TGD as a Generalized Number Theory I: p-Adicization Program

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Abstract

The vision about a number theoretic formulation of quantum TGD is based on the gradual accumulation of wisdom coming from different sources. The attempts to find a formulation allowing to understand real and p-adic physics as aspects of some more general scenario have been an important stimulus and generated a lot of, not necessarily mutually consistent ideas, some of which might serve as building blocks of the final formulation.

The first part of the 3-part chapter is devoted to the p-adicization program attempting to construct physics in various number fields as an algebraic continuation of physics in the field of rationals (or appropriate extension of rationals). The program involves in essential manner the generalization of number concept obtained by fusing reals and p-adic number fields to a larger structure by gluing them together along common rationals. Highly non-trivial number theoretic conjectures are an i outcome of the program.

1. Real and p-adic regions of the space-time as geometric correlates of matter and mind

The solutions of the equations determining space-time surfaces are restricted by the requirement that the embedding space coordinates are real. When this is not the case, one might apply instead of a real completion with some rational-adic or p-adic completion: this is how rational-adic p-adic physics could emerge from the basic equations of the theory. One could interpret the resulting rational-adic or p-adic regions as geometrical correlates for "mind stuff".

p-Adic non-determinism implies extreme flexibility and therefore makes the identification of the p-adic regions as seats of cognitive representations very natural. Unlike real completion, p-adic completions preserve the information about the algebraic extension of rationals and algebraic coding of quantum numbers must be associated with "mind like" regions of space-time. p-Adics and reals are in the same relationship as map and territory.

The implications are far-reaching and consistent with TGD inspired theory of consciousness: p-adic regions are present even at elementary particle level and provide some kind of geometric model of "self" and external world. In fact, p-adic physics must model the p-adic cognitive regions representing real elementary particle regions rather than elementary particles themselves!

2. The generalization of the notion of number

The unification of real physics of material work and p-adic physics of cognition and intentionality leads to the generalization of the notion of number field. Reals and various p-adic number fields are glued along their common rationals (and common algebraic numbers too) to form a fractal book like structure. Allowing all possible finite-dimensional extensions of p-adic numbers brings additional pages to this "Big Book".

At space-time level the book like structure corresponds to the decomposition of space-time surface to real and p-adic space-time sheets. This has deep implications for the view about cognition. For instance, two points infinitesimally near p-adically are infinitely distant in real sense so that cognition becomes a cosmic phenomenon.

3. Number theoretical Universality and number theoretical criticality

Number theoretic universality has been one of the basic guide lines in the construction of quantum TGD. There are two forms of the principle.

1. The strong form of number theoretical universality states that physics for any system should effectively reduce to a physics in algebraic extension of rational numbers at the level of M-matrix (generalization of S-matrix) so that an interpretation in both real and p-adic sense (allowing a suitable algebraic extension of p-adics) is possible. One can however worry whether this principle only means that physics is algebraic so that there would be no need to talk about real and p-adic physics at the level of M-matrix elements. It is not possible to get rid of real and p-adic numbers at the level of classical physics since calculus is a prerequisite for the basic variational principles used to formulate the theory. For this option the possibility of completion is what poses conditions on M-matrix.
2. The weak form of principle requires only that both real and p-adic variants of physics make sense and that the intersection of these physics consist of physics associated with various algebraic extensions of rational numbers. In this rational physics would be like rational numbers allowing infinite number of algebraic extensions and real numbers and p-adic number fields as its completions. Real and p-adic physics would be completions of rational physics. In this framework criticality with respect to phase transitions changing number field - number theoretical criticality - becomes a viable concept. This form of

principle allows also purely p-adic phenomena such as p-adic pseudo non-determinism assigned to imagination and cognition. Genuinely p-adic physics does not however allow definition of notions like conserved quantities since the notion of definite integral is lacking and only the purely local form of real physics allows p-adic counterpart.

Experience has taught that it is better to avoid too strong statements and perhaps the weak form of the principle is enough.

4. p-Adicization by algebraic continuation

One general idea which results as an outcome of the generalized notion of number is the idea of a universal function continuable from a function mapping rationals to rationals or to a finite extension of rationals to a function in any number field. It must be however emphasized that for weaker form of number theoretical universality this restriction applies only at number theoretical quantum criticality. This algebraic continuation is analogous to the analytical continuation of a real analytic function to the complex plane. Rational functions with rational coefficients are obviously functions satisfying this constraint. Algebraic functions with rational coefficients satisfy this requirement if appropriate finite-dimensional algebraic extensions of p-adic numbers are allowed. Exponent function is such a function.

For instance, residue calculus might be generalized so that the value of an integral along the real axis could be calculated by continuing it instead of the complex plane to any number field via its values in the subset of rational numbers forming the rim of the book like structure having number fields as its pages. If the poles of the continued function in the finitely extended number field allow interpretation as real numbers it might be possible to generalize the residue formula. One can also imagine of extending residue calculus to any algebraic extension. An interesting situation arises when the poles correspond to extended p-adic rationals common to different pages of the "great book". Could this mean that the integral could be calculated at any page having the pole common. In particular, could a p-adic residue integral be calculated in the ordinary complex plane by utilizing the fact that in this case numerical approach makes sense.

Algebraic continuation is the basic tool of p-adicization program. Entire physics of the TGD Universe should be algebraically continuable to various number fields. Real number based physics would define the physics of matter and p-adic physics would describe correlates of cognition and intentionality. The basic stumbling block of this program is integration and algebraic continuation should allow to circumvent this difficulty. Needless to say, the requirement that the continuation exists must pose immensely tight constraints on the physics.

Due to the fact that real and p-adic topologies are fundamentally different, ultraviolet and infrared cutoffs in the set of rationals are unavoidable notions and correspond to a hierarchy of different physical phases on one hand and different levels of cognition on the other hand. Two types of cutoffs are predicted: p-adic length scale cutoff and a cutoff due to phase resolution. Zero energy ontology provides a natural realization for the p-adic length scale cutoff. The latter cutoff seems to correspond naturally to the hierarchy of algebraic extensions of p-adic numbers and quantum phases $exp(i2\pi/n)$, $n \geq 3$, coming as roots of unity and defining extensions of rationals and p-adics allowing to define p-adically sensible trigonometric functions These phases relate closely to the hierarchy of quantum groups, braid groups, and II_1 factors of von Neumann algebra.

5. Number theoretic democracy

The interpretation allows all finite-dimensional extensions of p-adic number fields and perhaps even infinite-P p-adics. The notion arithmetic quantum theory generalizes to include Gaussian and Eisenstein variants of infinite primes and corresponding arithmetic quantum field theories. Also the notion of p-adicity generalizes: it seems that one can indeed assign to Gaussian and Eisenstein primes what might be called G-adic and E-adic numbers.

p-Adicization by algebraic continuation gives hopes of continuing quantum TGD from reals to various p-adic number fields. The existence of this continuation poses extremely strong constraints on theory.

1 Introduction

The vision about a number theoretic formulation of quantum TGD is based on the gradual accumulation of wisdom coming from different sources. The attempts to find a formulation allowing to understand real and p-adic physics as aspects of some more general scenario have been an important stimulus and generated a lot of, not necessarily mutually consistent ideas, some of which might serve as building blocks of the final formulation. The original chapter representing the number theoretic vision as a consistent narrative grew so massive that I decided to divide it to three parts.

The first part is devoted to the p-adicization program attempting to construct physics in various number fields as an algebraic continuation of physics in the field of rationals (or appropriate extension of rationals). The program involves in essential way the generalization of number concept obtained by fusing reals and p-adic number fields to a larger structure by gluing them together along common rationals. Highly non-trivial number theoretic conjectures are an i outcome of the program.

Second part focuses on the idea that the tangent spaces of space-time and embedding space can be regarded as 4- *resp.* 8-dimensional algebras such that space-time tangent space defines sub-algebra of embedding space. The basic candidates for the pair of algebras are hyper-quaternions and hyper-octonions.

The great idea is that space-time surfaces X^4 correspond to hyper-quaternionic or co-hyper-quaternionic sub-manifolds of $HO = M^8$. The possibility to assign to X^4 a surface in $M^4 \times CP_2$ means a number theoretic analog for spontaneous compactification. Of course, nothing dynamical is involved and dual relation between totally different descriptions of the physical world would be in question.

The third part is devoted to infinite primes. Infinite primes are in one-one correspondence with the states of super-symmetric arithmetic quantum field theories. The infinite-primes associated with hyper-quaternionic and hyper-octonionic numbers are the most natural ones physically because of the underlying Lorentz invariance, and the possibility to interpret them as momenta with mass squared equal to prime. Most importantly, the polynomials associated with hyper-octonionic infinite primes have automatically space-time surfaces as representatives so that space-time geometry becomes a representative for the quantum states.

1.1 The Painting Is The Landscape

The work with TGD inspired theory of consciousness has led to a vision about the relationship of mathematics and physics. Physics is not in this view a model of reality but objective reality itself: painting is the landscape. One can also equate mathematics and physics in a well defined sense and the often implicitly assumed Cartesian theory-world division disappears. Physical realities are mathematical ideas represented by configuration space spinor fields (quantum histories) and quantum jumps between quantum histories give rise to consciousness and to the subjective existence of mathematician.

The concrete realization for the notion algebraic hologram based on the notion of infinite prime is a second new element. The notion of infinite rationals leads to the generalization of also the notion of finite number since infinite-dimensional space of real units obtained from finite rational valued ratios q of infinite integers divided by q. These units are not units in p-adic sense. The generalization to the quaternionic and octonionic context means that ordinary space-time points become infinitely structured and space-time point is able to represent even the quantum physical state of the Universe in its algebraic structure. Single space-time point becomes the Platonia not visible at the level of real physics but essential for mathematical cognition.

In this view evolution becomes also evolution of mathematical structures, which become more and more self-conscious quantum jump by quantum jump. The notion of p-adic evolution is indeed a basic prediction of quantum TGD but even this vision might be generalized by allowing rational-adic topologies for which topology is defined by a ring with unit rather than number field.

1.2 Real And P-Adic Regions Of The Space-Time As Geometric Correlates Of Matter And Mind

One could end up with p-adic space-time sheets via field equations. The solutions of the equations determining space-time surfaces are restricted by the requirement that the coordinates are real. When this is not the case, one might apply instead of a real completion with some p-adic completion. It however seems that p-adicity is present at deeper level and automatically present via the generalization of the number concept obtained by fusing reals and p-adics along rationals and common algebraics.

p-Adic non-determinism due to the presence of non-constant functions with vanishing derivative implies extreme flexibility and therefore suggests the identification of the p-adic regions as seats of cognitive representations. Unlike the completion of reals to complex numbers, the completions of p-adic numbers preserve the information about the algebraic extension of rationals and algebraic coding of quantum numbers must be associated with "mind like" regions of space-time. p-Adics and reals are in the same relationship as map and territory.

The implications are far-reaching and consistent with TGD inspired theory of consciousness: p-adic regions are present even at elementary particle level and provide some kind of model of "self" and external world. In fact, p-adic physics must model the p-adic cognitive regions representing real elementary particle regions rather than elementary particles themselves!

1.3 The Generalization Of The Notion Of Number

The unification of real physics of material work and p-adic physics of cognition and intentionality leads to the generalization of the notion of number field. Reals and various p-adic number fields are glued along their common rationals (and common algebraic numbers too) to form a fractal book like structure. Allowing all possible finite-dimensional extensions of p-adic numbers brings additional pages to this "Big Book".

At space-time level the book like structure corresponds to the decomposition of space-time surface to real and p-adic space-time sheets. This has deep implications for the view about cognition. For instance, two points infinitesimally near p-adically are infinitely distant in real sense so that cognition becomes a cosmic phenomenon.

1.4 Zero Energy Ontology, Cognition, And Intentionality

One could argue that conservation laws forbid p-adic-real phase transitions in practice so that cognitions (intentions) realized as real-to-padic (p-adic-to-real) transitions would not be possible. The situation changes if one accepts zero energy ontology [K14, K13].

1.4.1 Zero energy ontology classically

In TGD inspired cosmology [K21] the imbeddings of Robertson-Walker cosmologies are vacuum extremals. Same applies to the imbeddings of Reissner-Nordström solution [K25] and in practice to all solutions of Einstein's equations imbeddable as extremals of Kähler action. Since four-momentum currents define a collection of vector fields rather than a tensor in TGD, both positive and negative signs for energy corresponding to two possible assignments of the arrow of the geometric time to a given space-time surface are possible. This leads to the view that all physical states have vanishing net energy classically and that physically acceptable universes are creatable from vacuum.

The result is highly desirable since one can avoid unpleasant questions such as "What are the net values of conserved quantities like rest mass, baryon number, lepton number, and electric charge for the entire universe?", "What were the initial conditions in the big bang?", "If only single solution of field equations is selected, isn't the notion of physical theory meaningless since in principle it is not possible to compare solutions of the theory?". This picture fits also nicely with the view that entire universe understood as quantum counterpart 4-D space-time is recreated in each quantum jump and allows to understand evolution as a process of continual re-creation.

1.4.2 Zero energy ontology at quantum level

Also the construction of S-matrix [K13] leads to the conclusion that all physical states possess vanishing conserved quantum numbers. Furthermore, the entanglement coefficients between positive and negative energy components of the state have interpretation as M-matrix identifiable as a "complex square root" of density matrix expressible as a product of positive diagonal square root of the density matrix and of a unitary S-matrix. S-matrix thus becomes a property of the zero energy state and physical states code by their structure what is usually identified as quantum dynamics.

The collection of M-matrices defines an orthonormal state basis for zero energy states and together they define unitary U-matrix charactering transition amplitudes between zero energy states. This matrix would not be however the counterpart of the usual S-matrix. Rather the unitary matrix phase of a given M-matrix would define the S-matrix measured in laboratory. U-matrix would also characterize the transitions between different number fields possible in the intersection of rel and p-adic worlds and having interpretation in terms of intention and cognition.

At space-time level this would mean that positive energy component and negative energy component are at a temporal distance characterized by the time scale of the causal diamond (CD) and the rational (perhaps integer) characterizing the value of Planck constant for the state in question. The scale in question would also characterize the geometric duration of quantum jump and the size scale of space-time region contributing to the contents of conscious experience. The interpretation in terms of a mini bang followed by a mini crunch suggests itself also. CDs are indeed important also in TGD inspired cosmology [K21].

1.4.3 Hyper-finite factors of type II$_1$ and new view about S-matrix

The representation of S-matrix as unitary entanglement coefficients would not make sense in ordinary quantum theory but in TGD the von Neumann algebra in question is not a type I factor as for quantum mechanics or a type III factor as for quantum field theories, but what is called hyper-finite factor of type II$_1$ [K26]. This algebra is an infinite-dimensional algebra with the almost defining, and at the first look very strange, property that the infinite-dimensional unit matrix has unit trace. The infinite dimensional Clifford algebra spanned by the configuration space gamma matrices (configuration space understood as the space of 3-surfaces, the " of classical worlds", WCW briefly) is indeed very naturally algebra of this kind since infinite-dimensional Clifford algebras provide a canonical representations for hyper-finite factors of type II$_1$.

1.4.4 The new view about quantum measurement theory

This mathematical framework leads to a new kind of quantum measurement theory. The basic assumption is that only a finite number of degrees of freedom can be quantum measured in a given measurement and the rest remain untouched. What is known as Jones inclusions $\mathcal{N} \subset \mathcal{M}$ of von Neumann algebras allow to realize mathematically this idea [K26]. $\mathcal{N}$ characterizes measurement resolution and quantum measurement reduces the entanglement in the non-commutative quantum space $\mathcal{M}/\mathcal{N}$. The outcome of the quantum measurement is still represented by a unitary S-matrix but in the space characterized by $\mathcal{N}$. It is not possible to end up with a pure state with a finite sequence of quantum measurements.

The obvious objection is that the replacement of a universal S-matrix coding entire physics with a state dependent unitary entanglement matrix is too heavy a price to be paid for the resolution of the above mentioned paradoxes. Situation could be saved if the S-matrices have fractal structure. The quantum criticality of TGD Universe indeed implies fractality. The possibility of an infinite sequence of Jones inclusions for hyperfinite type II_1 factors isomorphic as von Neumann algebras expresses this fractal character algebraically. Thus one can hope that the S-matrix appearing as entanglement coefficients is more or less universal in the same way as Mandelbrot fractal looks more or less the same in all length scales and for all resolutions. Whether this kind of universality must be posed as an additional condition on entanglement coefficients or is an automatic consequence of unitarity in type II$_1$ sense is an open question.

1.4.5 The S-matrix for p-adic-real transitions makes sense

In zero energy ontology conservation laws do not forbid p-adic-real transitions and one can develop a relatively concrete vision about what happens in these kind of transitions. The starting point is the generalization of the number concept obtained by gluing p-adic number fields and real numbers along common rationals (expressing it very roughly). At the level of the embedding space this means that p-adic and real space-time sheets intersect only along common rational points of the embedding space and transcendental p-adic space-time points are infinite as real numbers so that they can be said to be infinite distant points so that intentionality and cognition become cosmic phenomena.

In this framework the long range correlations characterizing p-adic fractality can be interpreted as being due to a large number of common rational points of embedding space for real space-time sheet and p-adic space-time sheet from which it resulted in the realization of intention in quantum jump. Thus real physics would carry direct signatures about the presence of intentionality. Intentional behavior is indeed characterized by short range randomness and long range correlations.

One can even develop a general vision about how to construct the S-matrix elements characterizing the process [K13]. The basic guideline is the vision that real and various p-adic physics as well as their hybrids are continuable from the rational physics. This means that these S-matrix elements must be characterizable using data at rational points of the embedding space shared by p-adic and real space-time sheets so that more or less same formulas describe all these S-matrix elements. Note that also $p_1 \rightarrow p_2$ p-adic transitions are possible.

1.5 What Number Theoretical Universality Might Mean?

Number theoretic universality has been one of the basic guide lines in the construction of quantum TGD. There are two forms of the principle.

1. The strong form of number theoretical universality states that physics for any system should effectively reduce to a physics in algebraic extension of rational numbers at the level of M-matrix so that an interpretation in both real and p-adic sense (allowing a suitable algebraic extension of p-adics) is possible. One can however worry whether this principle only means that physics is algebraic so that there would be no need to talk about real and p-adic physics at the level of M-matrix elements. It is not possible to get rid of real and p-adic numbers at the level of classical physics since calculus is a prerequisite for the basic variational principles used to formulate the theory. For this option the possibility of completion is what poses conditions on M-matrix.

2. The weak form of principle requires only that both real and p-adic variants of physics make sense and that the intersection of these physics consist of physics associated with various algebraic extensions of rational numbers. In this rational physics would be like rational numbers allowing infinite number of algebraic extensions and real numbers and p-adic number fields as its completions. Real and p-adic physics would be completions of rational physics. In this framework criticality with respect to phase transitions changing number field becomes a viable concept. This form of principle allows also purely p-adic phenomena such as p-adic pseudo non-determinism assigned to imagination and cognition. Genuinely p-adic physics does not however allow definition of notions like conserved quantities since the notion of definite integral is lacking and only the purely local form of real physics allows p-adic counterpart.

Experience has taught that it is better to avoid too strong statements and perhaps the weak form of the principle is enough. It is however clear that number theoretical criticality could provide important insights to quantum TGd. p-Adic thermodynamics [K7] is an excellent example of this. In consciousness theory the transitions transforming intentions to actions and actions to cognitions would be key applications. Needless to say, zero energy ontology is absolutely essential: otherwise this kind of transitions would not make sense.

1.6 P-Adicization By Algebraic Continuation

The basic challenges of the p-adicization program are following.

1. The first problem -the conceptual one- is the identification of preferred coordinates in which functions are algebraic and for which algebraic values of coordinates are in preferred position. This problem is encountered both at the level of space-time, embedding space, and configuration space. Here the group theoretical considerations play decisive role and the selection of preferred coordinates relates closely to the selection of quantization axes. This selection has direct physical correlates at the level of embedding space and the hierarchy of Planck constants has interpretation as a correlate for the selection of quantization axes [K15].

Algebraization does not necessarily mean discretization at space-time level: for instance, the coordinates characterizing partonic 2-surface can be algebraic so that algebraic point of the configuration space results and surface is not discretized. If this kind of function spaces are finite-dimensional, it is possible to fix X^2 completely data for a finite number of points only.

2. Local physics generalizes as such to p-adic context (field equations, etc...). The basic stumbling block of this program is integration already at space-time (Kähler action etc..). The problem becomes really horrible looking at configuration space level (functional integral). Algebraic continuation could allow to circumvent this difficulty. Needless to say, the requirement that the continuation exists must pose immensely tight constraints on the physics. Also the existence of the Kähler geometry does this and the solution to the constraint is that WCW is a union of symmetric spaces. In the case of symmetric spaces Fourier analysis generalizes to harmonics analysis and one can reduces integration to summation for functions allowing Fourier decomposition. In p-adic context the existence of plane waves requires an algebraic extension allowing roots of unity characterizing the measurement accuracy of angle like variables. This leads in the case of symmetric spaces to a general p-adicization recipe. One starts from a discrete variantnof the symmetric space defined for which points correspond to roots of unity and replaces each discrete point with is p-adic completion representing the p-adic variant of the symmetric space. There is infinite hierarchy of p-adicizations corresponding to measurement resolutions and to the choice of preferred coordinates and the interpretation is in terms of cognitive representations and refined view about General Coordinate Invariance taking into account the fact that cognition is also part of the quantum state.

One general idea which results as an outcome of the generalized notion of number is the idea of a universal function continuable from a function mapping rationals to rationals or to a finite extension of rationals to a function in any number field. This algebraic continuation is analogous to the analytical continuation of a real analytic function to the complex plane.

1. Rational functions with rational coefficients are obviously functions satisfying this constraint. Algebraic functions with rational coefficients satisfy this requirement if appropriate finite-dimensional algebraic extensions of p-adic numbers are allowed. Exponent function is also such a function.

2. For instance, residue calculus essential in the construction of N-point functions of conformal field theory might be generalized so that the value of an integral along the real axis could be calculated by continuing it instead of the complex plane to any number field via its values in the subset of rational numbers forming the rim of the book like structure having number fields as its pages. If the poles of the continued function in the finitely extended number field allow interpretation as real numbers it might be possible to generalize the residue formula. One can also imagine of extending residue calculus to any algebraic extension. An interesting situation arises when the poles correspond to extended p-adic rationals common to different pages of the "Big Book". Could this mean that the integral could be calculated at any page having the pole common. In particular, could a p-adic residue integral be calculated in the ordinary complex plane by utilizing the fact that in this case numerical approach makes sense.

3. Algebraic continuation is the basic tool of p-adicization program. Entire physics of the TGD Universe should be algebraically continuable to various number fields. Real number based physics would define the physics of matter and p-adic physics would describe correlates of cognition.

4. For instance, the idea that number theoretically critical partonic 2-surfaces are expressible in terms of rational functions with rational or algebraic coefficients so that also p-adic variants of these surfaces make sense, is very attractive.

5. Finite sums and products respect algebraic number property and the condition of finiteness is coded naturally by the notion of finite measurement resolution in terms of the notion of (number theoretic) braid. This simplifies dramatically the algebraic continuation since configuration space reduces to a finite-dimensional space and the space of configuration space spinor fields reduces to finite-dimensional function space.

The real configuration space can well contain sectors for which p-adicization does not make sense. For instance, if the exponent of Kähler function and Kähler function are not expressible in terms of algebraic functions with rational or at most algebraic functions or more general functions making sense p-adically, the continuation is not possible. p-Adic non-determinism in p-adic sectors makes also impossible the continuation to real sector. All this is consistent with vision about rational and algebraic physics as as analog of rational and algebraic numbers allowing completion to various continuous number fields.

Due to the fact that real and p-adic topologies are fundamentally different, ultraviolet and infrared cutoffs in the set of rationals are unavoidable notions and correspond to a hierarchy of different physical phases on one hand and different levels of cognition on the other hand. For instance, most points p-adic space-time sheets reside at infinity in real sense and p-adically infinitesimal is infinite in real sense. Two types of cutoffs are predicted p-adic length scale cutoff and a cutoff due to phase resolution related to the hierarchy of Planck constants. Zero energy ontology provides natural realization for the p-adic length scale cutoff. The latter cutoff seems to correspond naturally to the hierarchy of algebraic extensions of p-adic numbers and quantum phases $exp(i2\pi/n)$, $n \geq 3$, coming as roots of unity and defining extensions of rationals and p-adics allowing to define p-adically sensible trigonometric functions. These phases relate closely to the hierarchy of quantum groups, braid groups, and II_1 factors of von Neumann algebra.

1.7 For The Reader

Most of this chapter has been written for about decade before the above discussion of number theoretical universality and criticality. Therefore the chapter in its original form reflects the first violent burst of ideas of an innocent novice rather than the recent more balanced vision about the role of number theory in quantum TGD. For instance, in the original view about number theoretic universality is the strong one and is un-necessarily restricting. Although I have done my best to update the sections, the details of the representation may still reflect in many aspects quantum TGD as I understood it for a decade ago and the recent vision differs dramatically from this view.

The plan of the chapter is following. In the first one half I describe general ideas as they emerged years ago in a rather free flowing "Alice in the Wonderland" mood. I also describe phenomenological applications, such as conjectures about number theoretic anatomy of coupling constants which are now at rather firm basis. The chapter titled "The recent view about Quantum TGD" represents kind of turning point and introduces quantum TGD in its recent formulation in the real context. The remaining chapters are devoted to the challenge of understanding p-adic counterpart of this general theory.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L2].

2 How P-Adic Numbers Emerge From Algebraic Physics?

The new algebraic vision leads to several generalization of the p-adic philosophy. Besides p-adic topologies more general rational-adic topologies are possible. Topology is purely dynamically determined and ..-adic topologies are quite "real". There is a physics oriented review article by Brekke and Freund [A3]. The books of Gouvêa [A8] and Khrennikov [A2] give a more mathematics-oriented views about p-adics.

This section is written before the discovery that it is possible to generalize the notion of the number field by the fusion reals and various p-adic numbers fields and their extensions together along common rationals (and also common algebraic numbers) to form a book like structure. The interpretation of p-adic physics as physics of intention and cognition removes interpretational problems. This vision provides immediately an answer to many questions raised in the text. In particular, it leads naturally to a complete algebraic democracy. The introduction of infinite primes, which are discussed in next chapter, extends the algebraic democracy even further and gives hopes of describing mathematically also mathematical cognition.

2.1 Basic Ideas And Questions

It is good to list the basic ideas and pose the basic question before more detailed considerations.

2.1.1 Topology is dynamical

The dynamical emergence of p-adicity is strongly supported both by the applications of p-adic and algebraic physics. The solutions of polynomial equations involving more than one variable involve roots of polynomials. Only roots in the real algebraic extensions of rationals are allowed since the components of quaternions must be real numbers. When the root is complex in real topology, one can however introduce p-adic topology such that the root exists as a number in a real extension of p-adics. In p-adic context only a finite-dimensional algebraic extension of rational numbers is needed. The solutions of the derivative conditions guaranteeing Lagrange manifold property involve p-adic pseudo constants so that the p-adic solutions are non-deterministic. The interpretation is that real roots of polynomials correspond to geometric correlates of matter whereas p-adic regions are geometric correlates of mind in consistency with the p-adic non-determinism.

Does this picture imply the physically attractive working hypothesis stating that the decomposition of infinite prime into primes of lower level corresponds to a decomposition of the space-time surface to various p-adic regions appearing in the definition of the infinite prime? Generating infinite primes correspond to quaternionic rationals and these rationals contain powers of quaternionic primes defining the infinite prime. The convergence of the power series solution of the polynomial equations defining space-time surface might depend crucially on the norms of these rationals in the p-adic topology used. This could actually force in a given space-time region p-adic topology associated with some prime involved in the expansion. This is in complete accordance with the idea that p-adic topologies are topologies of sensory experience and real topology is the topology of reality.

2.1.2 Various generalizations of p-adic topologies

p-Adicized quaternions is not a number field anymore. One could allow also rational-adic extensions [A2] for which pinary expansions are replaced by expansions in powers of rational. These extensions give rise to rings with unit but not to number fields. In this approach p-adic, or more generally rational-adic, topology determined by the algebraic number field on a given space-time sheet would be absolutely "real" rather than mere effective topology. Space-time surface decomposes into regions which look like fractal dust when seen by an observer characterized by different number field unless the observer uses some resolution.

This approach suggests even further generalizations. The original observation stimulated by the work with Riemann hypothesis was that the primes associated with the algebraic extensions of rationals, in particular Gaussian primes and Eisenstein primes, have very attractive physical interpretation. Quaternionic primes and rationals might in turn define what might be regarded as non-commutative generalization of the p-adic and rational-adic topology.

2.1.3 ...-Adic topology measures the complexity of the quantum state

The higher the degree of the polynomial, and thus the number of particles in the physical state and its complexity, the higher the algebraic dimension of the rational quaternions. A complete algebraic and quaternion and octonion-dimensional democracy would prevail. Accordingly, space-time topology would be completely dynamical in the sense that space-time contains both rational-adic, p-adic regions, and real regions. Physical evolution could be seen as evolution of mathematical structures in this framework: p-adic topologies would be obviously winners over rational-adic topologies and p-adic length scale hypothesis would select the surviving p-adic topologies. For instance, Gaussian-adic and Eisenstein-adic topologies would in turn be higher level survivors possibly associated with biological systems.

Dimensional democracy would be realized in the sense that one can regard the space-time sheets defining n-sheeted topological condensate also as a $4n$-dimensional surface in H^n. This hypothesis fixes the interactions associated with the topological condensation, and the hierarchical structure of the topological condensate conforms with the hierarchical ordering of the quaternionic arguments of the polynomials to which infinite primes are mapped. Polynomials (infinite integers) at a given level of hierarchy in turn can be interpreted in terms of formation of bound states by the formation of join along boundaries bonds/flux tubes.

2.1.4 Is adelic principle consistent with the dynamical topology?

There is competing, and as it seems, almost diametrically opposite view. Just like adelic formula allows to express the norm of a rational number as product of its p-adic norms, various algebraic number fields and even more general structures such as quaternions allowing the notion of prime, provide a collection of incomplete but hopefully calculable views about physics. The net description gives rise to quantum TGD formulated using real numbers. These descriptions would be like summary over all experiences about world of conscious experiencers characterized by p-adic completions of various four-dimensional algebraic number rationals. What is important is that the descriptions using algebraic number fields or their generalization might be calculable. This view need not be conflict with the dynamical view and one could indeed claim that the p-adic physics associated with various algebraic extensions of rational quaternions provide a model about physics constructed by various conscious observers. For a given quantum state there would be however minimal algebraic extension containing all points of the space-time surface in it.

2.2 Are More General Adics Indeed Needed?

The considerations related to Riemann hypothesis inspired the notion of G- and E-adic numbers in which rational prime p is replaced with Gaussian or Eisenstein prime. The notion of Eisenstein prime is so attractive because it makes possible to circumvent the complexification of p-adic numbers for $p \ mod \ 4=1$ for which $\sqrt{-1}$ exists as a p-adic number. What forces to take the notion of G-adics very seriously is that Gaussian Mersennes correspond to the p-adic length scale of atomic nucleus and to important biological length scales in the range between 10 nanometers and few micrometers. Also the key role of Golden Mean τ in biology and self-organizing systems could be understood if $Q(\tau, i)$ defines D-adic topology. Thus there is great temptation to believe that the notion of p-adic number generalizes in these sense that any irreducible associated with real or complex algebraic extension defines generalization of p-adic numbers and that these extensions appear in the algebraic extensions of quaternions.

Thus one must consider seriously also generalized p-adic numbers, D-adics as they were called in [K20]. D-adics would correspond to powers series of a prime belonging to a complex algebraic extension of rationals. Quaternions decompose naturally in longitudinal and transversal part and transversal part can be interpreted as a complex algebraic extension of rationals in the case of both M^4 and CP_2. Thus some irreducibles of this complex extension could define a generalization of p-adic numbers used to define the algebraic extension of rational quaternions reduced to a pair of complex coordinates.

Perhaps one could go even further: quaternion-adics defined as power series of quaternionic primes of norm p suggest themselves. What would be nice that this prime could perhaps be interpreted as a representation for the momentum of corresponding space-time sheets. The components of the prime belong to algebraic extension of rationals and would even code information about external world if the proposed interpretations are correct. One can also ask whether quaternionic primes could define what might be called quaternion-adic algebras and whether these algebras might be a basic element of algebraic physics.

This would mean that space-time topology would code information about the quantum numbers of a physical state. Rings with unit rather than number fields are in question since the p-adic counterparts of quaternionic integers in general fail to have inverse. It must be emphasized that the field property might not be absolutely essential. For instance "rational-adics" [A2], for which prime p is replaced with a rational q such that norm comes as a power of q, exists as rings with unit and define topology. Rational-adic topologies could have also quaternionic counterparts.

The idea of q-rational topologies is supported by the physical picture about the correspondence between Fock states and space-time sheets. Single 3-surface can in principle carry arbitrarily high fermion and boson numbers but is unstable to a topological decay to 3-surfaces carrying single fermion and boson states. The translation of this statement to...-adic context would be that the Fock states associated with infinite primes which correspond to rational-adic quaternionic topologies are unstable against decay to states described by polynomial primes in which each factor corresponds to prime (bosons) or its inverse (fermions) in algebraic extension of quaternions. This tendency to evolve to prime-adic topologies could be seen also as a manifestation of p-adic evolution and self-organization. Rational-adic topologies would be simply losers in the fight for survival against topologies defining number fields. Since also quaternion-adic topologies fail to define number fields they are expected to be losers in the fight for survival. Winners would be...-adic topologies defining number fields. At the level of Fock states this would mean the instability of states which contain more than one prime: that this is indeed the case, is one of the basic assumptions of quantum TGD forced by the experimental fact that elementary particles correspond to simplest Fock states associated with WCW spinor s.

2.3 Why Completion To P-Adics Necessarily Occurs?

There is rather convincing argument in favor of...-adic physics. Typically one must find zeros of rational functions of several variables. Simplifying somewhat, at the first level one must find zeros of polynomials $P(x_1,x_2)$. Newton's theorem states that the monic polynomial $P_n(y,x) = y^n + a_{n-1}x^{n-1} + ..$ allows a factorization in an algebraically closed number field

$$P(y,x^m) = \prod_k (y - f_k(x)) \quad . \tag{2.1}$$

Here f_k are polynomials and m is integer which divides n and equals to n for an irreducible polynomial P. Since the multiplication of x by m: th root of unity (ζ_m) leaves left hand side invariant it must permute the factors on right hand side. Thus one can express the formula also as

$$P(y,x) = \prod_{k=1,..m} (y - f_k(\zeta_m^k x^{1/m})) \quad . \tag{2.2}$$

When number field is not algebraically closed this means that one must introduce an algebraic extension by m: th roots of all rationals.

The problem is that these roots are not real in general and one cannot solve the problem by using a completion to complex numbers since only real extensions for the components of quaternion are possible. Only in the region where some of the roots of the polynomial are real, this is possible. The only manner to achieve consistency with the reality requirement is to allow p-adic topology or possibly rational-adic topology: in this case also the algebraic extension allowing m: th roots is always finite-dimensional. For instance, for $m = 2$ p-adic extension of rationals would be 4-dimensional for $p > 2$. The situation is similar for rational-adic topology.

If this argument is correct, one can conclude that real topology is possible only in the regions where real roots of the polynomial equation are possible: in the regions where all roots are complex, p-adicization gives rise to roots in the algebraic extension of p-adics and p-adic topology emerges naturally. This picture provides a precise view about how the space-time surface defined by the polynomial of quaternions decomposes to real and p-adic regions. Also a connection with catastrophe theory [A6] emerges: the boundaries of the catastrophe regions where some roots coincide, serve also as boundaries between...-adic and real regions.

2.4 Decomposition Of Space-Time To...-Adic Regions

Number-theoretic constraints are important in determining which...-adic topologies are possible in a given space-time region. There is no hope of building any unique vision unless one poses some general principles. Complete algebraic and topological democracy and the generalization of the notion of p-adic evolution to what might be called rational-adic evolution allow to build plausible and sufficiently general working hypothesis not requiring too much ad hoc assumptions and allowing at least mathematical testing. A further natural principle states that the topology for a given region is such that complex extension of rationals is not needed and that the series defining the normal quaternionic coordinate as function of the space-time quaternionic coordinate converges and gives rise to a smooth surface.

2.4.1 The power series defining solutions of polynomial equations must converge in some topology

The roots of polynomials of several variables can be expressed as Taylor series. When the root is complex, real topology is not possible and some p-adic topology must be considered. This suggests a very attractive dynamical mechanism of p-adicization. In the regions where the root belongs to a complex extension of rationals in the real topology, one could find those values of p for which the series converges p-adically. The rational numbers characterizing the polynomials associated with the generating infinite primes certainly determine the convergence and the primes for which p-adic convergence occurs are certainly functions of these rationals. Hence it could occur that the p-adic topologies for which convergence occurs correspond to the primes appearing as factors in these rationals.

In this approach topology is a result of dynamics. Note that also the notion of symmetry depends on the region of space-time. Contrary to the basic working hypothesis, ...-adic topology of a given space-time sheet is its "real" topology rather than being only an effective topology and the topology of space-time is completely dynamical being dictated by algebraic physics and smoothness requirement.

It is also possible that convergence does not occur with respect to any ...-adic topology and in this case the topology would be discrete. This situation would correspond to primordial chaos but still the algebraic formulation and Fock space description of the theory would make sense.

2.4.2 Space-time surfaces must be smooth in the completion

The completion must give rise to a smooth or at least continuous ....-adic or real surface defining a critical extremal of Kähler action in the sense of having an infinite number of deformations for which the second variation of Kähler action vanishes. This requirement might allow only finite number of...-adic topologies for a given space-time region. If the completion involves functions expandable in powers of a (possibly quaternionic) rational $q = m/n$, then the prime factors of m define natural p-adic number fields for which completion is possible. Also q itself could define rational-adic topology. Since the space-time surface decomposes into regions labeled by rationals in an algebraic extension of rationals q_1, there is interesting possibility that q_1 as such defines the rational-adic topology so that there would be no need to understand why the space-time region labeled by q decomposes into space-time sheets labeled by the prime factors of q.

Whatever the details of the coding are, the coding would mean that the quantum numbers associated with the space-time sheet would determine the generalized...-adic topology associated with it. The information about quantum systems would be mapped to space-time physics and the coding of quantum numbers to...-adic topology would solve at a general level the problem how the information about quantum state is coded into the structure of space-time.

2.5 Universe As An Algebraic Hologram?

Quaternionic primes have a natural indentification as four-momenta. If the Minkowski norm for the quaternion is defined using the algebraic norm of the real extension of rationals involved with the state, mass squared is integer-valued as in super-conformal theories. The use of the algebraic norm means a loss of information carried by the units of the real algebraic extension $K(\theta)$ (see the appendix of this chapter). Hence one can say that besides ordinary elementary particle quantum numbers there are algebraic quantum numbers which presumably carry algebraic information. Very effective coding of information about quantum numbers becomes possible and these quantum numbers commute with ordinary quantum numbers. This information does not become manifest for matter-like regions where a real completion of rationals are used. In p-adic regions representing geometric correlates of mind the situation is different since p-adic number field in question is a finite algebraic extension of rationals.

Almost every calculation is approximation and completion to reals or p-adics makes possible to measure how good the approximation is. Real numbers are extremely practical in this respect but the failure of the real number based physics is that it reduces number to a mere quantity having a definite size but no number-theoretical properties. This is practical from the point of view of numerics but means huge loss of capacity for information storage and representation. In algebraic number theory number contains representation for its construction recipe. It seems that the correct manner to see numbers is as elements of the state space provided by the algebraic extension. p-Adic physics using p-adic versions of the algebraic extensions does not lead to a loss of this information unlike real physics. Thus the basic topology of the space-time sheet could code the quantum numbers associated with it.

Since the algebraic extension of rationals, and hence also of p-adics, depends on the number of particles present in the Fock state coded by the infinite prime, the only possible interpretation is that the additional quantum numbers code information about the many-particle state. Hence the idea about "cognitive representation" of the fractal quantum numbers of particles of the external world suggests itself naturally. In particular, the degree of the minimal polynomial for the real extension $Q(\theta)$ is n, where n is the number of particles in the Fock state in the case the resulting state represents infinite prime. This means that there are $n-1$ quantum numbers represented by fractal scalings (see Appendix for Dirichlet's unit theorem). The interpretation as a representation for the fractal quantum numbers representing information about states of other particles in the system suggests itself. One cannot exclude the possibility that the fractal quantum numbers represent momenta or some other quantum numbers of other particles.

If this rather un-orthodox interpretation is correct, then cognitive representations are present already at the elementary particle level in p-adic regions associated with particles and are realized as algebraic holograms. Universe as a Computer consisting of sub-computers mimicking each other would be realized already at the elementary particle level. This view is consistent with the TGD inspired theory of consciousness. Algebraic physics would also make possible kind of a Gödelian loop by providing a representation for how the information about the structure of a physical system is coded into its properties.

This view has also immediate implications for complexity theory. The dimension of the minimal algebraic extension containing the algebraic number is a unique measure for its complexity. More concretely: the degree of the minimal polynomial measures the complexity. Everyone can solve second order polynomial but very few of us remembers formulas for the roots of fourth order polynomials. For higher orders quadratures do not even exist. Of course, numbers represent typically coordinates and this is consistent with the general coordinate invariance only if some preferred coordinates exist. In TGD based physics these coordinates exist: embedding space allows (apart from isometries) unique coordinates in which the components of the metric tensor are rational functions of the coordinates.

Similar realization is fundamental in the second almost-proof of Riemann hypothesis described in [K20]. In this case ζ is interpreted as an element in an infinite-dimensional algebraic extension of rationals allowing all roots of rationals. The vanishing of ζ requires that all components of this infinite-dimensional vector contain a common rational factor which vanishes. This is possible only if an infinite number of partition functions in the product representation of the modulus squared of ζ are rational and their product vanishes. This implies Riemann hypothesis. The assumption that only square roots of rationals are needed is very probably wrong and must be replaced with the assumption that p^{iy} is algebraic numbers when $z = 1/2 + iy$ is zero of ζ for any prime p. It is quite possible that the almost-proof survives this generalization.

The notion of Platonia discussed already in the introduction adds cognition to this picture and allows to understand where all those mathematical structures continually invented by mathematicians but not realized physically in the conventional sense of the word reside. This notion takes also the notion of algebraic hologram to its extreme by making space-time points infinitely structured.

2.6 How To Assign A P-Adic Prime To A Given Real Space-Time Sheet?

p-Adic mass calculations force to assign p-adic prime also to the real space-time sheets and the longstanding problem is how this p-adic prime, or possibly many of them, are determined. Number theoretic view about information concept provides a possible solution of this long-standing problem.

2.6.1 Number theoretic information concept

The notion of information in TGD framework differs in some respects from the standard notion.

1. The definition of the entropy in p-adic context is based on the notion p-adic logarithm depending on the p-adic norm of the argument only ($Log_p(x) = Log_p(|x|_p) = n$) [K5]. For

rational- and even algebraic number valued probabilities this entropy can be regarded as a real number. The entanglement entropy defined in this manner can be negative so that the entanglement can carry genuine positive information. Rationally/algebraically entangled p-adic system has a positive information content only if the number of the entangled state pairs is proportional to a positive power of the p-adic prime p.

2. This kind of definition of entropy works also in the real-rational/algebraic case and makes always sense for finite ensembles. This would have deep implications. For ordinary definition of the entropy NMP [K5] states that entanglement is minimized in the state preparation process. For the number theoretic definition of entropy entanglement could be generated during state preparation for both p-adic and real sub-systems, and NMP forces the emergence of p-adicity (say the number of entangled state is power of prime). The fragility of quantum coherence is the basic problem of quantum computations and the good news would be that Nature itself (according to TGD) tends to stabilize quantum coherence both in the real and p-adic contexts.

3. Quantum-classical correspondence suggests that the notion of information is well defined also at the space-time level. In the presence of the classical non-determinism of Kähler action and p-adic non-determinism one can indeed define ensembles, and therefore also probability distributions and entropies. For a given space-time sheet the natural ensemble consists of the deterministic pieces of the space-time sheet regarded as different states of the same system.

2.6.2 Are living systems in the intersection of real and p-adic world?

NMP combined with number theoretic entropies leads to an important exception to the rule that the generation of bound state entanglement between system and its environment during U process leads to a loss of consciousness. When entanglement probabilities are rational (or even algebraic) numbers, the entanglement entropy defined as a number theoretic variant of Shannon entropy can be negative so that entanglement carries information. NMP favors the generation of algebraic entanglement. The attractive interpretation is that the generation of algebraic entanglement leads to an expansion of consciousness ("fusion into the ocean of consciousness") instead of its loss. Rational and even algebraic entanglement coefficients make sense in the intersection of real and p-adic words, which suggests that life and conscious intelligence reside in the intersection of the real and p-adic worlds. Life would represent number theoretically criticality so that the quantum criticality of TGD Universe would allow to understand also life.

1. To be in the intersection of real and p-adic worlds means that partonic 2-surfaces and their 4-D tangent planes representing the information about space-time sheet (holography) have a mathematical representation allowing an interpretation either as a real or p-adic surface (just like rationals can be regarded as being common to reals and p-adic numbers). Number theoretical criticality makes also possible the transformation of intentions to actions as transformations of a p-adic 2-surfaces to a real 2-surfaces via leakage through this common intersection. This process makes sense only in zero energy ontology. This would generalize the observation that rationals and algebraics in a well-defined sense represent islands of order in the seas of chaos defined by real and p-adic continua.

2. A more concrete interpretation for the intersection of real and p-adic worlds would be as the intersection of real and p-adic variants of space-time surface allowing interpretation in both number fields. This intersection is discrete set containing besides rational points also algebraic points common to reals and algebraic extension of p-adics involved.

3. These two interpretations for the intersection of real and p-adic worlds need not be independent. The absence of definite integral in p-adic number fields suggests that the transition amplitudes between p-adic and real sectors must be expressible using only the data associated with rational and common algebraic points (in the algebraic extension of p-adic numbers used) of embedding space. This intersection is discrete and could even consist of a finite number of points. For instance, Fermat's last theorem tells that the surface $x^n+y^n=z^n$ contains only origin as rational point for $n=3,4,...$ whereas for $n=2$ it contains all rational multiples of integer valued points defining Pythagorean triangles: this is due to the homogenuity of the

polynomial in question. Therefore p-adic-to real transition amplitudes would have a purely number theoretical interpretation. One could speak of number theoretical field theory as an analogy for topological field theory.

2.6.3 Does space-time sheet represent integer and its prime factorization?

A long-standing problem of quantum TGD is how to associate to a given real space-time sheet a (not necessarily) unique p-adic prime as required by the p-adic length scale hypothesis. One could achieve this by requiring that for this prime the negentropy associated with the ensemble is maximal. The simplest hypothesis is that a real space-time sheet consisting of N deterministic pieces corresponds to p-adic prime defining the largest factor of N. One could also consider a more general possibility. If N contains p^n as a factor, then the real fractality above n-ary p-adic length scale $L_p(n) = p^{(n-1)/2}L_p$ corresponds to smoothness in the p-adic topology. This option is more attractive since it predicts that the fundamental p-adic length scale L_p for a given p can be effectively replaced by any integer multiple NL_p, such that N is not divisible by p. There is indeed a considerable evidence for small p p-adicity in long length scales. For instance, genetic code and the appearance of binary pairs like cell membrane consisting of liquid layers suggests 2-adicity in nano length scales. This view means that the fractal structure of a given real space-time sheet represents both an integer N and its decomposition to prime factors physically. This obviously conforms with the physics as a generalized number theory vision.

Quantum-classical correspondence suggests that quantum computation processes might have counterparts at the level of space-time. An especially interesting process of this kind is the factorization of integers to prime factors. The classical cryptography relies on the fact that the factorization of large integers to prime factors is a very slow process using classical computation: the time needed to factor 100 digit number using modern computer would take more than the recent age of the universe. For quantum computers the factorization is achieved very rapidly using the famous Shor's algorithm. Does the factorization process indeed have a space-time counterpart?

Suppose that one can map the integer N to be factored to a real space-time sheet with N deterministic pieces. If one can measure the powers $p_i^{n_i}$ of primes p_i for which the fractality above the appropriate p-adic length scale looks smoothness in the p-adic topology, it is possible to deduce the factorization of N by direct physical measurements of the p-adic length scales characterizing the representative space-time sheet (say from the resonance frequencies of the radiation associated with the space-time sheet). If only the p-adic topology corresponding to the largest prime p_1 is realized in this manner, one can deduce first it, and repeat the process for N/p_1^n, and so on, until the full factorization is achieved. A possible test is to generate resonant radiation in a wave guide of having length which is an integer multiple of the fundamental p-adic length scale and to see whether frequencies which correspond to the factors of N appear spontaneously.

2.7 Gaussian And Eistenstein Primes And Physics

Gaussian and Eisenstein primes could give rise to what might be called G- and E-adicities and also these -adicities might be of physical interest.

2.7.1 Gaussian and Eisenstein primes and elementary particle quantum numbers

The properties of Gaussian and Eisenstein primes have intriguing parallels with quantum TGD at the level of elementary particle quantum numbers.

1. The lengths of the complex vectors defined by the non-degenerate Gaussian and Eisenstein primes are square roots of primes as are also the preferred p-adic length scales L_p: this suggests a direct connection with quantum TGD.

2. Each non-degenerate (purely real or imaginary) Gaussian prime of given norm p corresponds to 8 different complex numbers $G = \pm r \pm is$ and $G = \pm s \pm ir$. This is the number of different spin states for the embedding space spinors and also for the color states of massless gluons (note that in TGD quark color is not spin like quantum number but is analogous to orbital angular momentum). Complex conjugation might be interpreted as a representation of charge conjugation and multiplication by $\pm 1, \pm i$ could give rise to different spin states.

The 4-fold degeneracy associated with the $p\ mod\ 4 = 3$ Gaussian primes could correspond to the quartet of massless electro-weak gauge bosons with a given helicity $[(\gamma, Z^0) \leftrightarrow \pm p)$ and $(W^+, W^-) \leftrightarrow \pm ip]$.

3. For Eisenstein prime E_{p_1} the multiplication by $\pm i$ does not respect the rationality of the real part of $|Z_{p_1}|^2$ and the number of states is reduced to four. Eisenstein primes $r + isw$ and $s + irw$ have however the same norm squared so that also now the 8-fold degeneracy is present. When p_1^{iy} is of the general form $r + i\sqrt{k}s$ this degeneracy is not present.

4. The basic character of the quark color is triality realized as phases w which are third roots of unity. The fact that the phases are associated with the Eisenstein primes suggests that they might provide a representation of quark color. One can indeed multiply any Eisenstein prime in the product decomposition by factor 1, w or $\overline{w}$ and the interpretation is that the three primes represent three color states of quark. The obvious interpretation is that each factor Z_{p_1} with $p_1\ mod\ 4 = 1$ could represent 8 possible leptonic states. Each factor Z_{p_1} satisfying $p_1\ mod\ 4 = 3$ and $p_1\ mod\ 3 = 1$ conditions simultaneously would correspond to a product of Eisenstein prime with Eisenstein phase and each prime p_i associated with Eisenstein phase would correspond to one color state of quark. Even a number theoretical counterpart of color confinement could be imagined.

   There is also a further interesting analogy supporting the idea about number theoretical counterpart of the quark color. ζ decomposes into a product $\zeta_1 \times \zeta_3$, such that ζ_1 is the product of $p\ mod\ 4 = 1$ partition functions and ζ_3 the product of $p\ mod\ 4 = 3$ partition functions. This decomposition reminds of the leptonic color singlets and color triplet of quarks. Rather interestingly, leptons and quarks correspond to Ramond and Neveu-Schwartz type super Virasoro representations and the fields of N-S representation indeed contain square roots of complex variable existing p-adically for $p\ mod\ 4 = 3$.

5. What about the most general factors $r + is\sqrt{k}$? Can one assign some kind of color degeneracy also with these factors? It seems that this is the case. One can always find phase factors of type $U_\pm = (r \pm is\sqrt{k})/n$ with minimal values of n $(r^2 + s^2k = n^2)$. The factors $1, U_\pm$ clearly give rise to a 3-fold degeneracy analogous to color degeneracy.

6. What about interpretation of the components of the complex integers? For Super Virasoro representations basic quantum numbers of this kind correspond to energy and longitudinal momentum. This suggests the interpretation of $r^2 + s^2k$ as energy, $r^2 - s^2k$ as mass, and $2rs\sqrt{k}$ as momentum. For the squares $r^2 - s^2 + (2rs - s^2)w$ of Eisenstein primes $r^2 - s^2/2 - rs$ corresponds to mass, $r^2 + s^2 - rs$ to energy, and $(2rs - s^2)\sqrt{3}/2$ to momentum. Note that the sign of mass changes for Gaussian primes in the interchange $r \leftrightarrow s$. The fact that the hexagonal lattice defined by Eisenstein integers correspond to the root lattice of $SU(3)$ group means that energy, momentum and mass corresponds to the sides of the triangles in the root lattice of color group.

The following argument suggests that finite Gaussian and Eistenstein primes might be forced by zero energy ontology (ZEO)

1. In ZEO M-matrix is in a well-defined sense "complex" square root of density matrix reducing to a product of Hermitian square root of density matrix multiplied by unitary S-matrix. A natural guess is that p-adic thermodynamics possesses this kind of square root or better to say: is modulus squared for it.

2. For fermions the value of p-adic temperature is however $T = 1$ and thus maximal. It is not possible to construct real square root by simply taking the square root of thermodynamical probabilities for various conformal weights. One manner to solve the problem is to assume that one has quadratic algebraic extension of p-adic numbers in which the p-adic prime splits as $p = \pi\overline{\pi}$, $\pi = m + \sqrt{-k}n$. For $k = 1$ primes $p\ \ mod\ \ 4 = 1$ indeed allow a representation as product of Gaussian prime and its conjugate.

3. For primes $p\ \ mod\ \ 4 = 3$ this is not the case and Mersenne primes are important examples of these primes. Eisenstein primes provide the simplest extension of rationals splitting Mersenne

primes. For Eisenstein primes one has $k = 3$ and all ordinary primes satisfying either $p = 3$ or $p \ mod \ 3 = 1$ (true for Mersenne primes) allows this splitting. For the square root of p-adic thermodynamics the complex square roots of probabilities would be given by $\pi^{L_0/T}/\sqrt{Z}$, and the moduli squared would give thermodynamical probabilities as $p^{L_0/T}/Z$. Here Z is the partition function.

4. An interesting question is whether $T = 1$ for fermions means that complex square of thermodynamics is indeed complex and whether $T = 1/2$ for bosons means that the square root is actually real valued.

2.7.2 G-adic, E-adic and even more general fractals?

Still one line of thoughts relates to the possibility to generalize the notion of p-adicity so that could speak about G-adic and E-adic number fields. The properties of the Gaussian and Eisenstein primes indeed strongly suggest a generalization for the notion of p-adic numbers to include what might be called G-adic or E-adic numbers. In fact, the argument generalizes to the case of all nine $\sqrt{-d}$ type extensions of rationals allowing a unique prime decomposition so that one might perhaps speak about D-adics.

1. Consider for definiteness Gaussian primes. The basic point is that the decomposition into a product of prime factors is unique. For a given Gaussian prime one could consider the representation of the algebraic extension involved (complex integers in the case of Gaussian primes) as a ring formed by the formal power series

$$G = \sum_n z_n G_p^n \quad . \tag{2.3}$$

Here z_n is Gaussian integer with norm smaller than $|G_p|$, which equals to p for $p \ mod \ 4 = 3$ and $\sqrt{p}$ for $p \ mod \ 4 = 1$.

2. If any Gaussian integer z has a unique expansion in powers of G_p such that coefficients have norm squared smaller than p, modulo G arithmetics makes sense and one can construct the inverse of G and number field results. This is the case if Gaussian integers behave with respect to modulo G_p arithmetics like finite field $G(p, 2)$. For $p \ mod \ 4 = 1$ the extension of the p-adic numbers by introducing $\sqrt{-1}$ as a unit is not possible since $\sqrt{-1}$ exists as a p-adic number: the proposed structure might perhaps provide the counterpart of the p-adic complex numbers in the case$p \ mod \ 4 = 1$. Thus the question is whether one could regard Gaussian p-adic numbers as a natural complexification of p-adics for $p \ mod \ 4 = 1$, perhaps some kind of square root of R_p, and if they indeed form a number field, do they reduce to some known algebraic extension of R_p?

3. In the case of Eisenstein numbers one can identify the coefficients z_n in the formal power series $E = \sum z_n E_p^n$ as Eisenstein numbers having modulus square smaller than p associated with E_p and similar argument works also in this case.

4. As already noticed, in the case of complex extensions of form $r + \sqrt{-d}s$ a unique prime factorization is obtained only in nine cases corresponding to $d = 1, 2, 3, 7, 11, 19, 46, 67, 163$ [A7]. The poor man's argument above does not distinguish between G- and E-adics ($d = 1, 3$) and these extensions.One might perhaps call this extensions generally D-adics. This suggests that generalized p-adics could exist also in this case. In fact, the generalization p-adics could make sense also for higher-dimensional algebraic extensions allowing unique prime decomposition. For $d = 2$ complex algebraic primes are of form $r + s\sqrt{-2}$ satisfying the condition $r^2 + 2s^2 = p$. For $d > 2$ complex algebraic primes are of form $(r + s\sqrt{-d})/2$ such that both r and s are even or odd. Quite generally, the condition $p \ mod \ d = k^2$ must be satisfied. $\sqrt{-d}$ corresponds to a root of unity only for $d = 1$ and $d = 3$ so that the powers of a complex primes in this case have a finite number of possible phase angles: this might make G- and E-adics physically special.

TGD suggests rather interesting physical applications of D-adics.

1. What is interesting from the physics point of view is that for $p \ mod \ 4 = 1$ the points D_p^n are on the logarithmic spiral $z_n = p^{n/2} exp(in\phi_0/2)$, where ϕ is the phase associated with D_p^2. The logarithmic spiral can be written also as $\rho = exp(nlog(p)\phi/\phi_0)$. This reminds strongly of the logarithmic spirals, which are fractal structures frequently encountered in self-organizing systems: D-adics might provide the mathematics for the modelling of these structures.

2. p-Adic length scale hypothesis should hold true also for Gaussian primes, in particular, Gaussian Mersennes of form $(1 \pm i)^k - 1$ should be especially interesting from TGD point of view.

    (a) The integers k associated with the lowest Gaussian Mersennes are following: $2, 3, 5, 7, 11, 19, 29, 47, 73, 79,$ $k = 113$ corresponds to the p-adic length scale associated with the atomic nucleus and muon. Thus all known charged leptons, rather than only e and τ, as well as nuclear physics length scale, correspond to Mersenne primes in the generalized sense.

    (b) The primes $k = 151, 157, 163, 167$ define perhaps the most fundamental biological length scales: $k = 151$ corresponds to the thickness of the cell membrane of about ten nanometers and $k = 167$ to cell size about 2.56 μm. This strongly suggests that cellular organisms have evolved to their present form through four basic stages.

    (c) $k = 239, 241, 283, 353, 367, 379, 457$ associated with the next Gaussian Mersennes define astronomical length scales. $k = 239$ and $k = 241$ correspond to the p-adic time scales .55 ms and 1.1 ms: basic time scales associated with nerve pulse transmission are in question. $k = 283$ corresponds to the time scale of 38.6 min. An interesting question is whether this period could define a fundamental biological rhythm. The length scale $L(353)$ corresponds to about 2.6×10^6 light years, roughly the size scale of galaxies. The length scale $L(367) \simeq \times 3.3 \times 10^8$ light years is of same order of magnitude as the size scale of the large voids containing galaxies on their boundaries (note the analogy with cells). $T(379) \simeq 2.1 \times 10^{10}$ years corresponds to the lower bound for the order of the age of the Universe. $T(457) \sim 10^{22}$ years defines a completely super-astronomical time and length scale.

3. Eisenstein integers form a hexagonal lattice equivalent with the root lattice of the color group $SU(3)$. Microtubular surface defines a hexagonal lattice on the surface of a cylinder which suggests an interpretation in terms of E-adicity. Also the patterns of neural activity form often hexagonal lattices.

2.7.3 Gaussian and Eisenstein versions of infinite primes

The vision about quantum TGD as a generalized number theory generates a further line of thoughts.

1. As has been found, the zeros of ζ code for the physical states of a super-symmetric arithmetic quantum field theory. As a matter fact, the arithmetic quantum field theory in question can be identified as arithmetic quantum field theory in which single particle states are labeled by Gaussian primes. The properties of the Gaussian primes imply that the single particle states of this theory have 8-fold degeneracy plus the four-fold degeneracy related to the $\pm i$ or ± 1-factor which could be interpreted as a phase factor associated with any $p \ mod \ 4 = 3$ type Gaussian prime. Also Eisenstein primes could allow the construction of a similar arithmetic quantum field theory.

2. The construction of the infinite primes reduces to a repeated second quantization of an arithmetic quantum field theory. A straightforward generalization of the procedure of the previous chapter allows to define also the notion of infinite Gaussian and Eisenstein primes. Since each infinite prime is in a well-defined sense a composite of finite primes playing the role of elementary particles, this would mean that each composite prime in the expansion of an infinite prime has either four-fold degeneracy or eight-fold degeneracy. The interpretation of infinite primes could thus literally be as many-particle states of quantum TGD.

2.8 P-Adic Length Scale Hypothesis And Quaternionic Primality

p-Adic length scale hypothesis states that fundamental length scales correspond to the so called p-adic length scales proportional to $\sqrt{p}$, p prime. Even more: the p-adic primes $p \simeq 2^k$, k prime or possibly power of prime, are especially interesting physically. The so called elementary particle-blackhole analogy gives strong support for this hypothesis. Elementary particles correspond to the so called CP_2 type extremals in TGD. Elementary particle horizon can be defined as a surface at which the Euclidian signature of the metric of the space-time surface containing topologically condensed CP_2 type extremal, changes to Minkowskian signature. The generalization of the Hawking-Bekenstein formula relates the real counterpart of the p-adic entropy associated with the elementary particle to the area of the elementary particle horizon. If one requires that the radius of the elementary particle horizon corresponds to a p-adic length scale: $R = L(k)$ or $k^{n/2}L(k)$ where k is prime, then p is automatically near to 2^{k^n} and p-adic length scale hypothesis is reproduced! The proportionality of length scale to $\sqrt{p}$, rather than p, follows from p-adic thermodynamics for mass squared (!) operator and from Uncertainty Principle.

What Tony Smith [A11] suggested, was a beautiful connection with number theory based on the generalization of the concept of a prime number. In the so called D^4 lattice regarded as consisting of integer quaternions, one can identify prime quaternions as the generators of the multiplicative algebra of the integer quaternions. From the basic properties of the quaternion norm it follows directly that prime quaternions correspond to the 3-dimensional spheres $R^2 = p$, p prime. The crucial point from the TGD point of view is the appearance of the *square* of the norm instead of the norm. One can even define the product of spheres $R^2 = n_1$ and $R^2 = n_2$ by defining the product sphere with norm squared $R^2 = n_1n_2$ to consist of the quaternions, which are products of quaternions with norms squared n_1 and n_2 respectively. Prime spheres correspond to $n = p$. The powers of sphere p correspond to a multiplicatively closed structure consisting of powers p^n of the sphere p. It is also possible to speak about the multiplication of balls and prime balls in the case of integer quaternions.

p-Adic length scale hypothesis follows if one assumes that the Euclidian piece of the space-time surrounding the topologically condensed CP_2 type extremal can be approximated with a quaternion integer lattice with radius squared equal to $r^2 = k^n$, k prime. One manner to understand the finiteness in the time direction is that topological sum contacts of CP_2 type extremal are not static 3-dimensional topological sum contacts but genuinely four-dimensional: 3-dimensional contact is created, expands to a maximum size and is gradually reduced to point. The Euclidian space-time volume containing the contact would correspond to an Euclidian region $R^2 = k^n$ of space-time. The distances of the lattice points would be measured using the induced metric. These contacts could have arbitrarily long duration from the point of view of external observer since classical gravitational fields give rise to strong time dilation effects (strongest on the boundary of the Euclidian region where the metric becomes degenerate with the emergence of a light like direction).

Lattice structure is essential for the argument. Lattice structures of type D^4 indeed emerge naturally in the p-adic QFT limit of TGD as also in the construction of the p-adic counterparts of the space-time surfaces as p-adically analytic surfaces. The essential idea is to construct the p-adic surface by first discretizing space-time surface using a p-adic cutoff in k: th pinary digit and mapping this surface to its p-adic counterpart and complete this to a unique smooth p-adically analytic surface. This leads to a fractal construction in which a given interval is decomposed to p smaller intervals, when the resolution is increased. In the 4-dimensional case one naturally obtains a fractal hierarchy of nested D^4 lattices. The interior of the elementary particle horizon with Euclidian signature corresponds to some subset of the quaternionic integer lattice D^4: an attractive possibility is that the criticality of the Kähler action and the maximization of the Kähler function force this set to be a ball $R^2 \leq k^n$, k prime.

3 Scaling Hierarchies And Physics As A Generalized Number Theory

The scaling hierarchies defined by powers of Φ and primes p probably reflect something very profound. Mueller has proposed also a scaling law in powers of e [B1]. This scaling law can be however questioned since $\Phi^2 = 2.6180..$ is rather near to $e = 2.7183...$ Note that powers of e define p-dimensional extension of R_p since e^p exists as a p-adic number in this case.

The interpretation of the p-adic as physics of cognition and the vision about reduction of physics to rational physics continuable algebraically to various extensions of rationals and p-adic number fields is an attractive general framework allowing to understand how p-adic fractality could emerge in real physics. In this section it will be found that this vision provides a concrete tool in principle allowing to construct global solutions of field equations by reducing long length scale real physics to short length scale p-adic physics. Also p-adic length scale hypothesis can be understood and the notion of multi-p p-fractality can be formulated in precise sense in this framework. This vision leads also to a concrete quantum model for how intentions are transformed to actions and the S-matrix for the process has the same general form as the ordinary S-matrix.

The fractal hierarchy associated with Golden mean cannot be understood in a way analogous to p-adic fractal hierarchies. Rather, the understanding of Golden Mean and Fibonacci series could reduce to the hypothesis that space-time surfaces, and thus the geometry of physical systems, provide a representations for the hierarchy of Fibonacci numbers characterizing the Jones inclusions of infinite-dimensional Clifford sub-algebras of WCW spinors identifiable as infinite-dimensional von Neumann algebras known as hyper-finite factors of type II_1 (not that WCW corresponds here to the "world of classical worlds"). The emergence of powers of e has been discussed in [K20] and will not be discussed here.

3.1 P-Adic Physics And The Construction Of Solutions Of Field Equations

The number theoretic vision about physics relies on the idea that physics or, rather what we can know about it, is basically rational number based. One interpretation would be that space-time surfaces, the induced spinors at space-time surfaces, WCW spinor fields, S-matrix, etc..., can be obtained by algebraically continuing their values in a discrete subset of rational variant of the geometric structure considered to appropriate completion of rationals (real or p-adic). The existence of the algebraic continuation poses very strong additional constraints on physics but has not provided any practical means to solve quantum TGD.

In the following it is however demonstrated that this view leads to a very powerful iterative method of constructing global solutions of classical field equations from local data and at the same time gives justification for the notion of p-adic fractality, which has provided very successful approach not only to elementary particle physics but also physics at longer scales. The basic idea is that mere p-adic continuity and smoothness imply fractal long range correlations between rational points which are very close p-adically but far from each other in the real sense and vice versa.

3.1.1 The emergence of a rational cutoff

For a given p-adic continuation only a subset of rational points is acceptable since the simultaneous requirements of real and p-adic continuity can be satisfied only if one introduces ultraviolet cutoff length scale. This means that the distances between subset of rational points fixing the dynamics of the quantities involved are above some cutoff length scale, which is expected to depend on the p-adic number field R_p as well as a particular solution of field equations. The continued quantities coincide only in this subset of rationals but not in shorter length scales.

The presence of the rational cutoff implies that the dynamics at short scales becomes effectively discrete. Reality is however not discrete: discreteness and rationality only characterize the inherent limitations of our knowledge about reality. This conforms with the fact that our numerical calculations are always discrete and involve finite set of points.

The intersection points of various p-adic continuations with real space-time surface should code for all actual information that a particular p-adic physics can give about real physics in classical sense. There are reasons to believe that real space-time sheets are in the general case characterized by integers n decomposing into products of powers of primes p_i. One can expect that for p_i-adic continuations the sets of intersection points are especially large and that these p-adic space-time surfaces can be said to provide a good discrete cognitive mimicry of the real space-time surface.

Adelic formula represents real number as product of inverse of its p-adic norms. This raises the hope that taken together these intersections could allow to determine the real surface and thus classical physics to a high degree. This idea generalizes to quantum context too.

The actual construction of the algebraic continuation from a subset of rational points is of course something which cannot be done in practice and this is not even necessary since much more elegant approach is possible.

3.1.2 Hierarchy of algebraic physics

One of the basic hypothesis of quantum TGD is that it is possible to define exponent of Kähler action in terms of fermionic determinants associated with the Kähler-Dirac operator derivable from a Dirac action related super-symmetrically to the Kähler action.

If this is true, a very elegant manner to define hierarchy of physics in various algebraic extensions of rational numbers and p-adic numbers becomes possible. The observation is that the continuation to various p-adic numbers fields and their extensions for the fermionic determinant can be simply done by allowing only the eigenvalues which belong to the extension of rationals involved and solve field equations for the resulting Kähler function. Hence a hierarchy of fermionic determinants results. The value of the dynamical Planck constant characterizes in this approach the scale factor of the M^4 metric in various number theoretical variants of the embedding space $H = M^4 \times CP_2$ glued together along subsets of rational points of H. The values of $\hbar$ are determined from the requirement of quantum criticality [K26] meaning that Kähler coupling strength is analogous to critical temperature.

In this approach there is no need to restrict the embedding space points to the algebraic extension of rationals and to try to formulate the counterparts of field equations in these discrete embedding spaces.

3.1.3 p-Adic short range physics codes for long range real physics and vice versa

One should be able to construct global solutions of field equations numerically or by engineering them from the large repertoire of known exact solutions [K11]. This challenge looks formidable since the field equations are extremely non-linear and the failure of the strict non-determinism seems to make even in principle the construction of global solutions impossible as a boundary value problem or initial value problem.

The hope is that short distance physics might somehow code for long distance physics. If this kind of coding is possible at all, p-adicity should be crucial for achieving it. This suggests that one must articulate the question more precisely by characterizing what we mean with the phrases "short distance" and "long distance". The notion of short distance in p-adic physics is completely different from that in real physics, where rationals very close to each other can be arbitrary far away in the real sense, and vice versa. Could it be that in the statement "Short length scale physics codes for long length scale physics" the attribute "short"/"long" could refer to p-adic/real norm, real/p-adic norm, or both depending on the situation?

The point is that rational embedding space points very near to each other in the real sense are in general at arbitrarily large distances in p-adic sense and vice versa. This observation leads to an elegant method of constructing solutions of field equations.

1. Select a rational point of the embedding space and solve field equations in the real sense in an arbitrary small neighborhood U of this point. This can be done with an arbitrary accuracy by chosing U to be sufficiently small. It is possible to solve the linearized field equations or use a piece of an exact solution going through the point in question.

2. Select a subset of rational points in U and interpret them as points of p-adic embedding space and space-time surface. In the p-adic sense these points are in general at arbitrary large distances from each and real continuity and smoothness alone imply p-adic long range correlations. Solve now p-adic field equations in p-adically small neighborhoods of these points. Again the accuracy can be arbitrarily high if the neighborhoods are choose small enough. The use of exact solutions of course allows to overcome the numerical restrictions.

3. Restrict the solutions in these small p-adic neighborhoods to rational points and interpret these points as real points having arbitrarily large distances. p-Adic smoothness and continuity alone imply fractal long range correlations between rational points which are arbitrary distant in the real sense. Return to 1) and continue the loop indefinitely.

In this manner one obtains even in numerical approach more and more small neighborhoods representing almost exact p-adic and real solutions and the process can be continued indefinitely.

Some comments about the construction are in order.

1. Essentially two different field equations are in question: real field equations fix the local behavior of the real solutions and p-adic field equations fix the long range behavior of real solutions. Real/p-adic global behavior is transformed to local p-adic/real behavior. This might be the deepest reason why for the hierarchy of p-adic physics.

2. The failure of the strict determinism for the dynamics dictated by Kähler action and p-adic non-determinism due to the existence of p-adic pseudo constants give good hopes that the construction indeed makes it possible to glue together the (not necessarily) small pieces of space-time surfaces inside which solutions are very precise or exact.

3. Although the full solution might be impossible to achieve, the predicted long range correlations implied by the p-adic fractality at the real space-time surface are a testable prediction for which p-adic mass calculations and applications of TGD to biology provide support.

4. It is also possible to generalize the procedure by changing the value of p at some rational points and in this manner construct real space-time sheets characterized by different p-adic primes.

5. One can consider also the possibility that several p-adic solutions are constructed at given rational point and the rational points associated with p-adic space-time sheets labeled by $p_1, ..., p_n$ belong to the real surface. This would mean that real surface would be multi-p p-adic fractal.

I have earlier suggested that even elementary particles are indeed characterized by integers and that only particles for which the integers have common prime factors interact by exchanging particles characterized by common prime factors. In particular, the primes $p = 2, 3, ..., 23$ would be common to the known elementary particles and appear in the expression of the gravitational constant. Multi-p p-fractality leads also to an explanation for the weakness of the gravitational constant. The construction recipe for the solutions would give a concrete meaning for these heuristic proposals.

This approach is not restricted to space-time dynamics but is expected to apply also at the level of say S-matrix and all mathematical object having physical relevance. For instance, p-adic four-momenta appear as parameters of S-matrix elements. p-Adic four-momenta very near to each other p-adically restricted to rational momenta define real momenta which are not close to each other and the mere p-adic continuity and smoothness imply fractal long range correlations in the real momentum space and vice versa.

3.1.4 p-Adic length scale hypothesis

Approximate p_1-adicity implies also approximate p_2-adicity of the space-time surface for primes $p \simeq p_1^k$. p-Adic length scale hypothesis indeed states that primes $p \simeq 2^k$ are favored and this might be due to simultaneous $p \simeq 2^k$- and 2-adicity. The long range fractal correlations in real space-time implied by 2-adicity would indeed resemble those implied by $p \simeq 2^k$ and both $p \simeq 2^k$-adic and 2-adic space-time sheets have larger number of common points with the real space-time sheet.

If the scaling factor λ of $\hbar$ appearing in the dark matter hierarchy is in good approximation $\lambda = 2^{11}$ also dark matter hierarchy comes into play in a resonant manner and dark space-time sheets at various levels of the hierarchy tend to have many intersection points with each other.

There is however a problem involved with the understanding of the origin of the p-adic length scale hypothesis if the correspondence via common rationals is assumed.

1. The mass calculations based on p-adic thermodynamics for Virasoro generator L_0 predict that mass squared is proportional to $1/p$ and Uncertainty Principle implies that L_p is proportional to $\sqrt{p}$ rather than p, which looks more natural if common rationals define the correspondence between real and p-adic physics.

2. It would seem that length $d_p \simeq pR$, R or order CP_2 length, in the induced space-time metric must correspond to a length $L_p \simeq \sqrt{p}R$ in M^4. This could be understood if space-like geodesic lines at real space-time sheet obeying effective p-adic topology are like orbits of a particle performing Brownian motion so that the space-like geodesic connecting points with M^4 distance r_{M^4} has a length $r_{X^4} \propto r_{M^4}^2$. Geodesic random walk with randomness associated with the motion in CP_2 degrees of freedom could be in question. The effective p-adic topology indeed induces a strong local wiggling in CP_2 degrees of freedom so that r_{X^4} increases and can depend non-linearly on r_{M^4}.

3. If the size of the space-time sheet associated with the particle has size $d_p \sim pR$ in the induced metric, the corresponding M^4 size would be about $L_p \propto \sqrt{p}R$ and p-adic length scale hypothesis results.

4. The strongly non-perturbative and chaotic behavior $r_{X^4} \propto r_{M^4}^2$ is assumed to continue only up to L_p. At longer length scales the space-time distance d_p associated with L_p becomes the unit of space-time distance and geodesic distance r_{X^4} is in a good approximation given by

$$r_{X^4} = \frac{r_{M^4}}{L_p} d_p \propto \sqrt{p} \times r_{M^4} \quad , \tag{3.1}$$

and is thus linear in M^4 distance r_{M^4}.

3.1.5 Does cognition automatically solve real field equations in long length scales?

In TGD inspired theory of consciousness p-adic space-time sheets are identified as space-time correlates of cognition. Therefore our thoughts would have literally infinite size in the real topology if p-adics and reals correspond to each other via common rationals (also other correspondence based on the separate canonical identification of integers m and n in $q = m/n$ with p-adic numbers).

The cognitive solution of field equations in very small p-adic region would solve field equations in real sense in a discrete point set in very long real length scales. This would allow to understand why the notions of Universe and infinity are a natural part of our conscious experience although our sensory input is about an infinitesimally small region in the scale of universe.

The idea about Universe performing mimicry at all possible levels is one of the basic ideas of TGD inspired theory of consciousness. Universe could indeed understand and represent the long length scale real dynamics using local p-adic physics. The challenge would be to make quantum jumps generating p-adic surfaces having large number of common points with the real space-time surface. We are used to call this activity theorizing and the progress of science towards smaller real length scales means progress towards longer length scales in p-adic sense. Also real physics can represent p-adic physics: written language and computer represent examples of this mimicry.

3.2 A More Detailed View About How Local P-Adic Physics Codes For P-Adic Fractal Long Range Correlations Of The Real Physics

The vision just described gives only a rough heuristic view about how the local p-adic physics could code for the p-adic fractality of long range real physics. There are highly non-trivial details related to the treatment of M^4 and CP_2 coordinates and to the mapping of p-adic H-coordinates to their real counterparts and vice versa.

3.2.1 How real and p-adic space-time regions are glued together?

The first task is to visualize how real and p-adic space-time regions relate to each other. It is convenient to start with the extension of real axis to contain also p-adic points. For finite rationals $q = m/n$, m and n have finite power expansions in powers of p and one can always write $q = p^k \times r/s$ such that r and s are not divisible by p and thus have pinary expansion of in powers of p as $x = x_0 + \sum_1^N x_n p^n$, $x_i \in \{0, p\}$, $x_0 \neq 0$.

One can always express p-adic number as $x = p^n y$ where y has p-adic norm 1 and has expansion in non-negative powers of p. When x is rational but not integer the expansion contains infinite number of terms but is periodic. If the expansion is infinite and non-periodic, one can speak about *strictly p-adic* number having infinite value as a real number.

In the same manner real number x can be written as $x = p^n y$, where y is either rational or has infinite non-periodic expansion $y = r_0 + \sum_{n>0} r_n p^{-n}$ in negative powers of p. As a p-adic number y is infinite. In this case one can speak about strictly real numbers.

This gives a visual idea about what the solution of field equations locally in various number fields could mean and how these solutions are glued together along common rationals (see **Fig.** `http://tgdtheory.fi/appfigures/book.jpg` or **Fig. ??** in the appendix of this book). In the following I shall be somewhat sloppy and treat the rational points of the embedding space as if they were points of real axis in order to avoid clumsy formulas.

1. The p-adic variants of field equations can be solved in the strictly p-adic realm and by p-adic smoothness these solutions are well defined also in as subset of rational points. The strictly p-adic points in a neighborhood of a given rational point correspond as real points to infinitely distant points of M^4. The possibility of p-adic pseudo constants means that for rational points of M^4 having sufficiently large p-adic norm, the values of CP_2 coordinates or induced spinor fields can be chosen more or less freely.

2. One can solve the p-adic field equations in any p-adic neighborhood $U_n(q) = \{x = q + p^n y\}$ of a rational point q of M^4, where y has a unit p-adic norm and select the values of fields at different points q_1 and q_2 freely as long as the spheres $U_n(q_1)$ and $U_n(q_2)$ are disjoint (these spheres are either identical or disjoint by p-adic ultra-metricity).

   The points in the p-adic continuum part of these solutions are at an infinite distance from q in M^4. The points which are well-defined in real sense form a discrete subset of rational points of M^4. The p-adic space-time surface constructed in this manner defines a discrete fractal hierarchy of rational space-time points besides the original points inside the p-adic spheres. In real sense the rational points have finite distances and could belong to disjoint real space-time sheets. The failure of the strict non-determinism for the field equations in the real sense gives hopes for gluing these sheets partially together (say in particle reactions with particles represented as 3-surfaces).

3. All rational points q of the p-adic space-time sheet can be interpreted as real rational points and one can solve the field equations in the real sense in the neighborhoods $U_n(q) = \{x = q + p^n y\}$ corresponding to real numbers in the range $p^n \leq x \leq p^{n+1}$. Real smoothness and continuity fix the solutions at finite rational points inside $U_n(q)$ and by the phenomenon of p-adic pseudo constants these values can be consistent with p-adic field equations. Obviously one can can continue the construction process indefinitely.

3.2.2 p-Adic scalings act only in M^4 degrees of freedom

p-Adic fractality suggests that finite real space-time sheets around points $x + p^n$, $x = 0$, are obtained as by just scaling of the M^4 coordinates having origin at $x = 0$ by p^n of the solution defined in a neighborhood of x and leaving CP_2 coordinates as such. The known extremals of Kähler action indeed allow M^4 scalings as dynamical symmetries.

One can understand why no scaling should appear in CP_2 degrees of freedom. CP_2 is complex projective space for which points can be regarded as complex planes and for these p-adic scalings act trivially. It is worth of emphasizing that here could lie a further deep number theoretic reason for why the space S in $H = M^4 \times S$ must be a projective space.

3.2.3 What p-adic fractality for real space-time surfaces really means?

The identification of p-adic and real M^4 coordinates of rational points as such is crucial for p-adic fractality. On the other hand, the identification rational real and p-adic CP_2 coordinates as such would not be consistent with the idea that p-adic smoothness and continuity imply p-adic fractality manifested as long range correlations for real space-time sheets

The point is that p-adic fractality is not stable against small p-adic deformations of CP_2 coordinates as function of M^4 coordinates for solutions representable as maps $M^4 \to CP_2$. Indeed, if the rational valued p-adic CP_2 coordinates are mapped as such to real coordinates, the addition of large power p^n to CP_2 coordinate implies small modification in p-adic sense but large change in the real sense so that correlations of CP_2 at p-adically scaled M^4 points would be completely lost.

The situation changes if the map of p-adic CP_2 coordinates to real ones is continuous so that p-adically small deformations of the p-adic space-time points are mapped to small real deformations of the real space-time points.

1. Canonical identification $I : x = \sum x_n p^n \rightarrow \sum x_n p^{-n}$ satisfies continuity constraint but does not map rationals to rationals.

2. The modification of the canonical identification given by

$$I(q = p^k \times \frac{r}{s}) = p^k \times \frac{I(r)}{I(s)} \tag{3.2}$$

   is uniquely defined for rational points, maps rationals to rationals, has a symmetry under exchange of target and domain. This map reduces to a direct identification of rationals for $0 \leq r < p$ and $0 \leq s < p$.

3. The form of this map is not general coordinate invariant nor invariant under color isometries. The natural requirement is that the map should respect the symmetries of CP_2 maximally. Therefore the complex coordinates transforming linearly under U(2) subgroup of SU(3) defining the projective coordinates of CP_2 are a natural choice. The map in question would map the real components of complex coordinates to their p-adic variants and vice versa. The residual $U(2)$ symmetries correspond to rational unitary 2×2-matrices for which matrix elements are of form $U_{ij} = p^k r/s$, $r < p, s < p$. It would seem that these transformations must form a finite subgroup if they define a subgroup at all. In case of $U(1)$ Pythagorean phases define rational phases but sufficiently high powers fail to satisfy the conditions $r < p, s < p$. Also algebraic extensions of p-adic numbers can be considered.

4. The possibility of pseudo constant allows to modify canonical identification further so that it reduces to the direct identification of real and p-adic rationals if the highest powers of p in r and s ($q = p^n r/s$) are not higher than p^N. Write $x = \sum_{n\geq 0} x_n p^n = x^{N)} + p^{N+1} y$ with $x^{N)} = \sum_{n=0}^{N} x_n p^n$, $x_0 \neq 0$, $y_0 \neq 0$, and define $I_N(x) = x^{N)} + p^{N+1} I(y)$. For $q = p^n r/s$ define $I_N(q) = p^n I_N(r)/I_N(s)$. This map reduces to the direct identification of real and p-adic rationals for $y = 0$.

5. There is no need to introduce the imaginary unit explicitly. In case of spinors imaginary unit can be represented by the antisymmetric 2×2-matrix ϵ_{ij} satisfying $\epsilon_{12} = 1$. As a matter fact, the introduction of imaginary unit as number would lead to problems since for $p \mod 4 = 3$ imaginary unit should be introduced as an algebraic extension and CP_2 in this sense would be an algebraic extension of RP_2. The fact that the algebraic extension of p-adic numbers by $\sqrt{-1}$ is equivalent with an extension introducing $\sqrt{p-1}$ supports the view that algebraic imaginary unit has nothing to do with the geometric imaginary unit defined by Kähler form of CP_2. For $p \mod 4 = 1$ $\sqrt{-1}$ exists as a p-adic number but is infinite as a real number so that the notion of finite complex rational would not make sense.

3.2.4 Preferred CP_2 coordinates as a space-time correlate for the selection of quantization axis

Complex CP_2 coordinates are fixed only apart from the choice of the quantization directions of color isospin and hyper charge axis in SU(3) Lie algebra. Hence the selection of quantization axes seems to emerge at the level of the generalized space-time geometry as quantum classical correspondence indeed requires.

In a well-defined sense the choice of the quantization axis and a special coordinate system implies the breaking of color symmetry and general coordinate invariance. This breaking is induced by the presence of p-adic space-time sheets identified as correlates for cognition and intentionality. One could perhaps say that the cognition affects real physics via the embedding space points shared by real and p-adic space-time sheets and that these common points define discrete coordinatization of the real space-time surface analogous to discretization resulting in any numerical computation.

3.2.5 Relationship between real and p-adic induced spinor fields

Besides embedding space coordinates also induced spinor fields are fundamental variables in TGD. The free second quantized induced spinor fields define the fermionic oscillator operators in terms of which the gamma matrices giving rise to spinor structure of the "world of classical worlds" can be expressed.

p-Adic fractal long range correlations must hold true also for the induced spinor fields and they are in exactly the same role as CP_2 coordinates so that the variant of canonical identification mapping rationals to rationals should map the real and imaginary parts of real induced spinor fields to their p-adic counterparts and vice versa at the rational space-time points common to p-adic and real space-time sheets.

3.2.6 Could quantum jumps transforming intentions to actions really occur?

The idea that intentional action corresponds to a quantum jump in which p-adic space-time sheet is transformed to a real one traversing through rational points common to p-adic and real space-time sheet is consistent with the conservation laws since the sign of the conserved inertial energy can be also negative in TGD framework and the density of inertial energy vanishes in cosmological length scales [K21]. Also the non-diagonal transitions $p_1 \rightarrow p_2$ are in principle possible and would correspond to intersections of p-adic space-time sheets having a common subset of rational points. Kind of phase transitions changing the character of intention or cognition would be in question.

1. Realization of intention as a scattering process

The first question concerns the interpretation of this process and possibility to find some familiar counterpart for it in quantum field theory framework. The general framework of quantum TGD suggests that the points common to real and p-adic space-time sheets could perhaps be regarded as arguments of an n-point function determining the transition amplitudes for p-adic to real transition or $p_1 \rightarrow p_2$-adic transitions. The scattering event transforming an p-adic surface (infinitely distant real surface in real M^4) to a real finite sized surface (infinitely distant p-adic surface in p-adic M^4) would be in question.

2. Could S-matrix for realizations of intentions have the same general form as the ordinary S-matrix?

One might hope that the realization of intention as a number theoretic scattering process could be characterized by an S-matrix, which one might hope of being unitary in some sense. These S-matrix elements could be interpreted at fundamental level as probability amplitudes between intentions to prepare a define initial state and the state resulting in the process.

Super-conformal invariance is a basic symmetry of quantum TGD which suggests that the S-matrix in question should be constructible in terms of n-point functions of a conformal field theory restricted to a subset of rational points shared by real and p-adic space-time surfaces or their causal determinants. According to the general vision discussed in [K14], the construction of n-point functions effectively reduces to that at 2-dimensional sections of light-like causal determinants of space-time surfaces identified as partonic space-time sheets.

The idea that physics in various number fields results by algebraic continuation of rational physics serves as a valuable guideline and suggests that the form of the S-matrices between different number fields (call them non-diagonal S-matrices) could be essentially the same as that of diagonal S-matrices. If this picture is correct then the basic differences to ordinary real S-matrix would be following.

1. Intentional action could transform p-adic space-time surface to a real one only if the exponent of Kähler function for both is rational valued (or belongs to algebraic extension of rationals).

2. The points appearing as arguments of n-point function associated with the non-diagonal S-matrix are a subset of rational points of embedding space whereas in the real case, where the integration over these points is well defined, all values of arguments can be allowed.

Thus the difference between ordinary S-matrix and more general S-matrices would be that a continuous Fourier transform of n-point function in space-time domain is not possible in the latter case. The inherent nature of cognition would be that it favors localization in the position space.

3. Objection and its resolution

Exponent of Kähler function is the key piece of the configuration space spinor field. There is a strong counter argument against the existence of the Kähler function in the p-adic context. The basic problem is that the definite integral defining the Kähler action is not p-adically well-defined except in the special cases when it can be done algebraically. Algebraic integration is however very tricky and numerically completely unstable.

The definition of the exponent of Kähler function in terms of Dirac determinants or, perhaps equivalently, as a result of normal ordering of the Kähler-Dirac action for second quantized induced spinors might however lead to an elegant resolution of this problem. This approach is discussed in detail in [K6, K11]. The idea is that Dirac determinant can be defined as a product of eigenvalues of the Kähler-Dirac operator and one ends up to a hierarchy of theories based on the restriction of the eigenvalues to various algebraic extensions of rationals identified as a hierarchy associated with corresponding algebraic extensions of p-adic numbers. This hierarchy corresponds to a hierarchy of theories (and also physics!) based on varying values of Planck constant. The elegance of this approach is that no discretization at space-time level would be neededeverything reduces to the generalized eigenvalue spectrum of the Kähler-Dirac operator.

4. A more detailed view

Consider the proposed approach in more detail.

1. Fermionic oscillator operators are assigned with the generalized eigenvectors of the Kähler-Dirac operator defined at the light-like causal determinants:

$$\begin{aligned} \Psi &= \sum_n \Psi_n b_n \ , \\ D\Psi_n &= \Gamma^\alpha D_\alpha \Psi_n = \lambda_n O \Psi_n \ , \ \ O \equiv n_\alpha \Gamma^\alpha \ . \end{aligned} \tag{3.3}$$

   Here $\Gamma^\alpha = T^{\alpha k}\Gamma_k$ denote so called Kähler-Dirac gamma matrices expressible in terms of the energy momentum current $T^{\alpha k}$ assignable to Kähler action [K6]. The replacement of the ordinary gamma matrices with modified ones is forced by the requirement that the super-symmetries of the Kähler-Dirac action are consistent with the property of being an extremal of Kähler action. n_α is a light like vector assignable to the light-like causal determinant and $O = n_\alpha\Gamma^\alpha$ must be rational and have the same value at real and p-adic side at rational points. The integer n labels the eigenvalues λ_n of the Kähler-Dirac operator, and b_n corresponds to the corresponding fermionic oscillator operator.

2. The condition that the p-adic and real variants Ψ if the Ψ are identical at common rational points of real and p-adic space-time surface (the same applies to 4-surfaces corresponding to different p-adic number fields) poses a strong constraint on the algebraic continuation from rationals to p-adics and gives hopes of deriving implications of this approach.

3. Ordinary fermionic anti-commutation relations do not refer specifically to any number field. Super Virasoro (anti-)commutation relations involve only rationals. This suggest that fermionic Fock space spanned by the oscillator operators b_n is universal and same for reals and p-adic numbers and can be regarded as rational. Same would apply to Super Virasoro representations. Also the possibility to interpret WCW spinor fields as quantum superpositions of Boolean statements supports this kind of universality. This gives good hopes that the contribution of the inner produces between Fock states to the S-matrix elements are number field independent.

4. Dirac determinant can be defined as the product of the eigenvalues λ_n restricted to a given algebraic extension of rationals. The solutions of the Kähler-Dirac equation correspond to vanishing eigen values and define zero modes generating conformal super-symmetries and are not of course included.

5. Only those operators b_n for which λ_n belongs to the algebraic extension of rationals in question are used to construct physical states for a given algebraic extension of rationals. This might mean an enormous simplification of the formalism in accordance with the fact that WCW Clifford algebra corresponds as a von Neumann algebra to a hyper-finite factor of type II_1 for which finite truncations by definition allow excellent approximations [K26]. One can even ask whether this hierarchy of algebraic extensions of rationals could in fact define a hierarchy of finite-dimensional Clifford algebras. If so then the general theory of hyper-finite factors of type II_1 would provide an extremely powerful tool.

3.3 Cognition, Logic, And P-Adicity

There seems to be a nice connection between logic aspects of cognition and p-adicity. In particular, p-valued logic for $p = 2^k - n$ has interpretation in terms of ordinary Boolean logic with n "taboos" so that p-valued logic does not conflict with common sense in this case. Also an interpretation of projections of p-adic space-time sheets to an integer lattice of real Minkowski space M^4 in terms of generalized Boolean functions emerges naturally so that M^4 projections of p-adic space-time would represent Boolean functions for a logic with n taboos.

3.3.1 2-adic valued functions of 2-adic variable and Boolean functions

The binary coefficients f_{nk} in the 2-adic expansions of terms $f_n x^n$ in the 2-adic Taylor expansion $f(x) = \sum_{n=0} f_n x^n$, assign a sequence of truth values to a 2-adic integer valued argument $x \in \{0, 1, ..., 2^N\}$ defining a sequence of N bits. Hence $f(x)$ assigns to each bit of this sequence a sequence of truth values which are ordered in the sense that the truth values corresponding to bits are not so important p-adically: much like higher decimals in decimal expansion. If a binary cutoff in N: th bit of $f(x)$ is introduced, B^M-valued function in B^N results, where B denotes Boolean algebra fo 2 elements. The formal generalization to p-adic case is trivial: 2 possible truth values are only replaced by p truth values representable as $0, ..., p-1$.

3.3.2 p-Adic valued functions of p-adic variable as generalized Boolean functions

One can speak of a generalized Boolean function mapping finite sequences of p-valued Boolean arguments to finite sequences of p-valued Boolean arguments. The restriction to a subset $x = kp^n$, $k = 0, ..., p-1$ and the replacement of the function $f(x)$ with its lowest pinary digit gives a generalized Boolean function of a single p-valued argument. If $f(x)$ is invariant under the scalings by powers of p^k, one obtains a hologram like representation of the generalized Boolean function with same function represented in infinitely many length scales. This guarantees the robustness of the representation.

The special role of 2-adicity explaining p-adic length scale hypothesis $p \simeq 2^k$, k integer, in terms of multi-p-acidic fractality would correlate with the special role of 2-valued logic in the world order. The fact that all generalizations of 2-valued logic ultimately involve 2-adic logic at the highest level, where the generalization is formulated would be analog of p-adic length scale hypothesis.

3.3.3 $p = 2^k - n$-adicity and Boolean functions with taboos

It is difficult to assign any reasonable interpretation to $p > 2$-valued logic. Also the generalization of logical connectivesandOR is far from obvious. In the case $p = 2^k - n$ favored by the p-adic length scale hypothesis situation is however different. In this case one has interpretation in terms B^k with n Boolean statements dropped out so that one obtains what might be called $\hat{b^k}$. Since n is odd this set is not invariant under Boolean conjugation so that there is at least one statement, which is identically true and could be called taboo, axiom, or dogma: depending on taste. The allowed Boolean functions would be constructed in this case using standard Boolean functionsandOR with the constraint that taboos are respected in other words, both the inputs and values of functions belong to $\hat{b^k}$.

A unique manner to define the logic with taboos is to require that the number of taboos is maximal so that if statement is dropped its negation remains in the logic. This implies $n > B^k/2$.

3.3.4 The projections of p-adic space-time sheets to real embedding space as representations of Boolean functions

Quantum classical correspondence suggests that generalized Boolean functions should have space-time correlates. Since Boolean cognition involves free will, it should be possible to construct space-time representations of arbitrary Boolean functions with finite number of arguments freely. The non-determinism of p-adic differential equations guarantees this freedom.

p-Adic space-time sheets and p-adic non-determinism make possible to represent generalization of Boolean functions of four Boolean variables obtained by replacing both argument and function with p-valued pinary digit instead of bit. These representations result as discrete projections of p-adic space-time sheets to integer valued points of real Minkowski space M^4. The interpretation would be in terms of 4 sequences of truth values of p-valued logic associated with a finite 4-D integer lattice whose lattice points can be identified as sequences of truth values of a p-valued logic with a set of p-valued truth value at each point so that in the 2-adic case one has map $B^{4M} \to B^{4N}$. Here the number of lattice points in a given coordinate direction of M^4 is M and N is the number of bits allowed by binary cutoff for CP_2 coordinates. For $p = 2^k - n$ representing Boolean algebra with n taboos, the maps can be interpreted as maps $\hat{b}^{4M} \to \hat{b}^{4N}$.

These lattices can be seen as subsets of rational shadows of p-adic space-time sheets to Minkowski space. The condensed matter analog would be a lattice with a a sequence of p-valued dynamical variables (sequence of bits/spins for $p = 2$) at each lattice point. At a fixed spatial point of M^4 the lowest bits define a time evolution of a generalized Boolean function: $B \to B$.

These observations support the view that intentionality and logic related cognition could perhaps be regarded as 2-adic aspects of consciousness. The special role of primes $p = 2^k - n$ could also be understood as special role of Boolean logic among p-valued logics and $p = 2^k - n$ logic would correspond to B^k with n axioms representing logic respecting a belief system with n beliefs. Recall that multi-p p-adic fractality involving 2-adic fractality is possible for the solutions of field equations and explains p-adic length scale hypothesis.

Most points of the p-adic space-time sheets correspond to real points which are literally infinite as real points. Therefore cognition would be in quite literal sense outside the real cosmos. Perhaps this is a direct correlate for the basic experience that mind is looking the material world from outside.

3.3.5 Connection with the theory of computational complexity?

There are interesting questions concerning the interpretation of four generalized Boolean arguments. TGD explains the number $D = 4$ for space-time dimensions and also the dimension of embedding space. Could one also find explanation why $d = 4$ defines special value for the number of generalized Boolean inputs and outputs?

1. Could the general theory of computational complexity allow to understand $d = 4$ as a maximum number of inputs and outputs allowing the computation of something related to these functions in polynomial time? For instance, complexity theorist could probably immediately answer following questions. Could the computation of the 2-adic values of CP_2 coordinates as a function of 2-adic M^4 coordinates expressed in terms of fundamental logical connectives take a time which is polynomial as a function of the number of N^4 pinary digits of M^4 coordinates and N^4 pinary digits of CP_2 coordinates? Is this time non-polynomial for M^d and S_d, S_d d-dimensional internal space, $d > 4$. Unfortunately I do not possess the needed complexity theoretic knowhow to answer these questions.

2. The same question could make sense also for $p > 2$ if the notion of the logical connectives and functions generalizes as it indeed does for $p = 2^k - n$. Therefore the question would be whether p-adic length scale hypothesis and dimensions of embedding space and space-time

are implied by a polynomial computation time? This could be the case since essentially a restriction of values and arguments of Boolean functions to a subset of B^k is in question.

3.3.6 Some calculational details

In the following the details of p-adic non-determinism are described for a differential equation of single p-adic variable and some comments about the generalization to the realistic case are given.

1. One-dimensional case

To understand the essentials consider for simplicity a solution of a p-adic differential equation giving function $y = f(x)$ of one independent variable $x = \sum_{n\geq n_0} x_n p^n$.

1. p-Adic non-determinism means that the initial values $f(x)$ of the solution can be fixed arbitrarily up to $N+1$: th pinary digit. In other words, $f(x_N)$, where $x_N = \sum_{n_0\leq n\geq N} x_n p^n$ is a rational obtained by dropping all pinary digits higher than N in $x = \sum_{n\geq n_0} x_n p^n$ can be chosen arbitrarily.

2. Consider the projection of $f(x)$ to the set of rationals assumed to be common to reals and p-adics.

   i) Genuinely p-adic numbers have infinite number of positive pinary digits in their non-periodic expansion (non-periodicity guarantees non-rationality) and are strictly infinite as real numbers. In this regime p-adic differential equation fixes completely the solution. This is the case also at rational points $q = m/n$ having infinite number of pinary digits in their pinary expansion.

   ii) The projection of p-adic x-axis to real axis consists of rationals. The set in which solution of p-adic differential equations is non-vanishing can be chosen rather freely. For instance, p-adic ball of radius p^{-n} consisting of points $x = p^M y$, $y \neq 0$, $|y|_p \leq 1$, can be considered. Assume $N > M$. p-Adic nondeterminism implies that $f(q)$ for $q = \sum_{M\leq n\leq N} x_n p^n$, can be chosen arbitrarily. For $M \geq 0$ q is always integer valued and the scaling of x by a suitable power of p always allows to get a finite integer lattice at x-axis.

   iii) The lowest pinary digit in the expansion of $f(q)$ in powers of p in defines a pinary digit. These pinary digits would define a representation for a sequence of truth values of p-logic. $p = 2$ gives the ordinary Boolean logic. It is also interpret this pinary function as a function of pinary argument giving Boolean function of one variable in 2-adic case.

2. Generalization to the space-time level

This picture generalizes to space-time level in a rather straight forward manner. y is replaced with CP_2 coordinates, x is replaced with M^4 coordinates, and differential equation with field equations deducible from the Kähler action. The essential point is that p-adic space-time sheets have projection to real Minkowski space which consists of a discrete subset of integers when suitable scaling of M^4 coordinates is allowed. The restriction of 4 CP_2 coordinates to a finite integer lattice of M^4 defines 4 Boolean functions of four Boolean arguments or their generalizations for $p > 2$. Also the modes of the induce spinor field define a similar representation.

3.4 Fibonacci Numbers, Golden Mean, And Jones Inclusions

The picture discussed above does not apply in the case of Golden Mean since powers of Φ do not have any special role for the algebraic extension of rationals by $\sqrt{5}$. It is however possible to understand the emergence of Fibonacci numbers and Golden Mean using quantum classical correspondence and the fact that the Clifford algebra and its sub-algebras associated with configuration space spinors corresponds to the so called hyper-finite factor of type II_1 (WCW refers to the "world of classical worlds").

3.4.1 Infinite braids as representations of Jones inclusions

The appearance of hyper-finite factor of type II_1 at the level of basic quantum tGD justifies the expectation that Jones inclusions $\mathcal{N} \subset \mathcal{M}$ of these factors play a key role in TGD Universe. For instance, subsystem system inclusions could induce Jones inclusions.

For the Jones inclusion $\mathcal{N} \subset \mathcal{M}$ $\mathcal{M}$ can be regarded as an $\mathcal{N}$-module with fractal dimension given by Beraha number $B_n = 4cos^2(\pi/n)$, $n \geq 3$ or equivalently by the quantum group phases $exp(i\pi/n)$. B_5 satisfies $B_5 = 4cos^2(\pi/5) = \Phi^2 = \Phi + 1$ so that the special role of $n = 5$ inclusion could explain the special role of Golden Mean in Nature.

Hecke algebras H_n, which are also characterized by quantum phase $q = exp(i\pi/n)$ or the corresponding Beraha number $B_n = 4cos^2(\pi/n)$, characterize the anyonic quantum statistics of n-braid system. Braids are understood as threads which can get linked and define in this manner braiding. Braid group describes these braidings. Like any algebra, Hecke algebra H_n can be decomposed into a direct sum of matrix algebras. Fibonacci numbers characterize the dimensions of these matrix algebras for $n = 5$. Interestingly, topological quantum computation is based on the idea that computer programs can be coded into braidings. What is remarkable is that $n = 5$ characterizes the simplest universal quantum computer so that Golden Mean could indeed have very deep roots to quantum information processing.

The so called Bratteli diagrams characterize the inclusions of various direct summands of H_k to direct summands H_{k+1} in the sequence $H_3 \subset H_4 \subset ... \subset H_k \subset ...$ of Hecke algebras. Essentially the reduction of the representations of H_{k+1} to those of H_k is in question. The same Bratteli diagrams characterize also the Jones inclusions $\mathcal{N} \subset \mathcal{M}$ of hyper-finite factors of type II_1 with index n as a limit of a finite-dimensional inclusion. Thus Jones inclusion can be visualized as a system consisting of infinite number of braids. In TGD framework the braids could be represented by magnetic flux lines or flux tubes.

3.4.2 Logarithmic spirals as representations of Jones inclusions

The inclusion sequence for Hecke algebras has a representations as a logarithmic spiral. The angle $\pi/5$ can be identified as a limit for angles ϕ_n with $cos(\phi_n) = F_{n+1}/2F_n$ assignable to orthogonal triangle with hypothenuse $2F_n$ and short side F_{n+1} and $\sqrt{4F_n^2 - F_{n+1}^2}$. Fibonacci sequence defines via this prescription a logarithmic spiral as a symbolic representation of the $n = 5$ Jones inclusion representable also in terms of infinite number of braids.

3.4.3 DNA as a topological quantum computer?

Quantum classical correspondence encourages to think that space-time geometry could define a correlate for Jones inclusions of hyper-finite factors of Clifford sub-algebras associated with Clifford algebra of WCW spinor s. The appearance of Fibonacci series in living systems could represent one example of this correspondence. The angle $\pi/10$ closely related to Golden Mean characterizes the winding of DNA double strand. Could this mean that DNA allows to realize topological quantum computer programs as braidings? A possible realization would be based on the notion of super-genes [K4], which are like pages of a book identified as magnetic flux sheets containing genomes of sequences of cell nuclei as text lines. These text lines would represent line through which magnetic flux lines traverse.

The braiding of magnetic flux lines (or possibly flux sheets regarded as flattened tubes) would define the braiding and the particles involved would be anyons obeying dynamics having quantum group $SU(2)_q$, $q = exp(i\pi/5)$, as its symmetries. The anyons could be assigned with DNA nucleotides or triplets.

TGD predicts also different kind of new physics to DNA double strand. So called H_N-atoms consist of ordinary proton an N dark electrons at space-time sheet which is λ-fold covering of space-time sheet of ordinary hydrogen atom. The effective charge of H_N-atom is $1 - N/\lambda$ since the fine structure constant for dark electrons is scaled down by $1/\lambda$. H_λ-atoms have full electron shell and are therefore exceptionally stable. The proposal is that H_λ-atoms could replace ordinary hydrogen atoms in hydrogen bonds [K4, K17]. Single base pair corresponds to 2 or 3 hydrogen bonds. The question is whether λ-hydrogen atom might somehow relate to the anyons involved with topological quantum computation.

Anyons could be dark protons resulting in the formation dark hydrogen bond in the fusion of H_N atom and its conjugate H_{N_c}, $N_c = \lambda - N$. Neutron scattering and electron diffraction suggest

4 The Recent View About Quantum TGD

Before detailed discussion of what p-adicization of quantum TGD could mean, it is good to have an overall view about what quantum TGD in real context is.

4.1 Basic Notions

The notions of embedding space, 3-surface (and 4-surface), and WCW (world of classical worlds (WCW)) are central to quantum TGD. The original idea was that 3-surfaces are space-like 3-surfaces of $H = M^4 \times CP_2$ or $H = M^4_+ \times CP_2$, and WCW consists of all possible 3-surfaces in H. The basic idea was that the definition of Kähler metric of WCW assigns to each X^3 a unique space-time surface $X^4(X^3)$ allowing in this manner to realize general coordinate invariance. During years these notions have however evolved considerably. Therefore it seems better to begin directly from the recent picture.

4.1.1 The notion of embedding space

Two generalizations of the notion of embedding space were forced by number theoretical vision [K23, K22].

1. p-Adicization forced to generalize the notion of embedding space by gluing real and p-adic variants of embedding space together along rationals and common algebraic numbers. The generalized embedding space has a book like structure with reals and various p-adic number fields (including their algebraic extensions) representing the pages of the book.

2. With the discovery of zero energy ontology [K6, K14] it became clear that the so called causal diamonds (CDs) interpreted as intersections $M^4_+ \cap M^4_-$ of future and past directed light-cones of $M^4 \times CP_2$ define correlates for the quantum states. The position of the "lower" tip of CD characterizes the position of CD in H. If the temporal distance between upper and lower tip of CD is quantized power of 2 multiples of CP_2 length, p-adic length scale hypothesis [K19] follows as a consequence. The upper *resp.* lower light-like boundary $\delta M^4_+ \times CP_2$ *resp.* $\delta M^4_- \times CP_2$ of CD can be regarded as the carrier of positive *resp.* negative energy part of the state. All net quantum numbers of states vanish so that everything is creatable from vacuum. Space-time surfaces assignable to zero energy states would would reside inside $CD \times CP_2$s and have their 3-D ends at the light-like boundaries of $CD \times CP_2$. Fractal structure is present in the sense that CDs can contains CDs within CDs, and measurement resolution dictates the length scale below which the sub-CDs are not visible.

3. The realization of the hierarchy of Planck constants [K15] led to a further generalization of the notion of embedding space. Generalized embedding space is obtained by gluing together Cartesian products of singular coverings and factor spaces of CD and CP_2 to form a book like structure. The particles at different pages of this book behave like dark matter relative to each other. This generalization also brings in the geometric correlate for the selection of quantization axes in the sense that the geometry of the sectors of the generalized embedding space with non-standard value of Planck constant involves symmetry breaking reducing the isometries to Cartan subalgebra. Roughly speaking, each CD and CP_2 is replaced with a union of CDs and CP_2s corresponding to different choices of quantization axes so that no breaking of Poincare and color symmetries occurs at the level of entire WCW .

4.1.2 The notions of 3-surface and space-time surface

The question what one exactly means with 3-surface turned out to be non-trivial.

1. The original identification of 3-surfaces was as arbitrary space-like 3-surfaces subject to Equivalence implied by General Coordinate Invariance. There was a problem related to

the realization of GCI since it was not at all obvious why the preferred extremal $X^4(Y^3)$ for Y^3 at $X^4(X^3)$ and Diff^4 related X^3 should satisfy $X^4(Y^3) = X^4(X^3)$.

2. Much later it became clear that light-like 3-surfaces have unique properties for serving as basic dynamical objects, in particular for realizing the General Coordinate Invariance in 4-D sense (obviously the identification resolves the above mentioned problem) and understanding the conformal symmetries of the theory. On basis of these symmetries light-like 3-surfaces can be regarded as orbits of partonic 2-surfaces so that the theory is locally 2-dimensional. It is however important to emphasize that this indeed holds true only locally. At the level of WCW metric this means that the components of the Kähler form and metric can be expressed in terms of data assignable to 2-D partonic surfaces. It is however essential that information about normal space of the 2-surface is needed. The mathematical formulation of this vision is however highly nontrivial challenge: is it due to analogs of gauge symmetries or should effective 2-dimensionality formulated explicitly as assumed until 2014 when stringy formulation of WCW geometry emerged.

3. Rather recently came the realization that light-like 3-surfaces can have singular topology in the sense that they are analogous to Feynman diagrams. This means that the light-like 3-surfaces representing lines of Feynman diagram can be glued along their 2-D ends playing the role of vertices to form what I call generalized Feynman diagrams. The ends of lines are located at boundaries of sub-CDs. This brings in also a hierarchy of time scales: the increase of the measurement resolution means introduction of sub-CDs containing sub-Feynman diagrams. As the resolution is improved, new sub-Feynman diagrams emerge so that effective 2-D character holds true in discretized sense and in given resolution scale only.

The basic vision has been that space-time surfaces correspond to preferred extremals $X^4(X^3)$ of Kähler action. Kähler function $K(X^3)$ defining the Kähler geometry of the world of classical worlds would correspond to the Kähler action for the preferred extremal. The precise identification of the preferred extremals actually has however remained open.

1. The obvious guess motivated by physical intuition was that preferred extremals correspond to the absolute minima of Kähler action for space-time surfaces containing X^3. This choice had some nice implications. For instance, one can develop an argument for the existence of an infinite number of conserved charges. If X^3 is light-like surface- either light-like boundary of X^4 or light-like 3-surface assignable to a wormhole throat at which the induced metric of X^4 changes its signature- this identification circumvents the obvious objections.

   The identification of $X^3(X^3)$ as preferred extremal is however not consistent with quantum criticality suggesting in zero energy ontology (ZEO) a large number of space-time sheets associated with same 3-surface at the ends of causal diamond CD, and having same value of Kähler function. More technically, the Kähler action would have degenerate Hessian as a functional of X^4 with fixed ends X^3).

2. Much later number theoretical vision led to the conclusion that $X^4(X^3_{l,i})$, where $X^3_{l,i}$ denotes a connected component of the light-like 3-surfaces X^3_l, contain in their 4-D tangent space $T(X^4(X^3_{l,i}))$ a subspace $M^2_i \subset M^4$ having interpretation as the plane of non-physical polarizations. This means a close connection with super string models. Geometrically this would mean that the deformations of 3-surface in the plane of non-physical polarizations would not contribute to the line element of WCW . This is as it must be since complexification does not make sense in M^2 degrees of freedom.

   In number theoretical framework M^2_i has interpretation as a preferred hyper-complex subspace of hyper-octonions defined as 8-D subspace of complexified octonions with the property that the metric defined by the octonionic inner product has signature of M^8. A stronger condition would be that the condition holds true at all points of $X^4(X^3)$ for a global choice M^2 but this is un-necessary and leads to strong un-proven conjectures. The condition $M^2_i \subset T(X^4(X^3_{l,i}))$ in principle fixes the tangent space at $X^3_{l,i}$, and one has good hopes that the boundary value problem is well-defined and fixes $X^4(X^3)$ uniquely as a preferred extremal of Kähler action. This picture is rather convincing since the choice $M^2_i \subset M^3$ plays also other important roles.

3. The next step [K6] was the realization that the construction of WCW geometry in terms of modified Dirac action strengthens the boundary conditions to the condition that there exists space-time coordinates in which the induced CP_2 Kähler form and induced metric satisfy the conditions $J_{ni} = 0$, $g_{ni} = 0$ hold at X_l^3. One could say that at X_l^3 situation is static both metrically and for the Maxwell field defined by the induced Kähler form. There are reasons to hope that this is the final step in a long process.

4. The weakest form of number theoretic compactification [K23] states that light-like 3-surfaces $X^3 \subset X^4(X^3) \subset M^8$, where $X^4(X^3)$ hyper-quaternionic surface in hyper-octonionic M^8 can be mapped to light-like 3-surfaces $X^3 \subset X^4(X^3) \subset M^4 \times CP_2$, , where $X^4(X^3)$ is now preferred extremum of Kähler action. The natural guess is that $X^4(X^3) \subset M^8$ is a preferred extremal of Kähler action associated with Kähler form of E^4 in the decomposition $M^8 = M^4 \times E^4$, where M^4 corresponds to hyper-quaternions. The conjecture would be that the value of the Kähler action in M^8 is same as in $M^4 \times CP_2$: in fact that 2-surface would have identical induced metric and Kähler form so that this conjecture would follow trivial. $M^8 - H$ duality would in this sense be Kähler isometry.

4.1.3 The notion of WCW (“world of classical worlds”)

From the beginning there was a problem related to the precise definition of WCW (“world of classical worlds” (WCW)). Should one regard CH as the space of 3-surfaces of $M^4 \times CP_2$ or $M^4_+ \times CP_2$ or perhaps something more delicate.

1. For a long time I believed that the question “M^4_+ or M^4?” had been settled in favor of M^4_+ by the fact that M^4_+ has interpretation as empty Roberson-Walker cosmology. The huge conformal symmetries assignable to $\delta M^4_+ \times CP_2$ were interpreted as cosmological rather than laboratory symmetries. The work with the conceptual problems related to the notions of energy and time, and with the symmetries of quantum TGD, however led gradually to the realization that there are strong reasons for considering M^4 instead of M^4_+.

2. With the discovery of zero energy ontology it became clear that the so called causal diamonds (CDs) define excellent candidates for the fundamental building blocks of WCW or “world of classical worlds” (WCW). The spaces $CD \times CP_2$ regarded as subsets of H defined the sectors of WCW .

3. This framework allows to realize the huge symmetries of $\delta M^4_\pm \times CP_2$ as isometries of WCW . The gigantic symmetries associated with the $\delta M^4_\pm \times CP_2$ are also laboratory symmetries. Poincare invariance fits very elegantly with the two types of super-conformal symmetries of TGD. The first conformal symmetry corresponds to the light-like surfaces $\delta M^4_\pm \times CP_2$ of the embedding space representing the upper and lower boundaries of CD.

   The original long-held belief was that the second conformal symmetry corresponds to local embedding space isometries for light-like 3-surfaces X_l^3, which are either boundaries of X^4 (probably not: it seems that boundary conditions cannot be satisfied so that space-time surfaces must consists of regions defining at least double coverings of M^4) or light-like surfaces separating space-time regions with different signatures of the induced metric. This symmetry would be identifiable as the counterpart of the Kac Moody symmetry of string models.

   It has turned out that one can assume Kac-Moody algebra to be sub-algebra of symplectic algebra consisting of the symplectic isometries of $\delta M^4_\pm \times CP_2$. This super Kac-Moody algebra is generated by super-currents assignable to the modes of induced spinor fields other than right-handed neutrino and localized at string world sheets. The symplectic algebra would correspond to right-handed neutrino and one would have direct sum of these two. The beauty of this option is that localization would be inherent property of both algebras and with respect to the light-like coordinate of light-cone boundary rather than forced by hand. The issues related to diffeo-invariance would be avoided in this manner.

   Strong form of holography implied by strong form of GCI suggests the duality between space-like 3-surfaces at the end of CD and light-like 3-surfaces. By parallel translating the boundary of CD one can indeed define the action of symplectic algebra at the light-like 3-surfaces.

Therefore also the symplectic and Kac-Moody algebras associated with these surfaces could be used to generate zero energy states, and one would have effective 2-dimensionality in the sense that only the partonic 2-surfaces defined by the intersections of space-like and light-like 3-surfaces and their 4-D tangent space data would code for quantum physics.

A rather plausible conclusion is that WCW (WCW) is a union of sub- WCW s associated with the spaces $CD \times CP_2$. CDs can contain CDs within CDs so that a fractal like hierarchy having interpretation in terms of measurement resolution results. Since the complications due to p-adic sectors and hierarchy of Planck constants are not relevant for the basic construction, it reduces to a high degree to a study of a simple special case $\delta M^4_+ \times CP_2$.

A further piece of understanding emerged from the following observations.

1. The induced Kähler form at the partonic 2-surface X^2 - the basic dynamical object if holography is accepted- can be seen as a fundamental symplectic invariant so that the values of $\epsilon^{\alpha\beta} J_{\alpha\beta}$ at X^2 define local symplectic invariants not subject to quantum fluctuations in the sense that they would contribute to the WCW metric. Hence only induced metric corresponds to quantum fluctuating degrees of freedom at WCW level and TGD is a genuine theory of gravitation at this level.

2. WCW can be divided into slices for which the induced Kähler forms of CP_2 and $\delta M^4_\pm$ at the partonic 2-surfaces X^2 at the light-like boundaries of CDs are fixed. The symplectic group of $\delta M^4_\pm \times CP_2$ parameterizes quantum fluctuating degrees of freedom in given scale (recall the presence of hierarchy of CDs).

3. This leads to the identification of the coset space structure of the sub- WCW associated with given CD in terms of the generalized coset construction for super-symplectic and super Kac-Moody type algebras. WCW in quantum fluctuating degrees of freedom for given values of zero modes can be regarded as being obtained by dividing symplectic group with subgroup leaving the 3-surface invariant for a preferred 3-surface which could be chosen maximum/minimum of Kähler function. Equivalently, the local coset space associated with $S^2 \times CP_2$ is in question: this was one of the first ideas about WCW which I gave up as too naïve!

4. The original construction of WCW metric was in terms of flow Hamiltonians induced by those of $\delta M^4_\pm \times CP_2$. Matrix elements of WCW metric were constructed as anti-commutators of super-Hamiltonians having interpretation also as WCW Hamiltonians.

   The construction had problematic aspects. Flux Hamiltonians were strictly 2-D objects and also the fact that they contained very little explicit information about the dynamics of the Kähler-Dirac action. The realization of super-Hamiltonians in terms of conserved symplectic super-charges of Kähler-Dirac action labelled by the modes of the Kähler-Dirac operator cures the situation and the construction becomes 3-dimensional although effective 2-dimensionality still holds true. Anti-commutations are fixed completely and the construction works for dimension $D = 8$ of embedding space only. The stringy picture forced by the solutions of the Kähler-Dirac operator becomes very explicit at the level of WCW . 8-D embedding space only [K9].

5. Generalized coset construction and coset space structure have very deep physical meaning. Symmetric space structure requires involution and it corresponds to inversion in light-like radial coordinate r_M of $\delta M^4_\pm$ (determined only up to Lorentz transformation). Super Virasoro algebra realizes quantum criticality, and one obtains hierarchy of criticalities represented by the hierarchy of sub-algebras of Super Virasoro algebra.

4.1.4 Head aches from Equivalent Principle

Equivalence Princple (EP) has been continual source of headaches during years. It is not even clear whether the uncritical assumption that there gravitational and inertial masses exist at separate notions creates the problem as a pseudoproblem. Stringy description of graviton mediated scattering predicted also by TGD indeed suggests this.

1. A longstanding conjecture has been that coset representations could Equivalence Principle (EP) at quantum level: the identity of Super Virasoro generators for super-symplectic and super Kac-Moody algebras was proposed to imply that inertial and gravitational four-momenta are identical. This conjecture is probably wrong.

2. The equivalence of classical Noether momentum associated with Kähler action with eigenvalues of the corresponding quantal momentum for Kähler-Dirac action certainly realizes quantum classical correspondence. It could also realized EP. Zero energy ontology suggests an alternative formulation for the same idea.

3. A further option is that EP reduces to the identification of the four momenta assignable to Super Virasoro representations assignable to space-like and light-like 3-surfaces and therefore become part of strong form of holography and quantum classical correspondence (QCC).

So it seems that EP might reduce to holography, GCI, or QCC and it might well be that it is trivially true! At classical level the understanding of the relationship between TGD and GRT led to the final break through in the understanding of EP. The recent view is that EP at quantum level reduces to Quantum Classical Correpondence (QCC) in the sense that Cartan algebra Noether charges assignable to 3-surface in case of Kähler action (inertial charges) are identical with eigenvalues of the quantal variants of Noether charges for Kähler-Dirac action (gravitational charges). The well-definedness of the latter charges is due to the conformal invariance assignable to 2-D surfaces (string world sheets and possibly partonic 2-surfaces) at which the spinor modes are localized in generic case. This localization follows from the condition that em charge has well defined value for the spinor modes. The localization is possibly only for the Kähler-Dirac action and key role is played by the modification of gamma matrices to Kähler-Dirac gamma matrices. The gravitational four-momentum is thus completely analogous to stringy four-momentum.

At classical level EP follows from the interpretation of GRT space-time as effective space-time obtained by replacing many-sheeted space-time (see **Fig.** `http://tgdtheory.fi/appfigures/manysheeted.jpg` or **Fig.** 9 in the appendix of this book) with Minkowski space with effective metric determined as a sum of Minkowski metric and sum over the deviations of the induced metrices of space-time sheets from Minkowski metric. Poincare invariance suggests strongly classical EP for the GRT limit in long length scales at least. One can consider also other kinds of limits such as the analog of GRT limit for Euclidian space-time regions assignable to elementary particles. In this case deformations of CP_2 metric define anatural starting point and CP_2 indeed defines a gravitational instanton with very large cosmological constant in Einstein-Maxwell theory. Also gauge potentials of standard model correspond classically to superpositions of induced gauge potentials over space-time sheets.

4.2 The Most Recent Vision About Zero Energy Ontology

The generalization of the number concept obtained by fusing real and p-adics along rationals and common algebraics is the basic philosophy behind p-adicization. This however requires that it is possible to speak about rational points of the embedding space and the basic objection against the notion of rational points of embedding space common to real and various p-adic variants of the embedding space is the necessity to fix some special coordinates in turn implying the loss of a manifest general coordinate invariance. The isometries of the embedding space could save the situation provided one can identify some special coordinate system in which isometry group reduces to its discrete subgroup. The loss of the full isometry group could be compensated by assuming that WCW is union over sub- WCW : s obtained by applying isometries on basic sub- WCW with discrete subgroup of isometries.

The combination of zero energy ontology realized in terms of a hierarchy causal diamonds and hierarchy of Planck constants providing a description of dark matter and leading to a generalization of the notion of embedding space suggests that it is possible to realize this dream. The article [L1] provides a brief summary about recent state of quantum TGD helping to understand the big picture behind the following considerations.

4.2.1 Zero energy ontology briefly

1. The basic construct in the zero energy ontology is the space $CD \times CP_2$, where the causal diamond CD is defined as an intersection of future and past directed light-cones with time-like separation between their tips regarded as points of the underlying universal Minkowski space M^4. In zero energy ontology physical states correspond to pairs of positive and negative energy states located at the boundaries of the future and past directed light-cones of a particular CD. CD: s form a fractal hierarchy and one can glue smaller CD: s within larger CD along the upper light-cone boundary along a radial light-like ray: this construction recipe allows to understand the asymmetry between positive and negative energies and why the arrow of experienced time corresponds to the arrow of geometric time and also why the contents of sensory experience is located to so narrow interval of geometric time. One can imagine evolution to occur as quantum leaps in which the size of the largest CD in the hierarchy of personal CD: s increases in such a way that it becomes sub-CD of a larger CD. p-Adic length scale hypothesis follows if the values of temporal distance T between tips of CD come in powers of 2^n: a weaker condition would be $T_p = pT_0$, p prime, and would assign all p-adic time scales to the size scale hierarchy of CDs. All conserved quantum numbers for zero energy states have vanishing net values. The interpretation of zero energy states in the framework of positive energy ontology is as physical events, say scattering events with positive and negative energy parts of the state interpreted as initial and final states of the event.

2. In the realization of the hierarchy of Planck constants $CD \times CP_2$ is replaced with a Cartesian product of book like structures formed by almost copies of CD: s and CP_2: s defined by singular coverings and factors spaces of CD and CP_2 with singularities corresponding to intersection $M^2 \cap CD$ and homologically trivial geodesic sphere S^2 of CP_2 for which the induced Kähler form vanishes. The coverings and factor spaces of CD: s are glued together along common $M^2 \cap CD$. The coverings and factors spaces of CP_2 are glued together along common homologically non-trivial geodesic sphere S^2. The choice of preferred M^2 as subspace of tangent space of X^4 at all its points and having interpretation as space of non-physical polarizations, brings M^2 into the theory also in different manner. S^2 in turn defines a subspace of the much larger space of vacuum extremals as surfaces inside $M^4 \times S^2$.

3. WCW (the world of classical worlds, WCW) decomposes into a union of sub-WCW:s corresponding to different choices of M^2 and S^2 and also to different choices of the quantization axes of spin and energyand color isospin and hyper-charge for each choice of this kind. This means breaking down of the isometries to a subgroup. This can be compensated by the fact that the union can be taken over the different choices of this subgroup.

4. p-Adicization requires a further breakdown to discrete subgroups of the resulting sub-groups of the isometry groups but again a union over sub- WCW : s corresponding to different choices of the discrete subgroup can be assumed. Discretization relates also naturally to the notion of number theoretic braid.

Consider now the critical questions.

1. Very naïvely one could think that center of mass wave functions in the union of sectors could give rise to representations of Poincare group. This does not conform with zero energy ontology, where energy-momentum should be assignable to say positive energy part of the state and where these degrees of freedom are expected to be pure gauge degrees of freedom. If zero energy ontology makes sense, then the states in the union over the various copies corresponding to different choices of M^2 and S^2 would give rise to wave functions having no dynamical meaning. This would bring in nothing new so that one could fix the gauge by choosing preferred M^2 and S^2 without losing anything. This picture is favored by the interpretation of M^2 as the space of longitudinal polarizations.

2. The crucial question is whether it is really possible to speak about zero energy states for a given sector defined by generalized embedding space with fixed M^2 and S^2. Classically this is possible and conserved quantities are well defined. In quantal situation the presence of

the light-cone boundaries breaks full Poincare invariance although the infinitesimal version of this invariance is preserved. Note that the basic dynamical objects are 3-D light-like "legs" of the generalized Feynman diagrams.

4.2.2 Definition of energy in zero energy ontology

Can one then define the notion of energy for positive and negative energy parts of the state? There are two alternative approaches depending on whether one allows or does not allow wave-functions for the positions of tips of light-cones.

Consider first the naïve option for which four momenta are assigned to the wave functions assigned to the tips of CD: s.

1. The condition that the tips are at time-like distance does not allow separation to a product but only following kind of wave functions

$$\Psi = exp[ip\cdot(m_+ - m_-)]\Theta(T^2)\Theta(m_+^0 - m_-^0)\Phi(p) \ , \ T^2 = (m_+ - m_-)^2 \ . \tag{4.1}$$

   Here m_+ and m_- denote the positions of the light-cones and Θ denotes step function. Φ denotes WCW spinor field in internal degrees of freedom of 3-surface. One can introduce also the decomposition into particles by introducing sub-CD: s glued to the upper light-cone boundary of CD.

2. The first criticism is that only a local eigen state of 4-momentum operators $p_\pm = \hbar\nabla/i$ is in question everywhere except at boundaries and at the tips of the CD with exact translational invariance broken by the two step functions having a natural classical interpretation. The second criticism is that the quantization of the temporal distance between the tips to $T = 2^kT_0$ is in conflict with translational invariance and reduces it to a discrete scaling invariance.

The less naïve approach relying of super conformal structures of quantum TGD assumes fixed value of T and therefore allows the crucial quantization condition $T = 2^kT_0$.

1. Since light-like 3-surfaces assignable to incoming and outgoing legs of the generalized Feynman diagrams are the basic objects, can hope of having enough translational invariance to define the notion of energy. If translations are restricted to time-like translations acting in the direction of the future (past) then one has local translation invariance of dynamics for classical field equations inside $\delta M^4_\pm$ as a kind of semigroup. Also the M^4 translations leading to interior of X^4 from the light-like 2-surfaces surfaces act as translations. Classically these restrictions correspond to non-tachyonic momenta defining the allowed directions of translations realizable as particle motions. These two kinds of translations have been assigned to super-symplectic conformal symmetries at $\delta M^4_\pm \times CP_2$ and and super Kac-Moody type conformal symmetries at light-like 3-surfaces.

2. The condition selecting preferred extremals of Kähler action is induced by a global selection of M^2 as a plane belonging to the tangent space of X^4 at all its points [K14]. The M^4 translations of X^4 as a whole in general respect the form of this condition in the interior. Furthermore, if M^4 translations are restricted to M^2, also the condition itself - rather than only its general form - is respected. This observation, the earlier experience with the p-adic mass calculations, and also the treatment of quarks and gluons in QCD encourage to consider the possibility that translational invariance should be restricted to M^2 translations so that mass squared, longitudinal momentum and transversal mass squared would be well defined quantum numbers. This would be enough to realize zero energy ontology. Encouragingly, M^2 appears also in the generalization of the causal diamond to a book-like structure forced by the realization of the hierarchy of Planck constant at the level of the embedding space.

3. That the cm degrees of freedom for CD would be gauge like degrees of freedom sounds strange. The paradoxical feeling disappears as one realizes that this is not the case for sub-CD: s, which indeed can have non-trivial correlation functions with either upper or lower tip of the CD playing a role analogous to that of an argument of n-point function in QFT description. One can also say that largest CD in the hierarchy defines infrared cutoff.

5 P-Adicization At The Level Of Embedding Space And Space-time

In this section p-adicization program at the level if embedding space and space-time is discussed. The general problems of p-adicization, namely the selection of preferred coordinates and the problems caused by the non-existence of p-adic definite integral and algebraic continuation a solution of these problems has been discussed in the introduction.

5.1 P-Adic Variants Of The Embedding Space

Consider now the construction of p-adic variants of the embedding space.

1. Rational values of p-adic coordinates are non-negative so that light-cone proper time $a_{4,+} = \sqrt{t^2 - z^2 - x^2 - y^2}$ is the unique Lorentz invariant choice for the p-adic time coordinate near the lower tip of CD. For the upper tip the identification of a_4 would be $a_{4,-} = \sqrt{(t-T)^2 - z^2 - x^2 - y^2}$. In the p-adic context the simultaneous existence of both square roots would pose additional conditions on T. For 2-adic numbers $T = 2^n T_0$, $n \geq 0$ (or more generally $T = \sum_{k \geq n_0} b_k 2^k$), would allow to satisfy these conditions and this would be one additional reason for $T = 2^n T_0$ implying p-adic length scale hypothesis. Note however that also $T_p = pT_0$, p prime, can be considered. The remaining coordinates of CD are naturally hyperbolic cosines and sines of the hyperbolic angle $\eta_{\pm,4}$ and cosines and sines of the spherical coordinates θ and ϕ.

2. The existence of the preferred plane M^2 of un-physical polarizations would suggest that the 2-D light-cone proper times $a_{2,+} = \sqrt{t^2 - z^2}$ $a_{2,-} = \sqrt{(t-T)^2 - z^2}$ can be also considered. The remaining coordinates would be naturally $\eta_{\pm,2}$ and cylindrical coordinates (ρ, ϕ).

3. The transcendental values of a_4 and a_2 are literally infinite as real numbers and could be visualized as points in infinitely distant geometric future so that the arrow of time might be said to emerge number theoretically. For M^2 option p-adic transcendental values of ρ are infinite as real numbers so that also spatial infinity could be said to emerge p-adically.

4. The selection of the preferred quantization axes of energy and angular momentum unique apart from a Lorentz transformation of M^2 would have purely number theoretic meaning in both cases. One must allow a union over sub-WCWs labeled by points of $SO(1,1)$. This suggests a deep connection between number theory, quantum theory, quantum measurement theory, and even quantum theory of mathematical consciousness.

5. In the case of CP_2 there are three real coordinate patches involved [?]. The compactness of CP_2 allows to use cosines and sines of the preferred angle variable for a given coordinate patch.

$$
\begin{aligned}
\xi^1 &= tan(u)exp(i\frac{(\Psi+\Phi)}{2})cos(\frac{\Theta}{2}) \ , \\
\xi^2 &= tan(u)exp(i\frac{(\Psi-\Phi)}{2})sin(\frac{\Theta}{2}) \ .
\end{aligned}
\tag{5.1}
$$

   The ranges of the variables u, Θ, Φ, Ψ are $[0, \pi/2], [0, \pi], [0, 4\pi], [0, 2\pi]$ respectively. Note that u has naturally only the positive values in the allowed range. S^2 corresponds to the values $\Phi = \Psi = 0$ of the angle coordinates.

6. The rational values of the (hyperbolic) cosine and sine correspond to Pythagorean triangles having sides of integer length and thus satisfying $m^2 = n^2 + r^2$ ($m^2 = n^2 - r^2$). These conditions are equivalent and allow the well-known explicit solution [A1]. One can construct a p-adic completion for the set of Pythagorean triangles by allowing p-adic integers which are infinite as real integers as solutions of the conditions $m^2 = r^2 \pm s^2$. These angles correspond to genuinely p-adic directions having no real counterpart. Hence one obtains p-adic continuum

also in the angle degrees of freedom. Algebraic extensions of the p-adic numbers bringing in cosines and sines of the angles π/n lead to a hierarchy increasingly refined algebraic extensions of the generalized embedding space. Since the different sectors of WCW directly correspond to correlates of selves this means direct correlation with the evolution of the mathematical consciousness. Trigonometric identities allow to construct points which in the real context correspond to sums and differences of angles.

7. Negative rational values of the cosines and sines correspond as p-adic integers to infinite real numbers and it seems that one use several coordinate patches obtained as copies of the octant $(x \geq 0, y \geq 0, z \geq 0,)$. An analogous picture applies in CP_2 degrees of freedom.

8. The expression of the metric tensor and spinor connection of the embedding in the proposed coordinates makes sense as a p-adic numbers in the algebraic extension considered. The induction of the metric and spinor connection and curvature makes sense provided that the gradients of coordinates with respect to the internal coordinates of the space-time surface belong to the extensions. The most natural choice of the space-time coordinates is as subset of embedding space-coordinates in a given coordinate patch. If the remaining embedding space coordinates can be chosen to be rational functions of these preferred coordinates with coefficients in the algebraic extension of p-adic numbers considered for the preferred extremals of Kähler action, then also the gradients satisfy this condition. This is highly non-trivial condition on the extremals and if it works might fix completely the space of exact solutions of field equations. Space-time surfaces are also conjectured to be hyper-quaternionic [K23], this condition might relate to the simultaneous hyper-quaternionicity and Kähler extremal property. Note also that this picture would provide a partial explanation for the decomposition of the embedding space to sectors dictated also by quantum measurement theory and hierarchy of Planck constants.

5.2 P-Adicization At The Level Of Space-Time

Number theoretical Universality in weak sense does not seem to pose problems. The field equations defining the preferred extremals of Kähler action make sense also p-adically if the preferred extremals correspond to critical space-time sheets for which the second variation of Kähler action vanishes for some deformations [K6]: this guarantees that the Noether currents associated with the Kähler-Dirac action are conserved. In this case the matrix determined by second variations has rank which is not maximal. The interpretation is in terms of a generalized catastrophe theory: space-time surfaces are critical with respect to the variation of Kähler action. These conditions are algebraic and make sense also p-adically. Also the conditions implied by number theoretical compactification make sense p-adically. Therefore one can construct the p-adic variants of preferred extremals of Kähler action. The new element is the possibility of p-adic pseudo constants depending on finite number of pinary digits only.

The natural expectation is that the number of critical deformations is infinite and corresponds to conformal symmetries naturally assignable to criticality. The number n of conformal equivalence classes of the deformations can be finite and n would naturally relate to the hierarchy of Planck constants $h_{eff} = n \times h$ (see **Fig.** `http://tgdtheory.fi/appfigures/planckhierarchy.jpg`, or **Fig. ??** in the appendix of this book).

At number theoretical criticality it should be possible to assign to the real partonic 2-surface a unique p-adic counterpart. This might be true also for X_l^3 and even for the space-time sheet $X^4(X_l^3)$. This is possible if the objects in question are defined by algebraic equations making sense also p-adically. Also trigonometric functions and exponential functions can be considered. Obviously p-adic pseudo constants are genuine constants for the geometric objects being shared in algebraic sense by the worlds defined by different number fields.

1. The starting point are the algebraic equations defining light-like partonic 3-surfaces X_l^3 via the condition that the determinant of the induced metric vanishes. If the coordinate functions appearing in the determinant are algebraic functions with algebraic coefficients, p-adicization should make sense.

2. General Coordinate Invariance would suggest that this true also for the light-like 3-surfaces parallel to X_l^3 appearing in the slicing of $X^4(X_l^3)$ assumed in the quantization of induced spinor fields and suggested by the properties of known extremals.

3. If the 4-dimensional real space-time sheet is expressible as a hyper-quaternionic surface of hyper-octonionic variant M^8 of the embedding space as number-theoretic vision suggests [K23], it might be possible to construct also the p-adic variant of the space-time sheet by algebraic continuation in the case that the functions appearing in the definition of the space-time sheet are algebraic.

Some preferred space-time coordinates are necessary.

1. Standard Minkowski coordinates associated with $M^2 \times E^2$ decomposition are implied by the selection of quantization axes also also preferred CP_2 coordinates and preferred coordinates for geodesic sphere S_i^2, $i = I$ or II. These coordinates could be used to define coordinates also for X^4. Which combination of coordinate variables is good would be determined by the dimensions of projections to M^4 and CP_2.

2. The construction of solutions of field equations leads to the so called Hamilton-Jacobi coordinates for M^4, when the induced metric has Minkowski signature [K11]. These coordinates define a slicing of $X^4(X_l^3)$ by string world sheets and their partonic duals required also by the number theoretic compactification. For 4-D M^4 projection these coordinates could be used also as X^4 coordinates. The light-like coordinates u, v assigned with the string world sheets *resp.* complex coordinate w associated with the partonic 2-surface would give a candidate for preferred coordinates fixed apart from hyper-conformal *resp.* conformal transformations.

3. A good candidate for preferred coordinates for $X^2(v)$ is defined by the fluxes $J = \epsilon^{\alpha\beta} J_{\alpha\beta}\sqrt{g_2}$ and their canonical conjugates assignable to partonic 2-surfaces X^2 and their translates $X^2(v)$ along $X_l^3(X^2)$. Here J could correspond to either S^2 or CP_2 Kähler form. These coordinates are discussed in detail in the section about number theoretic braids.

4. For u, v coordinates the basic condition is that v varies along $X_l^3(u)$ and u labels these slices. This condition allows only scalings as hyper-complex analytic transformations and one might hope of fixing this scaling uniquely.

5.3 P-Adicization Of Second Quantized Induced Spinor Fields

Induction procedure makes it possible to geometrize the concept of a classical gauge field and also of the spinor field with internal quantum numbers. In the case of the electro-weak gauge fields induction means the projection of the H-spinor connection to a spinor connection on the space-time surface.

In the most recent formulation induced spinor fields appear only at light-like 3-surfaces and satisfy Kähler-Dirac action associated with Kähler action possibly complexified by addition imaginary CP breaking instanton term. The Kähler-Dirac equation makes sense also p-adically as also the anti-commutation relations of the induced spinor fields at different points of the (number theoretic) braid. Here discreteness is essential since delta functions are not easy to define in p-adic context.

Possible difficulties relate to the definition of p-adic variants of plane wave factors appearing in the construction and being defined with respect to the variable u labeling the slices in the slicing of $X^4(X_l^3)$ by light-like 3-surfaces $X_l^3(v)$ "parallel" to X_l^3. Exponent function as such is well-defined in p-adic context if the argument has p-adic norm smaller than one. It however fails to have the basic properties of its real variant failing to be periodic and having fixed unit p-adic norm for all values of its argument. Periodicity does not however seem to be essential for the formulation of quantum TGD in its recent form. The exponential functions involved are of form $exp(i\sqrt{n}u)$, and are not periodic even in real sense. The p-adic existence requires $u \ mod \ p = 0$ unless one introduces e and possibly also some roots of e to the extension of p-adics used (e^p exists so that the extension would be finite-dimensional).

These observations raise the hope that the continuation of the second quantized induced spinor fields to various p-adic number fields is a straightforward procedure at the level of principle.

6 P-Adicization At The Level Of WCW

This section is not a distilled final answer to the challenges involved with the p-adicization of WCW geometry and spinor structure. There are several questions. What is the precise meaning of concepts like number theoretical universality and criticality? What does p-adicization mean and is it needed/possible? Is algebraic continuation the manner to achieve it?

The notion of reduced WCW implied by the notion of finite measurement resolution is what gives hopes about performing this continuation in practice.

1. The weaker notion of reduced WCW emerges from finite measurement resolution and for given induced Kähler form at partonic 2-surfaces reduces WCW to a finite-dimensional space $(\delta M^4_\pm \times CP_2)^n/S_n$ for given number of points of number theoretic braid. The metric and Kähler structure of this space is determined dynamically in terms of the spectrum of the Kähler-Dirac operator.

2. The stronger notion of reduced WCW identified as the space of the maxima of Kähler function in quantum fluctuating degrees of freedom labeled by symplectic group is second key notion and suggests strongly discretization. The points of reduced configuration space with rational of algebraic coordinates would correspond to those 3-surfaces through which leakage between different sectors of WCW is possible. Reduced configuration space in this sense is the direct counterpart of the spin glass landscape known to obey ultra-metric topology naturally. This approach is reasonably concrete and relies heavily on the most recent, admittedly still speculative, view about quantum TGD.

6.1 Generalizing The Construction Of WCW Geometry To The P-Adic Context

A problematics analogous to that related with the entanglement between real and p-adic number fields is encountered also in the construction of WCW geometry. The original construction was performed in the real context. What is needed are Kähler geometry and spinor structure for the WCW , and a construction of the WCW spinor fields. What might solve these immense architectural challenges are the equally immense symmetries of WCW and algebraic continuation as the method of p-adicization.

What one can hope that everything of physical interest reduces to the level of algebra (rational or algebraic numbers) and that topology (be it real or p-adic) disappears totally at the level of the matrix elements of the metric and of U-matrix mediating transitions between sectors of WCW corresponding to different number fields. It is not necessary to require this to happen for M-matrix identified as time-like entanglement coefficients between positive and negative energy parts of zero energy states.

6.1.1 The notions of number theoretical universality and number theoretical criticality

An essential question is however what one means with the notions of number theoretical universality and criticality.

1. The weak form of the number theoretical universality means that there are sub- WCW s which can be regarded as real, those which are genuinely p-adic, and those which are algebraic in the sense that the representation of partonic 2-surface, perhaps also 3-surface, and perhaps even space-time surface is in terms of rational/algebraic functions allows the interpretation in terms of both real and p-adic numbers. These surfaces would be like rational and algebraic numbers common for the continua formed by reals and p-adics. This poses conditions on the representations of surfaces and typically rational functions with rational coefficients would represent these surfaces.

   For these surfaces - and only for these- physics should be expressible in terms of algebraic numbers and define as a completion the physics in real and p-adic number fields. This would allow p-adic non-determinism. Book analogy is convenient here: the physics corresponding to various number fields would be like pages of books glued together along rational and algebraic physics. If the transitions between states in different number field taking place via a leakage between different pages of the book are allowed, one can regard the algebraic sectors of the WCW as critical. This number theoretic criticality could be interpreted in terms of intentionality and cognition, and living matter would represent a school example about number theoretically critical phase. For this option it is not at all obvious whether it makes sense to speak about WCW geometry. The construction of WCW spinor structure reducing exponent of Kähler function to determinant is what gives some hopes.

2. A much stronger condition - which I adopted originally - is that all 3-surfaces allow interpretation as as both real and p-adic surfaces: in this case p-adic non-determinism would be excluded. The objection is that this kind of number theoretical universality might reduce to a purely algebraic physics. This condition has interpretation in terms of number theoretical criticality if the weaker notion of universality is adopted.

6.1.2 Generalizing the construction for WCW metric

It is not enough to generalize this construction to the p-adic context. 3-surfaces contain both real and p-adic regions and should be able to perform the construction for this kind of objects.

1. Very naïvely, one could start from the Riemannian construction of the line element which tells the length squared between infinitesimally close points at each point of the Riemann manifold. The notion of line element involves the notion of nearness and one obviously cannot do without topology here. The line element makes formally sense sense for real and p-adic contexts but since p-adic definite integral does not exist, the notions of p-adic length and volume do not exist naturally. Of course, p-adic norm defines very rough measure of distance in number theoretic sense. The notion of line-element is not needed in the quantum theory at WCW level since only the matrix elements of the WCW metric matter.

2. WCW metric can be constructed in terms if Dirac determinant identified as exponent of Kähler function and the formula for matrix elements is expressible in terms of derivatives of logarithms of the eigen values of the Kähler-Dirac operator with respect to complex coordinates of WCW . This means enormous simplification if the number of eigenvalues is finite as implied by finite measurement resolution realized in terms of braids defined by physical conditions. If eigenvalues are algebraic functions of complex coordinates of WCW then also the exponent of Kähler function and WCW covariant metric defining as its inverse as propagator in WCW degrees of freedom are algebraic functions.

I have also proposed a formula for the matrix elements of configuration space metric and Kähler form between the Killing vector fields of isometry generators. Isometries are identified as X^2 local symplectic symmetries. These expressions can be given also in terms of WCW Hamiltonians as "half Poisson brackets" in complex coordinates. Also the construction of quantum states involves WCW Hamiltonians and their super counterparts.

1. The definition of WCW s Hamiltonians involves definite integrals of corresponding complexified Hamiltonians of $(\delta M^4_\pm \times CP_2)^n$ over X^2. Definite integrals are problematic in the p-adic context, as is clear from the fact that in-numerable number of definitions of definite integral have been proposed.

2. Finite measurement resolution would reduce integrals to sums since WCW reduces to $(\delta M^4_\pm \times CP_2)^n/S_n$ for given CD. Furthermore, only the Hamiltonians corresponding to triplet *resp.* octet representations of $SO(3)$ *resp.* $SU(3)$ would be needed to coordinatize $S^2 \times CP_2$ part of the reduced WCW .

3. Without number theoretic braids the definition of these integrals seems really difficult in p-adic context. Residue calculus might give some hopes but One might however hope that one could reduce the construction in the real case to that for the representations of superconformal and symplectic symmetries, and analytically continue the construction from the real context to the p-adic contexts by *defining* the matrix elements of the metric to be what the symmetry respecting analytical continuation gives.

WCW integration should be also continued algebraically to the p-adic context. Quantum criticality realized as the vanishing of loop corrections associated with the WCW integral, would reduce WCW integration to purely algebraic process much like in free field theory and this would give could hopes about p-adicization. Matrix elements would be proportional to the exponent of Kähler function at its maximum plus matrix elements expressible as correlation functions of conformal field theory: the recent state of construction is considered in [K13]. This encourages further the hopes about complete algebraization of the theory so that the independence of the basic formulation on number field could be raised to a principle analogous to general coordinate invariance.

6.1.3 Is the exponential of the Kähler function rational function?

The simplest possibility that one can imagine are that the exponent e^{2K} of Kähler function appearing in WCW inner products is a rational or at most a simple algebraic function existing in a finite-dimensional algebraic extension of p-adic numbers.

The exponent of the CP_2 Kähler function is a rational function of the standard complex coordinates and thus rational-valued for all rational values of complex CP_2 coordinates. Therefore one is lead to ask whether this property holds true quite generally for symmetric spaces and even in the infinite-dimensional context. If so, then the continuation of the vacuum functional to the p-adic sectors of the WCW would be possible in the entire WCW . Also the spherical harmonics of CP_2 are rational functions containing square roots in normalization constants. That also WCW spinor fields could use rational functions containing square roots as normalization constant as basic building blocks would conform with general number theoretical ideas as well as with the general features of harmonic oscillator wave functions.

The most obvious manner to realize this idea relies on the restriction of light-like 3-surfaces X^3_l to those representable in terms of polynomials or rational functions with rational or at most algebraic coefficients serving as natural preferred coordinates of the WCW . This of course requires identification of preferred coordinates also for H. This would lead to a hierarchy of inclusions for sub- WCW s induced by algebraic extensions of rationals.

The presence of cutoffs for the degrees of polynomials involved makes the situation finite-dimensional and give rise to a hierarchy of inclusions also now. These inclusion hierarchies would relate naturally also to hierarchies of inclusions for hyperfinite factors of type II_1 since the spinor spaces associated with these finite-D versions of WCW would be finite-dimensional. Hyper-finiteness means that this kind of cutoff can give arbitrarily precise approximate representation of the infinite-D situation.

This vision is supported by the recent understanding related to the definition of exponent of Kähler function as Dirac determinant [K6]. The number of eigenvalues involved is necessarily finite, and if the eigenvalues of D_K are algebraic numbers for 3-surfaces X^3_l for which the coefficients characterizing the rational functions defining X^3_l are algebraic numbers, the exponent of Kähler function is algebraic number.

The general number theoretical conjectures implied by p-adic physics as physics of cognition also supportthis conjecture. Although one must take these arguments with a big grain of salt, the general idea might be correct. Also the elements of the configuration space metric would be rational functions as is clear from the fact that one can express the second derivatives of the Kähler function in terms of $F = exp(K)$ as

$$\partial_K \partial_{\overline{L}} K = \frac{\partial_K \partial_{\overline{L}} F}{F} - \frac{\partial_K F \partial_{\overline{L}} F}{F^2} \quad . \tag{6.1}$$

An expression of same form but with sum over eigenvalues of the Kähler-Dirac operator with F replaced with eigenvalue results if exponent of Kähler function is expressible as Dirac determinant:

$$\partial_K \partial_{\overline{L}} K = \frac{\partial_K \partial_{\overline{L}} \lambda_k}{\lambda_k} - \frac{\partial_K \lambda \partial_{\overline{L}} \lambda_k}{\lambda_k^2} \quad . \tag{6.2}$$

What is important that this formula in principles relates WCW geometry directly to quantum physics as represented by the Kähler-Dirac operator.

6.1.4 Generalizing the notion of WCW spinor field

One must also construct spinor structure. Also this construction relies crucially super Kac-Moody and super-symplectic symmetries. Spinors at a given point of WCW correspond to the Fock space spanned by fermionic oscillator operators and again one might hope that super-symmetries would allow algebraization of the whole procedure.

The identification of WCW gamma matrices as super Hamiltonians of WCW . The generators of various super-algebras are also needed in order to construction configuration space spinors at given point of WCW . In ideal measurement resolution these algebra elements are expressible as integrals of Hamiltonians and super-Hamiltonians of $\delta M^4_\pm \times CP_2$ and this leads to difficulties in p-adic context. It might be that finite measurement resolution which seems to be coded by the classical dynamics provides the only possible solution of these difficulties. In the case of reduced WCW the construction of orthonormalized based of WCW spinor fields looks a rather reasonable challenge and the continuation of this procedure to p-adic context might make sense.

6.2 WCW Functional Integral

One can make some general statements about WCW functional integral.

1. If only braid points are specified, there is a functional integral over a huge number of 2-surfaces meaning sum of perturbative contributions from very large number of partonic 2-surfaces selected as maxima of Kähler function or by stationary phase approximation. This kind of non-perturbative contribution makes it very difficult to understand what is involved so that it seems that some restrictions must be posed. Also all information about crucial vacuum degeneracy of Kähler action would be lost as a non-local information.

2. Induced Kähler form represents perhaps the most fundamental zero modes since it remains invariant under symplectic transformations acting as isometries of WCW . Therefore it seems natural organize WCW integral in such a way that each choice of the induced Kähler form represents its own quantized theory and functional integral is only over deformations leaving induced Kähler form invariant. The deformations of the partonic 2-surfaces would leave invariant both the induced areas and magnetic fluxes. The symplectic orbits of the partonic 2-surfaces (and 3-surfaces) would therefore define a slicing of WCW with separate quantization for each slice.

3. The functional integral would be over the symplectic group of CP_2 and over M^4 degrees of freedom -perhaps also in this case over the symplectic group of δM^4_+ - a rather well-defined mathematical structure. Symplectic transformations of CP_2 affect only the CP_2 part of the induced metric so that a nice separation of degrees of freedom results and the functional integral can be assigned solely to the gravitational degrees of freedom in accordance with the idea that fundamental quantum fluctuating bosonic degrees of freedom are gravitational.

4. WCW integration around a partonic 2-surface for which the Kähler function is maximum with respect to quantum fluctuating degrees of freedom should give only tree diagrams with propagator factors proportional to g_K^2 if loop corrections to the WCW integral vanish. One could hope that there exist preferred S^2 and CP_2 coordinates such that vertex factors involving finite polynomials of S^2 and CP_2 coordinates reduce to a finite number of diagrams just as in free field theory.

If WCW functional integral algebraizes by the vanishing of loop corrections, one has hopes that even p-adic variant of WCW functional integral might make sense. The exponent of Kähler function appears and if given by the Dirac determinant it would reduce to a finite product of eigenvalues of Kähler-Dirac operator which makes sense also p-adically.

6.2.1 Algebraization of WCW functional integral

WCW is a union of infinite-dimensional symmetric spaces labeled by zero modes. One can hope that the functional integral could be performed perturbatively around the maxima of the Kähler function. In the case of CP_2 Kähler function has only single maximum and is a monotonically decreasing function of the radial variable r of CP_2 and thus defines a Morse function. This suggests that a similar situation is true for all symmetric spaces and this might indeed be the case.

1. The point is that the presence of several maxima implies also saddle points at which the matrix defined by the second derivatives of the Kähler function is not positive definite. If the derivatives of type $\partial_K \partial_L K$ and $\partial_{\overline{K}} \partial_{\overline{L}} K$ vanish at the saddle point (this is the crucial assumption) in some complex coordinates holomorphically related to those in which the same holds true at maximum, the Kähler metric is not positive definite at this point. On the other hand, by symmetric space property the metric should be isometric with the positive define metric at maxima so that a contradiction results.

2. If this argument holds true, for given values of zero modes Kähler function has only one maximum, whose value depends on the values zero modes. Staying in the optimistic mood, one could go on to guess that the Duistermaat-Heckman theorem generalizes and the functional integral is simply the exponent of the Kähler function at the maximum (due to the compensation of Gaussian and metric determinants). Even more, one could bravely guess that for configuration space spinor fields belonging to the representations of symmetries the inner products reduces to the generalization of correlation functions of Gaussian free field theory. Each WCW spinor field would define a vertex from which lines representing the propagators defined by the contravariant WCW metric in isometry basis emanate.

If this optimistic line of reasoning makes sense, the definition of the p-adic WCW integral reduces to a purely algebraic one. What is needed is that the contravariant Kähler metric fixed by the symmetric space-property exists and that the exponent of the maximum of the Kähler function exists for rational values of zero modes or subset of them if finite-dimensional algebraic extension is allowed. This would give could hopes that the U-matrix elements resulting from the WCW integrals would exist also in the p-adic sense.

6.2.2 Should one p-adicize only the reduced configuration space?

An attractive approach to p-adicization might be characterized as minimalism and would involve geometrization of only the reduced WCW consisting of the maxima of Kähler function in quantum fluctuating degrees of freedom. A further reduction results from the finite measurement resolution replacing WCW effectively with $(\delta M^4_\pm \times CP_2)^n/S_n$. In zero modes discretization realizing quantum classical correspondence is attractive possibility.

1. If Duistermaat-Heckman theorem [A5] holds true in TGD context, one could express real WCW functional integral in terms of exactly calculable Gaussian integrals around the maxima of the Kähler function in quantum fluctuating degrees of freedom defining what might be called reduced WCW CH_{red}. The exponent of Kähler function and propagator identified as contravariant metric of WCW could be deduced from the spectrum of the modified Dirac operator.

2. The huge super-conformal symmetries raise the hope that the rest of M-matrix elements could be deduced using group theoretical considerations so that everything would become algebraic. If this optimistic scenario is realized, the p-adicization of CH_{red} might be enough to p-adicize all operations needed to construct the p-adic variant of M-matrix.

3. A possible problem of this reduction is that the number of degrees of freedom in functional integral is still infinite, which might mean problems in terms of algebraization. For instance, the inverse of covariant metric identified as algebraic function need not represent algebraic object. Finite measurement resolution improves the situation in this respect. Finite measurement resolution realized in terms of number theoretic braids would reduce WCW to $(\delta M^4_\pm \times CP_2)^n/S_n$ for given CD and this would reduce the situation to a finite dimensional one and maxima of Kähler function would form a discrete set, possibly only single point of $(\delta M^4_\pm \times CP_2)^n/S_n$. Also in this case exponent of Kähler function and the spectrum of Kähler-Dirac operator are needed. Also the values of $J = \epsilon^{\alpha\beta} J_{\alpha\beta}\sqrt{g_2}$ at the points of number theoretic braids labeled by $\delta M^4_\pm \times CP_2/S_n$ are needed.

Zero modes pose a further problem.

1. The absence of functional integral measure in zero modes would suggest that states depend on finite number of zero modes only and that there is localization in this degrees of freedom. Finite measurement resolution suggests the same. The extrema of the quantity $J = \epsilon^{\alpha\beta}J_{\alpha\beta}\sqrt{g_2}$ at the points of number theoretic represent finite set of values of fundamental zero modes assignable to X^2 forming a finite-dimensional space naturally. Non-local isometry invariants can be defined as Kähler magnetic fluxes if it is possible to define symplectic triangulation of X^2 with vertices identifiable naturally as points of number theoretic braid corresponding to the extrema of J. The notion of symplectic fusion algebra based on this kind of triangulation is discussed in [K12].

2. Kac-Moody group parameterizes zero modes assignable to X_l^3 and a correlation between these zero modes and the quantum numbers of quantum state is natural and could result by stationary phase approximation if finite-dimensional variant of functional integral can be defined. If there is localization in zero modes, this correspondence could be discrete and implied by classical equations of motion for braid points. A unique selection of preferred quantization axis would be made possible by the hierarchy of Planck constants selecting $M^2 \subset M^4$ and $S_i^2 \subset CP_2$ as critical manifolds with respect to the change of Planck constant.

What other difficulties can one imagine?

1. The optimal situation would be that M-matrix elements in real case are algebraic functions or at least functions continuable to the p-adic context in a form having sensible physical interpretation.

2. If one starts directly from Fourier transforms in p-adic context, difficulties are caused by trigonometric functions and exponent function whose p-adic counterparts do not behave in physically acceptable manner. It seems that it is phase factors defined by plane waves which should should restricted to roots of unity and continued to the p-adic realm as such. In p-adic context either momentum or position makes sense as p-adic number unless one introduces infinite-dimensional extension containing logarithms and π. Maybe the only manner to avoid problems is to accept discretization and algebraization of the phase factors.

Concerning number field changing transitions at number theoretical criticality possibly relevant for U-matrix some comments are in order. For real$\leftrightarrow$ p-adic transitions only the algebraic points of number theoretic braid common to both real and p-adic variant of partonic 2-surface are relevant and situation reduces to algebraic braid points in $(\delta M^4_{\pm} \times CP_2)/S_n$. Algebraic points in a given extension of rationals would be common to real and p-adic surfaces. It could happen that there are very few common algebraic points. For instance, Fermat's theorem says that the surface $x^n + y^n = z^n$ has no rational points for $n > 2$. The integral over reduced WCW should reduce to a sum over possible values of coordinates for these points. If only maxima of Kähler function an analytic continuation of real M-matrix to p-adic-real M-matrix could make sense.

If this picture is correct, the p-adicization of WCW would mean p-adicization of CH_{red} consisting of the maxima of the Kähler function with respect to both fiber degrees of freedom and zero modes acting effectively as control parameters of the quantum dynamics. Finite measurement resolution simplifies the situation dramatically. If CH_{red} is a discrete subset of CH or its finite-dimensional variant, ultra-metric topology induced from finite-p p-adic norm is indeed natural for it. "Discrete set in CH" need not mean a discrete set in the usual sense and the reduced WCW could be even finite-dimensional continuum. p-Adicization as a cognitive model would suggest that p-adicization in given point of CH_{red} is possible for all p-adic primes associated with the corresponding space-time surface (maximum of Kähler function) and represents a particular cognitive representation about CH_{red}.

6.3 Number Theoretic Constraints On M-Matrix

Assume that U-matrix assignable to quantum jump between zero energy states exists simultaneously in all number fields and perhaps even between different number fields at number theoretical quantum criticality (allowing finite-dimensional extensions of p-adics). If so the immediate question is whether also the construction procedure of the M-matrix defined as time-like entanglement coefficients between positive and negative energy parts of zero energy state could have a p-adic counterpart for each p, and whether the mere requirement that this is the case could provide non-trivial intuitions about the general structure of the theory. The identification of M-matrices as building blocks of U-matrix in the manner discussed in [K13] supports affirmative answer to the first question. Not only the WCW but also Kähler function and its exponent, Kähler metric, and WCW functional integral should have p-adic variants. In the following this challenge is discussed in a rather optimistic number theoretic mood using the ideas stimulated by the connections between number theory and cognition.

6.3.1 Number theoretical Universality and M-matrix

Number theoretic constraints on M-matrix are non-trivial even for the weaker notion of number theoretical universality. Number theoretical criticality (or number theoretical universality in strong sense) requires that M-matrix elements are algebraic numbers. This is achieved naturally if the definition of M-matrix elements involves only the data associated with the number theoretic braid. Note that this data is non-local since it involves information about tangent space of X^4 at the point so that discretization happens in geometric sense but not in information theoretic sense. Note also that for algebraic surfaces finite number of points of surface allows to deduce the parameters of the polynomials involved and thus to deduce the entire surface.

If quantum version of WCW is adopted one must perform quantization for $E^2 \subset M^4$ coordinates of points S_i^2 braid and CP_2 coordinates of M^2 braid. In this kind of situation it becomes unclear whether one can speak about braiding anymore. This might make sense if each braid topology corresponds to its own quantization containing information about the fact that deformations of X_l^3 respect the braiding topology.

The partonic vertices appearing in M-matrix elements should be expressible in terms of N-point functions of some rational super-conformal field theory but with the p-adically questionable N-fold integrals over string appearing in the definition of n-point functions. The most elegant manner to proceed is to replace them with their explicit expressions if they are algebraic functions- quite generally or at number theoretical criticality. Spin chain type string discretization is an alternative, not so elegant option.

Propagators, that is correlations between partonic 2-surfaces, would be due to the interior dynamics of space-time sheets which means a deviation from super string theory. Another function of interior degrees of freedom is to provide zero modes of metric of WCW identifiable as classical degrees of freedom of quantum measurement theory entangling with quantal degrees of freedom at partonic 3-surfaces.

6.3.2 Number theoretical criticality and M-matrix

Number theoretical criticality poses very strong conditions on the theory.

1. The p-adic variants of 4-D field equations associated with Kähler action make sense. Also the notion of preferred extremal makes sense in p-adic context if it corresponds to quantum criticality in the sense that second variation of Kähler action vanishes for dynamical symmetries. A natural further condition is that the surface is representable in terms of algebraic equations involving only rational or algebraic coefficients and thus making sense both in real and p-adic sense. In this case also Kähler action and classical charges could exist in some algebraic extension of p-adic numbers.

2. Also Kähler-Dirac equation makes sense p-adically. The exponent of Kähler function defining vacuum functional is well-defined notion p-adically if the identification as product of finite number of eigenvalues of the Kähler-Dirac operator is accepted and eigenvalues are algebraic. Also the notion of WCW metric expressible in terms of derivatives of the eigenvalues with respect to complex coordinates of WCW makes sense.

3. The functional integral over WCW can be defined only as an algebraic extension of real functional integral around maximum of Kähler function if the theory is integrable and gives

as a result an algebraic number. One might hope that algebraic p-adicization makes sense for the vacuum function at points corresponding to the maxima of Kähler function with respect to quantum fluctuating degrees of freedom (assuming they exist) and with respect to zero modes. As discussed already earlier, in the case of zero modes quantum classical correspondence allows to select preferred value of zero modes even if functional integral in zero modes does not make sense. The basic requirement is that the inverse of the matrix defined by the Kähler metric defining propagator is algebraic function of the complex coordinate of WCW . If the eigen-values of the modified Dirac operator satisfy this condition this is indeed the case.

4. Ordinary perturbation series based on Feynman diagrams makes sense also in p-adic sense since the presence of cutoff for the size of CD implies that the number of terms if finite. One must be however cautious with momentum integrations which should reduce to finite sum due to the presence of both IR and UV cutoff implied by the finite size of CD. The formulation in terms of number theoretic braids whose intersections with partonic 2-surfaces consist of finite number of points supports the possibility of number theoretic universality.

There are hopes that M-matrix make sense p-adically. As far M-matrix is considered, The most plausible interpretation relies on the weaker form of number theoretic universality so that genuinely p-adic M-matrices should exist.

1. Dirac determinant exists for any p-adic 3-surfaces since the eigenvalues of Kähler-Dirac operator represent a purely local notion sensible also in p-adic context. The reason is that finite measurement resolution - now deducible from the vacuum degeneracy of Kähler action- implies that the number of eigenvalues is finite. Preferred extremals of Kähler action obey quantum criticality condition meaning that the second variation of Kähler action vanishes. This condition makes sense also p-adically.

2. If loops vanish, WCW integration gives only contractions with propagator expressible as the contravariant WCW Kähler metric expressible in terms of derivatives of the Kähler function with respect to the preferred complex coordinates of WCW . If this function is algebraic function, it allows algebraic continuation to p-adic context and all that is needed for calculation of M-matrix elements makes sense p-adically. The crucial question is whether the Kähler metric is algebraic function in preferred coordinates.

3. N-point functions involve also symplectically invariant multiplicative factors discussed in [K12] in terms of symplectic fusion algebras. For them algebraic universality holds true. N-point functions of conformal field theory associated with the generalized vertices should also be algebraic functions.

4. Finite measurement resolution realized in terms of braids for given $J = \epsilon^{\alpha\beta} J_{\alpha\beta}$ means a reduction of a given sector of WCW in quantum fluctuating degrees of freedom to finite-dimensional space $\delta M^4_\pm \times CP_2/S_n$ associated with the boundaries of CD. For instance, configuration space Hamiltonians reduce apart from J factor to those assignable naturally to the reduced WCW . Finite-dimensionality gives hopes of algebraic continuation of M-matrix defined in terms of general Feynman diagrams in real context using finite purely algebraic operations due to the cutoff in the size of CDs. In zero modes the simplest option would be that quantum states correspond to sums over different values of zero modes, in particular J as function in X^2.

Also number theoretical criticality is consistent with this picture.

1. If partonic 2-surface X^2 is determined by algebraic equations involving only rational coefficients, same equations define real and p-adic variants of X^2.

2. Number theoretic criticality for braids means that their points are algebraic and common to real and p-adic partonic 2-surfaces. The extrema of J -determined by algebraic conditions- must be algebraic numbers.

3. At quantum criticality Dirac determinant is algebraic number if the number of eigenvalues is finite (implied by finite measurement resolution) and if they are algebraic numbers. If the p-adic counterpart of X_l^3 exists, this allows to assign to the p-adic counterpart of X_l^3 the exponent of Kähler function as Dirac determinant although Kähler action remains ill-defined p-adically.

6.3.3 The relationship between U-matrix and M-matrix

The following represents the latest result concerning the relationship between the notions of U-matrix and M-matrix and probably provides answer to some of the questions posed in the chapter. What is highly satisfactory that U-matrix dictates M-matrix completely via unitarity conditions. A more detailed discussion can be [K5] discussing Negentropy Maximization Principle, which is the basic dynamical principle of TGD inspired theory of consciousness and states that the information content of conscious experience is maximal.

If state function reduction associated with time-like entanglement leads always to a product of positive and negative energy states (so that there is no counterpart of bound state entanglement and negentropic entanglement possible for zero energy states: these notions are discussed below) U-matrix and can be regarded as a collection of M-matrices

$$U_{m_+n_-,r_+,s_-} = M(m_+,n_-)_{r_+,s_-} \tag{6.3}$$

labeled by the pairs (m_+,n_-) labelling zero energy states assumed to reduced to pairs of positive and negative energy states. M-matrix element is the counterpart of S-matrix element $S_{r,s}$ in positive energy ontology. Unitarity conditions for U-matrix read as

$$\begin{aligned}
(UU^\dagger)_{m_+n_-,r_+s_-} &= \sum_{k_+,l_-} M(m_+,n_-)_{k_+,l_-}\overline{M}(r_+,s_-)_{k_+,l_-} = \delta_{m_+r_+,n_-s_-} \ , \\
(U^\dagger U)_{m_+n_-,r_+s_-} &= \sum_{k_+,l_-} \overline{M}(k_+,l_-)_{m_+,n_-}M(k_+,l_-)_{r_+,s_-} = \delta_{m_+r_+,n_-s_-} \ .
\end{aligned} \tag{6.4}$$

The conditions state that the zero energy states associated with different labels are orthogonal as zero energy states and also that the zero energy states defined by the dual M-matrix

$$M^\dagger(m_+,n_-)_{k_+,l_-} \equiv \overline{M}(k_+l_-)_{m_+,n_-} \tag{6.5}$$

-perhaps identifiable as phase conjugate states- define an orthonormal basis of zero energy states.

When time-like binding and negentropic entanglement are allowed also zero energy states with a label not implying a decomposition to a product state are involved with the unitarity condition but this does not affect the situation dramatically. As a matter fact, the situation is mathematically the same as for ordinary S-matrix in the presence of bound states. Here time-like bound states are analogous to space-like bound states and by definition are unable to decay to product states (free states). Negentropic entanglement makes sense only for entanglement probabilities, which are rationals or belong to their algebraic extensions. This is possible in what might be called the intersection of real and p-adic worlds (partonic surfaces in question have representation making sense for both real and p-adic numbers). Number theoretic entropy is obtained by replacing in the Shannon entropy the logarithms of probabilities with the logarithms of their p-adic norms. They satisfy the same defining conditions as ordinary Shannon entropy but can be also negative. One can always find prime p for which the entropy is maximally negative. The interpretation of negentropic entanglement is in terms of formations of rule or association. Schrödinger cat knows that it is better to not open the bottle: open bottle-dead cat, closed bottle-living cat and negentropic entanglement measures this information.

7 How To Define Generalized Feynman Diagrams?

S-matrix codes to a high degree the predictions of quantum theories. The longstanding challenge of TGD has been to construct or at least demonstrate the mathematical existence of S-matrix- or actually M-matrix which generalizes this notion in ZEO (ZEO) [K8] . This work has led to the notion of generalized Feynman diagram and the challenge is to give a precise mathematical meaning for this object. The attempt to understand the counterpart of twistors in TGD framework [K24] has inspired several key ideas in this respect but it turned out that twistors themselves need not be absolutely necessary in TGD framework.

1. The notion of generalized Feyman diagram defined by replacing lines of ordinary Feynman diagram with light-like 3-surfaces (elementary particle sized wormhole contacts with throats carrying quantum numbers) and vertices identified as their 2-D ends - I call them partonic 2-surfaces is central. Speaking somewhat loosely, generalized Feynman diagrams (plus background space-time sheets) define the "world of classical worlds" (WCW). These diagrams involve the analogs of stringy diagrams but the interpretation is different: the analogs of stringy loop diagrams have interpretation in terms of particle propagating via two different routes simultaneously (as in the classical double slit experiment) rather than as a decay of particle to two particles. For stringy diagrams the counterparts of vertices are singular as manifolds whereas the entire diagrams are smooth. For generalized Feynman diagrams vertices are smooth but entire diagrams represent singular manifolds just like ordinary Feynman diagrams do. String like objects however emerge in TGD and even ordinary elementary particles are predicted to be magnetic flux tubes of length of order weak gauge boson Compton length with monopoles at their ends as shown in accompanying article. This stringy character should become visible at LHC energies.

2. ZEO (ZEO) and causal diamonds (intersections of future and past directed light-cones) define second key ingredient. The crucial observation is that in ZEO it is possible to identify off mass shell particles as pairs of on mass shell fermions at throats of wormhole contact since both positive and negative signs of energy are possible and one obtains also space-like total momenta for wormhole contact behaving as a boson. The localization of fermions to string world sheets and the fact that super-conformal generator G carries fermion number combined with twistorial consideration support the view that the propagators at fermionic lines are of form $(1/G)ip^k\gamma_k(1/G^\dagger + h.c.$ and thus hermitian. In strong models $1/G$ would serve as a propagator and this requires Majorana condition fixing the dimension of the target space to 10 or 11.

3. A powerful constraint is number theoretic universality requiring the existence of Feynman amplitudes in all number fields when one allows suitable algebraic extensions: roots of unity are certainly required in order to realize p-adic counterparts of plane waves. Also embedding space, partonic 2-surfaces and WCW must exist in all number fields and their extensions. These constraints are enormously powerful and the attempts to realize this vision have dominated quantum TGD for last two decades.

4. Representation of 8-D gamma matrices in terms of octonionic units and 2-D sigma matrices is a further important element as far as twistors are considered [K24] . Kähler-Dirac gamma matrices at space-time surfaces are quaternionic/associative and allow a genuine matrix representation. As a matter fact, TGD and WCW could be formulated as study of associative local sub-algebras of the local Clifford algebra of 8-D embedding space parameterized by quaternionic space-time surfaces.

5. A central conjecture has been that associative (co-associative) 4-surfaces correspond to preferred extremals of Kähler action [K6]. It took long time to realize that in ZEO the notion of preferred extremal might be un-necessary! The reason is that 3-surfaces are now pairs of 3-surfaces at boundaries of causal diamonds and for deterministic dynamics the space-time surface connecting them is expected to be more or less unique. Now the action principle is non-deterministic but the non-determinism would give rise to additional discrete dynamical degrees of freedom naturally assignable to the hierarchy of Planck constants $h_{eff} = n \times h$, n the number of space-time surface with same fixed ends at boundaries of CD and with same

values of Kähler action and of conserved quantities. One must be however cautions: this leaves the possibility that there is a gauge symmetry present so that the n sheets correspond to gauge equivalence classes of sheets. Conformal invariance is associated with criticality and is expected to be present also now.

One can of course also ask whether one can assume that the pairs of 3-surfaces at the ends of CD are totally un-correlated. If this assumption is not made then preferred extremal property would make sense also in ZEO and imply additional correlation between the members of these pairs. This kind of correlations would correspond to the Bohr orbit property, which is very attractive space-time correlate for quantum states. This kind of correlates are also expected as space-time counterpart for the correlations between initial and final state in quantum dynamics.

6. A further conjecture has been that preferred extremals are in some sense critical (second variation of Kähler action could vanish for infinite number of deformations defining a super-conformal algebra). The non-determinism of Kähler action implies this property for $n > 0$ in $h_{eff} = nh$. If the criticality is present, it could correspond to conformal gauge invariance defined by sub-algebras of conformal algebra with conformal weights coming as multiples of n and isomorphic to the conformal algebra itself.

7. As far as twistors are considered, the first key element is the reduction of the octonionic twistor structure to quaternionic one at space-time surfaces and giving effectively 4-D spinor and twistor structure for quaternionic surfaces.

Quite recently quite a dramatic progress took place in this approach [K6, K24] .

1. The progress was stimulated by the simple observation that on mass shell property puts enormously strong kinematic restrictions on the loop integrations. With mild restrictions on the number of parallel fermion lines appearing in vertices (there can be several since fermionic oscillator operator algebra defining SUSY algebra generates the parton states)- all loops are manifestly finite and if particles has always mass -say small p-adic thermal mass also in case of massless particles and due to IR cutoff due to the presence largest CD- the number of diagrams is finite. Unitarity reduces to Cutkosky rules [B3] automatically satisfied as in the case of ordinary Feynman diagrams.

2. Ironically, twistors which stimulated all these development do not seem to be absolutely necessary in this approach although they are of course possible. Situation changes if one does not assume small p-adically thermal mass due to the presence of massless particles and one must sum infinite number of diagrams. Here a potential problem is whether the infinite sum respects the algebraic extension in question.

This is about fermionic and momentum space aspects of Feynman diagrams but not yet about the functional (not path-) integral over small deformations of the partonic 2-surfaces. The basic challenges are following.

1. One should perform the functional integral over WCW degrees of freedom for fixed values of on mass shell momenta appearing in the internal lines. After this one must perform integral or summation over loop momenta. Note that the order is important since the space-time surface assigned to the line carries information about the quantum numbers associated with the line by quantum classical correspondence realized in terms of Kähler-Dirac operator.

2. One must define the functional integral also in the p-adic context. p-Adic Fourier analysis relying on algebraic continuation raises hopes in this respect. p-Adicity suggests strongly that the loop momenta are discretized and ZEO predicts this kind of discretization naturally.

It indeed seems that the functional integrals over WCW could be carried out at general level both in real and p-adic context. This is due to the symmetric space property (maximal number of isometries) of WCW required by the mere mathematical existence of Kähler geometry [K3] in infinite-dimensional context already in the case of much simpler loop spaces [A4] .

1. The p-adic generalization of Fourier analysis allows to algebraize integration- the horrible looking technical challenge of p-adic physics- for symmetric spaces for functions allowing the analog of discrete Fourier decomposition. Symmetric space property is indeed essential also for the existence of Kähler geometry for infinite-D spaces as was learned already from the case of loop spaces. Plane waves and exponential functions expressible as roots of unity and powers of p multiplied by the direct analogs of corresponding exponent functions are the basic building bricks and key functions in harmonic analysis in symmetric spaces. The physically unavoidable finite measurement resolution corresponds to algebraically unavoidable finite algebraic dimension of algebraic extension of p-adics (at least some roots of unity are needed). The cutoff in roots of unity is very reminiscent to that occurring for the representations of quantum groups and is certainly very closely related to these as also to the inclusions of hyper-finite factors of type II_1 defining the finite measurement resolution.

2. WCW geometrization reduces to that for a single line of the generalized Feynman diagram defining the basic building brick for WCW. Kähler function decomposes to a sum of "kinetic" terms associated with its ends and interaction term associated with the line itself. p-Adicization boils down to the condition that Kähler function, matrix elements of Kähler form, WCW Hamiltonians and their super counterparts, are rational functions of complex WCW coordinates just as they are for those symmetric spaces that I know of. This would allow a continuation to p-adic context.

In the following this vision about generalized Feynman diagrams is discussed in more detail.

7.1 Questions

The goal is a proposal for how to perform the integral over WCW for generalized Feynman digrams and the best manner to proceed to this goal is by making questions.

7.1.1 What does finite measurement resolution mean?

The first question is what finite measurement resolution means.

1. One expects that the algebraic continuation makes sense only for a finite measurement resolution in which case one obtains only finite sums of what one might hope to be algebraic functions. The finiteness of the algebraic extension would be in fact equivalent with the finite measurement resolution.

2. Finite measurement resolution means a discretization in terms of number theoretic braids. p-Adicization condition suggests that one must allow only the number theoretic braids. For these the ends of braid at boundary of CD are algebraic points of the embedding space. This would be true at least in the intersection of real and p-adic worlds.

3. The question is whether one can localize the points of the braid. The necessity to use momentum eigenstates to achieve quantum classical correspondence in the Kähler-Dirac action [K6] suggests however a de-localization of braid points, that is wave function in space of braid points. In real context one could allow all possible choices for braid points but in p-adic context only algebraic points are possible if one wants to replace integrals with sums. This implies finite measurement resolution analogous to that in lattice. This is also the only possibility in the intersection of real and p-adic worlds.

   A non-trivial prediction giving a strong correlation between the geometry of the partonic 2-surface and quantum numbers is that the total number $n_F + n_{\overline{F}}$ of fermions and anti-fermions is bounded above by the number n_{alg} of algebraic points for a given partonic 2-surface: $n_F + n_{\overline{F}} \leq n_{alg}$. Outside the intersection of real and p-adic worlds the problematic aspect of this definition is that small deformations of the partonic 2-surface can radically change the number of algebraic points unless one assumes that the finite measurement resolution means restriction of WCW to a sub-space of algebraic partonic surfaces.

4. Braids defining propagator lines for fundamental fermions (to be distinguished from observer particles) emerges naturally. Braid strands correspond to the boundaries of string world

sheets at which the modes of induced spinor fields are localized from the condition that em charge is well-defined: induced W field and above weak scale also Z^0 field vanish at them.

In order to obtain non-trivial fermion propagator one must add to Dirac action 1-D Dirac action in induced metric with the boundaries of string world sheets at the light-like parton orbits. Its bosonic counterpart is line-length in induced metric. Field equations imply that the boundaries are light-like geodesics and fermion has light-like 8-momentum. This suggests strongly a connection with quantum field theory and an 8-D generalization of twistor Grassmannian approach. By field equations the bosonic part of this action does not contribute to the Kähler action. The light-like 8-momenta p^k have same M^4 and CP_2 mass squared and latter correspond to the the eigenvalues of the CP_2 spinor d'Alembertian by quantum-classical correspondence.

5. One has also discretization of the relative position of the second tip of CD at the hyperboloid isometric with mass shell. Only the number of braid points and their momenta would matter, not their positions.

6. The quantum numbers characterizing positive and negative energy parts of zero energy states couple directly to space-time geometry via the measurement interaction terms in Kähler action expressing the equality of classical conserved charges in Cartan algebra with their quantal counterparts for space-time surfaces in quantum superposition. This makes sense if classical charges parametrize zero modes. The localization in zero modes in state function reduction would be the WCW counterpart of state function collapse.

7.1.2 How to define integration in WCW degrees of freedom?

The basic question is how to define the integration over WCW degrees of freedom.

1. What comes mind first is Gaussian perturbation theory around the maxima of Kähler function. Gaussian and metric determinants cancel each other and only algebraic expressions remain. Finiteness is not a problem since the Kähler function is non-local functional of 3-surface so that no local interaction vertices are present. One should however assume the vanishing of loops required also by algebraic universality and this assumption look unrealistic when one considers more general functional integrals than that of vacuum functional since free field theory is not in question. The construction of the inverse of the WCW metric defining the propagator is also a very difficult challenge. Duistermaat-Hecke theorem states that something like this known as localization might be possible and one can also argue that something analogous to localization results from a generalization of mean value theorem.

2. Symmetric space property is more promising since it might reduce the integrations to group theory using the generalization of Fourier analysis for group representations so that there would be no need for perturbation theory in the proposed sense. In finite measurement resolution the symmetric spaces involved would be finite-dimensional. Symmetric space structure of WCW could also allow to define p-adic integration in terms of p-adic Fourier analysis for symmetric spaces. Essentially algebraic continuation of the integration from the real case would be in question with additional constraints coming from the fact that only phase factors corresponding to finite algebraic extensions of rationals are used. Cutoff would emerge automatically from the cutoff for the dimension of the algebraic extension.

7.1.3 How to define generalized Feynman diagrams?

Integration in symmetric spaces could serve as a model at the level of WCW and allow both the understanding of WCW integration and p-adicization as algebraic continuation. In order to get a more realistic view about the problem one must define more precisely what the calculation of the generalized Feynman diagrams means.

1. WCW integration must be carried out separately for all values of the momenta associated with the internal lines. The reason is that the spectrum of eigenvalues λ_i of the Kähler-Dirac operator D depends on the momentum of line and momentum conservation in vertices translates to a correlation of the spectra of D at internal lines.

2. For tree diagrams algebraic continuation to the p-adic context if the expression involves only the replacement of the generalized eigenvalues of D as functions of momenta with their p-adic counterparts besides vertices. If these functions are algebraically universal and expressible in terms of harmonics of symmetric space , there should be no problems.

3. If loops are involved, one must integrate/sum over loop momenta. In p-adic context difficulties are encountered if the spectrum of the momenta is continuous. The integration over on mass shell loop momenta is analogous to the integration over sub-CDs, which suggests that internal line corresponds to a $sub - CD$ in which it is at rest. There are excellent reasons to believe that the moduli space for the positions of the upper tip is a discrete subset of hyperboloid of future light-cone. If this is the case, the loop integration indeed reduces to a sum over discrete positions of the tip. p-Adizication would thus give a further good reason why for ZEO.

4. Propagator is expressible in terms of the inverse of generalized eigenvalue and there is a sum over these for each propagator line. At vertices one has products of WCW harmonics assignable to the incoming lines. The product must have vanishing quantum numbers associated with the phase angle variables of WCW. Non-trivial quantum numbers of the WCW harmonic correspond to WCW quantum numbers assignable to excitations of ordinary elementary particles. WCW harmonics are products of functions depending on the "radial" coordinates and phase factors and the integral over the angles leaves the product of the first ones analogous to Legendre polynomials $P_{l,m}$, These functions are expected to be rational functions or at least algebraic functions involving only square roots.

5. In ordinary QFT incoming and outgoing lines correspond to propagator poles. In the recent case this would mean that incoming stringy lines at the ends of CD correspond to fermions satisfying the stringy mass formula serving as a generalization of masslessness condition.

7.2 Generalized Feynman Diagrams At Fermionic And Momentum SpaceLevel

Negative energy ontology has already led to the idea of interpreting the virtual particles as pairs of positive and negative energy wormhole throats. Hitherto I have taken it as granted that ordinary Feynman diagrammatics generalizes more or less as such. It is however far from clear what really happens in the verties of the generalized Feynman diagrams. The safest approach relies on the requirement that unitarity realized in terms of Cutkosky rules in ordinary Feynman diagrammatics allows a generalization. This requires loop diagrams. In particular, photon-photon scattering can take place only via a fermionic square loop so that it seems that loops must be present at least in the topological sense.

One must be however ready for the possibility that something unexpectedly simple might emerge. For instance, the vision about algebraic physics allows naturally only finite sums for diagrams and does not favor infinite perturbative expansions. Hence the true believer on algebraic physics might dream about finite number of diagrams for a given reaction type. For simplicity generalized Feynman diagrams without the complications brought by the magnetic confinement since by the previous arguments the generalization need not bring in anything essentially new.

The basic idea of duality in early hadronic models was that the lines of the dual diagram representing particles are only re-arranged in the vertices. This however does not allow to get rid of off mass shell momenta. ZEO encourages to consider a stronger form of this principle in the sense that the virtual momenta of particles could correspond to pairs of on mass shell momenta of particles. If also interacting fermions are pairs of positive and negative energy throats in the interaction region the idea about reducing the construction of Feynman diagrams to some kind of lego rules might work.

7.2.1 Virtual particles as pairs of on mass shell particles in ZEO

The first thing is to try to define more precisely what generalized Feynman diagrams are. The direct generalization of Feynman diagrams implies that both wormhole throats and wormhole contacts join at vertices.

1. A simple intuitive picture about what happens is provided by diagrams obtained by replacing the points of Feynman diagrams (wormhole contacts) with short lines and imagining that the throats correspond to the ends of the line. At vertices where the lines meet the incoming on mass shell quantum numbers would sum up to zero. This approach leads to a straightforward generalization of Feynman diagrams with virtual particles replaced with pairs of on mass shell throat states of type $++$, $--$, and $+-$. Incoming lines correspond to $++$ type lines and outgoing ones to $--$ type lines. The first two line pairs allow only time like net momenta whereas $+-$ line pairs allow also space-like virtual momenta. The sign assigned to a given throat is dictated by the sign of the on mass shell momentum on the line. The condition that Cutkosky rules generalize as such requires $++$ and $--$ type virtual lines since the cut of the diagram in Cutkosky rules corresponds to on mass shell outgoing or incoming states and must therefore correspond to $++$ or $--$ type lines.

2. The basic difference as compared to the ordinary Feynman diagrammatics is that loop integrals are integrals over mass shell momenta and that all throats carry on mass shell momenta. In each vertex of the loop mass incoming on mass shell momenta must sum up to on mass shell momentum. These constraints improve the behavior of loop integrals dramatically and give excellent hopes about finiteness. It does not however seem that only a finite number of diagrams contribute to the scattering amplitude besides tree diagrams. The point is that if a the reactions $N_1 \rightarrow N_2$ and $N_2 \rightarrow N_3$,, where N_i denote particle numbers, are possible in a common kinematical region for N_2-particle states then also the diagrams $N_1 \rightarrow N_2 \rightarrow N_2 \rightarrow N_3$ are possible. The virtual states N_2 include all all states in the intersection of kinematically allow regions for $N_1 \rightarrow N_2$ and $N_2 \rightarrow N_3$. Hence the dream about finite number possible diagrams is not fulfilled if one allows massless particles. If all particles are massive then the particle number N_2 for given N_1 is limited from above and the dream is realized.

3. For instance, loops are not possible in the massless case or are highly singular (bringing in mind twistor diagrams) since the conservation laws at vertices imply that the momenta are parallel. In the massive case and allowing mass spectrum the situation is not so simple. As a first example one can consider a loop with three vertices and thus three internal lines. Three on mass shell conditions are present so that the four-momentum can vary in 1-D subspace only. For a loop involving four vertices there are four internal lines and four mass shell conditions so that loop integrals would reduce to discrete sums. Loops involving more than four vertices are expected to be impossible.

4. The proposed replacement of the elementary fermions with bound states of elementary fermions and monopoles $X_\pm$ brings in the analog of stringy diagrammatics. The 2-particle wave functions in the momentum degrees of freedom of fermion and $X_\pm$ might allow more flexibility and allow more loops. Note however that there are excellent hopes about the finiteness of the theory also in this case.

7.2.2 Loop integrals are manifestly finite

One can make also more detailed observations about loops.

1. The simplest situation is obtained if only 3-vertices are allowed. In this case conservation of momentum however allows only collinear momenta although the signs of energy need not be the same. Particle creation and annihilation is possible and momentum exchange is possible but is always light-like in the massless case. The scattering matrices of supersymmetric YM theories would suggest something less trivial and this raises the question whether something is missing. Magnetic monopoles are an essential element of also these theories as also massivation and symmetry breaking and this encourages to think that the formation of massive states as fermion $X_\pm$ pairs is needed. Of course, in TGD framework one has also high mass excitations of the massless states making the scattering matrix non-trivial.

2. In YM theories on mass shell lines would be singular. In TGD framework this is not the case since the propagator is defined as the inverse of the 3-D dimensional reduction of the Kähler-Dirac operator D containing also coupling to four-momentum (this is required by quantum classical correspondence and guarantees stringy propagators),

$$\begin{aligned} D &= i\hat{\Gamma}^{\alpha}p_{\alpha} + \hat{\Gamma}^{\alpha}D_{\alpha} \ , \\ p_{\alpha} &= p_k\partial_{\alpha}h^k \ . \end{aligned} \tag{7.1}$$

The propagator does not diverge for on mass shell massless momenta and the propagator lines are well-defined. This is of course of essential importance also in general case. Only for the incoming lines one can consider the possibility that 3-D Dirac operator annihilates the induced spinor fields. All lines correspond to generalized eigenstates of the propagator in the sense that one has $D_3\Psi = \lambda\gamma\Psi$, where γ is Kähler-Dirac gamma matrix in the direction of the stringy coordinate emanating from light-like surface and D_3 is the 3-dimensional dimensional reduction of the 4-D Kähler-Dirac operator. The eigenvalue λ is analogous to energy. Note that the eigenvalue spectrum depends on 4-momentum as a parameter.

3. Massless incoming momenta can decay to massless momenta with both signs of energy. The integration measure $d^2k/2E$ reduces to dx/x where $x \geq 0$ is the scaling factor of massless momentum. Only light-like momentum exchanges are however possible and scattering matrix is essentially trivial. The loop integrals are finite apart from the possible delicacies related to poles since the loop integrands for given massless wormhole contact are proportional to dx/x^3 for large values of x.

4. Irrespective of whether the particles are massless or not, the divergences are obtained only if one allows too high vertices as self energy loops for which the number of momentum degrees of freedom is $3N-4$ for N-vertex. The construction of SUSY limit of TGD in [K16] led to the conclusion that the parallelly propagating N fermions for given wormhole throat correspond to a product of N fermion propagators with same four-momentum so that for fermions and ordinary bosons one has the standard behavior but for $N > 2$ non-standard so that these excitations are not seen as ordinary particles. Higher vertices are finite only if the total number N_F of fermions propagating in the loop satisfies $N_F > 3N - 4$. For instance, a 4-vertex from which $N = 2$ states emanate is finite.

7.2.3 Taking into account magnetic confinement

What has been said above is not quite enough. The weak form of electric-magnetic duality [B2] leads to the picture about elementary particles as pairs of magnetic monopoles inspiring the notions of weak confinement based on magnetic monopole force. Also color confinement would have magnetic counterpart. This means that elementary particles would behave like string like objects in weak boson length scale. Therefore one must also consider the stringy case with wormhole throats replaced with fermion-$X_{\pm}$ pairs ($X_{\pm}$ is electromagnetically neutral and $\pm$ refers to the sign of the weak isospin opposite to that of fermion) and their super partners.

1. The simplest assumption in the stringy case is that fermion-$X_{\pm}$ pairs behave as coherent objects, that is scatter elastically. In more general case only their higher excitations identifiable in terms of stringy degrees of freedom would be created in vertices. The massivation of these states makes possible non-collinear vertices. An open question is how the massivation fermion-$X_{\pm}$ pairs relates to the existing TGD based description of massivation in terms of Higgs mechanism and Kähler-Dirac operator.

2. Mass renormalization could come from self energy loops with negative energy lines as also vertex normalization. By very general arguments supersymmetry implies the cancellation of the self energy loops but would allow non-trivial vertex renormalization [K16] .

3. If only 3-vertices are allowed, the loops containing only positive energy lines are possible if on mass shell fermion-$X_{\pm}$ pair (or its superpartner) can decay to a pair of positive energy pair particles of same kind. Whether this is possible depends on the masses involved. For ordinary particles these decays are not kinematically possible below intermediate boson mass scale (the decays $F_1 \rightarrow F_2 + \gamma$ are forbidden kinematically or by the absence of flavor changing neutral currents whereas intermediate gauge bosons can decay to on mass shell fermion-anti-fermion pair).

4. The introduction of IR cutoff for 3-momentum in the rest system associated with the largest CD (causal diamond) looks natural as scale parameter of coupling constant evolution and p-adic length scale hypothesis favors the inverse of the size scale of CD coming in powers of two. This parameter would define the momentum resolution as a discrete parameter of the p-adic coupling constant evolution. This scale does not have any counterpart in standard physics. For electron, d quark, and u quark the proper time distance between the tips of CD corresponds to frequency of 10 Hz, 1280 Hz, and 160 Hz: all these frequencies define fundamental bio-rhythms [K2] .

These considerations have left completely untouched one important aspect of generalized Feynman diagrams: the necessity to perform a functional integral over the deformations of the partonic 2-surfaces at the ends of the lines- that is integration over WCW. Number theoretical universality requires that WCW and these integrals make sense also p-adically and in the following these aspects of generalized Feynman diagrams are discussed.

7.3 Harmonic Analysis In WCW As a way To Calculate WCWFunctional Integrals

Previous examples suggest that symmetric space property, Kähler and symplectic structure and the use of symplectic coordinates consisting of canonically conjugate pairs of phase angles and corresponding "radial" coordinates are essential for WCW integration and p-adicization. Kähler function, the components of the metric, and therefore also metric determinant and Kähler function depend on the "radial" coordinates only and the possible generalization involves the identification the counterparts of the "radial" coordinates in the case of WCW.

7.3.1 Conditions guaranteeing the reduction to harmonic analysis

The basic idea is that harmonic analysis in symmetric space allows to calculate the functional integral over WCW.

1. Each propagator line corresponds to a symmetric space defined as a coset space G/H of the symplectic group and Kac-Moody group and one might hope that the proposed p-adicization works for it- at least when one considers the hierarchy of measurement resolutions forced by the finiteness of algebraic extensions. This coset space is as a manifold Cartesian product $(G/H) \times (G/H)$ of symmetric spaces G/H associated with ends of the line. Kähler metric contains also an interaction term between the factors of the Cartesian product so that Kähler function can be said to reduce to a sum of "kinetic" terms and interaction term.

2. Effective 2-dimensionality and ZEO allow to treat the ends of the propagator line independently. This means an enormous simplification. Each line contributes besides propagator a piece to the exponent of Kähler action identifiable as interaction term in action and depending on the propagator momentum. This contribution should be expressible in terms of generalized spherical harmonics. Essentially a sum over the products of pairs of harmonics associated with the ends of the line multiplied by coefficients analogous to $1/(p^2 - m^2)$ in the case of the ordinary propagator would be in question. The optimal situation is that the pairs are harmonics and their conjugates appear so that one has invariance under G analogous to momentum conservation for the lines of ordinary Feynman diagrams.

3. Momentum conservation correlates the eigenvalue spectra of the Kähler-Dirac operator D at propagator lines [K6] . G-invariance at vertex dictates the vertex as the singlet part of the product of WCW harmonics associated with the vertex and one sums over the harmonics for each internal line. p-Adicization means only the algebraic continuation to real formulas to p-adic context.

4. The exponent of Kähler function depends on both ends of the line and this means that the geometries at the ends are correlated in the sense that Kähler form contains interaction terms between the line ends. It is however not quite clear whether it contains separate "kinetic" or self interaction terms assignable to the line ends. For Kähler function the kinetic

and interaction terms should have the following general expressions as functions of complex WCW coordinates:

$$\begin{aligned} K_{kin,i} &= \sum_n f_{i,n}(Z_i)\overline{f_{i,n}(Z_i)} + c.c \ , \\ K_{int} &= \sum_n g_{1,n}(Z_1)\overline{g_{2,n}(Z_2)} + c.c \ , i = 1,2 \ . \end{aligned} \tag{7.2}$$

Here $K_{kin,i}$ define "kinetic" terms and K_{int} defines interaction term. One would have what might be called holomorphic factorization suggesting a connection with conformal field theories.

Symmetric space property -that is isometry invariance- suggests that one has

$$f_{i,n} = f_{2,n} \equiv f_n \ , \ g_{1,n} = g_{2,n} \equiv g_n \tag{7.3}$$

such that the products are invariant under the group H appearing in G/H and therefore have opposite H quantum numbers. The exponent of Kähler function does not factorize although the terms in its Taylor expansion factorize to products whose factors are products of holomorphic and antiholomorphic functions.

5. If one assumes that the exponent of Kähler function reduces to a product of eigenvalues of the Kähler-Dirac operator eigenvalues must have the decomposition

$$\lambda_k = \prod_{i=1,2} exp\left[\sum_n c_{k,n} g_n(Z_i)\overline{g_n(Z_i)} + c.c\right] \times exp\left[\sum_n d_{k,n} g_n(Z_1)\overline{g_n(Z_2)} + c.c\right] \tag{7.4}$$

Hence also the eigenvalues coming from the Dirac propagators have also expansion in terms of G/H harmonics so that in principle WCW integration would reduce to Fourier analysis in symmetric space.

7.3.2 Generalization of WCW Hamiltonians

This picture requires a generalization of the view about configuration space Hamiltonians since also the interaction term between the ends of the line is present not taken into account in the previous approach.

1. The proposed representation of WCW Hamiltonians as flux Hamiltonians [K1, K6]

$$\begin{aligned} Q(H_A) &= \int H_A(1+K)Jd^2x \ , \\ J &= \epsilon^{\alpha\beta}J_{\alpha\beta} \ , \ J^{03}\sqrt{g_4} = KJ_{12} \ . \end{aligned} \tag{7.5}$$

works for the kinetic terms only since J cannot be the same at the ends of the line. The formula defining K assumes weak form of self-duality (^{03} refers to the coordinates in the complement of X^2 tangent plane in the 4-D tangent plane). K is assumed to be symplectic invariant and constant for given X^2. The condition that the flux of $F^{03} = (\hbar/g_K)J^{03}$ defining the counterpart of Kähler electric field equals to the Kähler charge g_K gives the condition $K = g_K^2/\hbar$, where g_K is Kähler coupling constant. Within experimental uncertainties one has $\alpha_K = g_K^{/}4\pi\hbar_0 = \alpha_{em} \simeq 1/137$, where α_{em} is finite structure constant in electron length scale and $\hbar_0$ is the standard value of Planck constant.

The assumption that Poisson bracket of WCW Hamiltonians reduces to the level of embedding space - in other words $\{Q(H_A), Q(H_B\} = Q(\{H_A, H_B\})$ - can be justified. One starts from the representation in terms of say flux Hamiltonians $Q(H_A)$ and *defines* $J_{A,B}$ as $J_{A,B} \equiv Q(\{H_A, H_B\})$. One has $\partial H_A/\partial t_B = \{H_B, H_A\}$, where t_B is the parameter associated with the exponentiation of H_B. The inverse J^{AB} of $J_{A,B} = \partial H_B/\partial t_A$ is expressible as $J^{A,B} = \partial t_A/\partial H_B$. From these formulas one can deduce by using chain rule that the bracket $\{Q(H_A), Q(H_B\} = \partial t_C Q(H_A) J^{CD} \partial t_D Q(H_B)$ of flux Hamiltonians equals to the flux Hamiltonian $Q(\{H_A, H_B\})$.

2. One should be able to assign to WCW Hamiltonians also a part corresponding to the interaction term. The symplectic conjugation associated with the interaction term permutes the WCW coordinates assignable to the ends of the line. One should reduce this apparently non-local symplectic conjugation (if one thinks the ends of line as separate objects) to a non-local symplectic conjugation for $\delta CD \times CP_2$ by identifying the points of lower and upper end of CD related by time reflection and assuming that conjugation corresponds to time reflection. Formally this gives a well defined generalization of the local Poisson brackets between time reflected points at the boundaries of CD. The connection of Hermitian conjugation and time reflection in quantum field theories is is in accordance with this picture.

3. The only manner to proceed is to assign to the flux Hamiltonian also a part obtained by the replacement of the flux integral over X^2 with an integral over the projection of X^2 to a sphere S^2 assignable to the light-cone boundary or to a geodesic sphere of CP_2, which come as two varieties corresponding to homologically trivial and non-trivial spheres. The projection is defined as by the geodesic line orthogonal to S^2 and going through the point of X^2. The hierarchy of Planck constants assigns to CD a preferred geodesic sphere of CP_2 as well as a unique sphere S^2 as a sphere for which the radial coordinate r_M or the light-cone boundary defined uniquely is constant: this radial coordinate corresponds to spherical coordinate in the rest system defined by the time-like vector connecting the tips of CD. Either spheres or possibly both of them could be relevant.

   Recall that also the construction of number theoretic braids and symplectic QFT [K13] led to the proposal that braid diagrams and symplectic triangulations could be defined in terms of projections of braid strands to one of these spheres. One could also consider a weakening for the condition that the points of the number theoretic braid are algebraic by requiring only that the S^2 coordinates of the projection are algebraic and that these coordinates correspond to the discretization of S^2 in terms of the phase angles associated with θ and ϕ.

   This gives for the corresponding contribution of the WCW Hamiltonian the expression

$$Q(H_A)_{int} = \int_{S^2_\pm} H_A X \delta^2(s_+, s_-) d^2 s_\pm = \int_{P(X^2_+)\cap P(X^2_-)} \frac{\partial(s^1, s^2)}{\partial(x^1_\pm, x^2_\pm)} d^2 x_\pm \ . \tag{7.6}$$

   Here the Poisson brackets between ends of the line using the rules involve delta function $\delta^2(s_+, s_-)$ at S^2 and the resulting Hamiltonians can be expressed as a similar integral of $H_{[A,B]}$ over the upper or lower end since the integral is over the intersection of S^2 projections.

   The expression must vanish when the induced Kähler form vanishes for either end. This is achieved by identifying the scalar X in the following manner:

$$\begin{aligned} X &= J^{kl}_+ J^-_{kl} \ , \\ J^{kl}_\pm &= (1 + K_\pm) \partial_\alpha s^k \partial_\beta s^l J^{\alpha\beta}_\pm \ . \end{aligned} \tag{7.7}$$

   The tensors are lifts of the induced Kähler form of $X^2_\pm$ to S^2 (not CP_2).

4. One could of course ask why these Hamiltonians could not contribute also to the kinetic terms and why the brackets with flux Hamiltonians should vanish. This relate to how one *defines* the Kähler form. It was shown above that in case of flux Hamiltonians the definition of Kähler form as brackets gives the basic formula $\{Q(H_A), Q(H_B)\} = Q(\{H_A, H_B\}$ and same should hold true now. In the recent case $J_{A,B}$ would contain an interaction term defined

in terms of flux Hamiltonians and the previous argument should go through also now by identifying Hamiltonians as sums of two contributions and by introducing the doubling of the coordinates t_A.

5. The quantization of the Kähler-Dirac operator must be reconsidered. It would seem that one must add to the super-Hamiltonian completely analogous term obtained by replacing $(1+K)J$ with $X\partial(s^1,s^2)/\partial(x^1_\pm,x^2_\pm)$. Besides the anti-commutation relations defining correct anti-commutators to flux Hamiltonians, one should pose anti-commutation relations consistent with the anti-commutation relations of super Hamiltonians. In these anti-commutation relations $(1+K)J\delta^2(x,y)$ would be replaced with $X\delta^2(s^+,s^-)$. This would guarantee that the oscillator operators at the ends of the line are not independent and that the resulting Hamiltonian reduces to integral over either end for $H_{[A,B]}$.

6. In the case of CP_2 the Hamiltonians generating isometries are rational functions. This should hold true also now so that p-adic variants of Hamiltonians as functions in WCW would make sense. This in turn would imply that the components of the WCW Kähler form are rational functions. Also the exponentiation of Hamiltonians make sense p-adically if one allows the exponents of group parameters to be functions $Exp_p(t)$.

7.3.3 Does the expansion in terms of partial harmonics converge?

The individual terms in the partial wave expansion seem to be finite but it is not at all clear whether the expansion in powers of K actually converges.

1. In the proposed scenario one performs the expansion of the vacuum functional $exp(K)$ in powers of K and therefore in negative powers of α_K. In principle an infinite number of terms can be present. This is analogous to the perturbative expansion based on using magnetic monopoles as basic objects whereas the expansion using the contravariant Kähler metric as a propagator would be in positive powers of α_K and analogous to the expansion in terms of magnetically bound states of wormhole throats with vanishing net value of magnetic charge. At this moment one can only suggest various approaches to how one could understand the situation.

2. Weak form of self-duality and magnetic confinement could change the situation. Performing the perturbation around magnetic flux tubes together with the assumed slicing of the space-time sheet by stringy world sheets and partonic 2-surfaces could mean that the perturbation corresponds to the action assignable to the electric part of Kähler form proportional to α_K by the weak self-duality. Hence by $K=4\pi\alpha_K$ relating Kähler electric field to Kähler magnetic field the expansion would come in powers of a term containing sum of terms proportional to α_K^0 and α_K. This would leave to the scattering amplitudes the exponents of Kähler function at the maximum of Kähler function so that the non-analytic dependence on α_K would not disappear.

A further reason to be worried about is that the expansion containing infinite number of terms proportional to α_K^0 could fail to converge.

1. This could be also seen as a reason for why magnetic singlets are unavoidable except perhaps for $\hbar<\hbar_0$. By the holomorphic factorization the powers of the interaction part of Kähler action in powers of $1/\alpha_K$ would naturally correspond to increasing and opposite net values of the quantum numbers assignable to the WCW phase coordinates at the ends of the propagator line. The magnetic bound states could have similar expansion in powers of α_K as pairs of states with arbitrarily high but opposite values of quantum numbers. In the functional integral these quantum numbers would compensate each other. The functional integral would leave only an expansion containing powers of α_K starting from some finite possibly negative (unless one assumes the weak form of self-duality) power. Various gauge coupling strengths are expected to be proportional to α_K and these expansions should reduce to those in powers of α_K.

2. Since the number of terms in the fermionic propagator expansion is finite, one might hope on basis of super-symmetry that the same is true in the case of the functional integral expansion. By the holomorphic factorization the expansion in powers of K means the appearance of terms with increasingly higher quantum numbers. Quantum number conservation at vertices would leave only a finite number of terms to tree diagrams. In the case of loop diagrams pairs of particles with opposite and arbitrarily high values of quantum numbers could be generated at the vertex and magnetic confinement might be necessary to guarantee the convergence. Also super-symmetry could imply cancellations in loops.

7.3.4 Could one do without flux Hamiltonians?

The fact that the Kähler functions associated with the propagator lines can be regarded as interaction terms inspires the question whether the Kähler function could contain only the interaction terms so that Kähler form and Kähler metric would have components only between the ends of the lines.

1. The basic objection is that flux Hamiltonians too beautiful objects to be left without any role in the theory. One could also argue that the WCW metric would not be positive definite if only the non-diagonal interaction term is present. The simplest example is Hermitian 2×2-matrix with vanishing diagonal for which eigenvalues are real but of opposite sign.

2. One could of course argue that the expansions of $exp(K)$ and λ_k give in the general powers $(f_n\overline{f_n})^m$ analogous to diverging tadpole diagrams of quantum field theories due to local interaction vertices. These terms do not produce divergences now but the possibility that the exponential series of this kind of terms could diverge cannot be excluded. The absence of the kinetic terms would allow to get rid of these terms and might be argued to be the symmetric space counterpart for the vanishing of loops in WCW integral.

3. In ZEO this idea does not look completely non-sensical since physical states are pairs of positive and negative energy states. Note also that in quantum theory only creation operators are used to create positive energy states. The manifest non-locality of the interaction terms and absence of the counterparts of kinetic terms would provide a trivial manner to get rid of infinities due to the presence of local interactions. The safest option is however to keep both terms.

7.3.5 Summary

The discussion suggests that one must treat the entire Feynman graph as single geometric object with Kähler geometry in which the symmetric space is defined as product of what could be regarded as analogs of symmetric spaces with interaction terms of the metric coming from the propagator lines. The exponent of Kähler function would be the product of exponents associated with all lines and contributions to lines depend on quantum numbers (momentum and color quantum numbers) propagating in line via the coupling to the Kähler-Dirac operator. The conformal factorization would allow the reduction of integrations to Fourier analysis in symmetric space. What is of decisive importance is that the entire Feynman diagrammatics at WCW level would reduce to the construction of WCW geometry for a single propagator line as a function of quantum numbers propagating on the line.

8 Appendix: Basic Facts About Algebraic Numbers, Quaternions And Octonions

To understand the detailed connection between infinite primes, polynomial primes and Fock states, some basic concepts of algebraic number theory related to the generalization of prime and prime factorization [A7, A9, A10] (the first reference is warmly recommended for a physicist because it teaches the basic facts through exercises; also second book is highly enjoyable reading because of its non-Bourbakian style of representation).

8.1 Generalizing The Notion Of Prime

Algebraic numbers are defined as roots of polynomial equations with rational coefficients. Algebraic integers are identified as roots of monic polynomials (highest coefficient equals to one) with integer coefficients. Algebraic number fields correspond to algebraic extensions of rationals and can have any dimension as linear spaces over rationals. The notion of prime is extremely general and involves rather abctract mathematics in general case.

Quite generally, commutative ring R called integral domain, if the product ab vanishes only if a or b vanishes. To a given integral domain one can assign a number field by essentially the same construction by which one assigns the field of rationals to ordinary integers. The integer valued function $a \to N(a)$ in R is called norm if it has the properties $N(ab) = N(a)N(b)$ and $N(1) = 1$. For instance, for the algebraic extension $Q(\sqrt{-D})$ of rationals consisting of points $z = r + \sqrt{-D}s$, the function $N(z) = r^2 + Ds^2$ defines norm. More generally, the determinant of the linear map defined by the action of z in algebraic number field defines norm function. This determinant reduces to the product of all conjugates of z in K and is n: th order polynomial with respect to the components of z when K is n-dimensional.

Irreducible elements (almost the counterparts of primes) can be defined as elements P of integral domain having the property that if one has $P = bc$, then either b or c has unit norm. Elements with unit norm are called units and elements differing by a multiplication with unit are called associates. Note that in the case of p-adics all p-adic numbers with unit norm are units.

8.2 Ufds, Pids And Eds

If the elements of R allow a unique factorization to irreducible elements, R is said to be unique factorization domain (UFD). Ordinary integers are obviously UFD. The field $Z(\sqrt{-5})$ is not UFd for instance, one has $6 = 2 \times 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$. The fact that prime factorization is not unique forces to generalize the notion of primeness such that ideals in the ring of algebraic integers take the role of integers. The counterparts of primes can be identified as irreducible elements, which generate prime ideals containing one and only one rational prime. Irreducible elements, such as $1 \pm \sqrt{-5}$ in $Z(\sqrt{-5})$, are not primes in this sense.

Principal ideal domain (PID) is defined as an integral domain for which all ideals are principal, that is are generated as powers of single element. In the case of ordinary integers powers of integers define PID.

Euclidian domain (ED) is integral domain with the property that for any pair a and b one can find pair (q, r) such that $a = bq + r$ with $N(r) < N(a)$. This guarantees that the Euclidian algorithm used in the division of rationals converges. Integers form an Euclidian domain but polynomials with integer coefficients do not (elements 2 and x do not allow decomposition $2 = q(x)x + r$). It can be shown that EDs are PIDs in turn are UFDs. For instance, for complex quadratic extensions of integers $Z(\sqrt{-d})$ there are only 9 UFDs and they correspond to $d = 1, 2, 3, 7, 11, 19, 43, 67, 163$. For extensions of type $Z(\sqrt{d})$ the number of UFD: s is infinite. There are not too many quadratic extensions which are ED: s and the possible values of d are $d = -1, \pm 2, \pm 3, 5, 6, \pm 7, \pm 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73$.

Any algebraic number field K is representable always as a polynomial ring $Q[\theta]$ obtained from the polynomial ring $Q[x]$ by replacing x with an algebraic number θ, which is a root of an irreducible polynomial with rational coefficients. This field has dimension n over rationals, where n is the degree of the polynomial in question.

8.3 The Notion Of Prime Ideal

As already noticed, a general algebraic number field K does not allow a unique factorization into irreducibles and one must generalize the notion of prime number and integer in order to achieve a unique factorization. The ideals of the ring O_K of algebraic integers in K take the role of integers whereas prime ideals take the role of primes. The factorization of an ideal to a product of prime ideals is unique and each prime ideal contains single rational prime characterizing it. One can assign to an ideal norm which orders the ideals: $N(a) < N(b) \leftrightarrow b \subset a$. The smaller the integer generating ideal, the larger the ideal is and the ideals generated by primes are maximal ones in PID. The equivalence classes of the ideals of O_K under equivalence defined by integer multiplication form a group. The number of classes is a characteristic of an algebraic number field. For class-one algebraic number fields prime factorization of ideals is equivalent with the factorization to irreducibles in K. $Z(\sqrt{-5})$, which is not UFD, allows two classes of prime ideals. Cyclotomic number fields $Q(\zeta_m)$, where ζ_m is m: th root of unity have class number one for $3 \leq m \leq 10$. In particular, the four-dimensional algebraic number fields $Q(\zeta_8)$ and $Q(\zeta_5) = Q(\zeta_{10})$ are ED and thus UFD.

8.3.1 Basic facts about primality for polynomial rings

The notion of primality can be abstracted to the level of polynomial algebras in field K and these polynomial algebras seem to be more or less identical with the algebra formed by infinite integers. The following two results are crucial for the argument demonstrating that this is indeed the case.

8.3.2 Polynomial ring associated with any number field is UFD

The elements in the ring $K[x_1, ..., x_n]$ formed by the polynomials having coefficients in *any* field K and x_i having values in K, allow a unique decomposition into prime factors. This means that things are much simpler at the next abstraction level, since there is no need for refined class theories needed in the case of algebraic number fields.

The number field K appearing as a coefficient field of polynomials could correspond to finite fields (Galois fields), rationals, any algebraic number field obtained as an extension of rational, p-adic numbers, reals or complex numbers. For $Q[x]$, where Q denotes rationals, the simplest prime factors are monomials of form $x - q$, q rational number. More complicated prime factors correspond to minimal polynomials having algebraic number α and its conjugates as their roots. In the case of complex number field only monomomials $x - z$, z complex number are the only prime polynomials. Clearly, the primes at the higher level of abstraction are generalized rationals of previous level plus numbers which are algebraic with respect to the generalized rationals.

8.3.3 The polynomial rings associated with any UFD are UFD

If R is a unique factorization domain (UFD), then also $R[x]$ is UFD: this holds also for $R[x_1, ..., x_n]$. Hence one obtains an infinite hierarchy of UFDs by a repeated abstraction process by starting from a given algebraic number field K. At the first step one obtains the ring $K[x]$ of polynomials in K. At the next step one obtains the ring of polynomials $K^{2)}[y]$ having as coefficient ring the ring $K[x] \equiv K^{1)}[x]$ of polynomials. At the next step one obtains $K^{2)}[z]$, etc.. Note that $O_K[x]$ is not ED in general and need not be UFD neither unless O_K is UFD. $O_K[x]$ is not however interesting from the viewpoint of TGD.

An element of $K^{2)}(y)$ corresponds to a polynomial $P(y, x)$ of y such that its coefficients are K-rational functions of x. A polynomial in $K^{3)}(z)$ corresponds to a polynomial of $P(z, y, z)$ such that the coefficients of z are K-rational functions of functions of y with coefficients which are K-rational functions of z. Note that as a special case, polynomials of all n variables result. Note also the hierarchical ordering of the variables. Thus the hierarchy of polynomials gives rise to a hierarchy of functions having increasingly number of independent variables.

8.4 Examples Of Two-Dimensional Algebraic Number Fields

The general two-dimensional (in algebraic sense) algebraic extension of rationals corresponds to $K(\theta)$, where $\theta = (-b \pm \sqrt{b^2 - 4c})/2$ is root of second order irreducible polynomial $x^2 + bx + c$. Depending on whether the discriminant $D = b^2 - 4c$ is positive or negative, one obtains real and complex extensions. θ and its conjugate generate equivalent extensions and all extensions can be obtained as extensions of form $Q(\sqrt{\pm d})$.

For $Q(\sqrt{d})$, d square-free integer, units correspond to powers of $x = \pm(p_{n-1} + q_{n-1}\sqrt{d})$, where n defines the period of the continued fraction expansion of $\sqrt{d}$ and p_k/q_k defines k: th convergent in the continued fraction expansion. For $Q(\sqrt{-d})$, $d > 1$ units form group Z_2. For $d = 1$ the group is Z_2^2 and for $Q(w)$ where $w = -1/2 + \sqrt{3}/2$ is the third root of unity ($w^3 = 1$), this group is $Z_2 \times Z^3$ (note that in this case the minimal polynomial is $(x^3 - 1)/(x - 1)$.

$Z(w)$ and $Z(i)$ are exceptional in the sense that the group of the roots of unity is exceptionally large. $Z(i)$ and $Z(w)$ allow a unique factorization of their elements into products of irreducibles.

The primes π of $Z(w)$ consist of rational primes p, $p\ mod\ 4 = 3$ and complex Gaussian primes satisfying $N(\pi) = \pi\overline{\pi} = p$, $p\ mod\ 4 = 1$. Squares of the Gaussian primes generate as their product complex numbers giving rise to Pythagorean phases. The primes π of $Z(w)$ consist of rational primes p, $p\ mod\ 3 = 2$ and complex Eisenstein primes satisfing $N(\pi) = \pi\overline{\pi} = p$, $p\ mod\ 3 = 1$.

8.5 Cyclotomic Number Fields As Examples Of Four-Dimensional Algebraic Number Fields

By the "theorem of primitive element" all algebraic number fields are obtained by replacing the polynomial algebra $Q[x]$, by $Q[\theta]$, where θ is a root of an irreducible minimal polynomial which is of fourth order. One can readily calculate the extensions associated with a given irreducible polynomial by using quadratures for 4: th order polynomials. These polynomials are of general form $P_4(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ and by a substitution $x = y - a_3/4)$ which does not change the nature of algbebraic number field, they can be reduced to a canonical form $P_4(x) = x^4 + a_2x^2 + a_1x + a_0$. Thus a very rough view is that three rationals parametrize the 4-dimensional algebraic number fields.

A second manner to represent extensions is in form $K(\theta_1, \theta, ..)$ such that the units θ_i have no common factors different from one. In this case the dimension of the extension is 2^n, where n is the number of units. Examples of four-dimensional extensions are the algebraic extensions $Q(\sqrt{\pm d_1}, \sqrt{\pm d_2})$ of rationals, where d_i are square-free integers, reduce to form $Q(\theta)$. The cyclic extension of rationals by the powers of the m: th root of unity with $m = 5, 8, 12$ are four-dimensional extensions called cyclotomic number fields. Also the extensions $Q((\pm)d)^{1/4})$ are simple four-dimensional extensions. These extensions allow completion to a corresponding p-adic algebraic extension for some p-adic primes.

Quite generally, cyclotomic number fields $Q(\zeta_m)$ are obtained from polynomial algebra $Q[x]$ by replacing x with the m: th primitive root of unity denoted by ζ_m and thus satisfying $\zeta_m^m = 1$. There are three cyclic extensions of dimension 4 and they correspond to $Q(\zeta_5) = Q(\zeta_{10})$, $Q(\zeta_8)$ and $Q(\zeta_{12})$. Cyclotomic extensions are higly symmetric since the roots of unity act as symmetries of the norm.

The units of cyclotomic field $Q(\zeta_m)$ form group $Z_2 \times Z_m \times Z$. Z corresponds to the powers of units for $Q(\zeta_m + 1/\zeta_m)$. These powers have unit norm only with respect to the norm of $Q(\zeta_m)$ whereas with respect to the ordinary complex norm they correspond to fractal scalings. What looks fractal obtained by repeated scalings of the same structure with respect to the real norm looks like a lattice when algebraic norm is used.

1. $Q(\zeta_8)$

The cyclotomic number field $Q(\zeta_8)$, $\zeta_8 = exp(i\pi/4)$ satisfying $\zeta_8^8 = 1$, consists of numbers of form $k = m + in + \sqrt{i}(r + is)$. All roots ($\pm i^{1/2}$ and $\pm i^{3/2}$) are complex. The group of units is $Z_2^4 \times Z$. Z corresponds in real topology to the fractal scalings generated by $L = 1 + \sqrt{2}$. The integer multiples of $log(L)$ could be interpreted as a quantized momentum. $Q(\zeta_8)$ can be generated by $\pm\zeta_8$ and $\pm i\zeta_8$. This means additional Z_2^2 Galois symmetry which does not define multiplicative quantum number.

2. $Q(\zeta_{12})$

The extension $Q(\sqrt{-1}, w)$, $w = \zeta_3$, can be regarded as a cyclic extension $Q(iw) = Q(\zeta_{12})$ as is clear from the fact that the six lowest powers of iw come as $iw, -w^2, -i, w = -1 - w^2, iw^2 = -iw - i, -1$. $Z(iw)$ is especially interesting because it contains $Q(i)$ and $Q(w)$ for which primes correspond to Gaussian and Eisenstein primes. A unique factorization to a product of irreducibles is possible only for $Q(\zeta_m)$ $m \leq 10$: thus the algebraic integers in $Z(iw)$ do not always allow a unique decomposition into irreducibles. The most obvious candidates for primes not allowing unique factorization are primes satisfying simultaneously the conditions $p\ mod\ 4 = 3 = 1$ implying decomposition into a product of Gaussian prime and its conjugate and $p\ mod\ 3 = 1$ guaranteeing the decomposition into a product of Eisenstein prime and its conjugate.

The group of units reduces to $Z_2^2 \times Z_3 \times Z$ might have something to do with the group of discrete quantum numbers C, P and $SU(3)$ triality telling the number of quarks modulo 3 in the state. For the extensions $Q(\sqrt{-1}, \sqrt{d})$ the roots of unity form the group Z_2^2: these extensions could correspond to gauge bosons and the quantum numbers would correspond to C and P. For real extensions the group of the roots of unity reduces to Z_2: in this case the interpretation inters of parity suggests itself.

The lattice defined by Z corresponds to the scalings by powers of $\sqrt{3}+2$. It could be also interpreted also as the lattice of longitudinal momenta for hadronic quarks which move collinearly inside space-time sheet which might be identified as a massless extremal (ME) for which longitudinal direction is a preferred spatial direction.

$Q(\zeta_{12})$ can be generated by $\pm iw, \pm iw^2$ and the replacement of iw with these alternatives generates Z_2^2 symmetry not realizable as a multiplication with units.

3. $Q(\zeta_5)$ and biology

$Q(\zeta_5)$ indeed gives 4-dimensional extension of rationals since one has $1+\zeta_5+...\zeta_5^4 = 0$ implying that $\zeta_5^4 = 1/\zeta_5$ is expressible as rational combination of other units. Both $Q(\zeta_5)$ and $Q(\zeta_8)$ allows a unique decomposition of rational integers into prime factors. The primes in $Q(\zeta_5)$ allow decomposition to a product of $r = 1, 2$ or 4 primes of $Q(\zeta_5)$ [A9]. The value of r for a given p is fixed by the requirement that $f = 4/r$ is the smallest natural number for which $p^f - 1 \ mod \ p = 0$ holds true. For instance, $p = 2, 3$ correspond to $f = 4$ and are primes of $Q(\zeta_5)$, $p = 11$ has decomposition into a product of four primes of $Q(\zeta_5)$, and $p = 19$ has decomposition into two primes of $Q(\zeta_5)$).

What makes this extension interesting is that the phase angle associated with ζ_5 corresponds to the angle of 72 degrees closely related with Golden Mean $\tau = (1+\sqrt{5})/2$ satisfying the equation $\tau^2 - \tau - 1 = 0$. The phase of the fifth root is given by $\zeta_5 = (\tau - 1 + i\sqrt{2+\tau})2$. The group of units is $Z_2 \times Z_5 \times Z$. Z corresponds to the fractal scalings by $\tau = (1+\sqrt{5})/2$. The conjugations $\zeta_5 \rightarrow \zeta_5^k$, $k = 1, 2, 3, 4$ leave the norm invariant and generate group Z_2^5.

Fractal scalings by Golden Mean and the closely related Fibonacci numbers are closely related with the fractal structures associated with living systems (botany is full of logarithmic spirals involving Golden Mean and the phase angle 36 is involved even with DNA). Of course, the very fact that Golden Mean emerges in biological length scales provides strongest evidence for its dynamical origin in algebraic framework.

$Q(\zeta_5)$ cannot be realized as an algebraic extension $K(\theta, i)$ naturally associated with the transversal part of quaternionic primes but can appear only as a subfield of the 8-dimensional extension $K(i, cos(2\pi/5), sin(2\pi/5)$ containing also 20: th root of unity as $\zeta_{20} = i\zeta_5$. In [K10] it is indeed found that Golden Mean plays a fundamental role in topological quantum computation and is indeed a fundamental constant in TGD Universe.

8.5.1 Fractal scalings

By Dirichlet's unit theorem the group of units quite generally reduces to $Z_m \times Z^r$, where Z_m is cyclic group of roots of unity and Z^r can be regarded as an r-dimensional lattice with latticed units determined by the extension. For real extensions Z_m reduces to Z_2 since the only real roots of unity are $\{\pm 1\}$. All components of four-momentum represented by a quaternionic prime can be multiplied by separate real units of $Q(\theta)$. For a given quaternionic prime, one can always factor out the common factor of the units of $Q(\theta)$ or $Q(\theta, i)$.

The units generate nontrivial transformations at the level of single quaternionic prime. If the dimension of the real extension is n, the transformations form an $n-1$-dimensional lattice of scalings. Alternative but less plausible interpretation is that the logarithms of the scalings represent $n-1$-dimensional momentum lattice. Particle would be like a part of an algebraic hologram carrying information about external world in accordance with the ideas about fractality. Of course, units represent fractal scalings only with respect to ordinary real norm, with respect to number theoretical norm they act like phase factors.

For instance, in the case of $Q(\sqrt{5})$ the units correspond to scalings by powers of Golden Mean $\tau = (1+\sqrt{5})/2$ having number theoretic norm equal to one. Bio-systems are indeed full of fractals with scaling symmetry. For $K = Q(\sqrt{3})$ the scalings correspond to powers of $L = 2+\sqrt{3}$. An interesting possibility is that hadron physics might reveal fractality in powers of L. More generally, for $Q(\sqrt{d})$, d square-free integer, the basic fractal scaling is $L = p_{n-1} + q_{n-1}\sqrt{d}$, where n defines the period of the continued fraction expansion of $\sqrt{d}$ and p_k/q_k defines k: th convergent in the continued fraction expansion.

Four-dimensional algebraic extensions are very interesting for several reasons. First, algebraic dimension four is a borderline in complexity in the sense that for higher-dimensional irreducible algebraic extensions there is no general quadratures analogous to the formulas associated with second order polynomials giving the roots of the polynomial. Secondly, in transversal degrees of freedom the minimal dimension for $K(\theta, i)$ is four. The units of K which are algebraic integers having a unit norm in K. Quite generally, the group of units is a product $Z_{2k} \times Z_r$ of two groups. $Z_{2k} = Z_2 \times Z_k$ is the cyclic group generated by k: th root of unity. For real extensions one has $k = 1$. In transversal degrees of freedom one can have $k > 1$ since extension is $Q(\theta, i)$. The roots of unity possible in four-dimensional case correspond to $k = 2, 4, 6, 8, 10, 12$. Corresponding cyclic groups are products of Z_2^i, Z_3 and Z_5. Z_2, Z_2 and Z_3 and act as symmetries of the root lattices of Cartan algebras.

Z_3 gives rise to the Cartan algebra of $SU(3)$ and an interesting question is whether color symmetry is generated dynamically or whether it can be regarded as a basic symmetry with the lattice of integer quaternions providing scaled-up version for the root lattice of color group. Note that in TGD quark color is not spin like quantum number but corresponds to CP_2 partial waves for quark like spinors.

8.5.2 Permutations of the real roots of the minimal polynomial of θ

The replacements of the primitive element θ of $K(\theta)$ with a new one obtained by acting in it with the elements of Galois group of the minimal polynomial of θ generate different internal states of number theoretic fermions and bosons. The subgroup G_1 of Galois group permuting the real roots of the minimal polynomial with each other acts also as a symmetry. The number of equivalent primitive elements is $n_1 = n - 2r_1$, where r_2 is the number of complex root pairs. For instance, for 2-dimensional extensions these symmetries permute the real roots of a second order polynomial irreducible in the set of rationals. Since the entire polynomial has rational coefficients, kind of G_1-confinement is realized. One could say that kind of algebraically confined n-color is in question.

8.6 Quaternionic Primes

Primeness makes sense for quaternions and octonions. The following considerations are however restricted to quaternionic primes but can be easily generalized to the octonionic case. Quaternionic primes have Euclidian norm squared equal to a rational prime. The number $N(p)$ of primes associated with a given rational p depends on p and each p allows at least two primes. Quaternionic primes correspond to points of 3-sphere with prime-valued radius squared. Prime-valued radius squared is consistent with p-adic length scale hypothesis, and one can indeed reduce p-adic length scale hypothesis to the assumption that the Euclidian region associated with CP_2 type extremal has prime-valued radius squared.

It is interesting to count the number of quaternionic primes with same prime valued length squared.

1. In the case of algebraic extensions the first definition of quaternionic norm is by using number theoretic norm either for entire quaternion squared or for each component of quaternion separately. The construction of infinite primes suggests that the first definition is more appropriate. Both definitions of norm are natural for four-momentum squared since they give integer valued mass squared spectrum associated with super-conformally invariant systems. One could also decompose quaternion to two parts as $q = (q_0 + Iq_1) + J(q_2 + Iq_3)$ and define number theoretic norm with respect to the algebraic extension $Q(\theta, I)$.

2. Quaternionic primes with the same norm are related by $SO(4)$ rotation plus a change of sign of the real component of quaternion. The components of integer quaternion are analogous to components of four-momentum.

3. There are 2^4 quaternionic $\pm E_i$ and multiplication by these units defines symmetries. Non-commutativity of the quaternionic multiplication makes the interpretation of units as parity like quantum numbers somewhat problematic since the net parity associated with a product of primes representing physical particles associated with the infinite primes depends on the order of quaternionic primes. For real algebraic extensions $K = Q(\theta)$ there is also the units defining a "momentum" lattice with dimension $n - 1$, where n is the degree of the minimal polynomial $P(\theta)$.

4. Quaternionic primes cannot be real so that a given quaternionic prime with $k \geq 2$ components has 2^k conjugates obtained by changing the signs of the components of quaternion. Basic conjugation changes the signs of imagy components of quaternion. This corresponds to group $Z_2^k \subset Z_2^4$, $2 \leq k \leq 4$.

5. The group S_4 of $4! = 24$ permutations of four objects preserves the norm of a prime quaternion: these permutations are representable as a multiplication with non-prime quaternion and thus identifiable as subgroup of $SO(4)$ and also as a subgroup of $SO(3)$ (invariance group of tetrahedron). In degenerate cases (say when some components of q are identical), some subgroup of S_4 leaves quaternionic prime invariant and the rotational degeneracy reduces from $D = 24$ to some smaller number which is some factor of 24 and equals to $4, 6$ or 12 as is easy to see. There are 16 quaternionic conjugations corresponding to change of sign of any quaternion unit but all these conjugations are obtained from single quaternionic conjugation changing the sign of the imaginary part of quaternion by combining them with a multiplication with unit and its inverse. Thus the restricted group of symmetries is $S_4 \times Z_2$.

6. It is possible to find for every prime p at least two quaternionic (primes with norm squared equal to p. For a given prime p there are in general several quaternionic primes not obtainable from each other by transformations of S_4. There must exist some discrete subgroup of $SO(4)$ relating these quaternionic primes to each other.

7. The maximal number of quaternionic primes generated by $S_4 \times Z_2$ is 24×2. In non-commutative situation it is not clear whether units can be regarded as parity type quantum numbers. In any case, one can divide the entire group with Z_2^4 to obtain Z_3. This group corresponds to cyclic permutations of imaginary quaternion units.

$D = 24$ is the number of physical dimensions in bosonic string model. In TGD framework a possible interpretation is based on the observation that infinite primes constructed from rational primes the product of all primes contains the first power of each prime having interpretation as a representation for a single filled state of the fermionic sea. In the case of quaternions the Fock vacuum defined as a product of all quaternionic primes gives rise to a vacuum state

$$X = \prod_p p^{N(p)/2} \quad ,$$

since each prime and its quaternionic conjugate contribute one power of p.

8.7 Embedding Space Metric And Vielbein Must Involve Only Rational Functions

Algebraization requires that embedding space exists in the algebraic sense containing only points for which preferred coordinate variables have values in some algebraic extension of rationals. Embedding space metric at the algebraic level can be defined as a quadratic form without any reference to metric concepts like line element or distance. The metric tensors of both M_+^4 and CP_2 are indeed represented by algebraic functions in the preferred coordinates dictated by the symmetries of these spaces.

One should also construct spinor structure and this requires the introduction of an algebraic extension containing square roots since vielbein vectors appearing in the definition of the gamma matrices involve square roots of the components of the metric. In CP_2 degrees of freedom this forces the introduction of square root function, and thus all square roots, unless one restricts the values of the radial CP_2 coordinate appearing in the vielbein in such a way that rationals result. What is interesting is that all components of spinor curvature and Kähler form of CP_2 are quadratic with respect to vierbein and algebraic functions of CP_2 complex coordinates. Also the square root of the determinant of the induce metric appears only as a multiplicative factor in the Euler-Lagrange equations so that one can get rid of the square roots.

Induced spinor structure and Dirac equation relies on the notion of the induced gamma matrices and here the projections of the vierbein of CP_2 containing square roots are unavoidable. In complex coordinates the components of CP_2 vielbein in complex coordinates ξ_1, ξ_2, in which the action of $U(2)$ is linear holomorphic transformation, involve the square roots $r = \sqrt{|\xi|^2 + |\xi_2|^2}$ and $\sqrt{1+r^2}$

(for detailed formulas see Appendix at the end of the book). If one has $r = m/n$, the requirement that $\sqrt{1+r^2}$ is rational, implies $m^2 + n^2 = k^2$ so that (m, n) defines Pythagorean square. Thus induced Dirac equation is rationalized if the allowed values of r correspond to Pythagorean phases. The notion of the phase preserving canonical identification [K18], crucial for the earlier formulation of TGD, is consistent with this assumption. The metric of $S^2 = CP_1$ is a simplified example of what happens. One can write the metric as $g_{z\bar{z}=r^2} = \frac{1}{1+r^2}$ and vielbein component is proportional to $1/\sqrt{1+r^2}$, this exists for $r = m/n$ as rational number if one has $m^2 + n^2 = k^2$, which indeed defines Pythagorean triangle.

The restriction of the phases associated with the CP_2 coordinates to Pythagorean ones has deeper coordinate-invariant meaning. Rational CP_2 can be defined as a coset space $SU_Q(3)/U_Q(2)$ of rational groups $SU_Q(3)$ and $U_Q(2)$: rationality is required in the linear matrix representation of these groups.

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