L/\partial(\partial_\alpha h^k)$ are contractions of the gamma matrices of H with the canonical impulse currents T^α_k determined by the action density L.

   Irrespective of action, field equations for the space-time surface reduce to the equations of a minimal surface, and are solved by the generalized holomorphy [L8]. The lower-dimensional singularities, at which the minimal surface equations fail, correspond to defects of the standard diffeomorphic structure and are analogs of poles and cuts to analytic functions [L3].

2. The induction of the second quantized spinor field of H on the space-time surface means only the restriction of the induced spinor field to X^4. This determines the fermionic propagators as H-propagators restricted to X^4. The induced spinor field can be expressed as a superposition of the modes associated with X^4. The modes should satisfy the modified Dirac equation, which should reduce by the generalized holomorphy to purely algebraic conditions as in the 2-D case. Is this possible without additional conditions that might fix the action principle? Or is this possible only at lower-dimensional surfaces such as string world sheets?

4.1 How to meet the challenges?

This section begins with an optimistic view of the solution of the problems followed by a critical discussion and detailed proposal for how the generalized holography would solve the modified Dirac equation.

4.1.1 Optimistic view of how holomorphy solves the modified Dirac equation

Consider first the notations: the coordinates for the 4-surface X^4 are the light-like coordinate pair (u,v) and the complex coordinate pair $(z,\overline{z})$. To simplify the notation, we take the notation $(u,v) \equiv (z_1,\overline{z}_1)$ for the light-like coordinate pair (u,v), so that the coordinates of the space-time surface can be denoted by (z_1,z_2) and $(\overline{z}_1,\overline{z}_2)$. As far as algebra is considered, one can consider E^4 instead of M^4, from which Minkowski's version is obtained by continuing analytically.

1. Let us optimistically assume that the H spinor modes can be expressed as superpositions of conformal X^4 spinor modes, which in their simplest form are products of powers of two "complex" variables $z_i^{n_i}$ or $\overline{z}_i^{n_i}$. Only four different types of modes: $z_1^{n_1}z_2^{n_2}$, $\overline{z}_1^{n_1}z_2^{n_2}$, $z_1^{n_1}\overline{z}_2^{n_2}$ and $\overline{z}_1^{n_1}\overline{z_2}^{n_2}$ should appear.

   The spinor modes of H are plane waves if M^4 has no Kähler structure. Could this mean that the modes can be expressed as products of exponentials $exp(ik_iz_i), exp(ik_i\overline{z}_i)$, $i=1,2$. More general analytical functions and their complex conjugates can also be thought of as building blocks of modes. In some cases, the complex coordinate of CP_2 comes into question as well as the complex coordinate of the homologous geodesic sphere.

2. The fermionic oscillator operators associated with X^4 are linear combinations of contributions from different H modes. They satisfy anticommutation relations. It is not clear whether the creation (annihilation) operators for X^4 spinor modes are sums of only creation (annihilation) operators for H spinor modes or wheter for instance sums of the fermion creation operator and the antifermion annihilation operator apppear.

4.1.2 Objections

Consider now the objections against the optimistic view.

1. Also non-holomorphic modes involving $z_i^{n_1}\overline{z_i}^{n_2}$ could be present and in this case both Γ^{z_i} and $\Gamma^{\overline{z}_i}$ should annihilate the mode. This is not possible unless the metric is degenerate.

2. The spinor modes of CP_2 could make the 4-D holomorphy impossible in the proposed sense. The spinor modes of CP_2 are not holomorphic with respect to the complex coordinates of CP_2 and only the covariantly constant right-handed neutrino satisfies massless Dirac equation in CP_2. Could this imply the presence of X^4 spinor modes, which are not holomorphic (antiholomorphic) with respect to the given coordinate z_i ($\overline{z}_i$) so that the modes involving $z_i^m\overline{z}_i^n$ are possible?

3. The general plane wave basis for M^4 without Kähler form in the transversal degrees of freedom is not consistent with the conformal invariance. Here the sum over this kind of modes should give vanishing non-holomorphic modes.

   Note that the Kähler structure for M^4 adds to the M^4 Dirac equation of H a coupling to the Kähler gauge potential of M^4 and implies a transversal mass squared so that the transversal basis does not consist of plane waves but is an analog of harmonic oscillator basis. Also now the failure of holomorphy takes place.

4. For the massive modes of CP_2 spinors, massivation takes place in M^4 degrees of freedom. This would suggest that the plane waves in longitudinal M^4 degrees of freedom cannot be massless.

   However, M^8-H duality implies an important difference between TGD and ordinary field theories. The choice of $M^4 \subset M^8$ is not unique and since particles are massless at the level of H one can always choose $M^4 \supset CD$ in such a way that the momentum has only M^4 component and is massless in M^4 sense. Could the holomorphy at the space-time level be seen as the M^8-H dual of this at the space-time level?

4.1.3 How could one overcome the objections?

One can consider two ways to overcome these objections.

1. The sum of the contributions of products of M^4 plane waves and CP_2 spinor harmonics is involved and could simply vanish for the non-holomorphic modes. This would look like a mathematical miracle transforming the symmetry under the isometries of H to a conformal symmetry at the level of X^4. This mechanism would not depend on the choice of action although the modified Dirac equation might hold only for a unique action.

2. The 4-D conformal invariance for fermions could degenerate to its 2-D version so that only the modified Dirac equation at 2-D string world sheets would allow conformal modes. Indeed, a longstanding question has been whether this is the case for physical reasons. The restriction of the induced spinors to 2-D string world sheets is consistent with the recent view of scattering amplitudes in which the boundaries of string world sheets at the light-like orbits of partonic 2-surfaces, which are metrically 2-dimensional, carry point-like fermions. If this is really true, then the 4-D conformal invariance would effectively reduce to ordinary conformal invariance.

4.1.4 Solution of the modified Dirac equation assuming the generalized holomorphy

Consider now the solution of the modified Dirac equation assuming that only holomorphic modes are present.

1. The modified Dirac equation reads a

$$(\Gamma^{z_i} D_{z_i} + \Gamma^{\overline{z}_i} D_{\overline{z}_i})\Psi = 0 \ .$$

   Γ matrices are modified gamma matrices. D_{z_i} denotes covariant derivative. Generalized conformal invariance produces the equations of the minimal surface almost independently of the action. It is however not clear whether in the modified Dirac equation the modified gammas can be replaced by the induced gamma matrices $\Gamma^\alpha = \gamma_k \partial_\alpha h^k$ (action as 4-volume). At least at the singularities that determine the vertices, this does not apply [L4].

2. The solution of the modified Dirac equation should reduce to the generalized holomorphy. This is achieved if one of the operators $D_{\overline{z}_i}$, D_{z_i}, Γ^{z_i}, $\Gamma^{\overline{z}_i}$ annihilates the given mode on the space-time surface. It follows that $\Gamma^{z_i} D_{z_i}$ and $\Gamma^{\overline{z}_i} D_{\overline{z}_i}$ for each index separately annihilate the spinor modes. Either Γ^{z_i} ($\Gamma^{\overline{z}_i}$) or D_{z_i} ($D_{\overline{z}_i}$) would do this.

   Two gamma matrices in the set $\{\Gamma^{z_i}, \Gamma^{\overline{z}_i} | i = 1, 2\}$ must eliminate a given X^4 spinor mode. Since modified gammas depend on the action, this condition might fix the action.

3. There are two cases to consider. The generalized complex structure of the 4-surface X^4 is induced from that of H [L8] or if the space-time surface is a product of Lagrange manifolds $X^2 \times Y^2 \subset M^4 \times CP_2$, is induced from the complex structures of the 2-D factors associated with their induced metrics [L13].

4. I have proposed that M^4 allows several generalized Kähler structures, which I have called Hamilton-Jacobi structures [L6]. The 4-surface could fix the Hamilton-Jacobi structure from the condition that the modified Dirac equation is valid. Since the modified gammas depend on the action, the annihilation conditions for the modified gamma matrices might fix the choice of the action, and this choice could correlate with the generalized complex structure of X^4.

To sum up, the above considerations are only an attempt to clarify the situation and it is not at all obvious that the generalized holomorphy trivializes the solution of the modified Dirac action.

4.2 Fermionic oscillator operators in X^4 as fermionic supersymmetry generators acting as gamma matrices of the "world of classical worlds" (WCW)

The challenge is to construct the fermionic oscillator operators in X^4 assignable to the modes of the induced spinor field in X^4.

1. By holography and the experience with quantum field theories one expects that the oscillator operators are expressible in terms of data at $t = constant$ surface and do not depend on the value of t chosen. Therefore the X^4 oscillator operators should be conserved quantities and the identification as supercharges is natural. These supercharges in turn would define the gamma matrices of "world of classical worlds" (WCW).

2. Modified Dirac equation indeed is constructed so that it has supersymmetry in the sense that conserved fermionic Noether charges associated with the isometries of H and generalized conformal transformations of H appearing as symmetries in the holography= holomorphy ansatz gave super counterparts.

   If the conserved Noether current associated with this kind of symmetry is of form $\overline{\Psi}O^\alpha\Psi$, the corresponding conserved supercurrent associated with the c-number valued mode Ψ_n of the modified Dirac equation is $\overline{\Psi}_nO\Psi$. The form of O can be deduced from the change of the modified Dirac action under the symmetry.

4.2.1 The Noether currents and their super counterparts associated with the modified Dirac action

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3. The action density associated with the modified Dirac action is given by

$$
\begin{array}{ll}
L_D = \overline{\Psi}D\Psi\sqrt{g} \ , & D = D^{\rightarrow} - D^{\leftarrow} \ , \\
D^{\rightarrow} = \Gamma^\alpha D^{\rightarrow}_\alpha & D^{\leftarrow} = D^{\leftarrow}_\alpha \Gamma^\alpha \ , \\
\Gamma^\alpha = \gamma^k T^\alpha_k & T^\alpha_k = \frac{\partial}{\partial(\partial_\alpha h^k)} L_B \ .
\end{array}
\tag{4.1}
$$

   Here L_B denotes the bosonic action density defining space-time surfaces as preferred extremals satisfying holography (analogs of Bohr orbits). The replacement of the ordinary induced gamma matrices as projections of the gamma matrices of H with the modified gamma matrices guarantees the hermicity of the modified Dirac operator and implies supersymmetry so that the conserved Noether currents for L_D are accompanied by the fermionic super counterparts.

4. The conserved Noether current associated with the symmetry $h^k \rightarrow h^k + \epsilon j^k$ can be deduced from the variation of L_D

$$\begin{array}{ll} J^{\alpha}_j = (X^{\alpha}_1 + X^{\alpha}_2 + X^{\alpha}_3 + X^{\alpha}_4)\sqrt{g_4} \ , & X^{\alpha}_1 = d_{\epsilon}\delta\overline{\Psi}\Gamma^{\alpha}\Psi - \overline{\Psi}\Gamma^{\alpha}d_{\epsilon}\delta\Psi \ , \\ X^{\alpha}_2 = \overline{\Psi}(j^k_A T^{\alpha\beta}_{kl}(\gamma^l D^{\rightarrow}_{\beta} - D^{\leftarrow}\gamma^l)T^{\alpha\beta}_{kl} j^k_A \Psi \ , & T^{\alpha\beta}_{kl} = \frac{\partial}{\partial(\partial_{\alpha}h^k)}T^{\beta}_l = \frac{\partial}{\partial(\partial_{\alpha}h^k)}\frac{\partial}{\partial(\partial_{\beta}h^l)}L_B \\ X^{\alpha}_3 = 2\overline{\Psi}\Gamma^{\alpha}A_k j^k_A\Psi \ , & X^{\alpha}_4 = L_D g^{\alpha\beta}\partial_{\beta}h^k h_{kl} j^l_A \ . \end{array} \quad (4.2)$$

5. The super current associated with J^{α}_j is obtained by replacing in the above currents either $\overline{\Psi}$ (or Ψ) with its c-number valued mode $\overline{\Psi}_n$ (Ψ_n).

   $\Delta\Psi$ and $\delta\overline{\Psi}$ can be deduced from the action of the symmetry transformation on spin degrees of freedom. For instance, rotations and Lorentz transformations induce spin rotation. Only the operator D has a direct dependence on h^k and $\partial_{\alpha}h^k$.

6. The conserved supercharges

$$Q_j = \int_{X^3} X^3 J_j d^3x \quad (4.3)$$

   defines the fermionic oscillator operators for X^4. Note that J_j contains the $\sqrt{g_4}$ factor defining the integration measures. By general coordinate invariance and conservation of these charges it is enough that X^3 is deformable to a section of causal diamond with constant M^4 time or light-cone proper time.

   associated with J^{α}_j defines a gamma matrix for WCW and a fermionic oscillator operator for the space-time surface. The oscillator operators of H spinor modes can in this way be transformed to oscillator operators of the induced spinor modes.

The modes of CP_2 Dirac operator without M^4 Kähler form have mass scale of order CP_2 mass with one exception: covariantly constant right-handed neutrino. In the presence of M^4 Kähler form also this state has mass of order CP_2 mass. Both the color quantum numbers and mass squared depend on the electroweak spin.

Unless the M^4 plane corresponds to a state, which is nearly at rest in the the rest frame of CD, its large spatial momentum implies very rapid wiggling and the contribution to the super charge as analog of Fourier component of Ψ is expected to be very small. If the state is at rest, the restriction to $t = constant$ surface guarantees that the contribution to the super charge is non-vanishing and does not depend on time t.

It should be noticed that the Feynmann propagator for an arbitrary massive fermion between a pair of points of M^4 becomes independent of the mass as the distance becomes light-like [K6] so that H spinor modes with arbitrarily high mass behave like massless particles at the boundaries of the string world sheets located at light-like partonic orbits. This would correspond to the assignment Chern-Simons-Kähler (CSK) action to the partonic orbits. The presence of M^4 part in the CSK action would allow nonvanishing light-like M^4 momenta.

4.3 About the relationship between supercharges and spinor modes of H

What can one say about the behavior of the modes of the induced spinor field? The most natural choice for the basis for holomorphic modes is such that it is of the same form as the planewave modes for H. Therefore the products of imaginary exponentials $exp(ih_iz_i)$ of "complex" coordinates $\tau_i = exp(z_i)$ and their complex conjugates assignable to the Hamilton-Jacobi structure looks like a natural choice.

The conformal weights h_i could be analogous to conformal weights. M^4 momenta would be replaced with a pair of conformal weights h_1 and h_2. For single conformal weight the natural interpretation is as mass squared and the challenge is to generalize this picture. Physical intuition would suggest h_i are real for the physical states whereas for "virtual" states h_i would be (possibly) complex algebraic numbers (I have talked about conformal confinement as a consequence of Galois confinement). If this is the case, there would be only 2 real conformal weights as opposed to 4 components for M^4 momenta (restricted by mass shell conditions).

The quantum numbers of H spinors are mapped to those of X^4. Could the conformal weights h_i correspond to the contributions of M^4 and CP_2 to the 8-momentum of M^8 and be identifiable as mass squared values for M^4 and CP_2? One cannot however assume that the M^4 and CP_2 mass squared values of H-spinors are mapped as such to h_i.

The identification $h_1 = m^2(M^4)$ and $h_2 = m^2(CP_2)$ combined with $m^2 = h_1 - h_2 = 0$ allows only massless states. $m^2 = h_1 - h_2 \geq 0$ for the physical mass squared is more plausible. p-Adic thermodynamics would give the physical mass as a thermodynamic expectation value so that positive values of $m^2 = h_1 - h_2$ are needed.

4.3.1 Does the presence of two conformal weights solve the tachyon problem of p-adic mass calculations

In p-adic mass calculations one assumes that physical fermion is created by the oscillator operator of H spinor mode. To this state super-Kac-Moody - or super-symplectic generator is applied to give a state with physical color quantum numbers.

One must also assume that the ground state is tachyonic with conformal weight $h = -3/2$ or $h = -5/2$. The action of Kac-Moody-/symplectic generators would compensate for the tachyonic conformal weight and give massless states as ground states. Their thermal excitations would give the physical mass as thermal mass squared. The challenge is to understand the origin of the tachyonic conformal weight.

1. For the 4-D generalization of conformal invariance, there would be two conformal weights h_1 and h_2 associated with longitudinal and transversal degrees of freedom of M^4 Hamilton-Jacobi structure [L6]. The conformal weights correspond physically to the mass squared and the identification $m^2 = h_1 - h_2 \geq 0$ for the physical mass squared could make sense. p-Adic thermodynamics would give the physical mass as a thermodynamic expectation value so that non-negative values of $m^2 = h_1 - h_2$ are needed. This would be the space-time analog for positive values of M^4 mass squared.

   Note that in the case of hadrons, longitudinal momenta of quarks are nearly massless but the transverse confinement gives rise to transversal momentum squared. The interpretation could be that the (dominating) contribution of the color magnetic body of the hadron mass makes the momentum of the state non-tachyonic.

2. In this framework, one could understand the construction of the physical states in the following way. The tachyonic ground state would correspond to a state having only the transversal contribution $-h_2$ to the mass squared and the action by Kac-Moody-/symplectic generators would add excitations with a nonvanishing h_1 and give a massless state as well as its excitations with positive mass squared. The replacement of 2-D string worlds sheets with 4-D space-time surface would solve the tachyon problem.

I have also considered an alternative approach to the tachyon problem and one can wonder if it is consistent with the proposed one.

1. As noticed, $M^8 - H$ duality involves a selection of $M^4 \subset M_c^8$. The octonionic automorphism group G_2 generates different choices of M^4. What could this freedom to choose $M^4 \subset M_c^8$ mean? How is it visible at the level of H? Since G_2 is an automorphism group, the states would be analogous to states differing by Lorentz boosts. Since these states are massless in M^8, it should be possible to find a choice of $M^4 \subset M_c^8$ for which the states are massless and thus also in $M^4 \subset H$. This choice is like going to the rest frame of a moving system in special relativity. How are these two states related at the level of H?

2. The natural proposal is that in $M^4 \subset M_c^8$ it is always possible to transform a given state with $m^2 \geq 0$ to a state with $m^2 = 0$. In the padic mass calculations this choice corresponds

to a construction of a massless state from a state which in absence of tachyons would have mass of order CP_2 mass.

The massless state would be obtained by an addition to the state of a transverse tachyonic contribution with a non-vanishing weight h_2 to give $h_1 = h_2$. The notion of mass defined as $m^2 = h_1 - h_2$ would be a relative notion like four-momentum in special relativity. Application of conformal generators would make it possible to generate states with different rest frames.

3. $SO(1,7)$ contains G_2 as a subgroup of the rotation group $SO(6) \subset SO(1,7)$. More general transformation of $SO(1,7)$ analogous to Lorentz boosts would not be allowed number-theoretically. The integer valued spectrum for m^2 allows only a discrete subgroup of G_2. In special relativity this would correspond to a discrete subgroup of the Lorentz group.

To sum up, the tachyon problem of the superstring models could be seen as the compelling reason for replacing string world sheets with 4-D space-time surfaces. The predicted two conformal weights would allow to get rid of tachyons, which also appeared in the p-adic mass calculations based on ordinary conformal invariance.

5 Challenging the existing view of symplectic symmetries in relation to WCW geometry

I have considered the possibility that also the symplectomorphisms of $\delta M^4 + \times CP_2$ could define WCW isometries. This actually the original proposal. One can imagine two options.

1. The continuation of symplectic transformations to transformations of the space-time surface from the boundary of light-cone or from the orbits partonic 2-surfaces should give rise to conserved Noether currents but it is not at all obvious whether this is the case.

2. One can assign conserved charges to the time evolution of the 3-D boundary data defining the holographic data: the time coordinate for the evolution would correspond to the light-like coordinate of light-cone boundary or partonic orbit. This option I have not considered hitherto. It turns out that this option works!

The conclusion would be that generalized holomorphies give rise to conserved charges for 4-D time evolution and symplectic transformations give rise to conserved charged for 3-D time evolution associated with the holographic data.

5.1 About extremals of Chern-Simons-Kähler action

Let us look first the general nature of the solutions to the extremization of Chern-Simons-Kähler action.

1. The light-likeness of the partonic orbits requires Chern-Simons action, which is equivalent to the topological action $J \wedge J$, which is total divergence and is a symplectic in variant. The field equations at the boundary cannot involve induced metric so that only induced symplectic structure remains. The 3-D holographic data at partonic orbits would extremize Cherns-Simons-Kähler action. Note that at the ends of the space-time surface about boundaries of CD one cannot pose any dynamics.

2. If the induced Kähler form has only the CP_2 part, the variation of Chern-Simons-Kähler form would give equations satisfied if the CP_2 projection is at most 2-dimensional and Chern-Simons action would vanish and imply that instanton number vanishes.

3. If the action is the sum of M^4 and CP_2 parts, the field equations in M^4 and CP_2 degrees of freedom would give the same result. If the induced Kähler form is identified as the sum of the M^4 and CP_2 parts, the equations also allow solutions for which the induced M^4 and CP_2 Kähler forms sum up to zero. This phase would involve a map identifying M^4 and CP_2 projections and force induce Kähler forms to be identical. This would force magnetic charge in M^4 and the question is whether the line connecting the tips of the CD makes non-trivial

homology possible. The homology charges and the 2-D ends of the partonic orbit cancel each other so that partonic surfaces can have monopole charge.

The conditions at the partonic orbits do not pose conditions on the interior and should allow generalized holomorphy. The following considerations show that besides homology charges as Kähler magnetic fluxes also Hamiltonian fluxes are conserved in Chern-Simons-Kähler dynamics.

5.2 Can one assign conserved charges with symplectic transformations or partonic orbits and 3-surfaces at light-cone boundary?

The geometric picture is that symplectic symmetries are Hamiltonian flows along the light-like partonic orbits generated by the projection A_t of the Kähler gauge potential in the direction of the light-like time coordinate. The physical picture is that the partonic 2-surface is a Kähler charged particle that couples to the Hamilton $H = A_t$. The Hamiltonians H_A are conserved in this time evolution and give rise to conserved Noether currents. The corresponding conserved charge is integral over the 2-surface defined by the area form defined by the induced Kähler form.

Let's examine the change of the Chern-Simons-Kähler action in a deformation that corresponds, for example, to the CP_2 symplectic transformation generated by Hamilton H_A. M^4 symplectic transformations can be treated in the same way:here however M^4 Kähler form would be involved, assumed to accompany Hamilton-Jacobi structure as a dynamically generated structure.

1. Instanton density for the induced Kähler form reduces to a total divergence and gives Chern-Simons-Kähler action, which is TGD analog of topological action. This action should change in infinitesimal symplectic transformations by a total divergence, which should vanish for extremals and give rise to a conserved current. The integral of the divergence gives a vanishing charge difference between the ends of the partonic orbit. If the symplectic transformations define symmetries, it should be possible to assign to each Hamiltonian H_A a conserved charge. The corresponding quantal charge would be associated with the modified Dirac action.

2. The conserved charge would be an integral over X^2. The surface element is not given by the metric but by the symplectic structure, so that it is preserved in symplectic transformations. The 2-surface of the time evolution should correspond to the Hamiltonian time transformation generated by the projection $A_\alpha = A_k \partial_\alpha s^k$ of the Kähler gauge potential A_k to the direction of light-like time coordinate $x^\alpha \equiv t$.

3. The effect of the generator $j_A^k = J^{kl} \partial_l H_A$ on the Kähler potential A_l is given by $j_A^k \partial_k A_l$. This can be written as $\partial_k A_l = J_{kl} + \partial_l A_k$. The first term gives the desired total divergence $\partial_\alpha (\epsilon^{\alpha\beta\gamma} J_{\beta\gamma} H_A)$.

   The second term is proportional to the term $\partial_\alpha H_A - \{A_\alpha, H\}$. Suppose that the induced Kähler form is transversal to the light-like time coordinate t, i.e. the induced Kähler form does not have components of form $J_{t\mu}$. In this kind of situation the only possible choice for α corresponds to the time coordinate t. In this situation one can perform the replacement $\partial_\alpha H_A - \{A_\alpha, H\} \to dH_A/dt - \{A_t, H\}$ This corresponds to a Hamiltonian time evolution generated by the projection A_t acting as a Hamiltonian. If this is really a Hamiltonian time evolution, one has $dH_A/dt - \{A, H\} = 0$. Because the Poisson bracket represents a commutator, the Hamiltonian time evolution equation is analogous to the vanishing of a covariant derivative of H_A along light-like curves: $\partial_t H_A + [A, H_A] = 0$. The physical interpretation is that the partonic surface develops like a particle with a Kähler charge. As a consequence the change of the action reduces to a total divergence.

   An explicit expression for the conserved current $J_A^\alpha = H_A \epsilon^{\alpha\beta\gamma} J_{\beta\gamma}$ can be derived from the vanishing of the total divergence. Symplectic transformations on X^2 generate an infinite-dimensional symplectic algebra. The charge is given by the Hamiltonian flux $Q_A = \int H_A J_{\beta\gamma} dx^\alpha \wedge dx^\beta$.

4. If the projection of the partonic path CP_2 or M^4 is 2-D, then the light-like geodesic line corresponds to the path of the parton surface. If A_l can be chosen parallel to the surface, its

projection in the direction of time disappears and one has $A_t = 0$. In the more general case, X^2 could, for example, rotate in CP_2. In this case A_t is nonvanishing. If J is transversal (no Kähler electric field), charge conservation is obtained.

Do the above observations apply at the boundary of the light-cone?

1. Now the 3-surface is space-like and Chern-Simons-Kähler action makes sense. It is not necessary but emerges from the "instanton density" for the Kähler form. The symplectic transformations of $\delta M^4_+ \times CP_2$ are the symmetries. The most time evolution associated with the radial light-like coordinate would be from the tip of the light-cone boundary to the boundary of CD. Conserved charges as homological invariants defining symplectic algebra would be associated with the 2-D slices of 3-surfaces. For closed 3-surfaces the total charges from the sheets of 3-space as covering of δM^4_+ must sum up to zero.

2. Interestingly, the original proposal for the isometries of WCW was that the Hamiltonian fluxes assignable to M^4 and CP_2 degrees of freedom at light-like boundary act define the charges associated with the WCW isometries as symplectic transformations so that a strong form of holography would have been be realized and space-time surface would have been effectively 2-dimensional. The recent view is that these symmetries pose conditions only on the 3-D holographic data. The holographic charges would correspond to additional isometries of WCW and would be well-defined for the 3-surfaces at the light-cone boundary.

To sum up, one can imagine many options but the following picture is perhaps the simplest one and is supported by mathematical facts. The isometry algebra of $\delta M^4_+ \times CP_2$ consists of generalized conformal and KM algebras at 3-surfaces in $\delta M^4_+ \times CP_2$ and symplectic algebras at the light cone boundary and 3-D light-like partonic orbits. The latter symmetries give constraints on the 3-D holographic data. It is still unclear whether one can assign generalized conformal and Kac-Moody charges to Chern-Simons-Kähler action. The isomorphic subalgebras labelled by a positive integer and their commutators with the entire algebra would annihilate the physical states. The isomorphic subalgebras labelled by a positive integer and their commutators with the entire algebra would annihilate the physical states. These two representations would generalize the notions of inertial and gravitational mass and their equivalence would generalize the Equivalence Principle.

5.2.1 Objection against the idea about theoretician friendly Mother Nature

One of the key ideas behind the TGD view of dark matter is that Nature is theoretician friendly [L5]. When the coupling strength proportional to $\hbar_{eff}$ becomes so large that perturbation series ceases to converge, a phase transition increasing the value of h_{eff} takes place so that the perturbation series converges.

One can however argue that this argument is quantum field-theoretic and does not apply in TGD since holography changes the very concept of perturbation theory. There is no path integral to worry about. Path integral is indeed such a fundamental concept that one expects it to have some approximate counterpart also in the TGD Universe. Bohr orbits are not completely deterministic: could the sum over the Bohr orbits however translate to an approximate description as a path integral at the QFT limit? The dynamics of light-like partonic orbits is indeed non-deterministic and could give rise to an analog of path integral as a finite sum.

1. The dynamics implied by Chern-Simons-Kähler action assignable to the partonic 3-surface with light-one coordinate in the role of time, is very topological in that the partonic orbits is light-like 3-surface and has 2-D CP_2 and M^4 projections unless the induced M^4 and CP_2 Kähler forms sum up to zero. The light-likeness of the projection is a very loose condition and and the sum over partonic orbits as possible representation of holographic data analogous to initial values (light-likeness!) is therefore analogous to the sum over all paths appearing as a representation of Schrödinger equation in wave mechanics.

   One would have an analog of 1-D QFT. This means that the infinities of quantum field theories are absent but for a large enough coupling strength $g^2/4\pi\hbar$ the perturbation series fails to converge. The increase of h_{eff} would resolve the problem. For instance, Dirac equation in atomic physics makes unphysical predictions when the value of nucler charge is larger than $Z \sim 137$.

2. I have also considered a discrete variant of this picture motivated by the fact that the presence of the volume term in the action implies that the M^4 projection of the CP_2 type extremal is a light-like geodesic line. The light-like orbits would consist of pieces of light-like geodesics implying that the average velocity would be smaller than c: this could be seen as a correlate for massivation.

   The points at which the direction of segment changes would correspond to points at which energy and momentum transfer between the partonic orbit and environment takes place. This kind of quantum number transfer might occur at least for the fermionic lines as boundaries of string world sheets. They could be described quantum mechanically as interactions with classical fields in the same way as the creation of fermion pairs as a fundamental vertex [L4]. The same universal 2-vertex would be in question.

   At these points the minimal surface property would fail and the trace of the second fundamental form would not vanish but would have a delta function-like singularity. The CP_2 part of the second fundamental form has quantum numbers of Higgs so that there would be an analogy with the standard description of massivation by the Higgs mechanism. Higgs would be only where the vertices are.

3. What is intriguing, that the light-likeness of the projection of the CP_2 type extremals in M^4 leads to Virasoro conditions assignable to M^4 coordinates and this eventually led to the idea of conformal symmetries as isometries as WCW. In the case of the partonic orbits, the light-like curve would be in $M^4 \times CP_2$ but it would not be surprising if the generalization of the Virasoro conditions would emerge also now.

   One can write M^4 and CP_2 coordinates for the light-like curve as Fourier expansion in powers of $exp(it)$, where t is the light-like coordinate. This gives $h^k = \sum h^k_n exp(int)$. If the CP_2 projection of the orbits of the partonic 2-surface is geodesic circle, CP_2 metric s_{kl} is constant, the light-likeness condition $h_{kl}\partial_t h^k \partial_t h^l = 0$ gives $Re[h_{kl} \sum_m h^k_{n-m} \overline{h}^l_m] = 0$. This does not give Virasoro conditions.

   The condition $d/dt(h_{kl}\partial_t h^k \partial_t h^l = 0) = 0$ however gives the standard Virasoro condition in quantization condition stating that the operator counterparts of quantities $L_n = Re[h_{kl} \sum_m (n-m) h^k_{n-m} \overline{h}^l_m]$ annihilate the physical states. What is interesting is that the latter condition also allows time-like (and even space-like) geodesics.

   Could massivation mean a failure of light-likeness? For piecewise light-like geodesics the light-likeness condition would be true only inside the segments. By taking Fourier transform one expects to obtain Virasoro conditions with a cutoff analogous to the momentum cutoff in condensed matter physics for crystals.

4. In TGD the Virasoro, Kac-Moody algebras and symplectic algebras are replaced by half-algebras and the gauge conditions are satisfied for conformal weights which are n-multiples of fundamentals with with n larger than some minimal value. This would dramatically reduce the effects of the non-determinism and could make the sum over all paths allowed by the light-likeness manifestly finite and reduce it to a sum with a finite number of terms. This cutoff in degrees of freedom would correspond to a genuinely physical cutoff due to the finite measurement resolution coded to the number theoretical anatomy of the space-time surfaces. This cutoff is analogous to momentum cutoff and could at the space-time picture correspond to finite minimum length for the light-like segments of the orbit of the partoic 2-surface.

5.2.2 Boundary conditions at partonic orbits and holography

TGD reduces coupling constant evolution to a number theoretical evolution of the coupling parameters of the action identified as Kähler function for WCW. An interesting question is how the 3-D holographic data at the partonic orbits relates to the corresponding 3-D data at the ends of space-time surfaces at the boundary of CD, and how it relates to coupling constant evolution.

1. The twistor lift of TGD strongly favours 6-D Kähler action, which dimensionally reduces to Kähler action plus volume term plus topological $\int J \wedge J$ term reducing to Chern Simons-Kähler action. The coefficients of these terms are proposed to be expressible in terms of number theoretical invariants characterizing the algebraic extensions of rationals and polynomials determining the space-time surfaces by $M^8 - H$ duality.

   Number theoretical coupling constant evolution would be discrete. Each extension of rationals would give rise to its own coupling parameters involving also the ramified primes characterizing the polynomials involved and identified as p-adic length scales.

2. The time evolution of the partonic orbit would be non-deterministic but subject to the light-likeness constraint and boundary conditions guaranteeing conservation laws. The natural expectation is that the boundary/interface conditions for a given action cannot be satisfied for all partonic orbits (and other singularities). The deformation of the partonic orbit requiring that boundary conditions are satisfied, does not affect X^3 but the time derivatives $\partial_t h^k$ at X^3 are affected since the form of the holomorphic functions defining the space-time surface would change. The interpretation would be in terms of duality of the holographic data associated with the partonic orbits *resp.* X^3.

   There can of course exist deformations, which require the change of the coupling parameters of the action to satisfy the boundary conditions. One can consider an analog of renormalization group equations in which the deformation corresponds to a modification of the coupling parameters of the action, most plausibly determined by the twistor lift. Coupling parameters would label different regions of WCW and the space-time surfaces possible for two different sets of coupling parameters would define interfaces between these regions.

In order to build a more detailed view one must fix the details related to the action whose value defines the WCW Kähler function.

1. If Kähler action is identified as Kähler action, the identification is unique. There is however the possibility that the imaginary exponent of the instanton term or the contribution from the Euclidean region is not included in the definition of Kähler function. For instance instanton term could be interpreted as a phase of quantum state and would not contribute.

2. Both Minkowskian and Euclidean regions are involved and the Euclidean signature poses problems. The definition of the determinant as $\sqrt{-g_4}$ is natural in Minkowskian regions but gives an imaginary contribution in Euclidean regions. $\sqrt{|g_4|}$ is real in both regions. $i\sqrt{g_4}$ is real in Minkowskian regions but imaginary in the Euclidean regions.

   There is also a problem related to the instanton term, which does not depend on the metric determinant at all. In QFT context the instanton term is imaginary and this is important for instance in QCD in the definition of CP breaking vacuum functional. Should one include only the 4-D or possibly only Minkowskian contribution to the Kähler function imaginary coefficient for the instanton/Euclidian term would be possible?

3. Boundary conditions guaranteeing the conservation laws at the partonic orbits must be satisfied. Consider the $\sqrt{|g_4|}$ case. Charge transfer between Euclidean and Minkowskian regions. If the C-S-K term is real, also the charge transfer between partonic orbit and 4-D regions is possible. The boundary conditions at the partonic orbit fix it to a high degree and also affect the time derivatives $\partial_t h^k$ at X^3. This option looks physically rather attractive because classical conserved charges would be real.

   If the C-S-K term is imaginary it behaves like a free particle since charge exchange with Minkowskian and Euclidean regions is not possible. A possible interpretation of the possible M^4 contribution to momentum could be in terms of decay width. The symplectic charges do not however involve momentum. The imaginary contribution to momentum could therefore come only from the Euclidean region.

4. If the Euclidean contribution is imaginary, it seems that it cannot be included in the Kähler function. Since in M^8 picture the momenta of virtual fermions are in general complex, one could consider the possibility that Euclidean contribution to the momentum is imaginary and allows an interpretation as a decay width.

5.3 The TGD counterparts of the gauge conditions of string models

The string model picture forces to ask whether the symplectic algebras and the generalized conformal and Kac-Moody algebras could act as gauge symmetries.

1. In string model picture conformal invariance would suggest that the generators of the generalized conformal and KM symmetries act as gauge transformations annihilate the physical states. In the TGD framework, this does not however make sense physically. This also suggests that the components of the metric defined by supergenerators of generalized conformal and Kac Moody transformations vanish. If so, the symplectomorphisms $\delta M^4_+ \times CP_2$ localized with respect to the light-like radial coordinate acting as isometries would be needed. The half-algebras of both symplectic and conformal generators are labelled by a non-negative integer defining an analog of conformal weight so there is a fractal hierarchy of isomorphic subalgebras in both cases.

2. TGD forces to ask whether only subalgebras of both conformal and Kac-Moody half algebras, isomorphic to the full algebras, act as gauge algebras. This applies also to the symplectic case. Here it is essential that only the half algebra with non-negative multiples of the fundamental conformal weights is allowed. For the subalgebra annihilating the states the conformal weights would be fixed integer multiples of those for the full algebra. The gauge property would be true for all algebras involved. The remaining symmetries would be genuine dynamical symmetries of the reduced WCW and this would reflect the number theoretically realized finite measurement resolution. The reduction of degrees of freedom would also be analogous to the basic property of hyperfinite factors assumed to play a key role in thee definition of finite measurement resolution.

3. For strong holography, the orbits of partonic 2-surfaces and boundaries of the spacetime surface at δM^4_+ would be dual in the information theoretic sense. Either would be enough to determine the space-time surface.

5.4 Could space-time or the space of space-time surfaces be a Lagrangian manifold in some sense?

Gary Ehlenberg sent a link to a tweet to X (see this) by Curt Jainmungal. The tweet has title "Everything is a Lagrangian submanifold". The title expresses the idea of Alan Weinstein (see this), which states that space-time is a Lagrangian submanifold (see this) of some symplectic manifold. Note that the phase space of classical mechanics represents a basic example of symplectic manifold.

Lagrangian manifolds emerge naturally in canonical quantization. They reduce one half of the degrees of freedom of the phase space. This realizes the Uncertainty Principle geometrically. Also holography= holomorphy principle realizes Uncertainty Principle by reducing the degrees of freedom by one half.

What about the situation in TGD [L11, L12, L9]. Does the proposal of Alan Weinstein have some analog in the TGD framework?

Consider first the formulation of Quantum TGD.

1. The original approach of TGD relied on the notion of Kähler action [K2, K3]. The reason was that it had exceptional properties. The Lagrangian manifolds L of CP_2 give rise to vacuum extremals for Kähler action: any 4-surface of $M^4 \times L \subset H = M^4 \times CP_2$ with M^4 is a vacuum extremal for this action. At these space-time surfaces, the induced Kähler form vanishes as also Kähler action as a non-linear analog of Maxwell action.

   The small variations of the Kähler action vanish in order higher than two so that the action would not have a kinetic term and the ordinary perturbation theory in QFT sense (based on path integral) would completely fail. The addition of a volume term to the action cures the situation and in the twistorialization of TGD it emerges naturally and does not bring in the analog of cosmological constant as a fundamental constant but as a dynamically generated parameter. Therefore scale invariance would not be broken at the level of action.

2. This was however not the only problem. The usual perturbation theory would be plagued by an infinite hierarchy of infinities much worse than those of ordinary QFTs: they would

be due to the extreme non-linearity of any general coordinate invariant action density as function of H coordinates and their partial derivatives.

These problems eventually led to the notion of the "world of classical worlds" (WCW) as an arena of dynamics identified as the space of 4-surfaces obeying what I call now holography and realized in some sense [K2, K1, K3, L8]. It took decades to understand in what sense the holography is realized.

1. The 4-D general coordinate invariance would be realized in terms of holography. The definition of WCW geometry assigns to a given 3-surface a unique or almost unique space-time surface at which general coordinate transformations can act. The space-time surfaces are therefore analogs of Bohr orbits so that the path integral disappears or reduces to a sum in the case that the classical dynamics is not completely deterministic. The counterparts of the usual QFT divergences disappear completely and Kähler geometry of WCW takes care of the remaining diverges.

   It should be noticed in passing, that year or two ago, I discussed space-times surfaces, which are Lagrangian manifolds of H with M^4 endowed with a generalization of the Kähler metric. This generalization was motivated by twistorialization.

2. Eventually emerged the realization of holography in terms of generalized holomorphy based on the idea that space-time surfaces are generalized complex surfaces of H having a generalized holomorphic structure based on 3 complex coordinates and one hyper complex coordinate associated which I call Hamilton-Jacobi structure.

   These 4-surfaces are universal extremals of *any* general coordinate invariant action constructible in terms of the induced geometry since the field equations reduce to a contraction of two complex tensors of different type having no common index pairs. Space-time surfaces are minimal surfaces and analogs of solutions of both massless field equations and of massless particles extended from point-like particles to 3-surfaces. Field particle duality is realized geometrically.

   It is now clear that the generalized 4-D complex submanifolds of H are the correct choice to realize holography [L9].

3. The universality realized as action independence, in turn leads to the view that the number theoretic view of TGD in principle could make possible purely number theoretic formulation of TGD [L10] There would be a duality between geometric and number theoretic views [L9], which is analogous to Langlands duality. The number theoretic view is extremely predictive: for instance, it allows to deduce the spectrum for the exponential of action defining vacuum functional for Bohr orbits does not depend on the action principle.

   The universality means enormous computational simplification as also does the possibility to construct space-time surfaces as roots for a pair of (f_1, f_2) of generalized analytic functions of generalized complex coordinates of H. The field equations, which are usually partial differential equations, reduce to algebraic equations. The function pairs form a hierarchy with an increasing complexity starting with polynomials and continuing with analytic functions: both have coefficients in some extension of rationals and even more general coefficients can be considered.

So, could Lagrangian manifolds appear in TGD in some sense?

1. The proposal that the WCW as the space of 4-surfaces obeying holography in some sense has symplectomorphisms of H as isometries, has been a basic idea from the beginning. If holography= holomorphy principle is realized, both generalized conformal transformations and generalized symplectic transformations of H would act as isometries of WCW [L8]. This infinite-dimensional group of isometries must be maximal possible to guarantee the existence of Riemann connection: this was already observed for loop spaces by Freed. In the case of loop spaces the isometries would be generated by a Kac-Moody algebra.

2. Holography, realized as Bohr orbit property of the space-time surfaces, suggests that one could regard WCW as an analog of a Lagrangian manifold of a larger symplectic manifold WCW_{ext} consisting of 4-surfaces of H appearing as extremals of some action principle. The Bohr orbit property defined by the holomorphy would not hold true anymore.

   If WCW can be regarded as a Lagrangian manifold of WCW_{ext}, then the group of Sp(WCW) of symplectic transformations of WCW_{ext} would indeed act in WCW . The group Sp(H) of symplectic transformations of H, a much smaller group, could define symplectic isometries of WCW_{ext} acting in WCW just as color rotations give rise to isometries of CP_2.

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