# Was von Neumann Right After All?

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**Abstract**

The work with TGD inspired model for topological quantum computation led to the realization that von Neumann algebras, in particular so called hyper-finite factors of type $II_1$, seem to provide the mathematics needed to develop a more explicit view about the construction of S-matrix. The original discussion has transformed during years from free speculation reflecting in many aspects my ignorance about the mathematics involved to a more realistic view about the role of these algebras in quantum TGD. The discussions of this chapter have been restricted to the basic notions are discussed and only short mention is made to TGD applications discussed in second chapter.

The goal of von Neumann was to generalize the algebra of quantum mechanical observables. The basic ideas behind the von Neumann algebra are dictated by physics. The algebra elements allow Hermitian conjugation $^*$ and observables correspond to Hermitian operators. Any measurable function $f(A)$ of operator $A$ belongs to the algebra and one can say that non-commutative measure theory is in question.

The predictions of quantum theory are expressible in terms of traces of observables. Density matrix defining expectations of observables in ensemble is the basic example. The highly non-trivial requirement of von Neumann was that identical a priori probabilities for a detection of states of infinite state system must make sense. Since quantum mechanical expectation values are expressible in terms of operator traces, this requires that unit operator has unit trace: $tr(Id) = 1$.

In the finite-dimensional case it is easy to build observables out of minimal projections to 1-dimensional eigen spaces of observables. For infinite-dimensional case the probably of projection to 1-dimensional sub-space vanishes if each state is equally probable. The notion of observable must thus be modified by excluding 1-dimensional minimal projections, and allow only projections for which the trace would be infinite using the straightforward generalization of the matrix algebra trace as the dimension of the projection.

The non-trivial implication of the fact that traces of projections are never larger than one is that the eigen spaces of the density matrix must be infinite-dimensional for non-vanishing projection probabilities. Quantum measurements can lead with a finite probability only to mixed states with a density matrix which is projection operator to infinite-dimensional sub-space. The simple von Neumann algebras for which unit operator has unit trace are known as factors of type $II_1$.

The definitions of adopted by von Neumann allow however more general algebras. Type $I_n$ algebras correspond to finite-dimensional matrix algebras with finite traces whereas $I_\infty$ associated with a separable infinite-dimensional Hilbert space does not allow bounded traces. For algebras of type $III$ non-trivial traces are always infinite and the notion of trace becomes useless being replaced by the notion of state which is generalization of the notion of thermodynamical state. The fascinating feature of this notion of state is that it defines a unique modular automorphism of the factor defined apart from unitary inner automorphism and the question is whether this notion or its generalization might be relevant for the construction of M-matrix in TGD. It however seems that in TGD framework based on Zero Energy Ontology identifiable as "square root" of thermodynamics a square root of thermodynamical state is needed.

The inclusions of hyper-finite factors define an excellent candidate for the description of finite measurement resolution with included factor representing the degrees of freedom below measurement resolution. The would also give connection to the notion of quantum group whose physical interpretation has remained unclear. This idea is central to the proposed applications to quantum TGD discussed in separate chapter.

# 1 Introduction

The work with TGD inspired model [K11] for topological quantum computation [B2] led to the realization that von Neumann algebras [A29, A41, A33, A20] , in particular so called hyper-finite factors of type $II_1$ [A14] , seem to provide the mathematics needed to develop a more explicit view about the construction of S-matrix. The lecture notes of R. Longo [A32] give a concise and readable summary about the basic definitions and results related to von Neumann algebras and I have used this material freely in this chapter.

The original discussion has transformed during years from a free speculation reflecting in many aspects my ignorance about the mathematics involved to a more realistic view about the role of these algebras in quantum TGD. In this chapter I will discuss various aspects of hyperfinite factors with only a brief digression to TGD inspired applications whose evolution discussed in separate chapter [K5].

## 1.1 Philosophical Ideas Behind Von Neumann Algebras

The goal of von Neumann was to generalize the algebra of quantum mechanical observables. The basic ideas behind the von Neumann algebra are dictated by physics. The algebra elements allow Hermitian conjugation $^*$ and observables correspond to Hermitian operators. Any measurable function $f(A)$ of operator $A$ belongs to the algebra and one can say that non-commutative measure theory is in question.

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The non-trivial implication of the fact that traces of projections are never larger than one is that the eigen spaces of the density matrix must be infinite-dimensional for non-vanishing projection probabilities. Quantum measurements can lead with a finite probability only to mixed states with a density matrix which is projection operator to infinite-dimensional subspace. The simple von Neumann algebras for which unit operator has unit trace are known as factors of type $II_1$ [A14] .

The definitions of adopted by von Neumann allow however more general algebras. Type $I_n$ algebras correspond to finite-dimensional matrix algebras with finite traces whereas $I_\infty$ associated with a separable infinite-dimensional Hilbert space does not allow bounded traces. For algebras of type $III$ non-trivial traces are always infinite and the notion of trace becomes useless being replaced by the notion of state which is generalization of the notion of thermodynamical state. The fascinating feature of this notion of state is that it defines a unique modular automorphism of the factor defined apart from unitary inner automorphism and the question is whether this notion or its generalization might be relevant for the construction of M-matrix in TGD. It however seems that in TGD framework based on Zero Energy Ontology identifiable as “square root” of thermodynamics a square root of thermodynamical state is needed.

The inclusions of hyper-finite factors define an excellent candidate for the description of finite measurement resolution with included factor representing the degrees of freedom below measurement resolution. The would also give connection to the notion of quantum group whose physical interpretation has remained unclear. This idea is central to the proposed applications to quantum TGD discussed in separate chapter.

## 1.2 Von Neumann, Dirac, And Feynman

The association of algebras of type $I$ with the standard quantum mechanics allowed to unify matrix mechanism with wave mechanics. Note however that the assumption about continuous momentum state basis is in conflict with separability but the particle-in-box idealization allows to circumvent this problem (the notion of space-time sheet brings the box in physics as something completely real).

Because of the finiteness of traces von Neumann regarded the factors of type $II_1$ as fundamental and factors of type III as pathological. The highly pragmatic and successful approach of Dirac [A30] based on the notion of delta function, plus the emergence of s [A35] , the possibility to formulate the notion of delta function rigorously in terms of distributions [A15, A22] , and the emergence of path integral approach [A34] meant that von Neumann approach was forgotten by particle physicists.

Algebras of type $II_1$ have emerged only much later in conformal and topological quantum field theories [A37, A12] allowing to deduce invariants of knots, links and 3-manifolds. Also algebraic structures known as bi-algebras, Hopf algebras, and ribbon algebras [A9] relate closely to type $II_1$

factors. In topological quantum computation [B2] based on braid groups [A42] modular S-matrices they play an especially important role.

In algebraic quantum field theory [B3] defined in Minkowski space the algebras of observables associated with bounded space-time regions correspond quite generally to the type $III_1$ hyper-finite factor [B1, B4].

I have restricted the considerations of this chapter mostly to the technical aspects and Appendix includes sections about inclusions of HFFs. The evolution of ideas about possible applications to quantum TGD is summarized in chapter, which was originally part of this chapter [K5].

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [?].

# 2 Von Neumann Algebras

In this section basic facts about von Neumann algebras are summarized using as a background material the concise summary given in the lecture notes of Longo [A32] .

## 2.1 Basic Definitions

A formal definition of von Neumann algebra [A41, A33, A20] is as a $^*$-subalgebra of the set of bounded operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ closed under weak operator topology, stable under the conjugation $J = ^*: x \rightarrow x^*$, and containing identity operator $Id$. This definition allows also von Neumann algebras for which the trace of the unit operator is not finite.

Identity operator is the only operator commuting with a simple von Neumann algebra. A general von Neumann algebra allows a decomposition as a direct integral of simple algebras, which von Neumann called factors. Classification of von Neumann algebras reduces to that for factors.

$\mathcal{B}(\mathcal{H})$ has involution $^*$ and is thus a $^*$-algebra. $\mathcal{B}(\mathcal{H})$ has order order structure $A \geq 0 : (Ax, x) \geq 0$. This is equivalent to $A = BB^*$ so that order structure is determined by algebraic structure. $\mathcal{B}(\mathcal{H})$ has metric structure in the sense that norm defined as supremum of $||Ax||$, $||x|| \leq 1$ defines the notion of continuity. $||A||^2 = inf\{\lambda > 0 : AA^* \leq \lambda I\}$ so that algebraic structure determines metric structure.

There are also other topologies for $\mathcal{B}(\mathcal{H})$ besides norm topology.

1. $A_i \rightarrow A$ strongly if $||Ax - A_i x|| \rightarrow 0$ for all $x$. This topology defines the topology of $C^*$ algebra. $\mathcal{B}(\mathcal{H})$ is a Banach algebra that is $||AB|| \leq ||A|| \times ||B||$ (inner product is not necessary) and also $C^*$ algebra that is $||AA^*|| = ||A||^2$.

2. $A_i \rightarrow A$ weakly if $(A_i x, y) \rightarrow (Ax, y)$ for all pairs $(x, y)$ (inner product is necessary). This topology defines the topology of von Neumann algebra as a sub-algebra of $\mathcal{B}(\mathcal{H})$.

Denote by $M'$ the commutant of $\mathcal{M}$ which is also algebra. Von Neumann's bicommutant theorem says that $\mathcal{M}$ equals to its own bi-commutant. Depending on whether the identity operator has a finite trace or not, one distinguishes between algebras of type $II_1$ and type $II_\infty$. $II_1$ factor allow trace with properties $tr(Id) = 1$, $tr(xy) = tr(yx)$, and $tr(x^*x) > 0$, for all $x \neq 0$. Let $L^2(\mathcal{M})$ be the Hilbert space obtained by completing $\mathcal{M}$ respect to the inner product defined $\langle x|y\rangle = tr(x^*y)$ defines inner product in $\mathcal{M}$ interpreted as Hilbert space. The normalized trace induces a trace in $M'$, natural trace $Tr_{M'}$, which is however not necessarily normalized. $JxJ$ defines an element of $M'$: if $\mathcal{H} = L^2(\mathcal{M})$, the natural trace is given by $Tr_{M'}(JxJ) = tr_M(x)$ for all $x \in M$ and bounded.

## 2.2 Basic Classification Of Von Neumann Algebras

Consider first some definitions. First of all, Hermitian operators with positive trace expressible as products $xx^*$ are of special interest since their sums with positive coefficients are also positive.

In quantum mechanics Hermitian operators can be expressed in terms of projectors to the eigen states. There is a natural partial order in the set of isomorphism classes of projectors by inclusion: $E < F$ if the image of $\mathcal{H}$ by $E$ is contained to the image of $\mathcal{H}$ by a suitable isomorph of $F$. Projectors are said to be metrically equivalent if there exist a partial isometry which maps the images $\mathcal{H}$ by them to each other. In the finite-dimensional case metric equivalence means that isomorphism classes are identical $E = F$.

The algebras possessing a minimal projection $E_0$ satisfying $E_0 \leq F$ for any $F$ are called type $I$ algebras. Bounded operators of n-dimensional Hilbert space define algebras $I_n$ whereas the bounded operators of infinite-dimensional separable Hilbert space define the algebra $I_\infty$. $I_n$ and $I_\infty$ correspond to the operator algebras of quantum mechanics. The states of harmonic oscillator correspond to a factor of type $I$.

The projection $F$ is said to be finite if $F < E$ and $F \equiv E$ implies $F = E$. Hence metric equivalence means identity. Simple von Neumann algebras possessing finite projections but no minimal projections so that any projection $E$ can be further decomposed as $E = F + G$, are called factors of type II.

Hyper-finiteness means that any finite set of elements can be approximated arbitrary well with the elements of a finite-dimensional sub-algebra. The hyper-finite $II_\infty$ algebra can be regarded as a tensor product of hyper-finite $II_1$ and $I_\infty$ algebras. Hyper-finite $II_1$ algebra can be regarded as a Clifford algebra of an infinite-dimensional separable Hilbert space sub-algebra of $I_\infty$.

Hyper-finite $II_1$ algebra can be constructed using Clifford algebras $C(2n)$ of $2n$-dimensional spaces and identifying the element $x$ of $2^n \times 2^n$ dimensional $C(n)$ as the element $diag(x,x)/2$ of $2^{n+1} \times 2^{n+1}$-dimensional $C(n+1)$. The union of algebras $C(n)$ is formed and completed in the weak operator topology to give a hyper-finite $II_1$ factor. This algebra defines the Clifford algebra of infinite-dimensional separable Hilbert space and is thus a sub-algebra of $I_\infty$ so that hyper-finite $II_1$ algebra is more regular than $I_\infty$.

von Neumann algebras possessing no finite projections (all traces are infinite or zero) are called algebras of type III. It was later shown by [A10] [A4] that these algebras are labeled by a parameter varying in the range $[0,1]$, and referred to as algebras of type $III_x$. $III_1$ category contains a unique hyper-finite algebra. It has been found that the algebras of observables associated with bounded regions of 4-dimensional Minkowski space in quantum field theories correspond to hyper-finite factors of type $III_1$ [A32] . Also statistical systems at finite temperature correspond to factors of type $III$ and temperature parameterizes one-parameter set of automorphisms of this algebra [B1] . Zero temperature limit correspond to $I_\infty$ factor and infinite temperature limit to $II_1$ factor.

## 2.3 Non-Commutative Measure Theory And Non-Commutative Topologies And Geometries

von Neumann algebras and $C^*$ algebras give rise to non-commutative generalizations of ordinary measure theory (integration), topology, and geometry. It must be emphasized that these structures are completely natural aspects of quantum theory. In particular, for the hyper-finite type $II_1$ factors quantum groups and Kac Moody algebras [B5] emerge quite naturally without any need for ad hoc modifications such as making space-time coordinates non-commutative. The effective 2-dimensionality of quantum TGD (partonic or stringy 2-surfaces code for states) means that these structures appear completely naturally in TGD framework.

### 2.3.1 *Non-commutative measure theory*

von Neumann algebras define what might be a non-commutative generalization of measure theory and probability theory [A32] .

1. Consider first the commutative case. Measure theory is something more general than topology since the existence of measure (integral) does not necessitate topology. Any measurable function $f$ in the space $L^\infty(X,\mu)$ in measure space $(X,\mu)$ defines a bounded operator $M_f$ in the space $\mathcal{B}(L^2(X,\mu))$ of bounded operators in the space $L^2(X,\mu)$ of square integrable functions with action of $M_f$ defined as $M_f g = fg$.

2. Integral over $\mathcal{M}$ is very much like trace of an operator $f_{x,y} = f(x)\delta(x,y)$. Thus trace is a natural non-commutative generalization of integral (measure) to the non-commutative case and defined for von Neumann algebras. In particular, generalization of probability measure results if the case $tr(Id) = 1$ and algebras of type $I_n$ and $II_1$ are thus very natural from the point of view of non-commutative probability theory.

The trace can be expressed in terms of a cyclic vector $\Omega$ or vacuum/ground state in physicist's terminology. $\Omega$ is said to be cyclic if the completion $\overline{M\Omega} = H$ and separating if $x\Omega$ vanishes only for $x = 0$. $\Omega$ is cyclic for $\mathcal{M}$ if and only if it is separating for $M'$. The expression for the trace given by

$$Tr(ab) \quad = \quad \left(\frac{(ab+ba)}{2}, \Omega\right) \tag{2.1}$$

is symmetric and allows to defined also inner product as $(a,b) = Tr(a^*b)$ in $\mathcal{M}$. If $\Omega$ has unit norm $(\Omega, \Omega) = 1$, unit operator has unit norm and the algebra is of type $II_1$. Fermionic oscillator operator algebra with discrete index labeling the oscillators defines $II_1$ factor. Group algebra is second example of $II_1$ factor.

The notion of probability measure can be abstracted using the notion of state. State $\omega$ on a $C^*$ algebra with unit is a positive linear functional on $\mathcal{U}$, $\omega(1) = 1$. By so called KMS construction [A32] any state $\omega$ in $C^*$ algebra $\mathcal{U}$ can be expressed as $\omega(x) = (\pi(x)\Omega, \Omega)$ for some cyclic vector $\Omega$ and $\pi$ is a homomorphism $\mathcal{U} \to \mathcal{B}(\mathcal{H})$.

### 2.3.2 *Non-commutative topology and geometry*

$C^*$ algebras generalize in a well-defined sense ordinary topology to non-commutative topology.

1. In the Abelian case Gelfand Naimark theorem [A32] states that there exists a contravariant functor $F$ from the category of unital abelian $C^*$ algebras and category of compact topological spaces. The inverse of this functor assigns to space $X$ the continuous functions $f$ on $X$ with norm defined by the maximum of $f$. The functor assigns to these functions having interpretation as eigen states of mutually commuting observables defined by the function algebra. These eigen states are delta functions localized at single point of $X$. The points of $X$ label the eigenfunctions and thus define the spectrum and obviously span $X$. The connection with topology comes from the fact that continuous map $Y \to X$ corresponds to homomorphism $C(X) \to C(Y)$.

2. In non-commutative topology the function algebra $C(X)$ is replaced with a general $C^*$ algebra. Spectrum is identified as labels of simultaneous eigen states of the Cartan algebra of $C^*$ and defines what can be observed about non-commutative space $X$.

3. Non-commutative geometry can be very roughly said to correspond to $^*$-subalgebras of $C^*$ algebras plus additional structure such as symmetries. The non-commutative geometry of Connes [A5] is a basic example here.

## 2.4 Modular Automorphisms

von Neumann algebras allow a canonical unitary evolution associated with any state $\omega$ fixed by the selection of the vacuum state $\Omega$ [A32] . This unitary evolution is an automorphism fixed apart form unitary automorphisms $A \to UAU^*$ related with the choice of $\Omega$.

Let $\omega$ be a normal faithful state: $\omega(x^*x) > 0$ for any $x$. One can map $\mathcal{M}$ to $L^2(\mathcal{M})$ defined as a completion of $\mathcal{M}$ by $x \to x\Omega$. The conjugation $^*$ in $\mathcal{M}$ has image at Hilbert space level as a map $S_0 : x\Omega \to x^*\Omega$. The closure of $S_0$ is an anti-linear operator and has polar decomposition $S = J\Delta^{1/2}$, $\Delta = SS^*$. $\Delta$ is positive self-adjoint operator and $J$ anti-unitary involution. The following conditions are satisfied

$$\begin{aligned} \Delta^{it}\mathcal{M}\Delta^{-it} &= \mathcal{M} \ , \\ J\mathcal{M}J &= \mathcal{M}' \ . \end{aligned} \tag{2.2}$$

$\Delta^{it}$ is obviously analogous to the time evolution induced by positive definite Hamiltonian and induces also the evolution of the expectation $\omega$ as $\pi \to \Delta^{it}\pi\Delta^{-it}$.

## 2.5 Joint Modular Structure And Sectors

Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion. The unitary operator $\gamma = J_N J_M$ defines a canonical endomorphisms $M \rightarrow N$ in the sense that it depends only up to inner automorphism on $\mathcal{N}$, $\gamma$ defines a sector of $\mathcal{M}$. The sectors of $\mathcal{M}$ are defined as $Sect(\mathcal{M}) = End(\mathcal{M})/Inn(\mathcal{M})$ and form a semi-ring with respected to direct sum and composition by the usual operator product. It allows also conjugation.

$L^2(\mathcal{M})$ is a normal bi-module in the sense that it allows commuting left and right multiplications. For $a, b \in M$ and $x \in L^2(\mathcal{M})$ these multiplications are defined as $axb = aJb^*Jx$ and it is easy to verify the commutativity using the factor $Jy^*J \in \mathcal{M}'$. [A10] [A5] has shown that all normal bi-modules arise in this way up to unitary equivalence so that representation concepts make sense. It is possible to assign to any endomorphism $\rho$ index $Ind(\rho) \equiv M : \rho(\mathcal{M})$. This means that the sectors are in 1-1 correspondence with inclusions. For instance, in the case of hyper-finite $II_1$ they are labeled by Jones index. Furthermore, the objects with non-integral dimension $\sqrt{[\mathcal{M} : \rho(\mathcal{M})]}$ can be identified as quantum groups, loop groups, infinite-dimensional Lie algebras, etc...

## 2.6 Basic Facts About Hyper-Finite Factors Of Type III

Hyper-finite factors of type $II_1$, $II_\infty$ and $III_1$, $III_0$, $III_\lambda$, $\lambda \in (0, 1)$, allow by definition hierarchy of finite approximations and are unique as von Neumann algebras. Also hyper-finite factors of type $II_\infty$ and type $III$ could be relevant for the formulation of TGD. HFFs of type $II_\infty$ and $III$ could appear at the level operator algebra but that at the level of quantum states one would obtain HFFs of type $II_1$. These extended factors inspire highly non-trivial conjectures about quantum TGD. The book of Connes [A5] provides a detailed view about von Neumann algebras in general.

### 2.6.1 *Basic definitions and facts*

A highly non-trivial result is that HFFs of type $II_\infty$ are expressible as tensor products $II_\infty = II_1 \otimes I_\infty$, where $II_1$ is hyper-finite [A5] .

*1. The existence of one-parameter family of outer automorphisms*

The unique feature of factors of type $III$ is the existence of one-parameter unitary group of outer automorphisms. The automorphism group originates in the following manner.

1. Introduce the notion of linear functional in the algebra as a map $\omega : \mathcal{M} \rightarrow C$. $\omega$ is said to be hermitian it respects conjugation in $\mathcal{M}$; positive if it is consistent with the notion of positivity for elements of $\mathcal{M}$ in which case it is called weight; state if it is positive and normalized meaning that $\omega(1) = 1$, faithful if $\omega(A) > 0$ for all positive $A$; a trace if $\omega(AB) = \omega(BA)$, a vector state if $\omega(A)$ is "vacuum expectation" $\omega_\Omega(A) = (\Omega, \omega(A)\Omega)$ for a non-degenerate representation $(\mathcal{H}, \pi)$ of $\mathcal{M}$ and some vector $\Omega \in \mathcal{H}$ with $||\Omega|| = 1$.

2. The existence of trace is essential for hyper-finite factors of type $II_1$. Trace does not exist for factors of type $III$ and is replaced with the weaker notion of state. State defines inner product via the formula $(x, y) = \phi(y^*x)$ and * is isometry of the inner product. *-operator has property known as pre-closedness implying polar decomposition $S = J\Delta^{1/2}$ of its closure. $\Delta$ is positive definite unbounded operator and $J$ is isometry which restores the symmetry between $\mathcal{M}$ and its commutant $\mathcal{M}'$ in the Hilbert space $\mathcal{H}_\phi$, where $\mathcal{M}$ acts via left multiplication: $\mathcal{M}' = J\mathcal{M}J$.

3. The basic result of Tomita-Takesaki theory is that $\Delta$ defines a one-parameter group $\sigma^t_\phi(x) = \Delta^{it} x \Delta^{-it}$ of automorphisms of $\mathcal{M}$ since one has $\Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}$. This unitary evolution is an automorphism fixed apart from unitary automorphism $A \rightarrow UAU^*$ related with the choice of $\phi$. For factors of type I and II this automorphism reduces to inner automorphism so that the group of outer automorphisms is trivial as is also the outer automorphism associated with $\omega$. For factors of type $III$ the group of these automorphisms divided by inner automorphisms gives a one-parameter group of $Out(\mathcal{M})$ of outer automorphisms, which does not depend at all on the choice of the state $\phi$.

More precisely, let $\omega$ be a normal faithful state: $\omega(x^*x) > 0$ for any $x$. One can map $\mathcal{M}$ to $L^2(\mathcal{M})$ defined as a completion of $\mathcal{M}$ by $x \rightarrow x\Omega$. The conjugation $^*$ in $\mathcal{M}$ has image at Hilbert space level as a map $S_0 : x\Omega \rightarrow x^*\Omega$. The closure of $S_0$ is an anti-linear operator and has polar decomposition $S = J\Delta^{1/2}$, $\Delta = SS^*$. $\Delta$ is positive self-adjoint operator and $J$ anti-unitary involution. The following conditions are satisfied

$$\begin{aligned} \Delta^{it}\mathcal{M}\Delta^{-it} &= \mathcal{M} \ , \\ J\mathcal{M}J &= \mathcal{M}' \ . \end{aligned} \quad (2.3)$$

$\Delta^{it}$ is obviously analogous to the time evolution induced by positive definite Hamiltonian and induces also the evolution of the expectation $\omega$ as $\pi \rightarrow \Delta^{it}\pi\Delta^{-it}$. What makes this result thought provoking is that it might mean a universal quantum dynamics apart from inner automorphisms and thus a realization of general coordinate invariance and gauge invariance at the level of Hilbert space.

*2. Classification of HFFs of type III*

Connes achieved an almost complete classification of hyper-finite factors of type $III$ completed later by others. He demonstrated that they are labeled by single parameter $0 \leq \lambda \leq 1]$ and that factors of type $III_\lambda$, $0 \leq \lambda < 1$ are unique. Haagerup showed the uniqueness for $\lambda = 1$. The idea was that the group has an invariant, the kernel $T(M)$ of the map from time like $R$ to $Out(M)$, consisting of those values of the parameter $t$ for which $\sigma_\phi^t$ reduces to an inner automorphism and to unity as outer automorphism. Connes also discovered also an invariant, which he called spectrum $S(\mathcal{M})$ of $\mathcal{M}$ identified as the intersection of spectra of $\Delta_\phi\backslash\{0\}$, which is closed multiplicative subgroup of $R^+$.

Connes showed that there are three cases according to whether $S(\mathcal{M})$ is

1. $R^+$, type $III_1$
2. $\{\lambda^n, n \in Z\}$, type $III_\lambda$.
3. $\{1\}$, type $III_0$.

   The value range of $\lambda$ is this by convention. For the reversal of the automorphism it would be that associated with $1/\lambda$.

Connes constructed also an explicit representation of the factors $0 < \lambda < 1$ as crossed product $II_\infty$ factor $\mathcal{N}$ and group $Z$ represented as powers of automorphism of $II_\infty$ factor inducing the scaling of trace by $\lambda$. The classification of HFFs of type $III$ reduced thus to the classification of automorphisms of $\mathcal{N} \otimes \mathcal{B}(\mathcal{H}$. In this sense the theory of HFFs of type $III$ was reduced to that for HFFs of type $II_\infty$ or even $II_1$. The representation of Connes might be also physically interesting.

### 2.6.2 *Probabilistic view about factors of type III*

Second very concise representation of HFFs relies on thermodynamical thinking and realizes factors as infinite tensor product of finite-dimensional matrix algebras acting on state spaces of finite state systems with a varying and finite dimension $n$ such that one assigns to each factor a density matrix characterized by its eigen values. Intuitively one can think the finite matrix factors as associated with $n$-state system characterized by its energies with density matrix $\rho$ defining a thermodynamics. The logarithm of the $\rho$ defines the single particle quantum Hamiltonian as $H = log(\rho)$ and $\Delta = \rho = exp(H)$ defines the automorphism $\sigma_\phi$ for each finite tensor factor as $exp(iHt)$. Obviously free field representation is in question.

Depending on the asymptotic behavior of the eigenvalue spectrum one obtains different factors [A5] .

1. Factor of type I corresponds to ordinary thermodynamics for which the density matrix as a function of matrix factor approaches sufficiently fast that for a system for which only ground state has non-vanishing Boltzmann weight.
2. Factor of type $II_1$ results if the density matrix approaches to identity matrix sufficiently fast. This means that the states are completely degenerate which for ordinary thermodynamics results only at the limit of infinite temperature. Spin glass could be a counterpart for this kind of situation.

3. Factor of type $III$ results if one of the eigenvalues is above some lower bound for all tensor factors in such a manner that neither factor of type I or $II_1$ results but thermodynamics for systems having infinite number of degrees of freedom could yield this kind of situation.

This construction demonstrates how varied representations factors can have, a fact which might look frustrating for a novice in the field. In particular, the infinite tensor power of $M(2,C)$ with state defined as an infinite tensor power of $M(2,C)$ state assigning to the matrix $A$ the complex number $(\lambda^{1/2}A_{11} + \lambda^{-1/2}\ \phi(A) = A_{22})/(\lambda^{1/2} + \lambda^{-1/2})$ defines HFF $III_\lambda$ [A5] , [C1] . Formally the same algebra which for $\lambda = 1$ gives ordinary trace and HFF of type $II_1$, gives $III$ factor only by replacing trace with state. This simple model was discovered by Powers in 1967.

It is indeed the notion of state or thermodynamics is what distinguishes between factors. This looks somewhat weird unless one realizes that the Hilbert space inner product is defined by the "thermodynamical" state $\phi$ and thus probability distribution for operators and for their thermal expectation values. Inner product in turn defines the notion of norm and thus of continuity and it is this notion which differs dramatically for $\lambda = 1$ and $\lambda < 1$ so that the completions of the algebra differ dramatically.

In particular, there is no sign about $I_\infty$ tensor factor or crossed product with $Z$ represented as automorphisms inducing the scaling of trace by $\lambda$. By taking tensor product of $I_\infty$ factor represented as tensor power with induces running from $-\infty$ to 0 and $II_1$ HFF with indices running from 1 to $\infty$ one can make explicit the representation of the automorphism of $II_\infty$ factor inducing scaling of trace by $\lambda$ and transforming matrix factors possessing trace given by square root of index $\mathcal{M} : \mathcal{N}$ to those with trace 2.

# 3 Braid Group, Von Neumann Algebras, Quantum TGD, And Formation Of Bound States

The article of Vaughan Jones in [A42] discusses the relation between knot theory, statistical physics, and von Neumann algebras. The intriguing results represented stimulate concrete ideas about how to understand the formation of bound states quantitatively using the notion of join along boundaries bond. All mathematical results represented in the following discussion can be found in [A42] and in the references cited therein so that I will not bother to refer repeatedly to this article in the sequel.

## 3.1 Factors Of Von Neumann Algebras

Von Neumann algebras M are algebras of bounded linear operators acting in Hilbert space. These algebras contain identity, are closed with respect to Hermitian conjugation, and are topologically complete. Finite-dimensional von Neuman algebras decompose into a direct sum of algebras $M_n$, which act essentially as matrix algebras in Hilbert spaces $\mathcal{H}_{nm}$, which are tensor products $C^n \otimes \mathcal{H}_m$. Here $\mathcal{H}_m$ is an m-dimensional Hilbert space in which $M_n$ acts trivially. $m$ is called the multiplicity of $M_n$.

A factor of von Neumann algebra is a von Neumann algebra whose center is just the scalar multiples of identity. The algebra of bounded operators in an infinite-dimensional Hilbert space is certainly a factor. This algebra decomposes into "atoms" represented by one-dimensional projection operators. This kind of von Neumann algebras are called type I factors.

The so called type $\mathrm{II}_1$ factors and type III factors came as a surprise even for Murray and von Neumann. $\mathrm{II}_1$ factors are infinite-dimensional and analogs of the matrix algebra factors $M_n$. They allow a trace making possible to define an inner product in the algebra. The trace defines a generalized dimension for any subspace as the trace of the corresponding projection operator. This dimension is however continuous and in the range $[0,1]$: the finite-dimensional analog would be the dimension of the sub-space divided by the dimension of $\mathcal{H}_n$ and having values $(0, 1/n, 2/n, ..., 1)$. $\mathrm{II}_1$ factors are isomorphic and there exists a minimal "hyper-finite" $\mathrm{II}_1$ factor is contained by every other $\mathrm{II}_1$ factor.

Just as in the finite-dimensional case, one can to assign a multiplicity to the Hilbert spaces where $\mathrm{II}_1$ factors act on. This multiplicity, call it $dim_M(\mathcal{H})$ is analogous to the dimension of the Hilbert space tensor factor $\mathcal{H}_m$, in which $\mathrm{II}_1$ factor acts trivially. This multiplicity can have all positive real values. Quite generally, von Neumann factors of type I and $II_1$ are in many respects analogous to the coefficient field of a vector space.

## 3.2 Sub-Factors

Sub-factors $N \subset M$, where $N$ and $M$ are of type $II_1$ and have same identity, can be also defined. The observation that $M$ is analogous to an algebraic extension of $N$ motivates the introduction of index $|M : N|$, which is essentially the dimension of $M$ with respect to $N$. This dimension is an analog for the complex dimension of $CP_2$ equal to 2 or for the algebraic dimension of the extension of p-adic numbers.

The following highly non-trivial results about the dimensions of the tensor factors hold true.

1. If $N \subset M$ are $II_1$ factors and $|M : N| < 4$, there is an integer $n \geq 3$ such $|M : N| = r = 4cos^2(\pi/n)$, $n \geq 3$.
2. For each number $r = 4cos^2(\pi/n)$ and for all $r \geq 4$ there is a sub-factor $R_r \subset R$ with $|R : R_r| = r$.

   One can say that $M$ effectively decomposes to a tensor product of $N$ with a space, whose dimension is quantized to a certain algebraic number $r$. The values of $r$ corresponding to $n = 3, 4, 5, 6...$ are $r = 1, 2, 1 + \Phi \simeq 2.61, 3, ...$ and approach to the limiting value $r = 4$. For $r \geq 4$ the dimension becomes continuous.

An even more intriguing result is that by starting from $N \subset M$ with a projection $e_N$: $M \rightarrow N$ one can extend $M$ to a larger $II_1$ algebra $\langle M, e_N \rangle$ such that one has

$$\begin{aligned} |\langle M, e_N \rangle : M| &= |M : N| \ , \\ tr(xe_N) &= |M : N|^{-1} tr(x) \ , \ x \in M \ . \end{aligned} \tag{3.1}$$

One can continue this process and the outcome is a tower of $II_1$ factors $M_i \subset M_{i+1}$ defined by $M_1 = N$, $M_2 = M$, $M_{i+1} = \langle M_i, e_{M_{i-1}} \rangle$. Furthermore, the projection operators $e_{M_i} \equiv e_i$ define a Temperley-Lieb representation of the braid algebra via the formulas

$$\begin{aligned} e_i^2 &= e_i \ , \\ e_i e_{i\pm1} e_i &= \tau e_i \ , \ \tau = 1/|M : N| \\ e_i e_j &= e_j e_i \ , \ |i - j| \geq 2 \ . \end{aligned} \tag{3.2}$$

Temperley Lieb algebra will be discussed in more detail later. Obviously the addition of a tensor factor of dimension $r$ is analogous with the addition of a strand to a braid.

The hyper-finite algebra $R$ is generated by the set of braid generators $\{e_1, e_2, .....\}$ in the braid representation corresponding to $r$. Sub-factor $R_1$ is obtained simply by dropping the lowest generator $e_1$, $R_2$ by dropping $e_1$ and $e_2$, etc..

## 3.3 $Ii_1$ Factors And The Spinor Structure Of WCW

The following observations serve as very suggestive guidelines for how one could interpret the above described results in TGD framework.

1. The discrete spectrum of dimensions $1, 2, 1 + \Phi, 3, ..$ below $r < 4$ brings in mind the discrete energy spectrum for bound states whereas the for $r \geq 4$ the spectrum of dimensions is analogous to a continuum of unbound states. The fact that $r$ is an algebraic number for $r < 4$ conforms with the vision that bound state entanglement corresponds to entanglement probabilities in an extension of rationals defining a finite-dimensional extension of p-adic numbers for every prime $p$.

2. The discrete values of $r$ correspond precisely to the angles $\phi$ allowed by the unitarity of Temperley-Lieb representations of the braid algebra with $d = -\sqrt{r}$. For $r \geq 4$ Temperley-Lieb representation is not unitary since $cos^2(\pi/n)$ becomes formally larger than one ($n$ would become imaginary and continuous). This could mean that $r \geq 4$, which in the generic case is a transcendental number, represents unbound entanglement, which in TGD Universe is not stable against state preparation and state function reduction processes.

3. The formula $tr(xe_N) = |M : N|^{-1}tr(x)$ is completely analogous to the formula characterizing the normalization of the link invariant induced by the second Markov move in which a new strand is added to a braid such that it braids only with the leftmost strand and therefore does not change the knot resulting as a link closure. Hence the addition of a single strand seems to correspond to an introduction of an r-dimensional sub-factor to II$_1$ factor.

In TGD framework the generation of bound state has the formation of (possibly braided join along boundaries bonds as a space-time correlate and this encourages a rather concrete interpretation of these findings. Also the I$_1$ factors themselves have a nice interpretation in terms of the WCW spinor structure.

*1. The interpretation of II$_1$ factors in terms of Clifford algebra of WCW*

The Clifford algebra of an infinite-dimensional Hilbert space defines a II$_1$ factor. The counterparts for $e_i$ would naturally correspond to the analogs of projection operators $(1 + \sigma_i)/2$ and thus to operators of form $(1 + \Sigma_{ij})/2$, defined by a subset of sigma matrices. The first guess is that the index pairs are $(i, j) = (1, 2), (2, 3), (3, 4), .....$ The dimension of the Clifford algebra is $2^N$ for $N$-dimensional space so that $\Delta N = 1$ would correspond to $r = 2$ in the classical case and to one qubit. The problem with this interpretation is $r > 2$ has no physical interpretation: the formation of bound states is expected to reduce the value of $r$ from its classical value rather than increase it.

One can however consider also the sequence $(i, j) = (1, 1+k), (1+k, 1+2k), (1+2k, 1+3k), ....$ For $k = 2$ the reduction of $r$ from $r = 4$ would be due to the loss of degrees of freedom due to the formation of a bound state and $(r = 4, \Delta N = 2)$ would correspond to the classical limit resulting at the limit of weak binding. The effective elimination of the projection operators from the braid algebra would reflect this loss of degrees of freedom. This interpretation could at least be an appropriate starting point in TGD framework.

In TGD Universe physical states correspond to WCW spinor fields, whose gamma matrix algebra is constructed in terms of second quantized free induced spinor fields defined at space-time sheets. The original motivation was the idea that the quantum states of the Universe correspond to the modes of purely classical free spinor fields in the infinite-dimensional configuration space of 3-surfaces (the "world of classical worlds", WCW) possessing general coordinate invariant (in 4-dimensional sense!) Kähler geometry. Quantum information-theoretical motivation could have come from the requirement that these fields must be able to code information about the properties of the point (3-surface, and corresponding space-time sheet). Scalar fields would treat the 3-surfaces as points and are thus not enough. Induced spinor fields allow however an infinite number of modes: according to the naive Fourier analyst's intuition these modes are in one-one correspondence with the points of the 3-surface. Second quantization gives much more. Also non-local information about the induced geometry and topology must be coded, and here quantum entanglement for states generated by the fermionic oscillator operators coding information about the geometry of 3-surface provides enormous information storage capacity.

In algebraic geometry also the algebra of the imbedding space of algebraic variety divided by the ideal formed by functions vanishing on the surface codes information about the surface: for instance, the maximal ideals of this algebra code for the points of the surface (functions of imbedding space vanishing at a particular point). The function algebra of the imbedding space indeed plays a key role in the construction of WCW-geometry besides second quantized fermions.

The Clifford algebra generated by the WCW gamma matrices at a given point (3-surface) of WCW of 3-surfaces could be regarded as a II$_1$-factor associated with the local tangent space endowed with Hilbert space structure (WCW Kähler metric). The counterparts for $e_i$ would naturally correspond to the analogs of projection operators $(1 + \sigma_i)/2$ and thus operators of form $(G_{\overline{A}B} \times 1 + \Sigma_{\overline{A}B})$ formed as linear combinations of components of the Kähler metric and of the sigma matrices defined by gamma matrices and contracted with the generators of the isometries of WCW. The addition of single complex degree of freedom corresponds to $\Delta N = 2$ and $r = 4$ and the classical limit and would correspond to the addition of single braid. $(r < 4, \Delta N < 2)$ would be due to the binding effects.

$r = 1$ corresponds to $\Delta N = 0$. The first interpretation is in terms of strong binding so that the addition of particle does not increase the number of degrees of freedom. In TGD framework $r = 1$ might also correspond to the addition of zero modes which do not contribute to the WCW metric and spinor structure but have a deep physical significance. $(r = 2, \Delta N = 1)$ would correspond to strong binding reducing the spinor and space-time degrees of freedom by a factor of half. $r = \Phi^2$ $(n = 5)$ *resp.* $r = 3$ $(n = 6)$ corresponds to $\Delta N_r \simeq 1.3885$ *resp.* $\Delta N_r = 1.585$. Using the terminology of quantum field theories, one might say that in the infinite-dimensional context a given complex bound state degree of freedom possesses anomalous real dimension $r < 2$. $r \geq 4$ would correspond to a unbound entanglement and increasingly classical behavior.

## 3.4 About Possible Space-Time Correlates For The Hierarchy Of $II_1$ Sub-Factors

By quantum classical correspondence the infinite-dimensional physics at WCW level should have definite space-time correlates. In particular, the dimension $r$ should have some fractal dimension as a space-time correlate.

*1. Quantum classical correspondence*

Join along boundaries bonds serve as correlates for bound state formation. The presence of join along boundaries bonds would lead to a generation of bound states just by reducing the degrees of freedom to those of connected 3-surface. The bonds would constrain the two 3-surfaces to single space-like section of imbedding space.

This picture would allow to understand the difficulties related to Bethe-Salpeter equations for bound states based on the assumption that particles are points moving in $M^4$. The restriction of particles to time=constant section leads to a successful theory which is however non-relativistic. The basic binding energy would relate to the entanglement of the states associated with the bonded 3-surfaces. Since the classical energy associated with the bonds is positive, the binding energy tends to be reduced as $r$ increases.

By spin glass degeneracy join along boundaries bonds have an infinite number of degrees of freedom in the ordinary sense. Since the system is infinite-dimensional and quantum critical, one expects that the number $r$ of degrees freedom associated with a single join along boundaries bond is universal. Since join along boundaries bonds correspond to the strands of a braid and are correlates for the bound state formation, the natural guess is that $r = 4cos^2(\pi/n)$, $n = 3, 4, 5, ...$ holds true. $r < 4$ should characterize both binding energy and the dimension of the effective tensor factor introduced by a new join along boundaries bond.

The assignment of 2 "bare" and $\Delta N \leq 2$ renormalized real dimensions to single join along boundaries bond is consistent with the effective two-dimensionality of anyon systems and with the very notion of the braid group. The picture conforms also with the fact that the degrees of freedom in question are associated with metrically 2-dimensional light-like boundaries (of say magnetic flux tubes) acting as causal determinants. Also vibrational degrees of freedom described by Kac-Moody algebra are present and the effective 2-dimensionality means that these degrees of freedom are not excited and only topological degrees of freedom coded by the position of the puncture remain.

$(r \geq 4, \Delta N \geq 2)$, if possible at all, would mean that the tensor factor associated with the join along boundaries bond is effectively more than 4-dimensional due to the excitation of the vibrational Kac-Moody degrees of freedom. The finite value of $r$ would mean that most of theme are eliminated also now but that their number is so large that bound state entanglement is not possible anymore.

The introduction of non-integer dimension could be seen as an effective description of an infinite-dimensional system as a finite-dimensional system in the spirit of renormalization group philosophy. The non-unitarity of $r \geq 4$ Temperley-Lieb representations could mean that they correspond to unbound entanglement unstable against state function reduction and preparation processes. Since this kind of entanglement does not survive in quantum jump it is not representable in terms of braid groups.

*2. Does $r$ define a fractal dimension of $CP_2$ projection of partonic 2-surface?*

On basis of the quantum classical correspondence one expects that $r$ should define some fractal dimension at the space-time level. Since $r$ varies in the range $1,..,4$ and corresponds to the fractal dimension of 2-D Clifford algebra the corresponding spinors would have dimension $d=\sqrt{r}$. There are two options.

1. $D=r/2$ is suggested on basis of the construction of quantum version of $M^d$.
2. $D=log_2(r)$ is natural on basis of the dimension $d=2^{D/2}$ of spinors in D-dimensional space.

$r$ can be assigned with $CP_2$ degrees of freedom in the model for the quantization of Planck constant based on the explicit identification of Josephson inclusions in terms of finite subgroups of $SU(2)\subset SU(3)$. Hence $D$ should relate to the $CP_2$ projection of the partonic 2-surface and one could have $D=D(X^2)$, the latter being the average dimension of the $CP_2$ projection of the partonic 2-surface for the preferred extremals of Kähler action.

Since a strongly interacting non-perturbative phase should be in question, the dimension for the $CP_2$ projection of the space-time surface must be at least $D(X^4)=2$ to guarantee that non-vacuum extremals are in question. This is true for $D(X^2)=r/2\geq 1$. The logarithmic formula $D(X^2)=log_2(r)\geq 0$ gives $D(X^2)=0$ for $n=3$ meaning that partonic 2-surfaces are vacua: space-time surface can still be a non-vacuum extremal.

As $n$ increases, the number of $CP_2$ points covering a given $M^4$ point and related by the finite subgroup of $G\subset SU(2)\subset SU(3)$ defining the inclusion increases so that the fractal dimension of the $CP_2$ projection is expected to increase also. $D(X^2)=2$ would correspond to the space-time surfaces for which partons have topological magnetic charge forcing them to have a 2-dimensional $CP_2$ projection. There are reasons to believe that the projection must be homologically non-trivial geodesic sphere of $CP_2$.

## 3.5 Could Binding Energy Spectra Reflect The Hierarchy Of Effective Tensor Factor Dimensions?

If one takes completely seriously the idea that join along boundaries bonds are a correlate of binding then the spectrum of binding energies might reveal the hierarchy of the fractal dimensions $r(n)$. Hydrogen atom and harmonic oscillator have become symbols for bound state systems. Hence it is of interest to find whether the binding energy spectrum of these systems might be expressed in terms of the "binding dimension" $x(n)=4-r(n)$ characterizing the deviation of dimension from that at the limit of a vanishing binding energy. The binding energies of hydrogen atom are in a good approximation given by $E(n)/E(1)=1/n^2$ whereas in the case of harmonic oscillator one has $E(n)/E_0=2n+1$. The constraint $n\geq 3$ implies that the principal quantum number must correspond $n-2$ in the case of hydrogen atom and to $n-3$ in the case of harmonic oscillator.

Before continuing one must face an obvious objection. By previous arguments different values of $r$ correspond to different values of $\hbar$. The value of $\hbar$ cannot however differ for the states of hydrogen atom. This is certainly true. The objection however leaves open the possibility that the states of the light-like boundaries of join along boundaries bonds correspond to reflective level and represent some aspects of the physics of, say, hydrogen atom.

In the general case the energy spectrum satisfies the condition

$$\frac{E_B(n)}{E_B(3)} = \frac{f(4-r(n))}{f(3)} \ , \tag{3.3}$$

where $f$ is some function. The simplest assumption is that the spectrum of binding energies $E_B(n)=E(n)-E(\infty)$ is a linear function of $r(n)-4$:

$$\frac{E_B(n)}{E_B(3)} = \frac{4-r(n)}{3}=\frac{4}{3}sin^2(\frac{\pi}{n})\rightarrow\frac{4\pi^2}{3}\times\frac{1}{n^2} \ . \tag{3.4}$$

In the linear approximation the ratio $E(n+1)/E(n)$ approaches $(n/n+1)^2$ as in the case of hydrogen atom but for small values the linear approximation fails badly. An exact correspondence results for

$$\frac{E(n)}{E(1)} = \frac{1}{n^2} \ ,$$

$$n = \frac{1}{\pi \arcsin\left(\sqrt{1-r(n+2)/4}\right)} - 2 \ .$$

Also the ionized states with $r \geq 4$ would correspond to bound states in the sense that two particle would be constrained to move in the same space-like section of space-time surface and should be distinguished from genuinely free states when particles correspond to disjoint space-time sheets.

For the harmonic oscillator one express $E(n) - E(0)$ instead of $E(n) - E(\infty)$ as a function of $x = 4 - r$ and one would have

$$\frac{E(n)}{E(0)} = 2n + 1 \ ,$$

$$n = \frac{1}{\pi \arcsin\left(\sqrt{1-r(n+3)/4}\right)} - 3 \ .$$

In this case ionized states would not be possible due to the infinite depth of the harmonic oscillator potential well.

## 3.6 Four-Color Problem, $II_1$ Factors, And Anyons

The so called four-color problem can be phrased as a question whether it is possible to color the regions of a plane map using only four colors in such a manner that no adjacent regions have the same color (for an enjoyable discussion of the problem see [A21] ). One might call this kind of coloring complete. There is no loss of generality in assuming that the map can be represented as a graph with regions represented as triangle shaped faces of the graph. For the dual graph the coloring of faces becomes coloring of vertices and the question becomes whether the coloring is possible in such a manner that no vertices at the ends of the same edge have same color. The problem can be generalized by replacing planar maps with maps defined on any two-dimensional surface with or without boundary and arbitrary topology. The four-color problem has been solved with an extensive use of computer [A7] but it would be nice to understand why the complete coloring with four colors is indeed possible.

There is a mysterious looking connection between four-color problem and the dimensions $r(n) = 4cos^2(\pi/n)$, which are in fact known as Beraha numbers in honor of the discoverer of this connection [A17] . Consider a more general problem of coloring two-dimensional map using $m$ colors. One can construct a polynomial $P(m)$, so called chromatic polynomial, which tells the number of colorings satisfying the condition that no neighboring vertices have the same color. The vanishing of the chromatic polynomial for an integer value of $m$ tells that the complete coloring using $m$ colors is not possible.

$P(m)$ has also other than integer valued real roots. The strange discovery due to Beraha is that the numbers $B(n)$ appear as approximate roots of the chromatic polynomial in many situations. For instance, the four non-integral real roots of the chromatic polynomial of the truncated icosahedron are very close to $B(5)$, $B(7)$, $B(8)$ and $B(9)$. These findings led Beraha to formulate the following conjecture. Let $P_i$ be a sequence of chromatic polynomials for a graph for which the number of vertices approaches infinity. If $r_i$ is a root of the polynomial approaching a well-defined value at the limit $i \to \infty$, then the limiting value of $r(i)$ is Beraha number.

A physicist's proof for Beraha's conjecture based on quantum groups and conformal theory has been proposed [A17] . It is interesting to look for the a possible physical interpretation of 4-color problem and Beraha's conjecture in TGD framework.

1. In TGD framework $B(n)$ corresponds to a renormalized dimension for a 2-spin system consisting of two qubits, which corresponds to 4 different colors. For $B(n) = 4$ two spin 1/2 fermions obeying Fermi statistics are in question. Since the system is 2-dimensional, the general case corresponds to two anyons with fractional spin $B(n)/4$ giving rise to $B(n) < 4$ colors and obeying fractional statistics instead of Fermi statistics. One can replace coloring problem with the problem whether an ideal antiferro-magnetic lattice using anyons with fractional spin $B(n)/4$ is possible energetically. In other words, does this system form a quantum mechanical bound state even at the limit when the lengths of the edges approach to zero.

2. The failure of coloring means that there are at least two neighboring vertices in the lattice with the property that the spins at the ends of the same edge are in the same direction. Lattice defect would be in question. At the limit of an infinitesimally short edge length the failure of coloring is certainly not an energetically favored option for fermionic spins ($m = 4$) but is allowed by anyonic statistics for $m = B(n) < 4$. Thus one has reasons to expect that when anyonic spin is $B(n)/4$ the formation of a purely 2-anyon bound states becomes possible and they form at the limit of an infinitesimal edge length a kind of topological macroscopic quantum phase with a non-vanishing binding energy. That $B(n)$ are roots of the chromatic polynomial at the continuum limit would have a clear physical interpretation.

3. Only $B(n) < 4$ defines a sub-factor of von Neumann algebra allowing unitary Temperley-Lieb representations. This is consistent with the fact that for $m = 4$ complete coloring must exists. The physical argument is that otherwise a macroscopic quantum phase with non-vanishing binding energy could result at the continuum limit and the upper bound for $r$ from unitarity would be larger than 4. For $m = 4$ the completely anti-ferromagnetic state would represent the ground state and the absence of anyon-pair condensate would mean a vanishing binding energy.

# 4 Inclusions Of $II_1$ And $III_1$ Factors

Inclusions $\mathcal{N} \subset \mathcal{M}$ of von Neumann algebras have physical interpretation as a mathematical description for sub-system-system relation. For type $I$ algebras the inclusions are trivial and tensor product description applies as such. For factors of $II_1$ and $III$ the inclusions are highly non-trivial. The inclusion of type $II_1$ factors were understood by Vaughan Jones [A2] and those of factors of type $III$ by Alain Connes [A4] .

Sub-factor $\mathcal{N}$ of $\mathcal{M}$ is defined as a closed $^*$-stable C-subalgebra of $\mathcal{M}$. Let $\mathcal{N}$ be a sub-factor of type $II_1$ factor $\mathcal{M}$. Jones index $\mathcal{M} : \mathcal{N}$ for the inclusion $\mathcal{N} \subset \mathcal{M}$ can be defined as $\mathcal{M} : \mathcal{N} = dim_N(L^2(\mathcal{M})) = Tr_{N'}(id_{L^2(\mathcal{M})})$. One can say that the dimension of completion of $\mathcal{M}$ as $\mathcal{N}$ module is in question.

## 4.1 Basic Findings About Inclusions

What makes the inclusions non-trivial is that the position of $\mathcal{N}$ in $\mathcal{M}$ matters. This position is characterized in case of hyper-finite $II_1$ factors by index $\mathcal{M} : \mathcal{N}$ which can be said to the dimension of $\mathcal{M}$ as $\mathcal{N}$ module and also as the inverse of the dimension defined by the trace of the projector from $\mathcal{M}$ to $\mathcal{N}$. It is important to notice that $\mathcal{M} : \mathcal{N}$ does not characterize either $\mathcal{M}$ or $\mathcal{M}$, only the imbedding.

The basic facts proved by Jones are following [A2] .

1. For pairs $\mathcal{N} \subset \mathcal{M}$ with a finite principal graph the values of $\mathcal{M} : \mathcal{N}$ are given by

$$\begin{aligned} &a)\ \ \mathcal{M} : \mathcal{N} = 4cos^2(\pi/h)\ \ ,\ \ \ h \geq 3\ \ ,\\ &b)\ \ \mathcal{M} : \mathcal{N} \geq 4\ \ . \end{aligned} \tag{4.1}$$

   the numbers at right hand side are known as Beraha numbers [A17] . The comments below give a rough idea about what finiteness of principal graph means.

2. As explained in [B5] , for $\mathcal{M} : \mathcal{N} < 4$ one can assign to the inclusion Dynkin graph of ADE type Lie-algebra $g$ with $h$ equal to the Coxeter number $h$ of the Lie algebra given in terms of its dimension and dimension $r$ of Cartan algebra $r$ as $h = (dimg(g) - r)/r$. The Lie algebras of $SU(n)$, $E_7$ and $D_{2n+1}$ are however not allowed. For $\mathcal{M} : \mathcal{N} = 4$ one can assign to the inclusion an extended Dynkin graph of type ADE characterizing Kac Moody algebra. Extended ADE diagrams characterize also the subgroups of SU(2) and the interpretation proposed in [A39] is following. The ADE diagrams are associated with the $n = \infty$ case having $\mathcal{M} : \mathcal{N} \geq 4$. There are diagrams corresponding to infinite subgroups: SU(2) itself,

circle group U(1), and infinite dihedral groups (generated by a rotation by a non-rational angle and reflection. The diagrams corresponding to finite subgroups are extension of $A_n$ for cyclic groups, of $D_n$ dihedral groups, and of $E_n$ with n=6,7,8 for tetrahedron, cube, dodecahedron. For $\mathcal{M}:\mathcal{N}<4$ ordinary Dynkin graphs of $D_{2n}$ and $E_6, E_8$ are allowed.

The interpretation of [A39] is that the subfactors correspond to inclusions $\mathcal{N}\subset\mathcal{M}$ defined in the following manner.

1. Let $G$ be a finite subgroup of SU(2). Denote by $R$ the infinite-dimensional Clifford algebras resulting from infinite-dimensional tensor power of $M_2(C)$ and by $R_0$ its subalgebra obtained by restricting $M_2(C)$ element of the first factor to be unit matrix. Let $G$ act by automorphisms in each tensor factor. $G$ leaves $R_0$ invariant. Denote by $R_0^G$ and $R^G$ the sub-algebras which remain element wise invariant under the action of $G$. The resulting Jones inclusions $R_0^G \subset R^G$ are consistent with the ADE correspondence.

2. The argument suggests the existence of quantum versions of subgroups of SU(2) for which representations are truncations of those for ordinary subgroups. The results have been generalized to other Lie groups.

3. Also $SL(2,C)$ acts as automorphisms of $M_2(C)$. An interesting question is what happens if one allows $G$ to be any discrete subgroups of SL(2,C). Could this give inclusions with $\mathcal{M}:\mathcal{N}>4$?. The strong analogy of the spectrum of indices with spectrum of energies with hydrogen atom would encourage this interpretation: the subgroup SL(2,C) not reducing to those of SU(2) would correspond to the possibility for the particle to move with respect to each other with constant velocity.

## 4.2 The Fundamental Construction And Temperley-Lieb Algebras

It was shown by Jones [A16] that for a given Jones inclusion with $\beta=\mathcal{M}:\mathcal{N}<\infty$ there exists a tower of finite $II_1$ factors $\mathcal{M}_k$ for $k=0,1,2,....$ such that

1. $\mathcal{M}_0=\mathcal{N}$, $\mathcal{M}_1=\mathcal{M}$,

2. $\mathcal{M}_{k+1}=End_{\mathcal{M}_{k-1}}\mathcal{M}_k$ is the von Neumann algebra of operators on $L^2(\mathcal{M}_k)$ generated by $\mathcal{M}_k$ and an orthogonal projection $e_k: L^2(\mathcal{M}_k)\rightarrow L^2(\mathcal{M}_{k-1})$ for $k\geq 1$, where $\mathcal{M}_k$ is regarded as a subalgebra of $\mathcal{M}_{k+1}$ under right multiplication.

It can be shown that $\mathcal{M}_{k+1}$ is a finite factor. The sequence of projections on $\mathcal{M}_\infty=\cup_{k\geq 0}\mathcal{M}_k$ satisfies the relations

$$\begin{array}{lll} e_i^2=e_i \ , & e_i^{=}e_i \ , & \\ e_i=\beta e_ie_je_i & \text{for } |i-j|=1 \ , & \\ e_ie_j=e_je_i & \text{for } |i-j|\geq 2 \ . & \end{array} \tag{4.2}$$

The construction of hyper-finite $II_1$ factor using Clifford algebra $C(2)$ represented by $2\times 2$ matrices allows to understand the theorem in $\beta=4$ case in a straightforward manner. In particular, the second formula involving $\beta$ follows from the identification of $x$ at $(k-1)^{th}$ level with $(1/\beta)diag(x,x)$ at $k^{th}$ level.

By replacing $2\times 2$ matrices with $\sqrt{\beta}\times\sqrt{\beta}$ matrices one can understand heuristically what is involved in the more general case. $\mathcal{M}_k$ is $\mathcal{M}_{k-1}$ module with dimension $\sqrt{\beta}$ and $\mathcal{M}_{k+1}$ is the space of $\sqrt{\beta}\times\sqrt{\beta}$ matrices $\mathcal{M}_{k-1}$ valued entries acting in $\mathcal{M}_k$. The transition from $\mathcal{M}_k$ to $\mathcal{M}_{k-1}$ linear maps of $\mathcal{M}_k$ happens in the transition to the next level. $x$ at $(k-1)^{th}$ level is identified as $(x/\beta)\times Id_{\sqrt{\beta}\times\sqrt{\beta}}$ at the next level. The projection $e_k$ picks up the projection of the matrix with $\mathcal{M}_{k-1}$ valued entries in the direction of the $Id_{\sqrt{\beta}\times\sqrt{\beta}}$.

The union of algebras $A_{\beta,k}$ generated by $1,e_1,...,e_k$ defines Temperley-Lieb algebra $A_\beta$ [A38] . This algebra is naturally associated with braids. Addition of one strand to a braid adds one generator to this algebra and the representations of the Temperley Lieb algebra provide link, knot, and 3-manifold invariants [A42] . There is also a connection with systems of statistical physics and with Yang-Baxter algebras [A8] .

A further interesting fact about the inclusion hierarchy is that the elements in $\mathcal{M}_i$ belonging to the commutator $\mathcal{N}'$ of $\mathcal{N}$ form finite-dimensional spaces. Presumably the dimension approaches infinity for $n \rightarrow \infty$.

## 4.3 Connection With Dynkin Diagrams

The possibility to assign Dynkin diagrams ($\beta < 4$) and extended Dynkin diagrams ($\beta = 4$ to Jones inclusions can be understood heuristically by considering a characterization of so called bipartite graphs [A40] , [B5] by the norm of the adjacency matrix of the graph.

Bipartite graphs $\Gamma$ is a finite, connected graph with multiple edges and black and white vertices such that any edge connects white and black vertex and starts from a white one. Denote by $w(\Gamma)$ ($b(\Gamma)$) the number of white (black) vertices. Define the adjacency matrix $\Lambda = \Lambda(\Gamma)$ of size $b(\Gamma) \times w(\Gamma)$ by

$$w_{b,w} = \left\{ \begin{array}{ll} m(e) & \text{if there exists } e \text{ such that } \delta e = b - w \ , \\ 0 & \text{otherwise} \ . \end{array} \right. \tag{4.3}$$

Here $m(e)$ is the multiplicity of the edge $e$.

Define norm $||\Gamma||$ as

$$\begin{aligned} ||X|| &= max\{||X||;\ ||x|| \leq 1\} \ , \\ ||\Gamma|| &= ||\Lambda(\Gamma)|| = \left|\left| \begin{array}{ll} 0 & \Lambda(\Gamma) \\ \Lambda(\Gamma)^t & 0 \end{array} \right|\right| \ . \end{aligned} \tag{4.4}$$

Note that the matrix appearing in the formula is $(m+n) \times (m+n)$ symmetric square matrix so that the norm is the eigenvalue with largest absolute value.

Suppose that $\Gamma$ is a connected finite graph with multiple edges (sequences of edges are regarded as edges). Then

1. If $||\Gamma|| \leq 2$ and if $\Gamma$ has a multiple edge, $||\Gamma|| = 2$ and $\Gamma = \tilde{A}_1$, the extended Dynkin diagram for $SU(2)$ Kac Moody algebra.

2. $||\Gamma|| < 2$ if and only $\Gamma$ is one of the Dynkin diagrams of A,D,E. In this case $||\Gamma|| = 2cos(\pi/h)$, where $h$ is the Coxeter number of $\Gamma$.

3. $||\Gamma|| = 2$ if and only if $\Gamma$ is one of the extended Dynkin diagrams $\tilde{A}, \tilde{D}, \tilde{E}$.

This result suggests that one can indeed assign to the Jones inclusions Dynkin diagrams. To really understand how the inclusions can be characterized in terms bipartite diagrams would require a deeper understanding of von Neumann algebras. The following argument only demonstrates that bipartite graphs naturally describe inclusions of algebras.

1. Consider a bipartite graph. Assign to each white vertex linear space $W(w)$ and to each edge of a linear space $W(b,w)$. Assign to a given black vertex the vector space $\oplus_{\delta e = b-w} W(b,w) \otimes W(w)$ where $(b,w)$ corresponds to an edge ending to $b$.

2. Define $\mathcal{N}$ as the direct sum of algebras $End(W(w))$ associated with white vertices and $\mathcal{M}$ as direct sum of algebras $\oplus_{\delta e = b-w} End(W(b,w)) \otimes End(W(w))$ associated with black vertices.

3. There is homomorphism $N \rightarrow M$ defined by imbedding direct sum of white endomorphisms $x$ to direct sum of tensor products $x$ with the identity endomorphisms associated with the edges starting from $x$.

It is possible to show that Jones inclusions correspond to the Dynkin diagrams of $A_n, D_{2n}$, and $E_6, E_8$ and extended Dynkin diagrams of ADE type. In particular, the dual of the bi-partite graph associated with $\mathcal{M}_{n-1} \subset \mathcal{M}_n$ obtained by exchanging the roles of white and black vertices describes the inclusion $\mathcal{M}_n \subset \mathcal{M}_{n+1}$ so that two subsequent Jones inclusions might define something fundamental (the corresponding space-time dimension is $2 \times log_2(\mathcal{M} : \mathcal{N}) \leq 4$.

## 4.4 Indices For The Inclusions Of Type $III_1$ Factors

Type $III_1$ factors appear in relativistic quantum field theory defined in 4-dimensional Minkowski space [B1] . An overall summary of basic results discovered in algebraic quantum field theory is described in the lectures of Longo [A32] . In this case the inclusions for algebras of observables are induced by the inclusions for bounded regions of $M^4$ in axiomatic quantum field theory. Tomita's theory of modular Hilbert algebras [A28] , [B4] forms the mathematical corner stone of the theory.

The basic notion is Haag-Kastler net [A27] consisting of bounded regions of $M^4$. Double cone serves as a representative example. The von Neumann algebra $\mathcal{A}(O)$ is generated by observables localized in bounded region $O$. The net satisfies the conditions implied by local causality:

1. Isotony: $O_1 \subset O_2$ implies $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.
2. Locality: $O_1 \subset O_2'$ implies $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)'$ with $O'$ defined as $\{x : \langle x, y\rangle < 0 \text{ for all } y \in O\}$.
3. Haag duality $\mathcal{A}(O')' = \mathcal{A}(O)$.

   Besides this Poincare covariance, positive energy condition, and the existence of vacuum state is assumed.

DHR (Doplicher-Haag-Roberts) [A11] theory allows to deduce the values of Jones index and they are squares of integers in dimensions $D > 2$ so that the situation is rather trivial. The 2-dimensional case is distinguished from higher dimensional situations in that braid group replaces permutation group since the paths representing the flows permuting identical particles can be linked in $X^2 \times T$ and anyonic statistics [D1, D2] becomes possible. In the case of 2-D Minkowski space $M^2$ Jones inclusions with $\mathcal{M} : \mathcal{N} < 4$ plus a set of discrete values of $\mathcal{M} : \mathcal{N}$ in the range $(4, 6)$ are possible. In [A32] some values are given ($\mathcal{M} : \mathcal{N} = 5, 5.5049..., 5.236...., 5.828...$).

At least intersections of future and past light cones seem to appear naturally in TGD framework such that the boundaries of future/past directed light cones serve as seats for incoming/outgoing states defined as intersections of space-time surface with these light cones. $III_1$ sectors cannot thus be excluded as factors in TGD framework. On the other hand, the construction of S-matrix at space-time level is reduced to $II_1$ case by effective 2-dimensionality.

# 5 TGD And Hyper-Finite Factors Of Type $\mathrm{II}_1$

By effective 2-dimensionality of the construction of quantum states the hyper-finite factors of type $II_1$ fit naturally to TGD framework. In particular, infinite dimensional spinors define a canonical representations of this kind of factor. The basic question is whether only hyper-finite factors of type $II_1$ appear in TGD framework. Affirmative answer would allow to interpret physical $M$-matrix as time like entanglement coefficients.

## 5.1 What Kind Of Hyper-Finite Factors One Can Imagine In TGD?

The working hypothesis has been that only hyper-finite factors of type $II_1$ appear in TGD. The basic motivation has been that they allow a new view about $M$-matrix as an operator representable as time-like entanglement coefficients of zero energy states so that physical states would represent laws of physics in their structure. They allow also the introduction of the notion of measurement resolution directly to the definition of reaction probabilities by using Jones inclusion and the replacement of state space with a finite-dimensional state space defined by quantum spinors. This hypothesis is of course just an attractive working hypothesis and deserves to be challenged.

### 5.1.1 *WCW spinors*

For WCW spinor s the HFF $II_1$ property is very natural because of the properties of infinite-dimensional Clifford algebra and the inner product defined by the WCW geometry does not allow other factors than this. A good guess is that the values of conformal weights label the factors appearing in the tensor power defining WCW spinor s. Because of the non-degeneracy and super-symplectic symmetries the density matrix representing metric must be essentially unit matrix for each conformal weight which would be the defining characteristic of hyper-finite factor of type $II_1$.

### 5.1.2 *Bosonic degrees of freedom*

The bosonic part of the super-symplectic algebra consists of Hamiltonians of $CH$ in one-one correspondence with those of $\delta M^4_\pm \times CP_2$. Also the Kac-Moody algebra acting leaving the light-likeness of the partonic 3-surfaces intact contributes to the bosonic degrees of freedom. The commutator of these algebras annihilates physical states and there are also Virasoro conditions associated with ordinary conformal symmetries of partonic 2-surface [K2] . The labels of Hamiltonians of WCW and spin indices contribute to bosonic degrees of freedom.

Hyper-finite factors of type $II_1$ result naturally if the system is an infinite tensor product finite-dimensional matrix algebra associated with finite dimensional systems [A5] . Unfortunately, neither Virasoro, symplectic nor Kac-Moody algebras do have decomposition into this kind of infinite tensor product. If bosonic degrees for super-symplectic and super-Kac Moody algebra indeed give $I_\infty$ factor one has HFF if type $II_\infty$. This looks the most natural option but threatens to spoil the beautiful idea about $M$-matrix as time-like entanglement coefficients between positive and negative energy parts of zero energy state.

The resolution of the problem is surprisingly simple and trivial after one has discovered it. The requirement that state is normalizable forces to project $M$-matrix to a finite-dimensional sub-space in bosonic degrees of freedom so that the reduction $I_\infty \to I_n$ occurs and one has the reduction $II_\infty \to II_1 \times I_n = II_1$ to the desired HFF.

One can consider also the possibility of taking the limit $n \to \infty$. One could indeed say that since $I_\infty$ factor can be mapped to an infinite tensor power of $M(2, C)$ characterized by a state which is not trace, it is possible to map this representation to HFF by replacing state with trace [A5] . The question is whether the forcing the bosonic foot to fermionic shoe is physically natural. One could also regard the $II_1$ type notion of probability as fundamental and also argue that it is required by full super-symmetry realized also at the level of many-particle states rather than mere single particle states.

### 5.1.3 *How the bosonic cutoff is realized?*

Normalizability of state requires that projection to a finite-dimensional bosonic sub-space is carried out for the bosonic part of the $M$-matrix. This requires a cutoff in quantum numbers of super-conformal algebras. The cutoff for the values of conformal weight could be formulated by replacing integers with $Z_n$ or with some finite field $G(p, 1)$. The cutoff for the labels associated with Hamiltonians defined as an upper bound for the dimension of the representation looks also natural.

Number theoretical braids which are discrete and finite structures would define space-time correlate for this cutoff. p-Adic length scale $p \simeq 2^k$ hypothesis could be interpreted as stating the fact that only powers of $p$ up to $p^k$ are significant in p-adic thermodynamics which would correspond to finite field $G(k, 1)$ if $k$ is prime. This has no consequences for p-adic mass calculations since already the first two terms give practically exact results for the large primes associated with elementary particles [K12] .

Finite number of strands for the theoretical braids would serve as a correlate for the reduction of the representation of Galois group $S_\infty$ of rationals to an infinite produce of diagonal copies of finite-dimensional Galois group so that same braid would repeat itself like a unit cell of lattice i condensed matter [A3] .

### 5.1.4 *HFF of type $III$ for field operators and HFF of type $II_1$ for states?*

One could also argue that the Hamiltonians with fixed conformal weight are included in fermionic $II_1$ factor and bosonic factor $I_\infty$ factor, and that the inclusion of conformal weights leads to a factor of type III. Conformal weight could relate to the integer appearing in the crossed product representation $III = Z \times_{cr} II_\infty$ of HFF of type $III$ [A5] .

The value of conformal weight is non-negative for physical states which suggests that $Z$ reduces to semigroup $N$ so that a factor of type $III$ would reduce to a factor of type $II_\infty$ since trace would become finite. If unitary process corresponds to an automorphism for $II_\infty$ factor, the action of automorphisms affecting scaling must be uni-directional. Also thermodynamical irreversibility suggests the same. The assumption that state function reduction for positive energy part of state implies unitary process for negative energy state and vice versa would only mean that the shifts for positive and negative energy parts of state are opposite so that $Z \rightarrow N$ reduction would still hold true.

### 5.1.5 *HFF of type $II_1$ for the maxima of Kähler function?*

Probabilistic interpretation allows to gain heuristic insights about whether and how hyper-finite factors of type type $II_1$ might be associated with WCW degrees of freedom. They can appear both in quantum fluctuating degrees of freedom associated with a given maximum of Kähler function and in the discrete space of maxima of Kähler function.

Spin glass degeneracy is the basic prediction of classical TGD and means that instead of a single maximum of Kähler function analogous to single free energy minimum of a thermodynamical system there is a fractal spin glass energy landscape with valleys inside valleys. The discretization of WCW in terms of the maxima of Kähler function crucial for the p-adicization problem, leads to the analog of spin glass energy landscape and hyper-finite factor of type $II_1$ might be the appropriate description of the situation.

The presence of the tensor product structure is a powerful additional constraint and something analogous to this should emerge in WCW degrees of freedom. Fractality of the many-sheeted space-time is a natural candidate here since the decomposition of the original geometric structure to parts and replacing them with the scaled down variant of original structure is the geometric analog of forming a tensor power of the original structure.

## 5.2 Direct Sum Of HFFs Of Type $II_1$ As A Minimal Option

HFF $II_1$ property for the Clifford algebra of WCW means a definite distinction from the ordinary Clifford algebra defined by the fermionic oscillator operators since the trace of the unit matrix of the Clifford algebra is normalized to one. This does not affect the anti-commutation relations at the basic level and delta functions can appear in them at space-time level. At the level of momentum space $I_\infty$ property requires discrete basis and anti-commutators involve only Kronecker deltas. This conforms with the fact that HFF of type $II_1$ can be identified as the Clifford algebra associated with a separable Hilbert space.

### 5.2.1 *$II_\infty$ factor or direct sum of HFFs of type $II_1$?*

The expectation is that super-symplectic algebra is a direct sum over HFFs of type $\mathrm{II}_1$ labeled by the radial conformal weight. In the same manner the algebra defined by fermionic anti-commutation relations at partonic 2-surface would decompose to a direct sum of algebras labeled by the conformal weight associated with the light-like coordinate of $X_l^3$. Super-conformal symmetry suggests that also the configuration space degrees of freedom correspond to a direct sum of HFFs of type $II_1$.

One can of course ask why not $II_\infty = I_\infty \times II_1$ structures so that one would have single factor rather than a direct sum of factors.

1. The physical motivation is that the direct sum property allow to decompose M-matrix to direct summands associated with various sectors with weights whose moduli squared have an interpretation in terms of the density matrix. This is also consistent with p-adic thermodynamics where conformal weights take the place of energy eigen values.

2. $II_\infty$ property would predict automorphisms scaling the trace by an arbitrary positive real number $\lambda \in R_+$. These automorphisms would require the scaling of the trace of the projectors of Clifford algebra having values in the range $[0, 1]$ and it is difficult to imagine how these automorphisms could be realized geometrically.

### 5.2.2 *How HFF property reflects itself in the construction of geometry of WCW?*

The interesting question is what HFF property and finite measurement resolution realizing itself as the use of projection operators means concretely at the level of WCW geometry.

Super-Hamiltonians define the Clifford algebra of the configuration space. Super-conformal symmetry suggests that the unavoidable restriction to projection operators instead of complex rays is realized also WCW degrees of freedom. Of course, infinite precision in the determination of the shape of 3-surface would be physically a completely unrealistic idea.

In the fermionic situation the anti-commutators for the gamma matrices associated with WCW individual Hamiltonians in 3-D sense are replaced with anti-commutators where Hamiltonians are replaced with projectors to subspaces of the space spanned by Hamiltonians. This projection is realized by restricting the anti-commutator to partonic 2-surfaces so that the anti-commutator depends only the restriction of the Hamiltonian to those surfaces.

What is interesting that the measurement resolution has a concrete particle physical meaning since the parton content of the system characterizes the projection. The larger the number of partons, the better the resolution about WCW degrees of freedom is. The degeneracy of WCW metric would be interpreted in terms of finite measurement resolution inherent to HFFs of type $II_1$, which is not due to Jones inclusions but due to the fact that one can project only to infinite-D subspaces rather than complex rays.

Effective 2-dimensionality in the sense that WCW Hamiltonians reduce to functionals of the partonic 2-surfaces of $X^3_l$ rather than functionals of $X^3_l$ could be interpreted in this manner. For a wide class of Hamiltonians actually effective 1-dimensionality holds true in accordance with conformal invariance.

The generalization of WCW Hamiltonians and super-Hamiltonians by allowing integrals over the 2-D boundaries of the patches of $X^3_l$ would be natural and is suggested by the requirement of discretized 3-dimensionality at the level of WCW.

By quantum classical correspondence the inclusions of HFFs related to the measurement resolution should also have a geometric description. Measurement resolution corresponds to braids in given time scale and as already explained there is a hierarchy of braids in time scales coming as negative powers of two corresponding to the addition of zero energy components to positive/negative energy state. Note however that particle reactions understood as decays and fusions of braid strands could also lead to a notion of measurement resolution.

## 5.3 Bott Periodicity, Its Generalization, And Dimension $D = 8$As An Inherent Property Of The Hyper-Finite $II_1$ Factor

Hyper-finite $II_1$ factor can be constructed as infinite-dimensional tensor power of the Clifford algebra $M_2(C) = C(2)$ in dimension $D = 2$. More precisely, one forms the union of the Clifford algebras $C(2n) = C(2)^{\otimes n}$ of $2n$-dimensional spaces by identifying the element $x \in C(2n)$ as block diagonal elements $diag(x, x)$ of $C(2(n + 1))$. The union of these algebras is completed in weak operator topology and can be regarded as a Clifford algebra of real infinite-dimensional separable Hilbert space and thus as sub-algebra of $I_\infty$. Also generalizations obtained by replacing complex numbers by quaternions and octions are possible.

1. The dimension 8 is an inherent property of the hyper-finite $II_1$ factor since Bott periodicity theorem states $C(n+8) = C_n(16)$. In other words, the Clifford algebra $C(n+8)$ is equivalent with the algebra of $16 \times 16$ matrices with entries in $C(n)$. Or articulating it still differently: $C(n+8)$ can be regarded as $16 \times 16$ dimensional module with $C(n)$ valued coefficients. Hence the elements in the union defining the canonical representation of hyper-finite $II_1$ factor are $16^n \times 16^n$ matrices having $C(0)$, $C(2)$, $C(4)$ or $C(6)$ valued valued elements.

2. The idea about a local variant of the infinite-dimensional Clifford algebra defined by power series of space-time coordinate with Taylor coefficients which are Clifford algebra elements fixes the interpretation. The representation as a linear combination of the generators of Clifford algebra of the finite-dimensional space allows quantum generalization only in the case of Minkowski spaces. However, if Clifford algebra generators are representable as gamma *matrices*, the powers of coordinate can be absorbed to the Clifford algebra and the local algebra is lost. Only if the generators are represented as quantum versions of octonions allowing no matrix representation because of their non-associativity, the local algebra makes sense. From this it is easy to deduce both quantum and classical TGD.

## 5.4 The Interpretation Of Jones Inclusions In TGD Framework

By the basic self-referential property of von Neumann algebras one can consider several interpretations of Jones inclusions consistent with sub-system-system relationship, and it is better to start by considering the options that one can imagine.

### 5.4.1 *How Jones inclusions relate to the new view about sub-system?*

Jones inclusion characterizes the imbedding of sub-system $\mathcal{N}$ to $\mathcal{M}$and $\mathcal{M}$ as a finite-dimensional $\mathcal{N}$-module is the counterpart for the tensor product in finite-dimensional context. The possibility to express $\mathcal{M}$ as $\mathcal{N}$ module $\mathcal{M}/\mathcal{N}$ states fractality and can be regarded as a kind of self-referential "Brahman=Atman identity" at the level of infinite-dimensional systems.

Also the mysterious looking almost identity $CH^2 = CH$ for the WCW would fit nicely with the identity $M \oplus M = M$. $M \otimes M \subset M$ in WCW Clifford algebra degrees of freedom is also implied and the construction of $\mathcal{M}$ as a union of tensor powers of $C(2)$ suggests that $M \otimes M$ allows $\mathcal{M} : \mathcal{N} = 4$ inclusion to $\mathcal{M}$. This paradoxical result conforms with the strange self-referential property of factors of $II_1$.

The notion of many-sheeted space-time forces a considerable generalization of the notion of sub-system and simple tensor product description is not enough. Topological picture based on the length scale resolution suggests even the possibility of entanglement between sub-systems of un-entangled sub-systems. The possibility that hyper-finite $II_1$-factors describe the physics of TGD also in bosonic degrees of freedom is suggested by WCW super-symmetry. On the other hand, bosonic degrees could naturally correspond to $I_\infty$ factor so that hyper-finite $II_\infty$ would be the net result.

The most general view is that Jones inclusion describes all kinds of sub-system-system inclusions. The possibility to assign conformal field theory to the inclusion gives hopes of rather detailed view about dynamics of inclusion.

1. The topological condensation of space-time sheet to a larger space-time sheet mediated by wormhole contacts could be regarded as Jones inclusion. $\mathcal{N}$ would correspond to the condensing space-time sheet, $\mathcal{M}$ to the system consisting of both space-time sheets, and $\sqrt{\mathcal{M} : \mathcal{N}}$ would characterize the number of quantum spinorial degrees of freedom associated with the interaction between space-time sheets. Note that by general results $\mathcal{M} : \mathcal{N}$ characterizes the fractal dimension of quantum group ($\mathcal{M} : \mathcal{N} < 4$) or Kac-Moody algebra ($\mathcal{M} : \mathcal{N} = 4$) [B5] .

2. The branchings of space-time sheets (space-time surface is thus homologically like branching like of Feynman diagram) correspond naturally to n-particle vertices in TGD framework. What is nice is that vertices are nice 2-dimensional surfaces rather than surfaces having typically pinch singularities. Jones inclusion would naturally appear as inclusion of operator spaces $\mathcal{N}_i$ (essentially Fock spaces for fermionic oscillator operators) creating states at various lines as sub-spaces $N_i \subset M$ of operators creating states in common von Neumann factor $\mathcal{M}$. This would allow to construct vertices and vertices in natural manner using quantum groups or Kac-Moody algebras.

   The fundamental $\mathcal{N} \subset \mathcal{M} \subset \mathcal{M} \otimes_N \mathcal{M}$ inclusion suggests a concrete representation based on the identification $N_i = M$, where $M$ is the universal Clifford algebra associated with incoming line and $\mathcal{N}$ is defined by the condition that $\mathcal{M}/\mathcal{N}$ is the quantum variant of Clifford algebra of $H$. $N$-particle vertices could be defined as traces of Connes products of the operators creating incoming and outgoing states. It will be found that this leads to a master formula for S-matrix if the generalization of the old-fashioned string model duality implying that all generalized Feynman diagrams reduce to diagrams involving only single vertex is accepted.

3. If 4-surfaces can branch as the construction of vertices requires, it is difficult to argue that 3-surfaces and partonic/stringy 2-surfaces could not do the same. As a matter fact, the master formula for S-matrix to be discussed later explains the branching of 4-surfaces as an apparent effect. Despite this one can consider the possibility that this kind of joins are possible so that a new kind of mechanism of topological condensation would become possible. 3-space-sheets and partonic 2-surfaces whose p-adic fractality is characterized by different p-adic primes could be connected by "joins" representing branchings of 2-surfaces. The structures formed by soap film foam provide a very concrete illustration about what would happen. In the TGD based model of hadrons [K7] it has been assumed that join along boundaries bonds (JABs) connect quark space-time space-time sheets to the hadronic space-time sheet. The problem is that, at least for identical primes, the formation of join along boundaries bond fuses two systems to single bound state. If JABs are replaced joins, this objection is circumvented.

4. The space-time correlate for the formation of bound states is the formation of JABs. Standard intuition tells that the number of degrees of freedom associated with the bound state is smaller than the number of degrees of freedom associated with the pair of free systems. Hence the inclusion of the bound state to the tensor product could be regarded as Jones inclusion. On the other hand, one could argue that the JABs carry additional vibrational degrees of freedom so that the idea about reduction of degrees of freedom might be wrong: free system could be regarded as sub-system of bound state by Jones inclusion. The self-referential holographic properties of von Neumann algebras allow both interpretations: any system can be regarded as sub-system of any system in accordance with the bootstrap idea.

5. Maximal deterministic regions inside given space-time sheet bounded by light-like causal determinants define also sub-systems in a natural manner and also their inclusions would naturally correspond to Jones inclusions.

6. The TGD inspired model for topological quantum computation involves the magnetic flux tubes defined by join along boundaries bonds connecting space-time sheets having light-like boundaries. These tubes condensed to background 3-space can become linked and knotted and code for quantum computations in this manner. In this case the addition of new strand to the system corresponds to Jones inclusion in the hierarchy associated with inclusion $\mathcal{N} \subset \mathcal{M}$. The anyon states associated with strands would be represented by a finite tensor product of quantum spinors assignable to $\mathcal{M}/\mathcal{N}$ and representing quantum counterpart of $H$-spinors.

One can regard $\mathcal{M} : \mathcal{N}$ degrees of freedom correspond to quantum group or Kac-Moody degrees of freedom. Quantum group degrees of freedom relate closely to the conformal and topological degrees of freedom as the connection of $II_1$ factors with topological quantum field theories and braid matrices suggests itself. For the canonical inclusion this factorization would correspond to factorization of quantum $H$-spinor from WCW spinor .

A more detailed study of canonical inclusions to be carried out later demonstrates what this factorization corresponds at the space-time level to a formation of space-time sheets which can be regarded as multiple coverings of $M^4$ and $CP_2$ with invariance group $G = G_a \times G_b \subset SL(2,C) \times SU(2)$, $SU(2) \subset SU(3)$. The unexpected outcome is that Planck constants assignable to $M^4$ and $CP_2$ degrees of freedom depend on the canonical inclusions. The existence of macroscopic quantum phases with arbitrarily large Planck constants is predicted.

It would seem possible to assign the $\mathcal{M} : \mathcal{N}$ degrees quantum spinorial degrees of freedom to the interface between subsystems represented by $\mathcal{N}$ and $\mathcal{M}$. The interface could correspond to the wormhole contacts, joins, JABs, or light-like causal determinants serving as boundary between maximal deterministic regions, etc... In terms of the bipartite diagrams representing the inclusions, joins (say) would correspond to the edges connecting white vertices representing sub-system (the entire system without the joins) to black vertices (entire system).

### 5.4.2 *About the interpretation of $\mathcal{M} : \mathcal{N}$ degrees of freedom*

The Clifford algebra $\mathcal{N}$ associated with a system formed by two space-time sheet can be regarded as $1 \leq \mathcal{M} : \mathcal{N} \leq 4$-dimensional module having $\mathcal{N}$ as its coefficients. It is possible to imagine several interpretations the degrees of freedom labeled by $\beta$.

1. The $\beta = \mathcal{M} : \mathcal{N}$ degrees of freedom could relate to the interaction of the space-time sheets. Beraha numbers appear in the construction of S-matrices of topological quantum field theories and an interpretation in terms of braids is possible. This would suggest that the interaction between space-time sheets can be described in terms of conformal quantum field theory and the S-matrices associated with braids describe this interaction. Jones inclusions would characterize the effective number of active conformal degrees of freedom. At $n = 3$ limit these degrees of freedom disappear completely since the conformal field theory defined by the Chern-Simons action describing this interaction would become trivial ($c = 0$ as will be found).

2. The interpretation in terms of imbedding space Clifford algebra would suggest that $\beta$-dimensional Clifford algebra of $\sqrt{\beta}$-dimensional spinor space is in question. For $\beta = 4$

the algebra would be the Clifford algebra of 2-dimensional space. $\mathcal{M}/\mathcal{N}$ would have interpretation as complex quantum spinors with components satisfying $z_1z_2 = qz_2z_1$ and its conjugate and having fractal complex dimension $\sqrt{\beta}$. This would conform with the effective 2-dimensionality of TGD. For $\beta < 4$ the fractal dimension of partonic quantum spinors defining the basic conformal fields would be reduced and become $d = 1$ for $n = 3$: the interpretation is in terms of strong correlations caused by the non-commutativity of the components of quantum spinor. For number theoretical generalizations of infinite-dimensional Clifford algebras $Cl(C)$ obtained by replacing $C$ with Abelian complexification of quaternions or octonions one would obtain higher-dimensional spinors.

## 5.5 WCW, Space-Time, Imbedding Space AndHyper-Finite Type $II_1$ Factors

The preceding considerations have by-passed the question about the relationship of WCW tangent space to its Clifford algebra. Also the relationship between space-time and imbedding space and their quantum variants could be better. In particular, one should understand how effective 2-dimensionality can be consistent with the 4-dimensionality of space-time.

### 5.5.1 *Super-conformal symmetry and WCW Poisson algebra as hyper-finite type $II_1$ factor*

It would be highly desirable to achieve also a description of the WCW degrees of freedom using von Neumann algebras. Super-conformal symmetry relating fermionic degrees of freedom and WCW degrees of freedom suggests that this might be the case. Super-symplectic algebra has as its generators configuration space Hamiltonians and their super-counterparts identifiable as $CH$ gamma matrices. Super-symmetry requires that the Clifford algebra of $CH$ and the Hamiltonian vector fields of $CH$ with symplectic central extension both define hyper-finite $II_1$ factors. By super-symmetry Poisson bracket corresponds to an anti-commutator for gamma matrices. The ordinary quantized version of Poisson bracket is obtained as $\{P_i, Q_j\} \rightarrow [P_i, Q_j] = J_{ij}Id$. Finite trace version results by assuming that $Id$ corresponds to the projector $CH$ Clifford algebra having unit norm. The presence of zero modes means direct integral over these factors.

WCW gamma matrices anti-commuting to identity operator with unit norm corresponds to the tangent space $T(CH)$ of $CH$. Thus it would be not be surprising if $T(CH)$ could be imbedded in the sigma matrix algebra as a sub-space of operators defined by the gamma matrices generating this algebra. At least for $\beta = 4$ construction of hyper-finite $II_1$ factor this definitely makes sense.

The dimension of WCW defined as the trace of the projection operator to the sub-space spanned by gamma matrices is obviously zero. Thus WCW has in this sense the dimensionality of single space-time point. This sounds perhaps absurd but the generalization of the number concept implied by infinite primes indeed leads to the view that single space-time point is infinitely structured in the number theoretical sense although in the real sense all states of the point are equivalen. The reason is that there is infinitely many numbers expressible as ratios of infinite integers having unit real norm in the real sense but having different p-adic norms.

### 5.5.2 *How to understand the dimensions of space-time and imbedding space?*

One should be able to understand the dimensions of 3-space, space-time and imbedding space in a convincing matter in the proposed framework. There is also the question whether space-time and imbedding space emerge uniquely from the mathematics of von Neumann algebras alone.

*1. The dimensions of space-time and imbedding space*

Two sub-sequent inclusions dual to each other define a special kind of inclusion giving rise to a quantum counterpart of $D = 4$ naturally. This would mean that space-time is something which emerges at the level of cognitive states.

The special role of classical division algebras in the construction of quantum TGD [K10] , $D = 8$ Bott periodicity generalized to quantum context, plus self-referential property of type $II_1$ factors might explain why 8-dimensional imbedding space is the only possibility.

State space has naturally quantum dimension $D \leq 8$ as the following simple argument shows. The space of quantum states has quark and lepton sectors which both are super-symmetric implying $D \leq 4$ for each. Since these sectors correspond to different Hamiltonian algebras (triality one for quarks and triality zero for leptonic sector), the state space has quantum dimension $D \leq 8$.

*2. How the lacking two space-time dimensions emerge?*

3-surface is the basic dynamical unit in TGD framework. This seems to be in conflict with the effective 2-dimensionality [K10] meaning that partonic 2-surface code for quantum states, and with the fact that hyper-finite $II_1$ factors have intrinsic quantum dimension 2.

A possible resolution of the problem is that the foliation of 3-surface by partonic two-surfaces defines a one-dimensional direct integral of isomorphic hyper-finite type $II_1$ factors, and the zero mode labeling the 2-surfaces in the foliation serves as the third spatial coordinate. For a given 3-surface the contribution to the WCW metric can come only from 2-D partonic surfaces defined as intersections of 3-D light-like CDs with $X^7_\pm$ [K3] . Hence the direct integral should somehow relate to the classical non-determinism of Kähler action.

1. The one-parameter family of intersections of light-like CD with $X^7_\pm$ inside $X^4 \cap X^7_\pm$ could indeed be basically due to the classical non-determinism of Kähler action. The contribution to the metric from the normal light-like direction to $X^3 = X^4 \cap X^7_\pm$ can cause the vanishing of the metric determinant $\sqrt{g_4}$ of the space-time metric at $X^2 \subset X^3$ under some conditions on $X^2$. This would mean that the space-time surface $X^4(X^3)$ is not uniquely determined by the minimization principle defining the value of the Kähler action, and the complete dynamical specification of $X^3$ requires the specification of partonic 2-surfaces $X^2_i$ with $\sqrt{g_4} = 0$.

2. The known solutions of field equations [K1] define a double foliation of the space-time surface defined by Hamilton-Jacobi coordinates consisting of complex transversal coordinate and two light-like coordinates for $M^4$ (rather than space-time surface). Number theoretical considerations inspire the hypothesis that this foliation exists always [K10] . Hence a natural hypothesis is that the allowed partonic 2-surfaces correspond to the 2-surfaces in the restriction of the double foliation of the space-time surface by partonic 2-surfaces to $X^3$, and are thus locally parameterized by single parameter defining the third spatial coordinate.

3. There is however also a second light-like coordinate involved and one might ask whether both light-like coordinates appear in the direct sum decomposition of $II_1$ factors defining $T(CH)$. The presence of two kinds of light-like CDs would provide the lacking two space-time coordinates and quantum dimension $D = 4$ would emerge at the limit of full non-determinism. Note that the duality of space-like partonic and light-like stringy 2-surfaces conforms with this interpretation since it corresponds to a selection of partonic/stringy 2-surface inside given 3-D CD whereas the dual pairs correspond to different CDs.

4. That the quantum dimension would be $2D_q = \beta < 4$ above $CP_2$ length scale conforms with the fact that non-determinism is only partial and time direction is dynamically frozen to a high degree. For vacuum extremals there is strong non-determinism but in this case there is no real dynamics. For $CP_2$ type extremals, which are not vacuum extremals as far action and small perturbations are considered, and which correspond to $\beta = 4$ there is a complete non-determinism in time direction since the $M^4$ projection of the extremal is a light-like random curve and there is full 4-D dynamics. Light-likeness gives rise to conformal symmetry consistent with the emergence of Kac Moody algebra [K1] .

*3. Time and cognition*

In a completely deterministic physics time dimension is strictly speaking redundant since the information about physical states is coded by the initial values at 3-dimensional slice of space-time. Hence the notion of time should emerge at the level of cognitive representations possible by to the non-determinism of the classical dynamics of TGD.

Since Jones inclusion means the emergence of cognitive representation, the space-time view about physics should correspond to cognitive representations provided by Feynman diagram states with zero energy with entanglement defined by a two-sided projection of the lowest level S-matrix.

These states would represent the "laws of quantum physics" cognitively. Also space-time surface serves as a classical correlate for the evolution by quantum jumps with maximal deterministic regions serving as correlates of quantum states. Thus the classical non-determinism making possible cognitive representations would bring in time. The fact that quantum dimension of space-time is smaller than $D = 4$ would reflect the fact that the loss of determinism is not complete.

*4. Do space-time and imbedding space emerge from the theory of von Neumann algebras and number theory?*

The considerations above force to ask whether the notions of space-time and imbedding space emerge from von Neumann algebras as predictions rather than input. The fact that it seems possible to formulate the S-matrix and its generalization in terms of inherent properties of infinite-dimensional Clifford algebras suggest that this might be the case.

### 5.5.3 *Inner automorphisms as universal gauge symmetries?*

The continuous outer automorphisms $\Delta^{it}$ of HFFs of type III are not completely unique and one can worry about the interpretation of the inner automorphisms. A possible resolution of the worries is that inner automorphisms act as universal gauge symmetries containing various super-conformal symmetries as a special case. For hyper-finite factors of type $II_1$ in the representation as an infinite tensor power of $M_2(C)$ this would mean that the transformations non-trivial in a finite number of tensor factors only act as analogs of local gauge symmetries. In the representation as a group algebra of $S_\infty$ all unitary transformations acting on a finite number of braid strands act as gauge transformations whereas the infinite powers $P \times P \times ...$, $P \in S_n$, would act as counterparts of global gauge transformations. In particular, the Galois group of the closure of rationals would act as local gauge transformations but diagonally represented finite Galois groups would act like global gauge transformations and periodicity would make possible to have finite braids as space-time correlates without a loss of information.

### 5.5.4 *Do unitary isomorphisms between tensor powers of $II_1$ define vertices?*

What would be left would be the construction of unitary isomorphisms between the tensor products of the HFFs of type $II_1 \otimes I_n = II_1$ at the partonic 2-surfaces defining the vertices. This would be the only new element added to the construction of braiding $M$-matrices.

As a matter fact, this element is actually not completely new since it generalizes the fusion rules of conformal field theories, about which standard example is the fusion rule $\phi_i = {c_i}^{jk} \phi_j \phi_k$ for primary fields. These fusion rules would tell how a state of incoming HFF decomposes to the states of tensor product of two outgoing HFFs.

These rules indeed have interpretation in terms of Connes tensor products $\mathcal{M} \otimes_{\mathcal{N}} ... \otimes_{\mathcal{N}} \mathcal{M}$ for which the sub-factor $\mathcal{N}$ takes the role of complex numbers [A13] so that one has $\mathcal{M}$ becomes $\mathcal{N}$ bimodule and "quantum quantum states" have $\mathcal{N}$ as coefficients instead of complex numbers. In TGD framework this has interpretation as quantum measurement resolution characterized by $\mathcal{N}$ (the group $G$ characterizing leaving the elements of $\mathcal{N}$ invariant defines the measured quantum numbers).

## 5.6 Quaternions, Octonions, And Hyper-Finite Type $II_1$Factors

Quaternions and octonions as well as their hyper counterparts obtained by multiplying imaginary units by commuting $\sqrt{-1}$ and forming a sub-space of complexified division algebra, are in in a central role in the number theoretical vision about quantum TGD [K10] . Therefore the question arises whether complexified quaternions and perhaps even octonions could be somehow inherent properties of von Neumann algebras. One can also wonder whether the quantum counterparts of quaternions and octonions could emerge naturally from von Neumann algebras. The following considerations allow to get grasp of the problem.

### 5.6.1 *Quantum quaternions and quantum octonions*

Quantum quaternions have been constructed as deformation of quaternions [A36] . The key observation that the Glebsch Gordan coefficients for the tensor product $3 \otimes 3 = 5 \oplus \oplus 3 \oplus 1$ of spin 1 representation of $SU(2)$ with itself gives the anti-commutative part of quaternionic product as spin 1 part in the decomposition whereas the commutative part giving spin 0 representation is identifiable as the scalar product of the imaginary parts. By combining spin 0 and spin 1 representations, quaternionic product can be expressed in terms of Glebsh-Gordan coefficients. By replacing GGC:s by their quantum group versions for group $sl(2)_q$, one obtains quantum quaternions.

There are two different proposals for the construction of quantum octonions [A25, A1] . Also now the idea is to express quaternionic and octonionic multiplication in terms of Glebsch-Gordan coefficients and replace them with their quantum versions.

1. The first proposal [A25] relies on the observation that for the tensor product of $j = 3$ representations of $SU(2)$ the Glebsch-Gordan coefficients for $7 \otimes 7 \rightarrow 7$ in $7 \otimes 7 = 9 \oplus 7 \oplus 5 \oplus 3 \oplus 1$ defines a product, which is equivalent with the antisymmetric part of the product of octonionic imaginary units. As a matter fact, the antisymmetry defines 7-dimensional Malcev algebra defined by the anti-commutator of octonion units and satisfying b definition the identity

$$[[x, y, z], x] = [x, y, [x, z]] \quad , \quad [x, y, z] \equiv [x, [y, z]] + [y, [z, x]] + [z, [x, y]] \quad . \tag{5.1}$$

   7-element Malcev algebra defining derivations of octonionic algebra is the only complex Malcev algebra not reducing to a Lie algebra. The $j = 0$ part of the product corresponds also now to scalar product for imaginary units. Octonions are constructed as sums of $j = 0$ and $j = 3$ parts and quantum Glebsch-Gordan coefficients define the octonionic product.

2. In the second proposal [A1] the quantum group associated with $SO(8)$ is used. This representation does not allow unit but produces a quantum version of octonionic triality assigning to three octonions a real number.

### 5.6.2 *Quaternionic or octonionic quantum mechanics?*

There have been numerous attempts to introduce quaternions and octonions to quantum theory. Quaternionic or octonionic quantum mechanics, which means the replacement of the complex numbers as coefficient field of Hilbert space with quaternions or octonions, is the most obvious approach (for example and references to the literature see for instance [A23] .

In both cases non-commutativity poses serious interpretational problems. In the octonionic case the non-associativity causes even more serious obstacles [B6, A23] , [B6] .

1. Assuming that an orthonormalized state basis with respect to an octonion valued inner product has been found, the multiplication of any basis with octonion spoils the orthonormality. The proposal to circumvent this difficulty discussed in [B6] , [B6] eliminates non-associativity by assuming that octonions multiply states one by one (rather than multiplying each other before multiplying the state). Effectively this means that octonions are replaced with $8 \times 8$-matrices.

2. The definition of the tensor product leads also to difficulties since associativity is lost (recall that Yang-Baxter equation codes for associativity in case of braid statistics [A9] ).

3. The notion of hermitian conjugation is problematic and forces a selection of a preferred imaginary unit, which does not look nice. Note however that the local selection of a preferred imaginary unit is in a key role in the proposed construction of space-time surfaces as

   hyper-quaternionic or co-hyper-quaternionic surfaces and allows to interpret space-time surfaces either as surfaces in 8-D Minkowski space $M^8$ of hyper-octonions or in $M^4 \times CP_2$. This selection turns out to have quite different interpretation in the proposed framework.

### 5.6.3 *Hyper-finite factor $II_1$ has a natural Hyper-Kähler structure*

In the case of hyper-finite factors of type $II_1$ quaternions a more natural approach is based on the generalization of the Hyper-Kähler structure rather than quaternionic quantum mechanics.

The reason is that also WCW tangent space should and is expected to have this structure [K3] . The Hilbert space remains a complex Hilbert space but the quaternionic units are represented as operators in Hilbert space. The selection of the preferred unit is necessary and natural. The identity operator representing quaternionic real unit has trace equal to one, is expected to give rise to the series of quantum quaternion algebras in terms of inclusions $\mathcal{N} \subset \mathcal{M}$ having interpretation as $N$-modules.

The representation of the quaternion units is rather explicit in the structure of hyper-finite $II_1$ factor. The $\mathcal{M} : \mathcal{N} \equiv \beta = 4$ hierarchical construction can be regarded as Connes tensor product of infinite number of 4-D Clifford algebras of Euclidian plane with Euclidian signature of metric ($diag(-1,-1)$). This algebra is nothing but the quaternionic algebra in the representation of quaternionic imaginary units by Pauli spin matrices multiplied by $i$.

The imaginary unit of the underlying complex Hilbert space must be chosen and there is whole sphere $S^2$ of choices and in every point of WCW the choice can be made differently. The space-time correlate for this local choice of preferred hyper-octonionic unit [K10] . At the level of WCW geometry the quaternion structure of the tangent space means the existence of Hyper-Kähler structure guaranteeing that WCW has a vanishing Einstein tensor. It it would not vanish, curvature scalar would be infinite by symmetric space property (as in case of loop spaces) and induce a divergence in the functional integral over 3-surfaces from the expansion of $\sqrt{g}$ [K3] .

The quaternionic units for the $II_1$ factor, are simply limiting case for the direct sums of $2\times 2$ units normalized to one. Generalizing from $\beta = 4$ to $\beta < 4$, the natural expectation is that the representation of the algebra as $\beta = \mathcal{M} : \mathcal{N}$-dimensional $\mathcal{N}$-module gives rise to quantum quaternions with quaternion units defined as infinite sums of $\sqrt{\beta}\times\sqrt{\beta}$ matrices.

At Hilbert space level one has an infinite Connes tensor product of 2-component spinor spaces on which quaternionic matrices have a natural action. The tensor product of Clifford algebras gives the algebra of $2\times 2$ quaternionic matrices acting on 2-component quaternionic spinors (complex 4-component spinors). Thus double inclusion could correspond to (hyper-)quaternionic structure at space-time level. Note however that the correspondence is not complete since hyper-quaternions appear at space-time level and quaternions at Hilbert space level.

#### 5.6.4 *Von Neumann algebras and octonions*

The octonionic generalization of the Hyper-Kähler manifold does not make sense as such since octonionic units are not representable as linear operators. The allowance of anti-linear operators inherently present in von Neumann algebras could however save the situation. Indeed, the Cayley-Dickson construction for the division algebras (for a nice explanation see [A21] ), which allows to extend any $^*$ algebra, and thus also any von Neumann algebra, by adding an imaginary unit it and identified as $^*$, comes in rescue.

The basic idea of the Cayley-Dickson construction is following. The $^*$ operator, call it $J$, representing a conjugation defines an *anti-linear* operator in the original algebra $A$. One can extend $A$ by adding this operator as a new element to the algebra. The conditions satisfied by $J$ are

$$a(Jb) = J(a^*b) \quad , \quad (aJ)b = (ab^*)J \quad , \quad (Ja)(bJ^{-1}) = (ab)^* \quad . \tag{5.2}$$

In the associative case the conditions are equivalent to the first condition.

It is intuitively clear that this addition extends the hyper-Kähler structure to an octonionic structure at the level of the operator algebra. The quantum version of the octonionic algebra is fixed by the quantum quaternion algebra uniquely and is consistent with the Cayley-Dickson construction. It is not clear whether the construction is equivalent with either of the earlier proposals [A25, A1] . It would however seem that the proposal is simpler.

#### 5.6.5 *Physical interpretation of quantum octonion structure*

Without further restrictions the extension by $J$ would mean that vertices contain operators, which are superpositions of linear and anti-linear operators. This would give superpositions of states and their time-reversals and mean that state could be a superposition of states with opposite values of say fermion numbers. The problem disappears if either the linear operators $A$ or anti-linear operators $JA$ can be used to construct physical states from vacuum. The fact, that space-time surfaces are either hyper-quaternionic or co-hyper-quaternionic, is a space-time correlate for this restriction.

The $HQ - coHQ$ duality discussed in [K10] states that the descriptions based on hyper-quaternionic and co-hyper-quaternionic surfaces are dual to each other. The duality can have two meanings.

1. The vacuum is invariant under $J$ so that one can use either complexified quaternionic operators $A$ or their co-counterparts of form $JA$ to create physical states from vacuum.

2. The vacuum is not invariant under $J$. This could relate to the breaking of $CP$ and $T$ invariance known to occur in meson-antimeson systems. In TGD framework two kinds of vacua are predicted corresponding intuitively to vacua in which either the product of all positive or negative energy fermionic oscillator operators defines the vacuum state, and these two vacua could correspond to a vacuum and its $J$ conjugate, and thus to positive and negative energy states. In this case the two state spaces would not be equivalent although the physics associated with them would be equivalent.

The considerations of [K10] related to the detailed dynamics of $HQ - coHQ$ duality demonstrate that the variational principles defining the dynamics of hyper-quaternionic and co-hyper-quaternionic space-time surfaces are antagonistic and correspond to world as seen by a conscientous book-keeper on one hand and an imaginative artist on the other hand. $HQ$ case is conservative: differences measured by the magnitude of Kähler action tend to be minimized, the dynamics is highly predictive, and minimizes the classical energy of the initial state. $coHQ$ case is radical: differences are maximized (this is what the construction of sensory representations would require). The interpretation proposed in [K10] was that the two space-time dynamics are just different predictions for what would happen (has happened) if no quantum jumps would occur (had occurred). A stronger assumption is that these two views are associated with systems related by time reversal symmetry.

What comes in mind first is that this antagonism follows from the assumption that these dynamics are actually time-reversals of each other with respect to $M^4$ time (the rapid elimination of differences in the first dynamics would correspond to their rapid enhancement in the second dynamics). This is not the case so that $T$ and $CP$ symmetries are predicted to be broken in accordance with the $CP$ breaking in meson-antimeson systems [K6] and cosmological matter-antimatter asymmetry [K8] .

## 5.7 Does The Hierarchy Of Infinite Primes Relate To The Hierarchy Of $II_1$ Factors?

The hierarchy of Feynman diagrams accompanying the hierarchy defined by Jones inclusions $\mathcal{M}_0 \subset \mathcal{M}_1 \subset ...$ gives a concrete representation for the hierarchy of cognitive dynamics providing a representation for the material world at the lowest level of the hierarchy. This hierarchy seems to relate directly to the hierarchy of space-time sheets.

Also the construction of infinite primes [K9] leads to an infinite hierarchy. Infinite primes at the lowest level correspond to polynomials of single variable $x_1$ with rational coefficients, next level to polynomials $x_1$ for which coefficients are rational functions of variable $x_2$, etc... so that a natural ordering of the variables is involved.

If the variables $x_i$ are hyper-octonions (subs-space of complexified octonions for which elements are of form $x + \sqrt{-1}y$, where $x$ is real number and $y$ imaginary octonion and $\sqrt{-1}$ is commuting imaginary unit, this hierarchy of states could provide a realistic representation of physical states as far as quantum numbers related to imbedding space degrees of freedom are considered in $M^8$ picture dual to $M^4 \times CP_2$ picture [K10] . Infinite primes are mapped to space-time surfaces in a manner analogous to the mapping of polynomials to the loci of their zeros so that infinite primes, integers, and rationals become concrete geometrical objects.

Infinite primes are also obtained by a repeated second quantization of a super-symmetric arithmetic quantum field theory. Infinite rational numbers correspond in this description to pairs of positive energy and negative energy states of opposite energies having interpretation as pairs of initial and final states so that higher level states indeed represent transitions between the states.

For these reasons this hierarchy has been interpreted as a correlate for a cognitive hierarchy coding information about quantum dynamics at lower levels. This hierarchy has also been assigned with the hierarchy of space-time sheets. Just as the hierarchy of generalized Feynman diagrams provides self representations of the lowest matter level and is coded by it, finite primes code the hierarchy of infinite primes.

Infinite primes, integers, and rationals have finite p-adic norms equal to 1, and one can wonder whether a Hilbert space like structure with dimension given by an infinite prime or integer makes sense, and whether it has anything to do with the Hilbert space for which dimension is infinite in the sense of the limiting value for a dimension of sub-space. The Hilbert spaces with dimension equal to infinite prime would define primes for the tensor product of these spaces. The dimension of this kind of space defined as any p-adic norm would be equal to one.

One cannot exclude the possibility that infinite primes could express the infinite dimensions of hyper-finite $III_1$ factors, which cannot be excluded and correspond to that part of quantum TGD which relates to the imbedding space rather than space-time surface. Indeed, infinite primes code naturally for the quantum numbers associated with the imbedding space. Secondly, the appearance of 7-D light-like causal determinants $X^7_\pm = M^4_\pm \times CP_2$ forming nested structures in the construction of S-matrix brings in mind similar nested structures of algebraic quantum field theory [B1] . If this is were the case, the hierarchy of Beraha numbers possibly associated with the phase resolution could correspond to hyper-finite factors of type $II_1$, and the decomposition of space-time surface to regions labeled by p-adic primes and characterized by infinite primes could correspond to hyper-finite factors of type $III_1$ and represent imbedding space degrees of freedom.

The state space would in this picture correspond to the tensor products of hyper-finite factors of type $II_1$ and $III_1$ (of course, also factors $I_n$ and $I_\infty$ are also possible). $III_1$ factors could be assigned to the sub-WCWs defined by 3-surfaces in regions of $M^4$ expressible in terms of unions and intersections of $X^7_\pm = M^4_\pm \times CP_2$. By conservation of four-momentum, bounded regions of this kind are possible only for the states of zero net energy appearing at the higher levels of hierarchy. These sub-WCWs would be characterized by the positions of the tips of light cones $M^4_\pm \subset M^4$ involved. This indeed brings in continuous spectrum of four-momenta forcing to introduce non-separable Hilbert spaces for momentum eigen states and necessitating $III_1$ factors. Infinities would be avoided since the dynamics proper would occur at the level of space-time surfaces and involve only $II_1$ factors.

# 6 HFFs Of Type $III$ And TGD

One can imagine several manners for how HFFs of type $III$ could emerge in TGD although the proposed view about $M$-matrix in zero energy ontology suggests that HFFs of type $III_1$ should be only an auxiliary tool at best. Same is suggested with interpretational problems associated with them. Both TGD inspired quantum measurement theory, the idea about a variant of HFF of type $II_1$ analogous to a local gauge algebra, and some other arguments, suggest that HFFs of type $III$ could be seen as a useful idealization allowing to make non-trivial conjectures both about quantum TGD and about HFFs of type III. Quantum fields would correspond to HFFs of type $III$ and $II_\infty$ whereas physical states ($M$-matrix) would correspond to HFF of type $II_1$. I have summarized first the problems of $III_1$ factors so that reader can decide whether the further reading is worth of it.

## 6.1 Problems Associated With The Physical Interpretation Of $III_1$ Factors

Algebraic quantum field theory approach [B3, B1] has led to a considerable understanding of relativistic quantum field theories in terms of hyper-finite $III_1$ factors. There are however several reasons to suspect that the resulting picture is in conflict with physical intuition. Also the infinities of non-trivial relativistic QFTs suggest that something goes wrong.

#### 6.1.1 *Are the infinities of quantum field theories due the wrong type of von Neumann algebra?*

The infinities of quantum field theories involve basically infinite traces and it is now known that the algebras of observables for relativistic quantum field theories for bounded regions of Minkowski space correspond to hyper-finite $III_1$ algebras, for which non-trivial traces are always infinite. This might be the basic cause of the divergence problems of relativistic quantum field theory.

On basis of this observations there is some temptation to think that the finite traces of hyper-finite $II_1$ algebras might provide a resolution to the problems but not necessarily in QFT context. One can play with the thought that the subtraction of infinities might be actually a process in which $III_1$ algebra is transformed to $II_1$ algebra. A more plausible idea suggested by dimensional regularization is that the elimination of infinities actually gives rise to $II_1$ inclusion at the limit $\mathcal{M} : \mathcal{N} \rightarrow 4$. It is indeed known that the dimensional regularization procedure of quantum field theories can be formulated in terms of bi-algebras assignable to Feynman diagrams and [A6] and the emergence of bi-algebras suggests that a connection with $II_1$ factors and critical role of dimension $D = 4$ might exist.

#### 6.1.2 *Continuum of inequivalent representations of commutation relations*

There is also a second difficulty related to type III algebras. There is a continuum of inequivalent representations for canonical commutation relations [A31] . In thermodynamics this is blessing since temperature parameterizes these representations. In quantum field theory context situation is however different and this problem has been usually put under the rug.

#### 6.1.3 *Entanglement and von Neumann algebras*

In quantum field theories where 4-D regions of space-time are assigned to observables. In this case hyper-finite type $III_1$ von Neumann factors appear. Also now inclusions make sense and has been studiedin fact, the parameters characterizing Jones inclusions appear also now and this due to the very general properties of the inclusions.

The algebras of type $III_1$ have rather counter-intuitive properties from the point of view of entanglement. For instance, product states between systems having space-like separation are not possible at all so that one can speak of intrinsic entanglement [A19] . What looks worse is that the decomposition of entangled state to product states is highly non-unique.

Mimicking the steps of von Neumann one could ask what the notion of observables could mean in TGD framework. Effective 2-dimensionality states that quantum states can be constructed using the data given at partonic or stringy 2-surfaces. This data includes also information about normal derivatives so that 3-dimensionality actually lurks in. In any case this would mean that observables are assignable to 2-D surfaces. This would suggest that hyper-finite $II_1$ factors appear in quantum TGD at least as the contribution of single space-time surface to S-matrix is considered. The contributions for WCW degrees of freedom meaning functional (not path-) integral over 3-surfaces could of course change the situation.

Also in case of $II_1$ factors, entanglement shows completely new features which need not however be in conflict with TGD inspired view about entanglement. The eigen values of density matrices are infinitely degenerate and quantum measurement can remove this degeneracy only partially. TGD inspired theory of consciousness has led to the identification of rational (more generally algebraic entanglement) as bound state entanglement stable in state function reduction. When an infinite number of states are entangled, the entanglement would correspond to rational (algebraic number) valued traces for the projections to the eigen states of the density matrix. The symplectic transformations of $CP_2$ are almost $U(1)$ gauge symmetries broken only by classical gravitation. They imply a gigantic spin glass degeneracy which could be behind the infinite degeneracies of eigen states of density matrices in case of $II_1$ factors.

## 6.2 Quantum Measurement Theory And HFFs Of Type III

The attempt to interpret the HFFs of type $III$ in terms of quantum measurement theory based on Jones inclusions leads to highly non-trivial conjectures about these factors.

### 6.2.1 *Could the scalings of trace relate to quantum measurements?*

What should be understood is the physical meaning of the automorphism inducing the scaling of trace. In the representation based of factors based on infinite tensor powers the action of $g$ should transform single $n \times n$ matrix factor with density matrix $Id/n$ to a density matrix $e_{11}$ of a pure state.

Obviously the number of degrees of freedom is affected and this can be interpreted in terms of appearance or disappearance of correlations. Quantization and emergence of non-commutativity indeed implies the emergence of correlations and effective reduction of degrees of freedom. In particular, the fundamental quantum Clifford algebra has reduced dimension $\mathcal{M} : \mathcal{N} = r \leq 4$ instead of $r = 4$ since the replacement of complex valued matrix elements with $\mathcal{N}$ valued ones implies non-commutativity and correlations.

The transformation would be induced by the shift of finite-dimensional state to right or left so that the number of matrix factors overlapping with $I_\infty$ part increases or is reduced. Could it have interpretation in terms of quantum measurement for a quantum Clifford factor? Could quantum measurement for $\mathcal{M}/\mathcal{N}$ degrees of freedom reducing the state in these degrees of freedom to a pure state be interpreted as a transformation of single finite-dimensional matrix factor to a type I factor inducing the scaling of the trace and could the scalings associated with automorphisms of HFFs of type $III$ also be interpreted in terms of quantum measurement?

This interpretation does not as such say anything about HFF factors of type $III$ since only a decomposition of $II_1$ factor to $I_2^k$ factor and $II_1$ factor with a reduced trace of projector to the latter. However, one can ask whether the scaling of trace for HFFs of type $III$ could correspond to a situation in which infinite number of finite-dimensional factors have been quantum measured. This would correspond to the inclusion $\mathcal{N} \subset \mathcal{M}_\infty = \cup_n \mathcal{M}_n$ where $\mathcal{N} \subset \mathcal{M} \subset ...\mathcal{M}_n...$ defines the canonical inclusion sequence. Physicist can of course ask whether the presence of infinite number of $I_2$-, or more generally, $I_n$-factors is at all relevant to quantum measurement and it has already become clear that situation at the level of $M$-matrix reduces to $I_n$.

### 6.2.2 *Could the theory of HHFs of type $III$ relate to the theory of Jones inclusions?*

The idea about a connection of HFFs of type $III$ and quantum measurement theory seems to be consistent with the basic facts about inclusions and HFFs of type $III_1$.

1. Quantum measurement would scale the trace by a factor $2^k/\sqrt{\mathcal{M} : \mathcal{N}}$ since the trace would become a product for the trace of the projector to the newly born $M(2,C)^{\otimes k}$ factor and the trace for the projection to $\mathcal{N}$ given by $1/\sqrt{\mathcal{M} : \mathcal{N}}$. The continuous range of values $\mathcal{M} : \mathcal{N} \geq 4$ gives good hopes that all values of $\lambda$ are realized. The prediction would be that $2^k\sqrt{\mathcal{M} : \mathcal{N}} \geq 1$ holds always true.

2. The values $\mathcal{M} : \mathcal{N} \in \{r_n = 4cos^2(\pi/n)\}$ for which the single $M(2,C)$ factor emerges in state function reduction would define preferred values of the inverse of $\lambda = \sqrt{\mathcal{M} : \mathcal{N}/4}$ parameterizing factors $III_\lambda$. These preferred values vary in the range $[1/2, 1]$.

3. $\lambda = 1$ at the end of continuum would correspond to HFF $III_1$ and to Jones inclusions defined by infinite cyclic subgroups dense in $U(1) \subset SU(2)$ and this group combined with reflection. These groups correspond to the Dynkin diagrams $A_\infty$ and $D_\infty$. Also the classical values of $\mathcal{M} : \mathcal{N} = n^2$ characterizing the dimension of the quantum Clifford $\mathcal{M} : \mathcal{N}$ are possible. In this case the scaling of trace would be trivial since the factor $n$ to the trace would be compensated by the factor $1/n$ due to the disappearance of $\mathcal{M}/\mathcal{N}$ factor $III_1$ factor.

4. Inclusions with $\mathcal{M} : \mathcal{N} = \infty$ are also possible and they would correspond to $\lambda = 0$ so that also $III_0$ factor would also have a natural identification in this framework. These factors correspond to ergodic systems and one might perhaps argue that quantum measurement in this case would give infinite amount of information.

5. This picture makes sense also physically. p-Adic thermodynamics for the representations of super-conformal algebra could be formulated in terms of factors of type $I_\infty$ and in excellent approximation using factors $I_n$. The generation of arbitrary number of type $II_1$ factors in quantum measurement allow this possibility.

#### 6.2.3 *The end points of spectrum of preferred values of $\lambda$ are physically special*

The fact that the end points of the spectrum of preferred values of $\lambda$ are physically special, supports the hopes that this picture might have something to do with reality.

1. The Jones inclusion with $q = exp(i\pi/n)$, $n = 3$ (with principal diagram reducing to a Dynkin diagram of group SU(3)) corresponds to $\lambda = 1/2$, which corresponds to HFF $III_1$ differing in essential manner from factors $III_\lambda$, $\lambda < 1$. On the other hand, $SU(3)$ corresponds to color group which appears as an isometry group and important subgroup of automorphisms of octonions thus differs physically from the ADE gauge groups predicted to be realized dynamically by the TGD based view about McKay correspondence [A3] .

2. For $r = 4$ SU(2) inclusion parameterized by extended ADE diagrams $M(2, C)^{\otimes 2}$ would be created in the state function reduction and also this would give $\lambda = 1/2$ and scaling by a factor of 2. Hence the end points of the range of discrete spectrum would correspond to the same scaling factor and same HFF of type III. SU(2) could be interpreted either as electro-weak gauge group, group of rotations of th geodesic sphere of $\delta M^4_\pm$, or a subgroup of SU(3). In TGD interpretation for McKay correspondence a phase transition replacing gauge symmetry with Kac-Moody symmetry.

3. The scalings of trace by factor 2 seem to be preferred physically which should be contrasted with the fact that primes near prime powers of 2 and with the fact that quantum phases $q = exp(i\pi/n)$ with $n$ equal to Fermat integer proportional to power of 2 and product of the Fermat primes (the known ones are 5, 17, 257, and $2^{16} + 1$) are in a special role in TGD Universe.

## 6.3 What Could One Say About $II_1$ Automorphism Associated With The $II_\infty$ Automorphism Defining Factor Of Type III?

An interesting question relates to the interpretation of the automorphisms of $II_\infty$ factor inducing the scaling of trace.

1. If the automorphism for Jones inclusion involves the generator of cyclic automorphism subgroup $Z_n$ of $II_1$ factor then it would seem that for other values of $\lambda$ this group cannot be cyclic. SU(2) has discrete subgroups generated by arbitrary phase $q$ and these are dense in $U(1) \subset SU(2)$ sub-group. If the interpretation in terms of Jones inclusion makes sense then the identification $\lambda = \sqrt{\mathcal{M} : \mathcal{N}}/2^k$ makes sense.

2. If HFF of type $II_1$ is realized as group algebra of infinite symmetric group [A3] , the outer automorphism induced by the diagonally imbedded finite Galois groups can induce only integer values of $n$ and $Z_n$ would correspond to cyclic subgroups. This interpretation conforms with the fact that the automorphisms in the completion of inner automorphisms of HFF of type $II_1$ induce trivial scalings. Therefore only automorphisms which do not belong to this completion can define HFFs of type III.

## 6.4 What Could Be The Physical Interpretation Of Two Kinds Of Invariants Associated With HFFs Type III?

TGD predicts two kinds of counterparts for $S$-matrix: $M$-matrix and $U$-matrix. Both are expected to be more or less universal.

There are also *two* kinds of invariants and automorphisms associated with HFFs of type III.

1. The first invariant corresponds to the scaling $\lambda \in ]0, 1[$ of the trace associated with the automorphism of factor of $II_\infty$. Also the end points of the interval make sense. The inverse of this scaling accompanies the inverse of this automorphism.

2. Second invariant corresponds to the time scales $t = T_0$ for which the outer automorphism $\sigma_t$ reduces to inner automorphism. It turns out that $T_0$ and $\lambda$ are related by the formula $\lambda^{iT_0} = 1$, which gives the allowed values of $T_0$ as $T_0 = n2\pi/log(\lambda)$ [A5] . This formula can

be understood intuitively by realizing that $\lambda$ corresponds to the eigenvalue of the density matrix $\Delta = e^H$ in the simplest possible realization of the state $\phi$.

The presence of two automorphisms and invariants brings in mind $U$ matrix characterizing the unitary process occurring in quantum jump and $M$-matrix characterizing time like entanglement.

1. If one accepts the vision based on quantum measurement theory then $\lambda$ corresponds to the scaling of the trace resulting when quantum Clifford algebra $\mathcal{M}/\mathcal{N}$ reduces to a tensor power of $M(2,C)$ factor in the state function reduction. The proposed interpretation for $U$ process would be as the inverse of state function reduction transforming this factor back to $\mathcal{M}/\mathcal{N}$. Thus $U$ process and state function reduction would correspond naturally to the scaling and its inverse. This picture might apply not only in single particle case but also for zero energy states which can be seen as states associated the a tensor power of HFFs of type $II_1$ associated with partons.

2. The implication is that $U$ process can occur only in the direction in which trace is reduced. This would suggest that the full $III_1$ factor is not a physical notion and that one must restrict the group $Z$ in the crossed product $Z \times_{cr} II_\infty$ to the group $N$ of non-negative integers. In this kind of situation the trace is well defined since the traces for the terms in the crossed product comes as powers $\lambda^{-n}$ so that the net result is finite. This would mean a reduction to $II_\infty$ factor.

3. Since time $t$ is a natural parameter in elementary particle physics experiment, one could argue that $\sigma_t$ could define naturally $M$-matrix. Time parameter would most naturally correspond to a parameter of scaling affecting all $M^4_\pm$ coordinates rather than linear time. This conforms also with the fundamental role of conformal transformations and scalings in TGD framework.

The identification of the full $M$-matrix in terms of $\sigma$ does not seem to make sense generally. It would however make sense for incoming and outgoing number theoretic braids so that $\sigma$ could define universal braiding $M$-matrices. Inner automorphisms would bring in the dependence on experimental situation. The reduction of the braiding matrix to an inner automorphism for critical values of $t$ which could be interpreted in terms of scaling by power of $p$. This trivialization would be a counterpart for the elimination of propagator legs from $M$-matrix element. Vertex itself could be interpreted as unitary isomorphism between tensor product of incoming and outgoing HFFs of type $II_1$ would code all what is relevant about the particle reaction.

## 6.5 Does The Time Parameter $T$ Represent Time Translation Or Scaling?

The connection $T_n = n2\pi/log(\lambda)$ would give a relationship between the scaling of trace and value of time parameter for which the outer automorphism represented by $\sigma$ reduces to inner automorphism. It must be emphasized that the time parameter $t$ appearing in $\sigma$ need not have anything to do with time translation. The alternative interpretation is in terms of $M^4_\pm$ scaling (implying also time scaling) but one cannot exclude even preferred Lorentz boosts in the direction of quantization axis of angular momentum.

### 6.5.1 *Could the time parameter correspond to scaling?*

The central role of conformal invariance in quantum TGD suggests that $t$ parameterizes scaling rather than translation. In this case scalings would correspond to powers of $(K\lambda)^n$. The numerical factor $K$ which cannot be excluded a priori, seems to reduce to $K = 1$.

1. The scalings by powers of $p$ have a simple realization in terms of the representation of HFF of type $II_\infty$ as infinite tensor power of $M(p,C)$ with suitably chosen densities matrices in factors to get product of $I_\infty$ and $II_1$ factor. These matrix algebras have the remarkable property of defining prime tensor power factors of finite matrix algebras. Thus p-adic fractality would reflect directly basic properties of matrix algebras as suggested already earlier. That scalings by powers of $p$ would correspond to automorphism reducing to inner automorphisms would conform with p-adic fractality.

2. Also scalings by powers $[\sqrt{\mathcal{M}:\mathcal{N}}/2^k]^n$ would be physically preferred if one takes previous arguments about Jones inclusions seriously and if also in this case scalings are involved. For $q = exp(i\pi/n)$, $n = 5$ the minimal value of $n$ allowing universal topological quantum computation would correspond to a scaling by Golden Mean and these fractal scalings indeed play a key role in living matter. In particular, Golden Mean makes it visible in the geometry of DNA.

#### 6.5.2 *Could the time parameter correspond to time translation?*

One can consider also the interpretation of $\sigma_t$ as time translation. TGD predicts a hierarchy of Planck constants parameterized by rational numbers such that integer multiples are favored. In particular, integers defining ruler and compass polygons are predicted to be in a very special role physically. Since the geometric time span associated with zero energy state should scale as Planck constant one expects that preferred values of time $t$ associated with $\sigma$ are quantized as rational multiples of some fundamental time scales, say the basic time scale defined by $CP_2$ length or p-adic time scales.

1. For $\lambda = 1/p$, $p$ prime, the time scale would be $T_n = nT_1$, $T_1 = T_0 = 2\pi/log(p)$ which is not what p-adic length scale hypothesis would suggest.

2. For Jones inclusions one would have $T_n/T_0 = n2\pi/log(2^{2k}/\mathcal{M}:\mathcal{N})$. In the limit when $\lambda$ becomes very small (the number $k$ of reduced $M(2,C)$ factors is large one obtains $T_n = (n/k)t_1$, $T_1 = T_0\pi/log(2)$. Approximate rational multiples of the basic length scale would be obtained as also predicted by the general quantization of Planck constant.

#### 6.5.3 *p-Adic thermodynamics from first principles*

Quantum field theory at non-zero temperature can be formulated in the functional integral formalism by replacing the time parameter associated with the unitary time evolution operator $U(t)$ with a complexified time containing as imaginary part the inverse of the temperature: $t \rightarrow t + i\hbar/T$. In the framework of standard quantum field theory this is a mere computational trick but the time parameter associated with the automorphisms $\sigma_t$ of HFF of type $III$ is a temperature like parameter from the beginning, and its complexification would naturally lead to the analog of thermal QFT.

Thus thermal equilibrium state would be a genuine quantum state rather than fictive but useful auxiliary notion. Thermal equilibrium is defined separately for each incoming parton braid and perhaps even braid (partons can have arbitrarily large size). At elementary particle level p-adic thermodynamics could be in question so that particle massivation would have first principle description. p-Adic thermodynamics is under relatively mild conditions equivalent with its real counterpart obtained by the replacement of $p^{L_0}$ interpreted as a p-adic number with $p^{-L_0}$ interpreted as a real number.

### 6.6 HFFs Of Type $III$ And The Dynamics In $M^4_\pm$ Degrees Of Freedom?

HFFs of type $III$ could be also assigned with the poorly understood dynamics in $M^4_\pm$ degrees of freedom which should have a lot of to do with four-dimensional quantum field theory. Hyper-finite factors of type $III_1$ might emerge when one extends $II_1$ to a local algebra by multiplying it with hyper-octonions replaced as analog of matrix factor and considers hyper-quaternionic subalgebra. The resulting algebra would be the analog of local gauge algebra and the elements of algebra would be analogous to conformal fields with complex argument replaced with hyper-octonionic, -quaternionic, or -complex one. Since quantum field theory in $M^4$ gives rise to hyper-finite $III_1$ factors one might guess that the hyper-quaternionic restriction indeed gives these factors.

The expansion of the local HFF $II_\infty$ element as $O(m) = \sum_n m^n O_n$, where $M^4$ coordinate $m$ is interpreted as hyper-quaternion, could have interpretation as expansion in which $O_n$ belongs to $\mathcal{N}g^n$ in the crossed product $\mathcal{N} \times_{cr} \{g^n, n \in Z\}$. The analogy with conformal fields suggests that the power $g^n$ inducing $\lambda^n$ fold scaling of trace increases the conformal weight by $n$.

One can ask whether the scaling of trace by powers of $\lambda$ defines an inclusion hierarchy of subalgebras of conformal sub-algebras as suggested by previous arguments. One such hierarchy would be the hierarchy of sub-algebras containing only the generators $O_m$ with conformal weight $m \geq n$, $n \in Z$.

It has been suggested that the automorphism $\Delta$ could correspond to scaling inside light-cone. This interpretation would fit nicely with Lorentz invariance and TGD in general. The factors $III_\lambda$ with $\lambda$ generating semi-subgroups of integers (in particular powers of primes) could be of special physical importance in TGD framework. The values of $t$ for which automorphism reduces to inner automorphism should be of special physical importance in TGD framework. These automorphisms correspond to scalings identifiable in terms of powers of p-adic prime $p$ so that p-adic fractality would find an explanation at the fundamental level.

If the above mentioned expansion in powers of $m^n$ of $M^4_\pm$ coordinate makes sense then the action of $\sigma^t$ representing a scaling by $p^n$ would leave the elements $O$ invariant or induce a mere inner automorphism. Conformal weight $n$ corresponds naturally to n-ary p-adic length scale by uncertainty principle in p-adic mass calculations.

The basic question is the physical interpretation of the automorphism inducing the scaling of trace by $\lambda$ and its detailed action in HFF. This scaling could relate to a scaling in $M^4$ and to the appearance in the trace of an integral over $M^4$ or subspace of it defining the trace. Fractal structures suggests itself strongly here. At the level of construction of physical states one always selects some minimum non-positive conformal weight defining the tachyonic ground state and physical states have non-negative conformal weights. The interpretation would be as a reduction to HHF of type $II_\infty$ or even $II_1$.

### 6.7 Could The Continuation Of Braidings To Homotopies Involve $\Delta^{It}$ Automorphisms

The representation of braidings as special case of homotopies might lead from discrete automorphisms for HFFs type $II_1$ to continuous outer automorphisms for HFFs of type $III_1$. The question is whether the periodic automorphism of $II_1$ represented as a discrete sub-group of $U(1)$ would be continued to $U(1)$ in the transition.

The automorphism of $II_\infty$ HFF associated with a given value of the scaling factor $\lambda$ is unique. If Jones inclusions defined by the preferred values of $\lambda$ as $\lambda = \sqrt{\mathcal{M}:\mathcal{N}}/2^k$ (see the previous considerations), then this automorphism could involve a periodic automorphism of $II_1$ factor defined by the generator of cyclic subgroup $Z_n$ for $\mathcal{M}:\mathcal{N} < 4$ besides additional shift transforming $II_1$ factor to $I_\infty$ factor and inducing the scaling.

### 6.8 HFFs Of Type $III$ As Super-Structures Providing Additional Uniqueness?

If the braiding $M$-matrices are as such highly unique. One could however consider the possibility that they are induced from the automorphisms $\sigma_t$ for the HFFs of type $III$ restricted to HFFs of type $II_\infty$. If a reduction to inner automorphism in HFF of type $III$ implies same with respect to HFF of type $II_\infty$ and even $II_1$, they could be trivial for special values of time scaling $t$ assignable to the partons and identifiable as a power of prime $p$ characterizing the parton. This would allow to eliminate incoming and outgoing legs. This elimination would be the counterpart of the division of propagator legs in quantum field theories. Particle masses would however play no role in this process now although the power of padic prime would fix the mass scale of the particle.

## 7 Appendix: Inclusions Of Hyper-Finite Factors Of Type $II_1$

Many names have been assigned to inclusions: Jones, Wenzl, Ocneacnu, Pimsner-Popa, Wasserman [A26] . It would seem to me that the notion Jones inclusion includes them all so that various names would correspond to different concrete realizations of the inclusions conjugate under outer automorphisms.

1. According to [A26] for inclusions with $\mathcal{M}:\mathcal{N} \leq 4$ (with $A_1^{(1)}$ excluded) there exists a countable infinity of sub-factors with are pairwise non inner conjugate but conjugate to $\mathcal{N}$.

2. Also for any finite group $G$ and its outer action there exists uncountably many sub-factors which are pairwise non inner conjugate but conjugate to the fixed point algebra of $G$ [A26] . For any amenable group $G$ the inclusion is also unique apart from outer automorphism [A13] .

Thus it seems that not only Jones inclusions but also more general inclusions are unique apart from outer automorphism.

Any *-endomorphism $\sigma$, which is unit preserving, faithful, and weakly continuous, defines a sub-factor of type $II_1$ factor [A26] . The construction of Jones leads to a standard inclusion sequence $\mathcal{N} \subset \mathcal{M} \subset \mathcal{M}^1 \subset ....$ This sequence means addition of projectors $e_i$, $i < 0$, having visualization as an addition of braid strand in braid picture. This hierarchy exists for all factors of type II. At the limit $\mathcal{M}^\infty = \cup_i \mathcal{M}^i$ the braid sequence extends from $-\infty$ to $\infty$. Inclusion hierarchy can be understood as a hierarchy of Connes tensor powers $\mathcal{M} \otimes_\mathcal{N} \mathcal{M} .... \otimes_\mathcal{N} \mathcal{M}$. Also the ordinary tensor powers of hyper-finite factors of type $II_1$ (HFF) as well as their tensor products with finite-dimensional matrix algebras are isomorphic to the original HFF so that these objects share the magic of fractals.

Under certain assumptions the hierarchy can be continued also in opposite direction. For a finite index an infinite inclusion hierarchy of factors results with the same value of index. $\sigma$ is said to be basic if it can be extended to *-endomorphisms from $\mathcal{M}^1$ to $\mathcal{M}$. This means that the hierarchy of inclusions can be continued in the opposite direction: this means elimination of strands in the braid picture. For finite factors (as opposed to hyper-finite ones) there are no basic *-endomorphisms of $\mathcal{M}$ having fixed point algebra of non-abelian $G$ as a sub-factor [A26] .

## 7.1 Jones Inclusions

For hyper-finite factors of type $II_1$ Jones inclusions allow basic *-endomorphism. They exist for all values of $\mathcal{M} : \mathcal{N} = r$ with $r \in \{4cos^2(\pi/n)|n \geq 3\} \cap [4, \infty)$ [A26] . They are defined for an algebra defined by projectors $e_i$, $i \geq 1$. All but nearest neighbor projectors commute. $\lambda = 1/r$ appears in the relations for the generators of the algebra given by $e_ie_je_i = \lambda e_i$, $|i-j| = 1$. $\mathcal{N} \subset \mathcal{M}$ is identified as the double commutator of algebra generated by $e_i$, $i \geq 2$.

This means that principal graph and its dual are equivalent and the braid defined by projectors can be continued not only to $-\infty$ but that also the dropping of arbitrary number of strands is possible [A26] . It would seem that ADE property of the principal graph meaning single root length codes for the duality in the case of $r \leq 4$ inclusions.

Irreducibility holds true for $r < 4$ in the sense that the intersection of $Q' \cap P = P' \cap P = C$. For $r \geq 4$ one has $dim(Q' \cap P) = 2$. The operators commuting with $Q$ contain besides identify operator of $Q$ also the identify operator of $P$. $Q$ would contain a single finite-dimensional matrix factor less than $P$ in this case. Basic *-endomorphisms with $\sigma(P) = Q$ is $\sigma(e_i) = e_{i+1}$. The difference between genuine symmetries of quantum TGD and symmetries which can be mimicked by TGD could relate to the irreducibility for $r < 4$ and raise these inclusions in a unique position. This difference could partially justify the hypothesis [K4] that only the groups $G_a \times G_b \subset SU(2) \times SU(2) \subset SL(2, C) \times SU(3)$ define orbifold coverings of $H_\pm = M^4_\pm \times CP_2 \rightarrow H_\pm / G_a \times G_b$.

## 7.2 Wassermann's Inclusion

Wasserman's construction of $r = 4$ factors clarifies the role of the subgroup of $G \subset SU(2)$ for these inclusions. Also now $r = 4$ inclusion is characterized by a discrete subgroup $G \subset SU(2)$ and is given by $(1 \otimes \mathcal{M})^G \subset (M_2(C) \times \mathcal{M})^G$. According to [A26] Jones inclusions are irreducible also for $r = 4$. The definition of Wasserman inclusion for $r = 4$ seems however to imply that the identity matrices of both $\mathcal{M}^G$ and $(M(2, C) \otimes \mathcal{M})^G$ commute with $\mathcal{M}^G$ so that the inclusion should be reducible for $r = 4$.

Note that $G$ leaves both the elements of $\mathcal{N}$ and $\mathcal{M}$ invariant whereas $SU(2)$ leaves the elements of $\mathcal{N}$ invariant. $M(2, C)$ is effectively replaced with the orbifold $M(2, C)/G$, with $G$ acting as automorphisms. The space of these orbits has complex dimension $d = 4$ for finite $G$.

For $r < 4$ inclusion is defined as $M^G \subset M$. The representation of $G$ as outer automorphism must change step by step in the inclusion sequence $... \subset \mathcal{N} \subset \mathcal{M} \subset ...$ since otherwise $G$ would act trivially as one proceeds in the inclusion sequence. This is true since each step brings in additional finite-dimensional tensor factor in which $G$ acts as automorphisms so that although $\mathcal{M}$ can be invariant under $G_{\mathcal{M}}$ it is not invariant under $G_{\mathcal{N}}$.

These two inclusions might accompany each other in TGD based physics. One could consider $r < 4$ inclusion $\mathcal{N} = \mathcal{M}^G \subset \mathcal{M}$ with $G$ acting non-trivially in $\mathcal{M}/\mathcal{N}$ quantum Clifford algebra. $\mathcal{N}$ would decompose by $r = 4$ inclusion to $\mathcal{N}_1 \subset \mathcal{N}$ with $SU(2)$ taking the role of $G$. $\mathcal{N}/\mathcal{N}_1$ quantum Clifford algebra would transform non-trivially under $SU(2)$ but would be $G$ singlet.

In TGD framework the $G$-invariance for $SU(2)$ representations means a reduction of $S^2$ to the orbifold $S^2/G$. The coverings $H_\pm \rightarrow H_\pm/G_a \times G_b$ should relate to these double inclusions and $SU(2)$ inclusion could mean Kac-Moody type gauge symmetry for $\mathcal{N}$. Note that the presence of the factor containing only unit matrix should relate directly to the generator $d$ in the generator set of affine algebra in the McKay construction [A3] . The physical interpretation of the fact that almost all ADE type extended diagrams ($D_n^{(1)}$ must have $n \geq 4$) are allowed for $r = 4$ inclusions whereas $D_{2n+1}$ and $E_6$ are not allowed for $r < 4$, remains open.

## 7.3 Generalization From $Su(2)$ To Arbitrary Compact Group

The inclusions with index $\mathcal{M} : \mathcal{N} < 4$ have one-dimensional relative commutant $\mathcal{N}' \cup \mathcal{M}$. The most obvious conjecture that $\mathcal{M} : \mathcal{N} \geq 4$ corresponds to a non-trivial relative commutant is wrong. The index for Jones inclusion is identifiable as the square of quantum dimension of the fundamental representation of $SU(2)$. This identification generalizes to an arbitrary representation of arbitrary compact Lie group.

In his thesis Wenzl [A18] studied the representations of Hecke algebras $H_n(q)$ of type $A_n$ obtained from the defining relations of symmetric group by the replacement $e_i^2 = (q-1)e_i + q$. $H_n$ is isomorphic to complex group algebra of $S_n$ if $q$ is not a root of unity and for $q = 1$ the irreducible representations of $H_n(q)$ reduce trivially to Young's representations of symmetric groups. For primitive roots of unity $q = exp(i2\pi/l)$, $l = 4, 5...$, the representations of $H_n(\infty)$ give rise to inclusions for which index corresponds to a quantum dimension of any irreducible representation of $SU(k)$, $k \geq 2$. For $SU(2)$ also the value $l = 3$ is allowed for spin 1/2 representation.

The inclusions are obtained by dropping the first $m$ generators $e_k$ from $H_\infty(q)$ and taking double commutant of both $H_\infty$ and the resulting algebra. The relative commutant corresponds to $H_m(q)$. By reducing by the minimal projection to relative commutant one obtains an inclusion with a trivial relative commutant. These inclusions are analogous to a discrete states superposed in continuum. Thus the results of Jones generalize from the fundamental representation of $SU(2)$ to all representations of all groups $SU(k)$, and in fact to those of general compact groups as it turns out.

The generalization of the formula for index to square of quantum dimension of an irreducible representation of $SU(k)$ reads as

$$\mathcal{M} : \mathcal{N} = \prod_{1 \leq r < s \leq k} \frac{sin^2\left((\lambda_r - \lambda_s + s - r)\pi/l\right)}{sin^2\left((s-r)n/l\right)} \quad . \tag{7.1}$$

Here $\lambda_r$ is the number of boxes in the $r^{th}$ row of the Yang diagram with $n$ boxes characterizing the representations and the condition $1 \leq k \leq l - 1$ holds true. Only Young diagrams satisfying the condition $l - k = \lambda_1 - \lambda_{r_{max}}$ are allowed.

The result would allow to restrict the generalization of the imbedding space in such a manner that only cyclic group $Z_n$ appears in the covering of $M^4 \rightarrow M^4/G_a$ or $CP_2 \rightarrow CP_2/G_b$ factor. Be as it may, it seems that quantum representations of any compact Lie group can be realized using the generalization of the imbedding space. In the case of SU(2) the interpretation of higher-dimensional quantum representations in terms of Connes tensor products of 2-dimensional fundamental representations is highly suggestive.

The groups $SO(3,1) \times SU(3)$ and $SL(2,C) \times U(2)_{ew}$ have a distinguished position both in physics and quantum TGD and the vision about physics as a generalized number theory implies them. Also the general pattern for inclusions selects these groups, and one can say that the condition that all possible statistics are realized is guaranteed by the choice $M^4 \times CP_2$.

1. $n > 2$ for the quantum counterparts of the fundamental representation of $SU(2)$ means that braid statistics for Jones inclusions cannot give the usual fermionic statistics. That Fermi

statistics cannot "emerge" conforms with the role of infinite-$D$ Clifford algebra as a canonical representation of HFF of type $II_1$. $SO(3,1)$ as isometries of $H$ gives $Z_2$ statistics via the action on spinors of $M^4$ and $U(2)$ holonomies for $CP_2$ realize $Z_2$ statistics in $CP_2$ degrees of freedom.

2. $n > 3$ for more general inclusions in turn excludes $Z_3$ statistics as braid statistics in the general case. $SU(3)$ as isometries induces a non-trivial $Z_3$ action on quark spinors but trivial action at the imbedding space level so that $Z_3$ statistics would be in question.

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