Twistors, $N = 4$ Super-Conformal Symmetry, and Quantum TGD

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Abstract

Twistors - a notion discovered by Penrose - have provided a fresh approach to the construction of perturbative scattering amplitudes in Yang-Mills theories and in $N = 4$ supersymmetric Yang-Mills theory. This approach was pioneered by Witten. The latest step in the progress was the proposal by Nima Arkani-Hamed and collaborators that super Yang Mills and super gravity amplitudes might be formulated in 8-D twistor space possessing real metric signature $(4, 4)$. The questions considered in this chapter are following.

1. Could twistor space could provide a natural realization of $N = 4$ super-conformal theory requiring critical dimension $D = 8$ and signature metric $(4, 4)$? Could string like objects in TGD sense be understood as strings in twistor space? More concretely, could one in some sense lift quantum TGD from $M^4 \times CP_2$ to 8-D twistor space T so that one would have three equivalent descriptions of quantum TGD.
2. Could one construct the preferred extremals of Kähler action in terms of twistors -may be by mimicking the construction of hyper-quaternionic *resp.* co-hyper-quaternionic surfaces in M^8 as surfaces having hyper-quaternionic tangent space *resp.* normal space at each point with the additional property that one can assign to each point x a plane $M^2(x) \subset M^4$ as sub-space or as sub-space defined by light-like tangent vector in M^4. Could one mimic this construction by assigning to each point of X^4 regarded as a 4-surface in T a 4-D plane of twistor space satisfying some conditions making possible the interpretation as a tangent plane and guaranteing the existence of a map of X^4 to a surface in $M^4 \times CP_2$. Could twistor formalism help to resolve the integrability conditions involved?
3. Could one define 8-D counterpart of twistors in order to avoid the problems posed by the description of massive states by regarding them as massless states in 8-D context. Could the octonionic realization of 8-D gamma matrices allow to define twistors in 8-D framework? Could associativity constraint reducing twistors to quaternionic twistors locally imply effective reduction to four-dimensional twistors.

The arguments of this chapter suggest that some these questions might have affirmative answers.

1 Introduction

Twistors - a notion discovered by Penrose [B5] - have provided a fresh approach to the construction of perturbative scattering amplitudes in Yang-Mills theories and in $N = 4$ supersymmetric Yang-Mills theory. This approach was pioneered by Witten [B6] . The latest step in the progress was the proposal by Nima Arkani-Hamed and collaborators [B3] that super Yang Mills and super gravity amplitudes might be formulated in 8-D twistor space possessing real metric signature $(4, 4)$. The questions considered below are following.

1. Could twistor space provide a natural realization of $N = 4$ super-conformal theory requiring critical dimension $D = 8$ and signature metric $(4, 4)$? Could string like objects in TGD sense be understood as strings in twistor space? More concretely, could one in some sense lift quantum TGD from $M^4 \times CP_2$ to 8-D twistor space T so that one would have three equivalent descriptions of quantum TGD.

2. Could one construct the preferred extremals of Kähler action in terms of twistors -may be by mimicking the construction of hyper-quaternionic *resp.* co-hyper-quaternionic surfaces in M^8 as surfaces having hyper-quaternionic tangent space *resp.* normal space at each point with the additional property that one can assign to each point x a plane $M^2(x) \subset M^4$ as sub-space or as sub-space defined by light-like tangent vector in M^4. Could one mimic this construction by assigning to each point of X^4 regarded as a 4-surface in T a 4-D plane of twistor space satisfying some conditions making possible the interpretation as a tangent plane and guaranteing the existence of a map of X^4 to a surface in $M^4 \times CP_2$. Could twistor formalism help to resolve the integrability conditions involved?

3. Could one define 8-D counterpart of twistors in order to avoid the problems posed by the description of massive states by regarding them as massless states in 8-D context. Could the octonionic realization of 8-D gamma matrices allow to define twistors in 8-D framework? Could associativity constraint reducing twistors to quaternionic twistors locally imply effective reduction to four-dimensional twistors.

4. Are 8-D counterparts of twistors needed at all? Could the reduction of the dynamics to that for 4-D surfaces and effective 2-dimensionality have twistorial counterparts in the sense that 4-D twistors or their suitable generalization or even 2-D twistors could make sense at the fundamental level? Number theoretical vision based on the requirement of not only associativity but also of commutativity would suggest a reduction to M^2-valued momenta having description in terms of 2-D twistors. The preferred $M^2 \subset M^4$ identified as hyper-complex plane plays also a key role in the realization of the zero energy ontology and hierarchy of Planck constants.

The arguments of this chapter suggest that some these questions might have affirmative answers. It must be of course emphasized that all considerations are highly speculative first thoughts of an innocent novice. The proposals to be discussed do not form a single coherent picture but are just alternatives between which one might choose in the lack of anything better. In the next chapter [K14] a proposal for the realization of twistor program inspired by the Yangian symmetry [A3] to the twistor Grasmannian program [B4] and looks much more realistic. I have however decided to keep this chapter as a document about the development of ideas.

1.1 Twistors and classical TGD

Consider first the twistorialization at the classical space-time level.

1. One can assign twistors to only 4-D Minkowski space (also to other than Lorentzian signature). One of the challenges of the twistor program is how to define twistors in the case of a general curved space-time. In TGD framework the structure of the imbedding space allows to circumvent this problem.

2. The lifting of classical TGD to twistor space level is a natural idea. Consider space-time surfaces representable as graphs of maps $M^4 \rightarrow CP_2$. At classical level the Hamilton-Jacobi structure [K2] required by the number theoretic compactification means dual slicings of the M^4 projection of the space-time surface X^4 by stringy word sheets and partonic two-surfaces. Stringy slicing allows to assign to each point of the projection of X^4 two light-like tangent vectors U and V parallel to light-like Hamilton-Jacobi coordinate curves. These vectors define components $\tilde{\mu}$ and λ of a projective twistor, and twistor equation assigns to this pair a point m of M^4. The conjecture is that for preferred extremals of Kähler action this point corresponds to the M^4 projection of the point in the natural M^4 coordinates associated with the upper or lower tip of causal diamond CD. If this conjecture is correct one can lift the M^4 projection of the space-time surface in $CD \times CP_2 \subset M^4 \times CP_2$ to a surface in $PT \times CP_2$, where CP_3 is projective twistor space $PT = CP_3$. Also induced spinor fields and induced gauge fields can be lifted to twistor space.

3. If one can fix the scales of the tangent vectors U and V and fix the phase of spinor λ one can consider also the lifting to 8-D twistor space T rather than 6-D projective twistor space PT. Kind of symmetry breaking would be in question. The proposal for how to achieve this relies on the notion of finite measurement resolution. The scale of V at partonic 2-surface $X^2 \subset \delta CD \times X^3_l$ would naturally correlate with the energy of the massless particle assignable to the light-like curve beginning from that point and thus fix the scale of V coordinate. Symplectic triangulation discussed in [K3] in turn allows to assign a phase factor to each strand of the number theoretic braid as the Kähler magnetic flux associated with the triangle having the point at its center. This allows to lift the stringy world sheets associated with number theoretic braids to their twistor variants but not the entire space-time surface. String model in twistor space is obtained in accordance with the fact that $N = 4$ super-conformal invariance is realized as a string model in a target space with $(4, 4)$ signature of metric. Note however that CP_2 defines additional degrees of freedom for the target space so that 12-D space is actually in question.

4. One can consider also a more general problem of identifying the counterparts for the preferred extremals of Kähler action with arbitrary dimensions of M^4 and CP_2 projections in 10-D space $PT \times CP2$. The key idea is the reduction of field equations to holomorphy as in Penrose's twistor representation of solutions of positive and negative frequency parts of free fields in M^4. A very helpful observation is that CP_2 as a sub-manifold of PT corresponds to the 2-D space of

null rays of the complexified Minkowski space M^4_c. For the 5-D space $N \subset PT$ of null twistors this 2-D space contains 1-dimensional light ray in M^4 so that N parameterizes the light-rays of M^4. The idea is to consider holomorphic surfaces in $PT_\pm \times CP_2$ ($\pm$ correlates with positive and negative energy parts of zero energy state) having dimensions $D = 6, 8, 10$; restrict them to $N \times CP_2$, select a sub-manifold of light-rays from N, and select from each light-ray subset of points which can be discrete or portion of the light-ray in order to get a 4-D space-time surface. If integrability conditions for the resulting distribution of light-like vectors U and V can be satisfied (in other words they are gradients), a good candidate for a preferred extremal of Kähler action is obtained. Note that this construction raises light-rays to a role of fundamental geometric object.

1.2 Twistors and Feynman diagrams

The recent successes of twistor concept in the understanding of 4-D gauge theories and $N = 4$ SYM motivate the question of how twistorialization could help to understand construction of M-matrix in terms of Feynman diagrammatics or its generalization.

1. One of the basic problems of twistor program is how to treat massive particles. Massive four-momentum can be described in terms of two twistors but their choice is uniquely only modulo $SO(3)$ rotation. This is ugly and one can consider several cures to the situation.

   (a) Number theoretic compactification and hierarchy of Planck constants leading to a generalization of the notion of imbedding space assign to each sector of configuration space defined by a particular CD a unique plane $M^2 \subset M^4$ defining quantization axes. The line connecting the tips of the CD selects also unique rest frame (time axis). The representation of a light-like four-momentum as a sum of four-momentum in this plane and second light-like momentum is unique and same is true for the spinors λ apart from the phase factors (the spinor associated with M^2 corresponds to spin up or spin down eigen state).

   (b) The tangent vectors of braid strands define light-like vectors in H and their M^4 projection is time-like vector allowing a representation as a combination of U and V. Could also massive momenta be represented as unique combinations of U and V?

   (c) One can consider also the possibility to represent massive particles as bound states of massless particles.

   It will be found that one can lift ordinary Feynman diagrams to spinor diagrams and integrations over loop momenta correspond to integrations over the spinors characterizing the momentum.

2. One assign to ordinary momentum eigen states spinor λ but it is not clear how to identify the spinor $\tilde{\mu}$ needed for a twistor.

   (a) Could one assign $\tilde{\mu}$ to spin polarization or perhaps to the spinor defined by the light-like M^2 part of the massive momentum? Or could λ and $\tilde{\mu}$ correspond to the vectors proportional to V and U needed to represent massive momentum?

   (b) Or is something more profound needed? The notion of light-ray is central for the proposed construction of preferred extremals. Should momentum eigen states be replaced with light ray momentum eigen states with a complete localization in degrees of freedom transversal to light-like momentum? This concept is favored both by the notion of number theoretic braid and by the massless extremals (MEs) representing "topological light rays" as analogs of laser beams and serving as space-time correlates for photons represented as wormhole contacts connecting two parallel MEs. The transversal position of the light ray would bring in $\tilde{\mu}$. This would require a modification of the perturbation theory and the introduction of the ray analog of Feynman propagator. This generalization would be M^4 counterpart for the highly successful twistor diagrammatics relying on twistor Fourier transform but making sense only for the (2,2) signature of Minkowski space.

1.3 Massive particles and the generalization of twistors to 8-D case

The basic problem of the twistor approach is that one cannot represent massive momenta in terms of twistors in elegant manner. This problem might be circumvented.

1. In quantum TGD massive states in M^4 can be regarded as massless states in M^8 and CP_2 (recall $M^8 - H$ duality), and one can map any massive M^4 momentum to a light-like M^8 momentum and hope that this association could be made in a unique manner.

2. One should assign to a massless 8-momentum an 8-dimensional spinor of fixed chirality. The spinor assigned with the light-like four-momentum is not unique without additional conditions. The existence of covariantly constant right-handed neutrino in CP_2 degrees generating the super-conformal symmetries could allow to eliminate the non-uniqueness. 8-dimensional twistor in M^8 would be a pair of this kind of spinors fixing the momentum of massless particle and the point through which the corresponding light-geodesic goes through: the set of these points forms 8-D light-cone and one can assign to each point a spinor. In $M^4 \times CP_2$ definitions makes also in the case of $M^4 \times CP_2$ and twistor space would also now be a lifting of the space of light-like geodesics.

3. The possibility to interpret M^8 as hyperoctonionic space suggests also the possibility to define the 8-D counterparts of sigma matrices to hyperoctonions to obtain a representation of sigma matrix algebra which is not a matrix representation. The mapping of gamma matrices to this representation allows to define a notion of hyper-quaternionicity in terms of the modified gamma matrices both in M^8 and H. In this case however hyper-quaternionic 4-plane associated with a given point of X^4 is not tangent plane in the general case. This approach allows to deduce an ansatz to the modified Dirac equation working also in the general case.

1.4 Twistors and electric-magnetic duality

The vision involves the notions of bosonic emergence, the identification of virtual states as pairs of on mass shell states assignable to wormhole throats inspired by zero energy ontology and the associated realization of Cutkosky rules in terms of manifestly finite Feynman diagrammatics, and as the latest and most important piece the weak form of electric-magnetic duality and the notion of M^2-valued pseudo-momentum associated with the generalized eigen states of the Chern-Simons Dirac operator. There must be a correlation between pseudo-momenta and real momenta and the identification of the difference of pseudo-momenta of wormhole throats representing virtual particle as the difference of corresponding on-mass-shell momenta is what gives a connection between ordinary virtual momenta and pseudo-momenta. One would obtain not only 4-D twistors but much simpler 2-D twistors with a discrete pseudo-momentum spectrum containing possibly only a finite number of momenta.

To sum up, the ideas about twistors are just ideas and it takes years to transform them to a genuine theory. At this moment the simplest and most promising approach is the one inspired by zero energy ontology combined with the implications of electric-magnetic duality and the combination of this approach with the twistor Grassmannian program discussed in the next chapter looks much more realistic than the considerations of this chapter.

2 Could the target space be identified in terms of twistors?

The problem of quantum theory in $(2,2)$ signature and corresponding real twistors is that a spacetime with this metric signature does not conform with the standard view about causality. The challenge is to find a physical interpretation consistent with the metric signature of Minkowski space: somehow M^4 or at least light-cone boundary should be lifted to twistor space. The (2,2) *resp.* (4,4) signature of the metric of the target space is a problem of also $N = 2$ *resp.* $N = 4$ super-conformal string theories, and $N = 4$ super-conformal string theory could be relevant for quantum TGD since TGD has $N = 4$ superconformal symmetries as broken symmetries. The identification of the target space of $N = 4$ theory as twistor space T looks natural.

Number theoretical compactification implies dual slicings of the space-time surface to string world sheets and partonic 2-surfaces. Finite measurement resolution reduces light-like 3-surfaces to braids defining boundaries of string world sheets. String model in T is obtained if one can lift the string world sheets from $CD \times CP_2$ to T. It turns out that this is possible and one can also find an interpretation for the phases associated with the spinors defining the twistor.

A physically attractive realization of the braids - and more generally- of slicings of space-time surface by 3-surfaces and string world sheets, is discussed in [K8] by starting from the observation that TGD defines an almost topological QFT of braids, braid cobordisms, and 2-knots. The boundaries of the string world sheets at the space-like 3-surfaces at boundaries of CDs and wormhole throats would define space-like and time-like braids uniquely.

The idea relies on a rather direct translation of the notions of singular surfaces and surface operators used in gauge theory approach to knots [A4] to TGD framework. It leads to the identification of slicing by three-surfaces as that induced by the inverse images of $r = constant$ surfaces of CP_2, where r is $U(2)$ invariant radial coordinate of CP_2 playing the role of Higgs field vacuum expectation value in gauge theories. $r = \infty$ surfaces correspond to geodesic spheres and define analogs of fractionally magnetically charged Dirac strings identifiable as preferred string world sheets. The union of these sheets labelled by subgroups $U(2) \subset SU(3)$ would define the slicing of space-time surface by string world sheets. The choice of $U(2)$ relates directly to the choice of quantization axes for color quantum numbers characterizing CD and would have the choice of braids and string world sheets as a space-time correlate.

2.1 General remarks

Some remarks are in order before considering a detailed proposal for how to achieve the above described goal.

1. Penrose ends up with the notion of twistor by expressing Pauli-Lubanski vector and four-momentum vector of massless particle in terms of two spinors and their conjugates. Twistor Z^A consists of a pair $(\tilde{\mu}^{\dot{a}}, \lambda_a)$ of spinors in representations $(1/2, 0)$ and $(0, 1/2)$ of Lorentz group. The antisymmetric tensor ϵ^{ab} defines Kähler form in the space of 2-spinors and $i\epsilon^{ab}$ defines Kähler metric which reduces to the $(1, 1, -1, -1)$ diagonal form in real representation. The hermitian matrix defined by the tensor product of $\lambda_{\dot{a}}$ and its conjugate characterizes the four-momentum of massless particle in the representation $p^a\sigma_a$ using Pauli's sigma matrices. In Penrose's original approach $\tilde{\mu}^{\dot{a}}$ characterizes the angular angular momentum of the particle: spin is given by $s = Z^\alpha \overline{Z}\alpha$. The representation is not unique since λ_a is fixed only apart from a phase factor, which might be called "twist". The phases of two spinors are completely correlated.

2. This interpretation is not equivalent with that discussed mostly in [B6] and [B3] . Scattering amplitudes are not functions of momenta and polarizations but of a spinor, its conjugate defining light-like momentum, and helicity having values ± 1. In Minkowski space with Lorentz signature the momentum as kinematic variable is replaced with spinor and its conjugate and spinor is defined apart from a phase factor. In the latter article the signature of Minkowski space is taken to be (2,2) so that the situation changes dramatically. Light rays assignable to twistors are 2-D light-like light-like surfaces and the spinor associated with light-like point decomposes to two independent real spinors replacing light-like momentum as a kinematic variable. The phase factor as an additional kinematic variable is replaced by a real scaling factors t and $1/t$ for the two spinors. Fourier transform with respect to the real spinor or its conjugate is possible and gives scattering amplitude as a function of a twistor variable. In Lorentz signature the twistor Fourier transform in this sense is not possible so one cannot replace spinor and its conjugate by a twistor.

3. The space of 2-spinors has a Hermitian metric with real signature (2,2) since the Lorentz invariant Hermitian metric $i\epsilon^{ab}$ has diagonal form $(1, -1)$ in complex coordinates. Twistors consist of two spinors and the 8-D twistor space -call it T- has Kähler metric with complex metric signature (2,2) and real metric signature (4,4), and could correspond to the target space of $N = 4$ super-conformally symmetric theory and might define the target space of $N = 4$ super-conformally symmetric string theory with strings identified as T lifts of the string world sheets having braid strands at their ends. The minimum requirement is that one can assign to each point of string world sheet a twistor.

2.2 What twistor Fourier transform could mean in TGD framework?

For the existence of twistor Fourier transform the reality and independence of the spinors λ and $\tilde{\mu}$ is essential and are satisfied for (2,2) signature. In Lorentzian signature these conditions fail. The question is whether TGD framework could allow to construct twistor amplitudes.

1. From Witten's paper [B6] one learns that twistor-space scattering amplitudes obtained as Fourier-transforms with respect to the real conjugate spinor in Minkowski space with (2,2) signature correspond to incoming and outgoing states for which the wave functions are not plane waves but are located to 2-D sub-spaces of Minkowski space defined by the equation

$$\tilde{\mu}_{\dot{a}} + x_{\dot{a}a}\lambda^a = 0 \ . \tag{2.1}$$

   In a more familiar notation one has $x^\mu\sigma_\mu\lambda = \tilde{\mu}$. This condition follows directly from twistor Fourier transform.

2. In Lorentz signature similar equation is obtained from Penrose transform relating the solutions of free wave equations for various spins to the elements of sheaf cohomology assignable to projective twistor space (see the appendix of [B6]). In this case the solution is unique apart from the shift $x^\mu \to x^\mu + kp^\mu$, where p^μ is the light-like momentum associated with λ identified as a solution of massless Dirac equation. Hence twistor corresponds to a wave function localized at light ray.

3. If the equivalent of twistor Fourier transform exists in some sense in Lorentz signature, the geometric interpretation would be as a decomposition of massless plane wave to a superposition of wave functions localized to light-like rays in the direction of momentum. Uncertainty Principle does not deny the existence of this kind of wave functions. These highly singular wave functions would be labeled by momentum and one point at the light ray or equivalently (apart from the phase factor) by λ_a and $\tilde{\mu}^{\dot{a}}$ defining the twistor. The wave functions would be constant at the rays and thus wave functions in a 3-dimensional sub-manifold of M^4 labeling the light rays. This sub-manifold could be taken light-cone boundary as is easy to see so that the overlap of wave function with different direction of 3-momentum would take place only at the tip of the light-cone. Fields in twistor space would be fields in the space of light-rays characterized by a wave vector.

4. Light-likeness fixes x and μ for given λ uniquely if one assumes that μ is in the plane M^2 defined by λ and thus light-like dual of the momentum vector satisfying $x \cdot p = -1$. Clearly, momentum conservation gives to conservation of x and one can interpret x as a geometric representation of momentum analogous to the representation momentum increment in X-ray scattering at "heavenly sphere". Quantum classical correspondence encourages to consider at least half seriously this kind of coding of momentum to a position of braid point at light-cone boundary. Since twistor Fourier transform does not work, one must invent some other manner to introduce these wave functions. Here the lifting of space-time surface to twistor space suggests itself.

5. The basic challenge is to assign to space-time surface or to each point of space-time surface a momentum like quantity. If this is achieved one can can assign to the point also λ and $\tilde{\mu}$.

   (a) One can assign to space-time sheet a conserved four-momentum identifiable by quantum classical correspondence as its quantal variant. This option would fix λ to be same at each point of the space-time surface about from a possible phase factor depending on space-time point. The resulting surfaces in twistor space would be rather boring.

   (b) Hamilton-Jacobi coordinates [K2] suggest the possibility of defining λ as a quantity depending on space-time point. The two light-like M^4 coordinates u, v define preferred coordinates for the string world sheets Y^2 appearing in the slicing of $X^4(X^3_l)$, and the light-like tangent vectors U and V of these curves define a pair $(\lambda, \tilde{\mu})$ of spinors defining twistor Z. The vector V defining the tangent vector of the braid strand is analogous to four-momentum. Twistor equation defines a point m of M^4 apart from a shift along the light ray defined by V and the consistency implying that the construction is not mere triviality is that m

corresponds to the projection of space-time point to M^4 in coordinates having origin at the tip of CD. One could distinguish between negative and positive energy extremals according to whether the tip is upper or lower one. One can assign to λ and $\tilde{\mu}$ also two polarization vectors by a standard procedure [B6] to be discussed later having identification as tangent vectors of coordinate curves of transversal Hamilton-Jacobi coordinates. This would give additional consistency conditions.

6. In this manner space-time surface representable as a graph of a map from M^4 to CP_2 would be mapped to a 4-surface in twistor space apart from the non-uniqueness related to the phase factor of λ. Also various field quantities, in particular induced spinor fields at space-time surface, could be lifted to fields restricted to a 4-dimensional surface of the twistor space so that the classical dynamics in twistor space would be induced from that in imbedding space.

7. This mapping would induce also a mapping of the string world sheets $Y^2 \subset P_{M^4}(X^4(X^3_l))$ to twistor space. V would determine λ and U -taking the role of light-cone point m - would determine $\tilde{\mu}$ in terms of the twistor equation. 2-surfaces in twistor space would be defined as images of the 2-D string world sheets if the integrability of the distribution for (U, V) pairs implies the integrability of $(\lambda, \tilde{\mu})$ pairs.

8. Twistor scattering amplitude would describe the scattering of a set of incoming light-rays to a set of outgoing light-rays so that the non-locality of interactions is obvious. Discretization of partonic 2-surfaces to discrete point sets would indeed suggest wave functions localized at light-like rays going through the braid points at the ends of X^3_l as a proper basis so that problems with Uncertainty Principle would be overcome. The incoming and outgoing twistor braid points would be determined by M^4 projections of the braid points at the ends of X^3_l. By quantum classical correspondence the conservation law of classical four-momentum would apply to the total classical four-momentum although for individual braid strands classical four-momenta would not conserved. The interpretation would be in terms of interactions. The orbits of stringy curves connecting braid points wold give string like objects in T required by $N = 4$ super-conformal field theory.

2.3 Could one define the phase factor of the twistor uniquely?

The proposed construction says nothing about the phase of the spinors assigned to the tangent vectors V and U. One can consider two possible interpretations.

1. Since the tangent vectors U and V are determined only apart from over all scaling the phase indeterminacy could be interpreted by saying that projective twistors are in question.

2. If one can fix the absolute magnitude of U and V -say by fixing the scale of Hamilton Jacobi coordinates by some physical argument- then the map is to twistors and one should be able to fix the phase.

It turns out that the twistor formulation of field equations taking into account also CP_2 degrees of freedom to be discussed latter favors the first option. The reason why the following argument deserves a consideration is that it would force braid picture and thus replacement of space-time sheets by string world sheets in twistor formulation.

1. The phase of the spinor λ_a associated with the light-like four-momentum and light-like point of $\delta M^4_\pm$ should represent genuine physical information giving the twistor its "twist". Algebraically twist corresponds to a $U(1)$ rotation along closed orbit with a physical significance, possibly a gauge rotation. Since the induced CP_2 Kähler form plays a central role in the construction of quantum TGD, the "twist" could correspond to the non-integrable phase factor defined as the exponent of Kähler magnetic flux (to achieve symplectic invariance and thus zero mode property) through an area bounded by some closed curve assignable with the point of braid strand at X^2. Both CP_2 and $\delta M^4_\pm$ Kähler forms define fluxes of this kind so that two kinds of phase factors are available but CP_2 Kähler flux looks more natural.

2. The symplectic triangulation defined by CP_2 Kähler form allows to identify the closed curve as the triangle defined by the nearest three vertices to which the braid point is connected by edges. Since each point of $X^4(X^3_l)$ belongs to a unique partonic 2-surface X^2, this identification can be made for the braid strands contained by any light-like 3-surface Y^3_l parallel to X^3_l so that phase factors can be assigned to all points of string world sheets having braid strands as their ends. One cannot assign phases to all points of $X^4(X^3_l)$. The exponent of this phase factor is proportional to the coupling of Kähler gauge potential to fermion and distinguishes between quarks and leptons.

3. The phase factor associated with the light-like four-momentum defined by V could be identified as the non-integrable phase factor defined by -say- CP_2 Kähler form. The basic condition would fix the phase of $\tilde{\mu}$. The phases could be permuted but the assignment of $\delta M^4_\pm$ Kähler form with m is natural. Note that the phases of the twistors are symplectic invariants and not subject to quantum fluctuations in the sense that they would contribute to the line element of the metric of the world of classical worlds. This conforms with the interpretation as kinematical variables.

4. Rather remarkably, this construction can assign the non-integrable phase factor only to the points of the number theoretic braid for each Y^3_l parallel to X^3_l so that one obtains only a union of string world sheets in T rather than lifting of the entire $X^4(X^3_l)$ to T^2. The phases of the twistors would code for non-local information about space-time surface coded by the tangent space of $X^4(X^3_l)$ at the points of stringy curves.

3 Could one regard space-time surfaces as surfaces in twistor space?

Twistors are used to construct solutions of free wave equations with given spin and self-dual solutions of both YM theories and Einstein's equations [B5] . Twistor analyticity plays a key role in the construction of construction of solutions of free field equations. In General Relativity the problem of the twistor approach is that twistor space does not make sense for a general space-time metric [B5] . In TGD framework this problem disappears and one can ask how twistors could possibly help to construct preferred extremals. In particular, one can ask whether it might be possible to interpret space-time surfaces as surfaces - not necessarily four-dimensional - in twistor space.

3.1 How $M^4 \times CP_2$ emerges in twistor context?

The finding that CP_2 emerges naturally in twistor space considerations is rather encouraging.

1. Twistor space allows two kinds of 2-planes in complexified M^4 known as α- and β-planes and assigned to twistor and its dual [B5] . This reflects the fundamental duality of the twistor geometry stating that the points Z of PT label also complex planes (CP_2) of PT via the condition

$$Z_a W^a = 0 . \tag{3.1}$$

To the twistor Z one can assign via twistor equation complex α-plane, which contains only null vectors and correspond to the plane defined by the twistors intersecting at Z.

For null twistors (5-D sub-space N of PT) satisfying $Z^a\tilde{Z}_a = 0$ and identifiable as the space of light-like geodesics of M^4 α-plane contains single real light-ray. β-planes in turn correspond to dual twistors which define 2-D null plane CP_2 in twistor space via the equation $Z_aW^a = 0$ and containing the point $W = \tilde{Z}$. Since all lines CP_1 of CP_2 intersect, also they parameterize a 2-D null plane of complexified M^4. The β-planes defined by the duals of null twistors Z contain single real light-like geodesic and intersection of two CP_2:s defined by two points of line of N define CP_1 coding for a point of M^4.

2. The natural appearance of CP_2 in twistor context suggests a concrete conjecture concerning the solutions of field equations. Light rays of M^4 are in 1-1 correspondence with the 5-D space $N \subset P$ of null twistors. Compactified M^4 corresponds to the real projective space PN. The dual of the null twistor Z defines 2-plane CP_2 of PT.

3. This suggests the interpretation of the counterpart of $M^4 \times CP_2$ as a bundle like structure with total space consisting of complex 2-planes CP_2 determined by the points of N. Fiber would be CP_2 and base space 5-D space of light-rays of M^4. The fact that N does not allow holomorphic structure suggests that one should extend the construction to PT and restrict it to N. The twistor counterparts of space-time surfaces in T would be holomorphic surfaces of $PT \times CP_2$ or possibly of $PT_\pm$ (twistor analogs of lower and upper complex plane and assignable to positive and negative frequency parts of classical and quantum fields) restricted to $N \times CP_2$.

3.2 How to identify twistorial surfaces in $PT \times CP_2$ and how to map them to $M^4 \times CP_2$?

The question is whether and how one could construct the correspondence between the points of M^4 and CP_2 defining space-time surface from a holomorphic correspondence between points of PT and CP_2 restricted to N.

1. The basic constraints are that space-time surfaces with varying values for dimensions of M^4 and CP_2 projections are possible and that these surfaces should result by a restriction from $PT \times CP_2$ to $N \times CP_2$ followed by a map from N to M^4 either by selecting some points from the light ray or by identifying entire light rays or their portions as sub-manifolds of X^4.

2. Quantum classical correspondence would suggest that surfaces holomorphic only in PT_+ or PT_- should be used so that one could say that positive and negative energy states have space-time correlates. This would mean an analogy with the construction of positive and negative energy solutions of free massless fields. The corresponding space-time surfaces would emerge from the lower and upper light-like boundaries of the causal diamond CD.

3. A rather general approach is based on an assignment of a sub-manifold of CP_2 to each light ray in $PT_\pm$ in holomorphic manner that is by n equations of form

$$F_i(\xi^1, \xi^2, Z) = 0 \ , \ i = 1, ..., n \leq 2 \ . \tag{3.2}$$

The dimension of this kind of surface in $PT \times CP_2$ is $D = 10 - 2n$ and equals to 6, 8 or 10 so that a connection or at least analogy with M-theory and branes is suggestive. For $n = 0$ entire CP_2 is assigned with the point Z (CP_2 type vacuum extremals with constant M^4 coordinates): this is obviously a trivial case. For $n = 1$ 8-D manifold is obtained. In the case that Z is expressible as a function of CP_2 coordinates, one could obtain CP_2 type vacuum extremals or their deformations. Cosmic strings could be obtained in the case that there is no Z dependence. For $n = 4$ discrete set of points of CP_2 are assigned with Z and this would correspond to field theory limit, in particular massless extremals. If the dimension of CP_2 projection for fixed Z is n, one must construct $4 - n$-dimensional subset of M^4 for given point of CP_2.

4. If one selects a discrete subset of points from each light ray, one must consider a $4-n$-dimensional subset of light rays. The selection of points of M^4 must be carried out in a smooth manner in this set. The light rays of M^4 with given direction can be parameterized by the points of light-cone boundary having a possible interpretation as a surface from which the light rays emerge (boundary of CD).

5. One could also select entire light rays of portions of them. In this case a $4 - n - 1$-dimensional subset of light rays must be selected. This option could be relevant for the simplest massless extremals representing propagation along light-like geodesics (in a more general case the first option must be considered). The selection of the subset of light rays could correspond to a choice of $4-n-1$-dimensional sub-manifold of light-cone boundary identifiable as part of the boundary

of CD in this case. In this case one could worry about the intersections of selected light rays. Generically the intersections occur in a discrete set of points of H so that this problem does not seem to be acute. The lines of generalized Feynman diagrams interpreted as space-time surfaces meet at 3-D vertex surfaces and in this case one must pose the condition that CP_2 projections at the 3-D vertices are identical.

6. The use of light rays as the basic building bricks in the construction of space-time surfaces would be the space-time counterpart for the idea that light ray momentum eigen states are more fundamental than momentum eigen states.

$M^8 - H$ duality is Kähler isometry in the sense that both induced metric and induced Kähler form are identical in M^8 and $M^4 \times CP_2$ representations of the space-time surface. In the recent case this would mean that the metric induced to the space-time surface by the selection of the subset of light-rays in N and subsets of points at them has the same property. This might be true trivially in the recent case.

3.3 How to code the basic parameters of preferred extremals in terms of twistors?

One can proceed by trying to code what is known about preferred extremals to the twistor language.

1. A very large class of preferred extremals assigns to a given point of X^4 two light-like vectors U and V of M^4 and two polarization vectors defining the tangent vectors of the coordinate lines of Hamilton-Jacobi coordinates of M^4 [K2] . As already noticed, given null-twistor defines via λ and $\tilde{\mu}$ two light-like directions V and U and twistor equation defines M^4 coordinate m apart from a shift in the direction of V. The polarization vectors ϵ_i in turn can be defined in terms of U and V. $\lambda = \mu$ corresponds to a degenerate case in which U and V are conjugate light-like vectors in plane M^2 and polarization vector is also light-like. This could correspond to the situation for CP_2 type vacuum extremals. For the simplest massless extremals light-like vector U is constant and the solution depends on U and transverse polarization ϵ vector only. More generally, massless extremals depend only on two M^4 coordinates defined by U coordinate and the coordinate varying in the direction of local polarization vector ϵ.

2. Integrable distribution of these light-like vectors and polarization vectors required. This means that these vectors are gradients of corresponding Hamilton-Jacobi coordinate variables. This poses conditions on the selection of the subset of light rays and the selection of M^4 points at them. Hyper-quaternionic and co-hyper-quaternionic surfaces of M^8 are also defined by fixing an integrable distribution of 4-D tangent planes, which are parameterized by points of CP_2 provided one can assign to the tangent plane $M^2(x)$ either as a sub-space or via the assignment of light-like tangent vector of x.

3. Positive (negative) helicity polarization vector [B6] can be constructed by taking besides λ arbitrary spinor μ_a and defining

$$\epsilon_{a\dot{a}} = \frac{\lambda_a \tilde{\mu}_{\dot{a}}}{\left[\tilde{\lambda},\tilde{\mu}\right]} \ , \ \left[\tilde{\lambda},\tilde{\mu}\right] \equiv \epsilon_{\dot{a}\dot{b}}\lambda^{\dot{a}}\mu^{\dot{b}} \tag{3.3}$$

for negative helicity and

$$\epsilon_{a\dot{a}} = \frac{\mu_a \tilde{\lambda}_{\dot{a}}}{\langle\lambda,\mu\rangle} \ , \ \langle\lambda,\mu\rangle \equiv \epsilon_{ab}\mu^a\lambda^b \tag{3.4}$$

for positive helicity. Real polarization vectors correspond to sums and differences of these vectors. In the recent case a natural identification of μ would be as the second light-like vector defining point of m. One should select one light-like vector and one real polarization vector at each point and find the corresponding Hamilton-Jacobi coordinates. These vectors could also code for directions of tangents of coordinate curves in transversal degrees of freedom.

The proposed construction seems to be consistent with the proposed lifting of preferred extremals representable as a graph of some map $M^4 \to CP_2$ to surfaces in twistor space. What was done in one variant of the construction was to assign to the light-like tangent vectors U and V spinors $\tilde{\mu}$ and λ assuming that twistor equation gives the M^4 projection m of the point of $X^4(X^3_l)$. This is the inverse of the process carried out in the recent construction and would give CP_2 coordinates as functions of the twistor variable in a 4-D subset of N determined by the lifting of the space-time surface. The facts that tangent vectors U and V are determined only apart from overall scaling factor and the fact that twistor is determined up to a phase, imply that projective twistor space PT is in question. This excludes the interpretation of the phase of the twistor as a local Kähler magnetic flux. The next steps would be extension to entire N and a further continuation to holomorphic field in PT or $PT_{\pm}$.

To summarize, although these arguments are far from final or convincing and are bound to reflect my own rather meager understanding of twistors, they encourage to think that twistors are indeed natural approach in TGD framework. If the recent picture is correct, they code only for a distribution of tangent vectors of M^4 projection and one must select both a subset of light rays and a set of M^4 points from each light-ray in order to construct the space-time surface. What remains open is how to solve the integrability conditions and show that solutions of field equations are in question. The possibility to characterize preferred extremal property in terms of holomorphy and integrability conditions would mean analogy with both free field equations in M^4 and minimal surfaces. For known extremals holomorphy in fact guarantees the extremal property.

3.4 Hyper-quaternionic and co-hyper-quaternionic surfaces and twistor duality

In TGD framework space-time surface decomposes into two kinds of regions corresponding to hyper-quaternionic and co-hyper-quaternionic regions of the space-time surface in M^8 (hyper-quaternionic regions were considered in preceding arguments). The regions of space-time with M^4 (Euclidian) signature of metric are identified tentatively as the counterparts of hyper-quaternionic (co-hyper-quaternionic) space-time regions. Pieces CP_2 type vacuum extremals representing generalized Feynman diagrams and having light-like random curve as M^4 projection represent the basic example here. Also these space-time regions should have any twistorial counterpart and one can indeed assign to M^4 projection of CP_2 type vacuum extremal a spinor λ as its tangent vector and spinor μ via twistor equation once M^4 projection is known.

The first guess would the correspondence hyper-quaternionic $\leftrightarrow$ α and co-hyper-quaternionic $\leftrightarrow$ β. Previous arguments in turn suggest that hyper-quaternionic space-time surfaces are mapped to surfaces for which two null twistors are assigned with given point of M^4 whereas co-hyper-quaternionic space-time surfaces are mapped to the surfaces for which only single twistor corresponds to a given M^4 point.

4 Could one lift Feynman diagrams to twistor space?

In [B3] the possibility of twistor diagrammatics is considered and it is interesting to look this from TGD perspective where standard beliefs about what quantum theory is must be given up.

1. The arguments start from ordinary momentum space perturbation theory. The amplitudes for the scattering of massless particles are expressed in terms of twistors after which one performs twistor Fourier transform obtaining amazingly simple expressions for the amplitudes. For instance, the 4-point one loop amplitude in N=4 SYM is extremely simple in twistor space having only values '1' and '0' in twistor space and vanishes for generic momenta.

2. Also IR divergences are absent in twistor transform of the scattering amplitude but are generated by the transform to the momentum space. Since plane waves are replaced with light rays, it is not surprising that the IR divergences coming from transversal degrees of freedom are absent. Interestingly, TGD description of massless particles as wormhole throats connecting two massless extremals extends ideal light-ray to massless extremal having finite transversal thickness so that IR cutoff emerges purely dynamically.

3. This approach fails at the level of loops unless one just uses the already calculated loops. The challenge would be a generalization of the ordinary perturbation theory so that loops could be calculated in twistor space formulation.

The vision about lifting TGD from 8-D $M^4 \times CP_2$ to 8-D twistor space suggests that it should be possible to lift also ordinary M^4 propagators to propagators to twistor space. The first problem is that the momenta of massive virtual particles do not allow any obvious unique representation in terms of twistors. Second problem relates to massive incoming momenta necessarily encountered in stringy picture even if one forgets massivation of light states by p-adic thermodynamics.

4.1 The treatment of massive case in terms of twistors

Massive incoming momenta and loop momenta are problematic from the point of view of twistor description. TGD suggests two alternative approaches two the problem.

1. One can express arbitrary four-momentum as a sum of two light-like momenta. What makes this representation inelegant is its non-uniqueness. For time-like momentum the two light-like momenta in opposite directions can have any direction so that sphere $SO(3)/SO(2) = S^2$ labels the degeneracy and for space-like case the degeneracy corresponds to the hyperboloid $SL(2,R)/SO(2)$ of M^3. This degeneracy has no obvious physical meaning unless virtual momentum corresponds physically to a pair of light-like momenta which can have also opposite signs of energy. This would however mean effectively introduction of two light-like loop momenta instead of one and therefore doubling of the loop. A possible interpretation would be as an introduction of an additional braid strand.

2. Also massive particles should be treated in practical approach. The existence of preferred $M^2 \subset M^4$ forced both by the number theoretic compactification and by the hierarchy of Planck constants would allow to express massive four-momenta uniquely as sums of two light-like momenta, with second momentum in the plane M^2. This would bring in two twistors with second twistor corresponding to a spin $\pm 1/2$ spinor depending on the direction of the momentum. Whether it is possible to interpret the momentum in terms of a genuine composition to a state of two massless particles with second particle moving in the preferred plane M^2 remains an open question. This would allow also to treat massive particles by assuming that loop momenta are on shell momenta. For both stringy excitations and particles receiving their mass by p-adic thermodynamics this might be an appropriate approach.

3. From the twistor point of view a more satisfactory description would be the identification of the massive states as bound states of massless fermions associated with braid strands. If braid strands carry light-like momenta which are not parallel, one can obtain massive off mass shell momenta. For conformal excitations it would be natural to assign the action of the Kac-Moody generators and corresponding Virasoro generators creating the state to separate braid strands. In QCD description of hadrons in terms of massless partons this kind of description is of course already applied.

4. A further possibility making sense in massless theories is the restriction of the momenta rotating in loops to be light-like. This idea turned out to be short lived but led to a first quantitatively precise proposal for how QFT like Feyman diagrammatics could emerge from TGD framework.

4.2 Purely twistorial formulation of Feynman graphs

In the following twistorial formulation of Feynman diagrammatics in TGD framework is considered. If only light-like loop momenta are allowed one can lift the 3-dimensional integral $d^3k/2E$ appearing in the propagators to an integral over twistor variables, which means that complete twistorialization of Feynman diagrams is possible if the loop integrals involve only light-like momenta. This formulation generalizes to the case when loop momenta are massive but requires the introduction of an auxiliary twistor corresponding to momenta restricted to the preferred plane $M^2 \subset M^4$predicted by the number theoretical compactification and hierarchy of Planck constants.

1. It is convenient to introduce double cylindrical coordinates $\lambda_i = \rho_i exp(i(\phi \pm \psi))$ in twistor space. The integration over overall phase ϕ gives only a 2π factor since ordinary Feynman amplitude has no dependence on this variable so that the non-redundant variables are ρ_1, ρ_2, ψ.

2. The condition is that the integral measure d^4uX of the spinor space with a suitable weight function X is equivalent with the measure $d^3k/2E$ in cylindrical coordinates. This gives

$$d^4uX = d\phi \frac{d^3k}{2E} \tag{4.1}$$

   when the integrand does not depend on ϕ.

3. In cylindrical coordinates this gives

$$\begin{aligned} & 2\rho_1\rho_2 d\rho_1 d\rho_2 d\psi X \delta(U - k_z)\delta(V - k_x)\delta(W - k_y) = 1 \ , \\ & U = \tfrac{\rho_1^2 - \rho_2^2}{2} \ , \ V = \tfrac{\rho_1\rho_2 cos(\psi)}{2} \ , \ W = \tfrac{\rho_1\rho_2 sin(\psi)}{2} \ . \end{aligned} \tag{4.2}$$

   Here the functions U, V, and W are obtained form the representations of k_z, k_x, k_y in terms of spinor and its conjugate.

4. Taking U, V, W as integration variables one has

$$2\rho_1\rho_2 \frac{\partial(\rho_1, \rho_2, \psi)}{\partial(U, V, W)} X = 1 \ . \tag{4.3}$$

5. The calculation of the Jacobian gives $X = (\rho_1^2 + \rho_2^2)/4 = E/2$ so that one has the equivalence

$$\frac{1}{4\pi} d^4u \leftrightarrow \frac{d^3k}{2E} \ . \tag{4.4}$$

6. Similar lifting can be carried out for the integration measure defined at light-cone boundary in M^4. If the integrations in generalized Feynman diagrams are over amplitudes depending on light-like momenta and coordinates of the light-like boundaries of CDs in given length scales coming as $T_n = 2^n T_0$ or $T_p = pT_0$ the integrals of momentum space and light-one can be transformed to integrals over twistor space in given length scale. Twistorialization requirement obviously gives a justification for the basic assumption of zero ontology that all transition amplitudes can be formulated in terms of data at the intersections of light-like 3-surfaces with the boundaries of CDs.

7. It should be emphasized that there is no need to keep the phase angle ϕ as a redundant variable is the interpretation as Kähler magnetic flux is accepted. In fact, Kähler magnetic fluxes are expected to appear as zero modes define external parameters in the amplitudes.

One can carry out similar calculation for d^4k assuming the representation of p as a sum of two light-like momenta k_1 and k_2 with another one lying in the preferred plane M^2. The representation is unique and given by

$$\begin{aligned} p &= k_1 + k \ , \\ k_1 &= (|p_T|cosh(\eta), |p_T|sinh(\eta), p_T) \ , \ k = |k|(1, \epsilon, 0, 0) \ , \ \epsilon = \pm 1 \ , \\ exp(\eta) &= [\frac{|p_T|}{p_0 - \epsilon p_z}]^{\epsilon} \ , \\ |k| &= p_0 - |p_T|cosh(\eta) \ . \end{aligned} \tag{4.5}$$

Both signs of $\epsilon = k_2^0/k_2^z$ are needed and correspond to spin up and spin down spinor μ with an indefinite phase whereas k_1 corresponds to λ as in previous example. The 6-dimensional volume element in the space of the spinors is

$$dV = \rho_1\rho_2\rho_3 d\rho_1 d\rho_2 d\rho_3 d\Psi d\Phi_1 d\Phi_2 \ . \tag{4.6}$$

Φ_1 and Φ_2 represent the phases of the spinors λ and μ and are redundant variables in the momentum integration. The expression for d^4k in terms of spinor variables reads as

$$d^4k = \frac{1}{16\pi^2}\left[\rho_1^2(1-\epsilon)+\rho_2^2(1+\epsilon)\right] \ timesdV \ . \tag{4.7}$$

Here the redundant integral over $d\Phi_i$ is included. The integration measure does not have so nice structure as in the case of light-cone. Whether one might combine the spinors to single twistor is an interesting question: conformal invariance does not encourage this. Second option is to combine spinors and their complex conjugates to twistors.

4.3 What could be the propagator in twistor space?

The mere lifting of Feynman diagrams is probably not enough since the propagator in momentum space corresponds to momentum eigen states whereas in TGD framework a more natural notion is the propagator in the space of light-rays, which correspond to states totally localized in the direction of light-like momentum and thus could be seen as superpositions of momentum eigen states with virtual momentum components in transversal directions so that all momenta would be actually space-like in standard sense. Topological light rays (massless extremals) are the direct space-time correlate for this picture and also braid picture and direct physical intuition about what particles are support the idea about ray propagator.

What could propagation mean assuming that one allows only the propagation of light-like momenta in loops in order to achieve an elegant expression of loop diagrams in terms of spinors λ?

1. The points of M^4 are effectively replaced with parallel light-rays for given four-momentum and so that it does not make sense to speak about propagation in the direction of light-like four-momentum. Rather, the propagation would be in the space defined by transversal degrees of freedom which can be parameterized by the points of light-cone. x is fixed uniquely if one assumes it to lie in the plane defined by p as dual of p and conservation p gives rise to conservation of x with the already suggests interpretation as a geometric representation of momentum. One could construct oscillator operators basis creating light-ray states. The task is to guess an expression for the commutators $[a^\dagger(p_1,m_1),a(p_2,m_2)]$.

   If one accepts the parametrization of the space of parallel light rays in terms of points m_1 and m_2 of light-cone, one can argue that only the complete overlap of light rays occurring for $m_1 = m_2$ should contribute to the commutator. This would give

   $$[a^\dagger(p_1,m_1),a(p_2,m_2)] = i2E_1 \times 2|m_1^0| \times \delta^3(p_1-p_2)\delta^3(m_1-m_2) \ .$$

   This picture is consistent with the classical intuitive picture and also with the idea that signals propagate only along light-rays. In twistor space this would give commutation relations which are completely local and there would be no propagation. Note the complete symmetry between momentum space and x-space.

2. This would give for the counterpart of massless scalar propagator G_- allowing only the propagation of light-like virtual momenta the expression

$$G_-(p_1,p_2,m_1,m_2) = i\delta^3(p_1-p_2)\delta^3(m_1-m_2)4E_1 \times |m_1^0| \ . \tag{4.8}$$

3. From this one can construct the counterpart of G_- in the twistor space. This would give

$$G_-(\lambda_1,\tilde{\mu}_1,\lambda_2,\tilde{\mu}_2) \quad = \quad i\delta(\lambda_1\tilde{\lambda}_1-\lambda_2\tilde{\lambda}_2)\delta(\mu_1\tilde{\mu}_1-\mu_2\tilde{\mu}_2)\times 4E|m_1^0| \ . \qquad (4.9)$$

   Note that m_1 and therefore also m_1^0 can be fixed uniquely from the basic twistor equation by using the constraint that $m_1 \equiv x$ is light-like so that one has $x_{a\dot{b}} = \mu_a\tilde{\mu}_{\dot{b}}$ if $\mu_a\lambda^a = 1$ is satisfied.

4. One can express the momentum conserving delta function in terms of delta function $\delta^4(\lambda_1-\lambda_2)$ if one assumes that the irrelevant phase $exp(i\phi)$ of λ (as far as ordinary Feynman diagrams are considered) is conserved. The alternative is the the propagator does not depend at all on the phase difference $\phi_1-\phi_2$. The proposed interpretation of the phase in terms of Kähler magnetic flux which can be interpreted as non-quantum fluctuating zero mode given for all points of braids as classical variable would suggest that it does not make sense to speak about correlation function for ϕ in quantal sense. Going to the cylindrical coordinates $(\rho_1,\rho_2,\psi,\phi)$ repeating the calculation of the Jacobian for the transformations $\lambda \rightarrow (\rho_1,\rho_2,\psi,\phi) \rightarrow (k_1,k_2,k_3,\phi)$ and its variant for m coordinate, one obtains that for massless virtual states the propagator for the two options is apart from normalization constants equal to

$$G_-(\lambda_1,\tilde{\mu}_1,\lambda_2,\tilde{\mu}_2) \quad = \quad i\frac{\delta^4(\lambda_1-\lambda_2)}{\delta(\phi_1-\phi_2)}\frac{\delta^4(\tilde{\mu_1}-\tilde{\mu_2})}{\delta(\phi_1-\phi_2)} \ . \qquad (4.10)$$

The division by $\delta(\phi_1-\phi_2)$ symbolizes the assumption of that ϕ is not quantum fluctuating variable.

Consider next the twistor counterpart of Feynman propagator

$$G_F(p_1,p_2) \quad = \quad i\delta^4(p_1-p_2)\frac{1}{p_1^2+i\epsilon} \ . \qquad (4.11)$$

p_1 can be expressed as a sum of $p_1 = p_{1a}+p_{1b}$ of light-like momenta expressible in terms of λ_{1a} and λ_{2a}. One can assign to p_{1a} and p_{1b} also light-cone points m_{1a} and m_{1b} as their duals and thus also $\tilde{\mu}_{1\dot{a}}$ and $\mu_{2\dot{a}}$. Note that the momentum defined by m would be conserved and provide a geometric space-time representation for the real momentum.

It is however not clear whether twistor space counterpart of Feynman propagator makes sense. Should one assume that the two light-like momenta propagate independently so that the ray propagator would be proportional to the product of delta functions $\delta(m_{1a}-m_{2a})\delta(m_{1b}-m_{2b})$ and $\delta(p_{1a}-p_{2a})\delta(p_{1b}-p_{2b})$? These expressions could be translated to delta functions in twistor degrees of freedom just as above and the only difference would be the presence of $1/p_1^2$ factor. One could perhaps say that effectively the off mass shell particle is a state of two massless particles with correlation between them characterized by the $1/p_1^2$ factor.

4.4 What to do with the perturbation theory?

The basic question is whether one should replace the perturbation theory based on momentum eigenstates with a perturbation theory relying on ray momentum eigen states completely localized in transverse degrees of freedom and allowing only light-like loop momenta or just restrict the loop momenta of ordinary Feynman diagrams to be light-like? Depending on answer to this question one ends up with different scenarios raising further questions.

1. Suppose that one uses ordinary momentum eigen states. The minimum option of the ordinary perturbation theory or of its stringy variant in TGD framework means the replacement of loop momenta with light-like momenta using G_- instead of G_F. In this approach spinors λ are enough and one can do without μ and m. One could of course introduce them but m would be simply the light-like dual of p in the minimal scenario and completely constrained.

2. If one introduces ray eigen states, then also m and $\tilde{\mu}$ emerge naturally. In TGD based perturbation theory m can be assumed to reside at light-cone boundary (at δCD). Since braid points at X^2 vary it seems that one must allow m to be dynamical so that μ is also dynamical. If m and p are duals then braid points come representatives of momenta and m and μ disappear again from the theory. This hypothesis is however ad hoc and un-necessary. For this option the naive generalization of Feynman diagrammatics is not enough. A possible guess for the generalization has been already proposed.

5 Could one generalize the notion of twistor to 8-D case?

The basic problem of the twistor approach is that one cannot represent massive momenta in terms of twistors in elegant manner. I have proposed a possible representation of massive states based on the existence of preferred plane of M^2 in the basic definition of theory allowing to express four-momentum as some of two light-like momenta allowing twistor description. One could however ask whether some more elegant representation of massive M^4 momenta might be possible by generalizing the notion of twistor -perhaps by starting from the number theoretic vision.

The basic idea is obvious: in quantum TGD massive states in M^4 can be regarded as massless states in M^8 and CP_2 (recall M^8-H duality). One can therefore map any massive M^4 momentum to a light-like M^8 momentum and hope that this association could be made in a unique manner. One should assign to a massless 8-momentum an 8-dimensional spinor of fixed chirality. The spinor assigned with the light-like four-momentum is not unique without additional conditions. The existence of covariantly constant right-handed neutrino in CP_2 degrees generating the super-conformal symmetries could allow to eliminate the non-uniqueness. 8-dimensional twistor in M^8 would be a pair of this kind of spinors fixing the momentum of massless particle and the point through which the corresponding light-geodesic goes through: the set of these points forms 8-D light-cone and one can assign to each point a spinor. In $M^4 \times CP_2$ definitions makes also in the case of $M^4 \times CP_2$ and twistor space would also now be a lifting of the space of light-like geodesics.

The possibility to interpret M^8 as hyperoctonionic space suggests also the possibility to define the 8-D counterparts of sigma matrices to hyperoctonions to obtain a representation of sigma matrix algebra which is not a matrix representation. The mapping of gamma matrices to this representation allows to define a notion of hyper-quaternionicity in terms of the modified gamma matrices both in M^8 and H.

5.1 Octo-twistors defined in terms of ordinary spinors

It is possible to define octo-twistors in terms of ordinary spinors of M^8 or H.

1. The condition for the octo-twistor makes sense also for ordinary spinors and the explicit representation can be obtained by using triality. The ansatz is $p^k = \overline{\Psi}\gamma^k\Psi$. The condition $p^k p_k = 0$ gives Dirac equation $p^k\gamma_k\Psi = 0$ and its conjugate solved by $\Psi = p^k\gamma_k\Psi_0$. The expression of p^k in turn gives the normalization condition $\overline{\Psi}_0\gamma^k p_k\Psi_0 = 1/2$.

2. Without further conditions almost any Ψ_0 not annihilated by $\gamma^k p_k$ is possible solution. One can map the spinor basis to hyper-octonion basis and assume $\Psi_0 \rightarrow 1 = \sigma_0$. This would give octo-twistor spinors as $\Psi = p^k\gamma_k\Psi_0$ and its conjugate and there would be natural mapping to $p^k\sigma_k$ so that Ψ and p^k would correspond to each other in 1-1 manner apart from the phase factor of Ψ.

3. A highly unique choice for Ψ_0 is the covariantly constant (with respect to CP_2 coordinates) right-handed neutrino spinor of $M^4 \times CP_2$ since the Dirac operators of M^8, H, and X^4 reduce to free Dirac operator when acting on it in both M^8 and H and giving also rise to super-conformal symmetry. The choice is unique apart from $SO(3)$ rotation but the condition that spin eigen state is in question for the choice of quantization axis fixed by the choice of hyper-octonion units and also by the definition of the hierarchy of Planck constants fixes Ψ_0 apart from the sign of the spin if reality is assumed. When $p^k\gamma_k\Psi_0 = 0$ holds true for fixed Ψ_0, the ansatz fails so that the gauge choice is not global. There are two gauge patches corresponding to the two signs of the spin of Ψ_0. Right handed neutrino spinor reflects directly the homological magnetic

monopole character of the Kähler form of CP_2 so that the monopole property is in well defined sense transferred from CP_2 to M^4. Note that this argument fails for quark spinors which do not allow any covariantly constant spinor.

4. For ordinary twistors the existence of the antisymmetric tensor ϵ acting as Kähler form in the space of spinors is what allows to define second spinor and these spinors together form twistor. Ordinary twistors are pairs of spinors and also in the recent case one would have pairs of octo-spinors. The geometric interpretation would be as a light-like geodesic of M^8 or tangent vector of light-like geodesic of $M^4 \times CP_2$ and the two spinors would code for the momentum associated with the ray and the transverse position of the ray expressible in terms of a light-like vector. This would double the dimension to D=16 which happens to be the dimension of complexified octonions. The standard definition of twistors would suggest that one has 2 triplets of this kind so that Dirac equation and above argument would reduce the situation to 16-dimensional one. Twistors space would be C^8 and 14-D projective twistor space would correspond to CP_7.

5. 2-D spinor and its conjugate as independent representations of Lorentz group define twistor. In an analogous manner M^8 vector, M^8-spinor, and its conjugate define a triplet as independent representations. One can therefore ask whether a triplet of these independent representations could define octo-twistor so that two triplets would not be needed. Together they would form an entity with 24 components when the overall complex phase is eliminated and if no gauge choice fixing Ψ_0 is made apart from the assumption Ψ_0 has real components. If the overall phase is allowed, the number of components is 26 (the momentum constraint of course reduces the number of degrees of freedom to 8). It seems that the magic dimensions of string models are unavoidable! Perhaps it might be a possible to reduce 26-D string theory to 8-D theory by posing triality symmetry and additional gauge symmetry. The problem of this identification is that one does not geometric interpretation as a lifting of the space of light-like geodesics. One could of course define octo-twistors as a pair of triplets with the members of triplet obtained from each other via triality symmetry.

5.2 Could right handed neutrino spinor modes define octo-twistors?

There is no absolute need to interpret induced spinor fields as parts of octo-twistors. One can however ask whether this might make sense for the solutions of the modified Dirac equation $D\Psi = 0$ representing right-handed neutrino and expressible as $\Psi = D\Psi_0$.

1. In the modified Dirac equation gamma matrices are replaced by the modified gamma matrices defined by the variation of Kähler action and the massless momentum $p^k\sigma_k$ is replaced with the modified Dirac operator D. In plane wave basis the derivatives in D reduce to an algebraic multiplication operators in the case of right handed neutrino since right-handed neutrino has no gauge couplings.

2. A non-trivial consistency condition comes from the condition $D^2\Psi_0 = 0$ giving sum of two terms.

   (a) The first term is the analog of scalar d'Alembertian and given by

   $$G^{\mu\nu}D_\mu D_\nu \Psi_0 \ , \quad G^{\mu\nu} = h_{kl}T^{\mu k}T^{\nu l} \ , \quad T^{\mu k} = \frac{\partial L_K}{\partial h^k_\alpha} \ ,$$

   and has quantum numbers of right handed neutrino as it should.

   (b) Second term is given by

   $$T^{\mu k}D_\mu T^{\nu l}\Sigma_{kl}D_\nu \Psi_0 \ ,$$

   and in the general case contains charged components. Only electromagnetically neutral CP_2 sigma matrices having right handed neutrino as eigen state are allowed if one wants twistor interpretation. This is not be true in the general case but might be implied by the preferred extremal property.

(c) This property would allow to choose the induced spinor fields to be eigenstates of electromagnetic charge globally and would be therefore physically very attractive. After all, one of the basic interpretational problems has been the fact that classical W fields seems to induce mixing of quarks and leptons with different electro-magnetic charges. If this is the case one could assign to each point of the space-time surface octo-twistor like abstract entity as the triplet $(\overline{\Psi}_0 D, D, D\Psi_0)$. This would map space-time sheet to a 4-D surface (in real sense) in the space of 8-D (in complex sense) leptonic spinors.

5.3 Octo-twistors and modified Dirac equation

Classical number fields define one vision about quantum TGD. This vision about quantum TGD has evolved gradually and involves several speculative ideas.

1. The hard core of the vision is that space-time surfaces as preferred extremals of Kähler action can be identified as what I have called hyper-quaternionic surfaces of M^8 or $M^4 \times CP_2$. This requires only the mapping of the modified gamma matrices to octonions or to a basis of subspace of complexified octonions. This means also the mapping of spinors to octonionic spinors. There is no need to assume that imbedding space-coordinates are octonionic.

2. I have considered also the idea that quantum TGD might emerge from the mere associativity.

   (a) Consider Clifford algebra of WCW. Treat "vibrational" degrees of freedom in terms second quantized spinor fields and add center of mass degrees of freedom by replacing 8-D gamma matrices with their octonionic counterparts - which can be constructed as tensor products of octonions providing alternative representation for the basis of 7-D Euclidian gamma matrix algebra - and of 2-D sigma matrices. Spinor components correspond to tensor products of octonions with 2-spinors: different spin states for these spinors correspond to leptons and baryons.

   (b) Construct a local Clifford algebra by considering Clifford algebra elements depending on point of M^8 or H. The octonionic 8-D Clifford algebra and its local variant are non-accociative. Associative sub-algebra of 8-D Clifford algebra is obtained by restricting the elements so any quaternionic 4-plane. Doing the same for the local algebra means restriction of the Clifford algebra valued functions to any 4-D hyper-quaternionic sub-manifold of M^8 or H which means that the gamma matrices span complexified quaternionic algebra at each point of space-time surface. Also spinors must be quaternionic.

   (c) The assignment of the 4-D gamma matrix sub-algebra at each point of space-time surface can be done in many manners. If the gamma matrices correspond to the tangent space of space-time surface, one obtains just induced gamma matrices and the standard definition of quaternionic sub-manifold. In this case induced 4-volume is taken as the action principle. If Kähler action defines the space-time dynamics, the modified gamma matrices do not span the tangent space in general.

   (d) An important additional element is involved. If the M^4 projection of the space-time surface contains a preferred subspace M^2 at each point, the quaternionic planes are labeled by points of CP_2 and one can equivalently regard the surfaces of M^8 as surfaces of $M^4 \times CP_2$ (number-theoretical "compactification"). This generalizes: M^2 can be replaced with a distribution of planes of M^4 which integrates to a 2-D surface of M^4 (for instance, for string like objects this is necessarily true). The presence of the preferred local plane M^2 corresponds to the fact that octonionic spin matrices Σ_{AB} span 14-D Lie-algebra of $G_2 \subset SO(7)$ rather than that 28-D Lie-algebra of $SO(7,1)$ whereas octonionic imaginary units provide 7-D fundamental representation of G_2. Also spinors must be quaternionic and this is achieved if they are created by the Clifford algebra defined by induced gamma matrices from two preferred spinors defined by real and preferred imaginary octonionic unit. Therefore the preferred plane $M^3 \subset M^4$ and its local variant has direct counterpart at the level of induced gamma matrices and spinors.

   (e) This framework implies the basic structures of TGD and therefore leads to the notion of world of classical worlds (WCW) and from this one ends up with the notion WCW spinor field and WCW Clifford algebra and also hyper-finite factors of type II_1 and III_1. Note that M^8 is exceptional: in other dimensions there is no reason for the restriction of the local Clifford algebra to lower-dimensional sub-manifold to obtain associative algebra.

The above line of ideas leads naturally to (hyper-)quaternionic sub-manifolds and to basic quantum TGD (note that the "hyper" is un-necessary if one accepts just the notion of quaternionic sub-manifold formulated in terms of modified gamma matrices). One can pose some further questions.

1. Quantum TGD reduces basically to the second quantization of the induced spinor fields. Could it be that the theory is integrable only for 4-D hyper-quaternionic space-time surfaces in M^8 (equivalently in $M^4 \times CP_2$) in the sense than one can solve the modified Dirac equation exactly only in these cases?
2. The construction of quantum TGD -including the construction of vacuum functional as exponent of Kähler function reducing to Kähler action for a preferred extremal - should reduce to the modified Dirac equation defined by Kähler action. Could it be that the modified Dirac equation can be solved exactly only for Kähler action.
3. Is it possible to solve the modified Dirac equation for the octonionic gamma matrices and octonionic spinors and map the solution as such to the real context by replacing gamma matrices and sigma matrices with their standard counterparts? Could the associativity conditions for octospinors and modified Dirac equation allow to pin down the form of solutions to such a high degree that the solution can be constructed explicitly?
4. Octonionic gamma matrices provide also a non-associative representation for 8-D version of Pauli sigma matrices and encourage the identification of 8-D twistors as pairs of octonionic spinors conjectured to be highly relevant also for quantum TGD. Does the quaternionicity condition imply that octo-twistors reduce to something closely related to ordinary twistors as the fact that 2-D sigma matrices provide a matrix representation of quaternions suggests?

In the following I will try to answer these questions by developing a detailed view about the octonionic counterpart of the modified Dirac equation and proposing explicit solution ansätze for the modes of the modified Dirac equation.

5.3.1 The replacement of $SO(7,1)$ with G_2

The basic implication of octonionization is the replacement of $SO(7,1)$ as the structure group of spinor connection with G_2. This has some rather unexpected consequences.

1. Octonionic representation of 8-D gamma matrices

Consider first the representation of 8-D gamma matrices in terms of tensor products of 7-D gamma matrices and 2-D Pauli sigma matrices.

1. The gamma matrices are given by

$$\gamma^0 = 1 \times \sigma_1 \quad , \quad \gamma^i = \gamma^i \otimes \sigma_2 \quad , i = 1,..,7 \quad . \tag{5.1}$$

   7-D gamma matrices in turn can be expressed in terms of 6-D gamma matrices by expressing γ^7 as

$$\gamma^{7)}_{i+1} = \gamma^{6)}_i , i = 1,...,6 \quad , \quad \gamma^{7)}_1 = \gamma^{6)}_7 = \prod_{i=1}^{6} \gamma^{6)}_i \quad . \tag{5.2}$$

2. The octonionic representation is obtained as

$$\gamma_0 = 1 \times \sigma_1 \quad , \quad \gamma_i = e_i \otimes \sigma_2 \quad . \tag{5.3}$$

where e_i are the octonionic units. $e_i^2 = -1$ guarantees that the M^4 signature of the metric comes out correctly. Note that $\gamma_7 = \prod \gamma_i$ is the counterpart for choosing the preferred octonionic unit and plane M^2.

3. The octonionic sigma matrices are obtained as commutators of gamma matrices:

$$\Sigma_{0i} = e_i \times \sigma_3 \ , \ \Sigma_{ij} = f_{ij}{}^k e_k \otimes 1 \ . \tag{5.4}$$

These matrices span G_2 algebra having dimension 14 and rank 2 and having imaginary octonion units and their conjugates as the fundamental representation and its conjugate. The Cartan algebra for the sigma matrices can be chosen to be Σ_{01} and Σ_{23} and belong to a quaternionic sub-algebra.

4. The lower dimension of the G_2 algebra means that some combinations of sigma matrices vanish. All left or right handed generators of the algebra are mapped to zero: this explains why the dimension is halved from 28 to 14. From the octonionic triangle expressing the multiplication rules for octonion units [A2] one finds $e_4e_5 = e_1$ and $e_6e_7 = -e_1$ and analogous expressions for the cyclic permutations of e_4, e_5, e_6, e_7. From the expression of the left handed sigma matrix $I_L^3 = \sigma_{23} + \sigma^{30}$ representing left handed weak isospin (see the Appendix about the geometry of CP_2 [L1] , [L1]) one can conclude that this particular sigma matrix and left handed sigma matrices in general are mapped to zero. The quaternionic sub-algebra $SU(2)_L \times SU(2)_R$ is mapped to that for the rotation group $SO(3)$ since in the case of Lorentz group one cannot speak of a decomposition to left and right handed subgroups. The elements of the complement of the quaternionic sub-algebra are expressible in terms of Σ_{ij} in the quaternionic sub-algebra.

2. Some physical implications of $SO(7,1) \rightarrow G_2$ reduction

This has interesting physical implications if one believes that the octonionic description is equivalent with the standard one.

1. If $SU(2)_L$ is mapped to zero only the right-handed parts of electro-weak gauge field survive octonization. The right handed part is neutral containing only photon and Z^0 so that the gauge field becomes Abelian. Z^0 and photon fields become proportional to each other ($Z^0 \rightarrow sin^2(\theta_W)\gamma$) so that classical Z^0 field disappears from the dynamics, and one would obtain just electrodynamics. This might provide a deeper reason for why electrodynamics is an excellent description of low energy physics and of classical physics. This is consistent with the fact that CP_2 coordinates define 4 field degrees of freedom so that single Abelian gauge field should be enough to describe classical physics. This would remove also the interpretational problems caused by the transitions changing the charge state of fermion induced by the classical W boson fields.

   Also the realization of $M^8 - H$ duality led to the conclusion M^8 spinor connection should have only neutral components. The isospin matrix associated with the electromagnetic charge is $e_1 \times 1$ and represents the preferred imaginary octonionic unit so that that the image of the electro-weak gauge algebra respects associativity condition. An open question is whether octonionization is part of M^8-H duality or defines a completely independent duality. The objection is that information is lost in the mapping so that it becomes questionable whether the same solutions to the modified Dirac equation can work as a solution for ordinary Clifford algebra.

2. If $SU(2)_R$ were mapped to zero only left handed parts of the gauge fields would remain. All classical gauge fields would remain in the spectrum so that information would not be lost. The identification of the electro-weak gauge fields as three covariantly constant quaternionic units would be possible in the case of M^8 allowing Hyper-Kähler structure [A1] , which has been speculated to be a hidden symmetry of quantum TGD at the level of WCW. This option would lead to difficulties with associativity since the action of the charged gauge potentials would lead out from the local quaternionic subspace defined by the octonionic spinor.

3. The gauge potentials and gauge fields defined by CP_2 spinor connection are mapped to fields in $SO(2) \subset SU(2) \times U(1)$ in quaternionic sub-algebra which in a well-defined sense corresponds to M^4 degrees of freedom! Since the resulting interactions are of gravitational character, one might say that electro-weak interactions are mapped to manifestly gravitational interactions. Since $SU(2)$ corresponds to rotational group one cannot say that spinor connection would give rise only to left or right handed couplings, which would be obviously a disaster.

3. Octo-spinors and their relation to ordinary imbedding space spinors

Octo-spinors are identified as octonion valued 2-spinors with basis

$$\begin{aligned} \Psi_{L,i} &= e_i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \\ \Psi_{q,i} &= e_i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad . \end{aligned} \tag{5.5}$$

One obtains quark and lepton spinors and conjugation for the spinors transforms quarks to leptons. Note that octospinors can be seen as 2-dimensional spinors with components which have values in the space of complexified octonions.

The leptonic spinor corresponding to real unit and preferred imaginary unit e_1 corresponds naturally to the two spin states of the right handed neutrino. In quark sector this would mean that right handed U quark corresponds to the real unit. The octonions decompose as $1 + 1 + 3 + \overline{3}$ as representations of $SU(3) \subset G_2$. The concrete representations are given by

$$\begin{array}{ll} \{1 \pm ie_1\} \ , & e_R \text{ and } \nu_R \text{ with spin } 1/2 \ , \\ \{e_2 \pm ie_3\} \ , & e_R \text{ and } \nu_L \text{ with spin -}1/2 \ , \\ \{e_4 \pm ie_5\} & e_L \text{ and } \nu_L \text{ with spin } 1/2 \ , \\ \{e_6 \pm ie_7\} & e_L \text{ and } \nu_L \text{ with spin } 1/2 \ . \end{array} \tag{5.6}$$

Instead of spin one could consider helicity. All these spinors are eigenstates of e_1 (and thus of the corresponding sigma matrix) with opposite values for the sign factor $\epsilon = \pm$. The interpretation is in terms of vectorial isospin. States with $\epsilon = 1$ can be interpreted as charged leptons and D type quarks and those with $\epsilon = -1$ as neutrinos and U type quarks. The interpretation would be that the states with vanishing color isospin correspond to right handed fermions and the states with non-vanishing SU(3) isospin (to be not confused with QCD color isospin) and those with non-vanishing SU(3) isospin to left handed fermions. The only difference between quarks and leptons is that the induced Kähler gauge potentials couple to them differently.

The importance of this identification is that it allows a unique map of the candidates for the solutions of the octonionic modified Dirac equation to those of ordinary one. There are some delicacies involved due to the possibility to chose the preferred unit e_1 so that the preferred subspace M^2 can corresponds to a sub-manifold $M^2 \subset M^4$.

5.3.2 Octonionic counterpart of the modified Dirac equation

The solution ansatz for the octonionic counterpart of the modified Dirac equation discussed below makes sense also for ordinary modified Dirac equation which raises the hope that the same ansatz, and even same solution could provide a solution in both cases.

1. The general structure of the modified Dirac equation

In accordance with quantum holography and the notion of generalized Feynman diagram, the modified Dirac equation involves two equations which must be consistent with each other.

1. There is 3-dimensional generalized eigenvalue equation for which the modified gamma matrices are defined by Chern-Simons action defined by the sum $J_{tot} = J + J_1$ of Kähler forms of S^2 and CP_2 [K4, K6] .

$$\begin{aligned} D_3\Psi &= [D_{C-S} + Q_{C-S}]\,\Psi = \lambda^k\gamma_k\Psi \ , \\ Q_{C-S} &= Q_\alpha\hat{\Gamma}^\alpha_{C-S} \ , \ \ Q_\alpha = Q_A g^{AB} j_{B\alpha} \ . \end{aligned} \tag{5.7}$$

The gamma matrices γ_k are M^4 gamma matrices in standard Minkowski coordinates and thus constant. Given eigenvalue λ_k defines pseudo momentum which is some function of the genuine momenta p_k and other quantum numbers via the boundary conditions associated with the generalized eigenvalue equation.

The charges Q_A correspond to real four-momentum and charges in color Cartan algebra. The term Q can be rather general since it provides a representation for the measurement interaction by mapping observables to Cartan algebra of isometry group and to the infinite hierarchy of conserved currents implied by quantum criticality. The operator O characterizes the quantum critical conserved current. The surface Y^3_l can be chosen to be any light-like 3-surface "parallel" to the wormhole throat in the slicing of X^4: this means an additional symmetry. Formally the measurement interaction term can be regarded as an addition of a gauge term to the Kähler gauge potential associated with the Kähler form J_{tot} of $S^2 \times CP_2$.

The square of the equation gives the spinor analog of d'Alembert equation and generalized eigenvalue as the analog of mass squared. The propagator associated with the wormhole throats is formally massless Dirac propagator so that standard twistor formalism applies also without the octonionic representation of the gamma matrices although the physical particles propagating along the opposite wormhole throats are massive on mass shell particles with both signs of energy [K6] .

2. Second equation is the 4-D modified Dirac equation defined by Kähler action.

$$D_K\Psi = 0 \ . \tag{5.8}$$

The dimensional reduction of this operator to a sum corresponding to $D_{K,3}$ acting on light-like 3-surfaces and 1-D operator $D_{K,1}$ acting on the coordinate labeling the 3-D light-like 3-surfaces in the slicing would allow to assign eigenvalues to $D_{K,3}$ as analogs of energy eigenvalues for ordinary Schrödinger equation. One proposal has been that Dirac determinant could be identified as the product of these eigen values. Another and more plausible identification is as the product of pseudo masses assignable to D_3 defined by Chern-Simons action [B1] . It must be however made clear that the identification of the exponent of the Kähler function to Chern-Simons term makes the identification as Dirac determinant un-necessary.

3. There are two options depending on whether one requires that the eigenvalue equation applies only on the wormhole throats and at the ends of the space-time surface or for all 3-surfaces in the slicing of the space-time surface by light-like 3-surfaces. In the latter case the condition that the pseudo four-momentum is same for all the light-like 3-surfaces in the slicing gives a consistency condition stating that the commutator of the two Dirac operators vanishes for the solutions in the case of preferred extremals, which depend on the momentum and color quantum numbers also:

$$[D_K, D_3]\,\Psi = 0 \ . \tag{5.9}$$

This condition is quite strong and there is no deep reason for it since λ_k does not correspond to the physical conserved momentum so that its spectrum could depend on the light-like 3-surface in the slicing. On the other hand, if the eigenvalues of D_3 belong to the preferred hyper-complex plane M^2, D_3 effectively reduces to a 2-dimensional algebraic Dirac operator $\lambda^k\gamma_k$ commuting with D_K: the values of λ^k cannot depend on slice since this would mean that D_K does not commute with D_3.

2. About the hyper-octonionic variant of the modified Dirac equation

What gives excellent hopes that the octonionic variant of modified Dirac equation could lead to a provide precise information about the solution spectrum of modified Dirac equation is the condition that everything in the equation should be associative. Hence the terms which are by there nature non-associative should vanish automatically.

1. The first implication is that the besides octonionic gamma matrices also octonionic spinors should belong to the local quaternionic plane at each point of the space-time surface. Spinors are also generated by quaternionic Clifford algebra from two preferred spinors defining a preferred plane in the space of spinors. Hence spinorial dynamics seems to mimic very closely the space-time dynamics and one might even hope that the solutions of the modified Dirac action could be seen as maps of the space-time surface to surfaces of the spinor space. The reduction to quaternionic sub-algebra suggest that some variant of ordinary twistors emerges in this manner in matrix representation.

2. The octonionic sigma matrices span G_2 where as ordinary sigma matrices define $SO(7,1)$. On the other hand, the holonomies are identical in the two cases if right-handed charge matrices are mapped to zero so that there are indeed hopes that the solutions of the octonionic Dirac equation cannot be mapped to those of ordinary Dirac equation. If left-handed charge matrices are mapped to zero, the resulting theory is essentially the analog of electrodynamics coupled to gravitation at classical level but it is not clear whether this physically acceptable. It is not clear whether associativity condition leaves only this option under consideration.

3. The solution ansatz to the modified Dirac equation is expected to be of the form $\Psi = D_K(\Psi_0 u_0 + \Psi_1 u_1)$, where u_0 and u_1 are constant spinors representing real unit and the preferred unit e_1. Hence constant spinors associated with right handed electron and neutrino and right-handed d and u quark would appear in Ψ and Ψ_i could correspond to scalar coefficients of spinors with different charge. This ansatz would reduce the modified Dirac equation to $D_K^2\Psi_i = 0$ since there are no charged couplings present. The reduction of a d'Alembert type equation for single scalar function coupling to $U(1)$ gauge potential and $U(1)$ "gravitation" would obviously mean a dramatic simplification raising hopes about integrable theory.

4. The condition $D_K^2\Psi = 0$ involves products of three octonions and involves derivatives of the modified gamma matrices which might belong to the complement of the quaternionic sub-space. The restriction of Ψ to the preferred hyper-complex plane M^2 simplifies the situation dramatically but $(D_K^2)D_K\Psi = D_K(D_K^2)\Psi = 0$ could still fail. The problem is that the action of D_K is not algebraic so that one cannot treat reduce the associativity condition to $(AA)A = A(AA)$.

5.3.3 Could the notion of octo-twistor make sense?

Twistors have led to dramatic successes in the understanding of Feynman diagrammatics of gauge theories, $N = 4$ SUSYs, and $N = 8$ supergravity [B5, B6, B3] . This motivated the question whether they might be applied in TGD framework too [K13] - at least in the description of the QFT limit. The basic problem of the twistor program is how to overcome the difficulties caused by particle massivation and TGD framework suggests possible clues in this respect.

1. In TGD it is natural to regard particles as massless particles in 8-D sense and to introduce 8-D counterpart of twistors by relying on the geometric picture in which twistors correspond to a pair of spinors characterizing light-like momentum ray and a point of M^8 through which the ray traverses. Twistors would consist of a pair of spinors and quark and lepton spinors define the natural candidate for the spinors in question. This approach would allow to handle massive on-mass-shell states but cannot cope with virtual momenta massive in 8-D sense.

2. The emergence of pseudo momentum λ_k from the generalized eigenvalue equation for D_{C-S} suggest a dramatically simpler solution to the problem. Since propagators are effectively massless propagators for pseudo momenta, which are functions of physical on shell momenta (with both signs of energy in zero energy ontology) and of other quantum numbers, twistor formalism can be applied in its standard form. An attractive assumption is that also λ^k are conserved in the

vertices but a good argument justifying this is lacking. One can ask whether also $N = 4$ SUSY, $N = 8$ super-gravity, and even QCD could have similar interpretation.

This picture should apply also in the case of octotwistors with minor modifications and one might hope that octotwistors could provide new insights about what happens in the real case.

1. In the case of ordinary Clifford algebra unit matrix and six-dimensional gamma matrices γ_i, $i = 1, ..., 6$ and $\gamma_7 = \prod_i \gamma_i$ would define the variant of Pauli sigma matrices as $\sigma_0 = 1$, $\sigma_k = \gamma_k$, $k = 1, .., 7$ The problem is that masslessness condition does not correspond to the vanishing of the determinant for the matrix $p_k\sigma^k$.

2. In the case of octo-twistors Pauli sigma matrices σ^k would correspond to hyper-octonion units $\{\sigma_0, \sigma_k\} = \{1, ie^k\}$ and one could assign to $p_k\sigma^k$ a matrix by the linear map defined by the multiplication with $P = p_k\sigma^k$. The matrix is of form $P_{mn} = p^k f_{kmn}$, where f_{kmn} are the structure constants characterizing multiplication by hyper-octonion. The norm squared for octonion is the fourth root for the determinant of this matrix. Since $p_k\sigma^k$ maps its octonionic conjugate to zero so that the determinant must vanish (as is easy to see directly by reducing the situation to that for hyper-complex numbers by considering the hyper-complex plane defined by P).

3. Associativity condition for the octotwistors requires that the gamma matrix basis appearing in the generalized eigenvalue equation for Chern-Simons Dirac operator must differs by a local G_2 rotation from the standard hyper-quaternionic gamma matrix for M^4 so that it is always in the local hyper-quaternionic plane. This suggests that octo-twistor can be mapped to an ordinary twistor by mapping the basis of hyper-quaternions to Pauli sigma matrices. A stronger condition guaranteing the commutativity of D_3 with $\lambda^k\gamma_k$ is that λ_k belongs to a preferred hyper-complex plane M^2 assignable to a given CD. Also the two spinors should belong to this plane for the proposed solution ansatz for the modified Dirac equation. Quaternionization would also allow to assign momentum to the spinors in standard manner.

   The spectrum of pseudo-momenta would be 2-dimensional (continuum at worst) and this should certainly improve dramatically the convergence properties for the sum over the non-conserved pseudo-momenta in propagators which in the worst possible of worlds might destroy the manifest finiteness of the theory based on the generalized Feynman diagrams with the throats of wormholes carrying always on mass shell momenta. This effective 2-dimensionality should apply also in the real case and would have no catastrophic consequences since pseudo momenta are in question.

   As a matter fact, the assumption the decomposition of quark momenta to longitudinal and transversal parts in perturbative QCD might have interpretation in terms of pseudo-momenta if they are conserved.

4. $M^8 - H$ duality suggests a possible interpretation of the pseudo-momenta as M^8 momenta which by purely number theoretical reasons must be commutative and thus belong to M^2 hyper-complex plane. One ends up with the similar outcome as one constructs a representation for the quantum states defined by WCW spinor fields as superpositions of real units constructed as ratios of infinite hyper-octonionic integers with precisely defined number theoretic anatomy and transformation properties under standard model symmetries having number theoretic interpretation [K11] .

6 What one really means with a virtual particle?

Massive particles are the basic problem of the twistor program. The twistorialization of massive particles does not seem to be a problem in TGD framework thanks to the possibility to interpret them as massless particles in 8-D sense but the situation is unsatisfactory for virtual particles.

The ideas possibly allowing to circumvent this problem emerged from a totally unexpected direction. The inspiration came from the finding of Martin Grusenick [E1] who discovered that a Mickelson-Morley interferometer rotating in plane gives rise a non-trivial interference pattern when the plane is orthogonal to the Earth's surface but no effect when parallel to the Earth's surface. The effect could be due to a contraction of the system in the vertical direction caused by the own weight of the system and would thus involve no new physics. If not, then one must try to find General Relativistic explanation for it. Schwartschild metric predicts this kind of effect but it is by a factor 10^{-4} too small.

In TGD framework one can however consider an explanation of the effect [K12] .

1. By relaxing the empty space assumption to the assumption that only the energy density (that is G^{tt}) vanishes but the other diagonal components of Einstein tensor in Schwartschild coordinates can be non-vanishing allows to explain the effect in terms of the deviation of the radial component g_{rr} of the metric from Schwartschild metric. The predicted deviation decreases as $1/r$ and does not affect planetary orbits appreciably even if present for all astrophysical objects. The value of G determined from radial acceleration at the surface of Earth is predicted to deviate from the actual value as a consequence. The deviation of the metric from empty space metric could also explain the known surprisingly large variation in the measured values of G since nearby gravitational fields are involved.

2. The Einstein tensor in regions with vanishing energy density would obviously correspond to a tachyonic matter. This led to a series of ideas allowing to sharpen the physical meaning of Einstein's equations in TGD framework. The basic result would be the extension of quantum classical correspondence. The Einstein tensor in matter free regions would describe the presence of virtual particles and would fail to satisfy causality constraint since it corresponds to the space-like momentum exchange of the system with the external world (space-likeness follows if the scattering is elastic).

3. It is difficult to understand how the energy momentum tensor of matter could behave like $G^{\alpha\beta}$ does if the latter describes tachyons. The resolution of the problem could be very simple in zero energy ontology. In zero energy ontology bosons (and their super counterparts) correspond to wormhole contacts carrying fermion and antifermion numbers at the light-like wormhole throats and having opposite signs of energy. This allows the possibility that the fermions at the throats are on mass shell and the sum of their momenta gives rise to off mass shell momentum which can be also space-like. In zero energy ontology $G^{\alpha\beta}$ would naturally correspond to the sum of on mass shell energy energy momentum tensors $T_{\pm}^{\alpha\beta}$ associated with positive and negative energy fermions and their super-counterparts. Note that for the energy momentum tensor $T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta - pg^{\alpha\beta}$ of fluid with $u^\alpha u_\alpha = 1$ constraint stating on mass shell condition the allowance of virtual particles would mean giving up the condition $u^\alpha u_\alpha = 1$ for the velocity field.

6.1 Could virtual particles be regarded as pairs of on mass shell particles with opposite energies?

This identification suggests a concrete identification of virtual particle as pairs of positive and negative energy on mass shell particles allowing an elegant formulation of the twistor program in the case of virtual particles [K13, K7] .

1. The basic idea is that massive on mass shell states can be regarded as massless states in 8-dimensional sense so that twistor program generalizes to the case of massive on mass shell states associated with the representations of super-conformal algebras. One has however allow now also off mass shell states, in particular those with space-like momenta, and the question is how to describe them in terms of generalized twistors. In the case of wormhole contacts the answer looks obvious. Bosons and their super partners could correspond to pairs of positive and negative energy on mass shell states and could be described using a pair of twistors associated with composite momenta massless in 8-D sense.

2. It took some time to realize that the most elegant identification of the on mass shall bosons would be as wormhole contacts for which both throats have either positive or negative energy. This would imply automatically on-mass shell property. The basic objection against this has been that one cannot construct massless spin 1 states in this manner. Dirac equation in M^4 implies that the momenta are parallel and for fermion and antifermion the helicities are therefore

opposite and only longitudinal polarization representing pure gauge degree of freedom is possible. It is amazing how long time it required to realize that I had swallowed this objection completely uncritically. After all, the first thing that I learned from the Dirac equation for massless induced spinors is that it mixes unavoidably M^4 chiralities except for very special vacuum extremals like canonically imbedded M^4. Same applies to the modified Dirac equation. Therefore there is no problem! Of course, also the p-adic mass calculations involve imbedding spaced spinors for which M^4 helicities are mixed strongly since only covariantly constant right handed neutrino is massless and possesses a well defined M^4 helicity. At space-time level a pair of massless extremals (topological light rays) with same (opposite) energies and connected by wormhole contacts could serve as a space-time correlate for on (off) mass shell boson.

3. How can one then identify virtual fermions and their super-counterparts? These particles have been assumed to consist of single wormhole throat associated with a deformation of CP_2 vacuum extremal so that the proposed definition would allow only on mass shell states. A possible resolution of the problem is the identification of also virtual fermions and their super-counterparts as wormhole contacts in the sense that the second wormhole throats is fermionic Fock vacuum carrying purely bosonic quantum numbers and corresponds to a state generated by purely bosonic generators of the super-symplectic algebra whose elements are in 1-1 correspondence with Hamiltonians of $\delta M^4_{\pm} \times CP_2$. Thus the distinction between on mass shell and of mass shell states would be purely topological for fermions and their super partners.

4. The concrete physical interpretation would be that particle scattering event involves at least two parallel space-time sheets. Incoming (outgoing) fermion is topologically condensed at positive energy (negative energy) sheet and corresponds to single throat. In the interaction region fermionic spaced-time sheet touches with a high probably the large space-time sheet sheet since the distance between sheets is about 10^4 Planck lengths. The touching (topological sum) generates a second wormhole throat with a spherical topology and carrying no fermion number but having on mass shell momentum. Virtual fermions would be interacting fermions. Since only topological sum contacts are formed, also virtual fermions are labeled by the genus g of the 2-D wormhole throat whereas bosons are labeled by the pair (g_1, g_2) of the genera of two wormhole throats. This classification is consistent with the mechanism giving rise to virtual bosons.

The proposed identification of virtual and on mass shell particles is beautiful but it is of course far from obvious whether it really make sense. Bosonic emergence means that the fundamental loop integrals are for fermionic loops. One could in principle get rid of bosonic loop integrals by using generalized Cutkosky rules [K9, K7] but it would be highly satisfying to have a concrete physical interpretation for the loops. It interesting to see whether the proposed picture picture works in practice. Bosonic emergence means that one path integrates first over fermions to get bosonic action as radiative corrections. Only 3-vertices (or rather, 3 momenta are associated with the vertex [K7]) are involved at the fermion level whereas at the bosonic level arbitrary high vertiecs appear.

6.2 How to treat the new degrees of freedom?

The identification of off mass shell states as on mass shell states of positive and negative energy throats brings in new degrees of freedom. Let us first look what happens if the momenta of the two throats of wormhole contact are completely uncorrelated apart from the condition $p_1 - p_2 = p$ coming from the energy conservation in the 3-vertex. Here p_1 $(-p_2)$ is the momentum of on mass shell positive (negative) energy throat and p is the momentum of outgoing (incoming) wormhole contact. On mass shell conditions eliminate two degrees of freedom so that in absence of correlations the 4-D integral over loop momenta should be extended to a 6-D integral. For a given time-like virtual momentum p these degrees of freedom corresponds to 2-dimensional sphere as one finds by looking the situation in the rest system of p (the direction of $\overline{p}_1 = -\overline{p}_2$ is arbitrary) so that additional loop integration is finite. For light-like p the additional degrees of freedom correspond to 2-D light-cone boundary δM^3_+ defined by the condition $t^2 - x^2 - y^2 = 0$: δM^3_+ $SO(1,2)$ invariant 2-volume does not exist. This is not a catastrophe since massless momenta define lower-dimensional sub-manifold of the momentum space. For space-like p one has hyperboloid $t^2 - x^2 - y^2 = -1$ and the 2-D loop integral would be infinite in absence of additional constraints.

A 2-dimensional integral appears at each line of Feynman diagram and if the only constraint comes from $p_1 - p_2 = p$ one obtains new divergences for space-like momenta p. One can imagine several approaches to the problem.

1. The most conservative approach assumes that the freedom to select the decomposition $p = p_1 + p_2$ is completely analogous to a gauge symmetry. This is the case if the propagators are just the usual ones. Although this decomposition would take place it would not have any physical consequences since scattering amplitudes do not depend on the choices of these decompositions. For each line the integral over the decompositions normalized by the volume of S^2 or hyperboloid would give the same result as an arbitrary gauge choice fixing the decompositions.

2. For the second option the new degrees of freedom would be present for each line of the generalized Feynman diagram in a non-trivial manner, and the dependence of the emission vertices on the decompositions should allow to avoid the infinities for space-like p. The vertices would depend on Lorentz invariant quantities such as $k \cdot \cdot p_1$ and $k \cdot \cdot p_2$, where k denotes the momentum of any line coming to the vertex, and in an optimist mood one could ask whether this dependence could allow to smooth out also the standard loop divergences by bringing in the effective momentum cutoff through the new momentum degrees of freedom. In twistorial description this kind of dependence could allow especially elegant realization. Note that also a sum over mass shells is involved and can cause divergences.

3. For the third option the new degrees of freedom would be eliminated by some physical mechanism fixing the direction of the projection of p_1 (and p_2) in the hyperplane normal to p. The minimum option would eliminate the additional 2-dimensional integral but would not pose conditions on the loop momenta p_1 and p_2. One should be able to fix the direction of the projection of p_1 in the hyperplane $P(p)$ whose normal is p by some rule having a physical justification. As a matter fact, this option would be special case of the first one.

Bosonic sector (with super partners included) poses additional conditions. N-boson vertices are defined by fermionic loops and N-boson vertices with arbitrary large value of N are possible. Bosonic propagators emerge as inverses of 2-boson vertices defined by fermionic loops. Let $p_B = p_1 + p_2$ denote the sought for decomposition to on mass shell momenta. For the first and second options there are no obvious problems in the bosonic sector. For the third option there is a serious difficulty involvedthe decompositions $p_B = p_1 + p_2$ defined by the vertices at the opposite ends of the boson line are not in general consistent. This kind of conditions lead to a hopelessly clumsy formalism.

6.3 Could additional degrees of freedom allow natural cutoff in loop integrals?

Second option involving two new degrees of freedom for each internal line deserves a more detailed discussion. The masses assignable to on mass shell throats define an inherent momentum cutoff allowing to get rid of infinities without giving up conformal invariance. Of course, mass squared cutoff comes also from the breakdown of the QFT limit at CP_2 length scale but one might hope that this cutoff is not actually needed.

1. To see what is involved, consider a BFF vertex with the fermionic momenta $p_1 = p_{11} + p_{12}$ and $p_2 = p_{21} + p_{22}$, and bosonic momentum $p_3 = p_{31} + p_{32}$. As a concrete example, one might consider the calculation of bosonic propagator as the inverse of the bosonic 2-vertex involving fermion loop for which a model was discussed in [K9] . For definiteness restrict the consideration to the decomposition of the fermionic momentum p_1. The natural direction in the orthogonal complement $P(p_1)$ of p_1 is defined by p_2 (equivalently by p_3). The corresponding momentum projections

$$P_{i1} = p_i - \frac{p_i \cdot p_1 p_1}{p_1^2} \quad , \quad i = 2, 3$$

are the same. P_{i1} in general diverges for $p_1^2 = 0$.

2. Conformal invariance allows only dimensionless Lorentz invariants constructed from the momenta. Strong form of the conformal invariance does not allow dependence on the masses of the throats. For time-like (space-like) p_1 the dimensionless variable

$$c_{12} \equiv \frac{p_{11} \cdot P_{21}}{\sqrt{p_{11}^2}\sqrt{P_{21}^2}} = c_{13}$$

describes the cosine (hyperbolic cosine) of the angle (hyperbolic angle) between p_{11} and P_{21}. The corresponding sine (hyperbolic sine) $s_{i,i+1}$ vanishes when p_{11} is parallel to the projection of p_2 (p_3) in $P(p_1)$. Similar variables can be assigned to p_2 and p_3. Together with the three analogous variables

$$c_{i,i} = \frac{p_{i1} \cdot p_i}{\sqrt{p_{i1}^2}\sqrt{p_i^2}}$$

measuring the hyperbolic angle between between p_{i1} and p_i, one has 6 variables. p_{i1}^2 and p_i^2 can have both signs and also vanish and this might lead difficulties if one wants Gaussians and analyticity.

3. The on mass shell property for throats allows to consider a milder form of conformal invariance for which one has variables

$$C_{12} \equiv \frac{p_{11} \cdot P_{21}}{m_1 m_2} = C_{13} \ ,$$

where m_i, $i = 1, 2$ denote that throat masses. This introduces a cutoff in P_{21} when p_1 is space-like. These variables have infinite values for massless throats so that massless throats cannot appear as building bricks of the virtual particles. The assumption that on mass-shell bosons involve massless wormhole throat would distinguish them from virtual bosons in a unique manner.

4. One can also identify dimensionless quantities formed from the loop momenta. Strong form of conformal invariance allows only

$$d_{ij} = \frac{p_i \cdot p_j}{\sqrt{p_i^2}\sqrt{p_j^2}}$$

possible also for ordinary loops. These variables give hope about cutoff with respect to Lorentz boost for p_i in the rest system of p_j but again the signs are problematic. The weaker form of conformal invariance allows also the variables

$$D_{ij} = \frac{p_i \cdot p_j}{m_i m_j}$$

not plagued by the sign problems and giving hopes also about mass squared cutoff. Indeed, if on mass shell throats are present they should take a key role in the physics of the virtual particles.

The following two simple examples give an idea about what might be involved.

1. Consider first a vertex factor which is a Gaussian of form $exp(-\sum_{ij} S_{ij}^2) = exp(-2\sum_i (S_{i,i+1})^2 - \sum_i S_{i,i}^2)$ suppressing the the momenta p_{i1} for which the projections in $P(p_i)$ are not parallel to those of p_j and also large boosts of p_{i1} in the rest system of p_i. Massless throats would not appear at all in internal lines. The additional 2-D integrals together with the correlation between p_k and p_{i1} do not probably smooth out the standard loop divergences in momentum squared and hyperbolic angle. The replacement of S_{ij} with s_{ij} together with analyticity leads to difficulties since s_{ij} does not have a definite sign.

2. The exponential $exp(-\sum_{i\neq j} D_{ij}^2)$ forces the decoupling of massless throats from virtual states, is free of the sign difficulties, and allowes a stronger hyperbolic cutoff as well as mass scale cutoff. The replacement of D_{ij} with d_{ij} leads to the same problems as encountered in the first example. The simple model for the hyperbolic cutoff discussed in [K9] could allow a more refined formulation in this framework. It is however important to realize that this kind of cutoffs look rather adhoc for the generalization of supersymmetric action for fermions [K7] . They might be present in the radiatively generated bosonic action.

6.4 Could quantum classical correspondence fix the correct option?

Concerning the dynamics in the new degrees of freedom the above argument lead two options under consideration. The first option assumes M^2 gauge invariance and can be criticized as being somewhat ad hoc unless one can find a convincing interpretation for the restriction of the momenta p_1 and p_2 to $M^2 \cap P(p)$, where M^2 denotes a sub-space of M^4 defining the space of non-physical polarizations and $P(p)$ is the orthogonal complement of $p = p_1 + p_2$. For both options one can argue that the decomposition $p = p_1 + p_2$ should have same space-time correlate.

1. Preferred extremals of Kähler action are characterized by a local choice of $M^2(x) \subset M^4$ in such a manner that the subspaces $M^2(x)$ integrate to a 2-D surface in M^4. $M^2(x)$ has a physical interpretation as the sub-space of non-physical polarizations. Number theoretical interpretation is as a hyper-complex plane of complexified octonions. In the generalized Feynman diagrammatics only the choice of $M^2(x)$ at the 2-D partonic 2-surfaces X^2 identified as the ends the 3-D light-like wormhole throats X_l^3 matters. For a given line one can also restrict the consideration to single point x of X^2 since fermion numbers is carried by a light-like curve along X_l^3: the is an integral over possible choices of course. The additional degrees of freedom would therefore have a concrete interpretation in terms of space-time surfaces. The effective two-dimensionality states that M-matrix depends only the partonic 2-surfaces and their 4-D tangent spaces containing $M^2(x)$ at the ends of the lines of generalized Feynman diagrams.

2. The first option would mean a complete independence on $M^2(x)$ at partonic 2-surface implied by the first option would mean actual 2-dimensionality instead of only effective one. This is not quite in spirit of quantum TGD although it might make sense at QFT limit.

3. For the second option preferred extremals would reflect in their properties the decomposition $p = p_1 + p_2$ for the internal lines and the dependence of vertices on the decomposition could correspond to the value of the vacuum functional for a given distribution of the planes $M^2(x)$. The locality of the choice $M^2(x)$ would mean that p_1 and p_2 are not separately conserved during the propagation along the internal line and physical picture suggests that the choice $M^2(x)$ is constant for light-like 3-surfaces representing lines of the generalized Feynman diagrams.

6.5 Could the formulation of SUSY limit of TGD allow the new view about off mass shell particles?

Could the proposed heuristic ideas about off mass shell particles and diagram-wise finiteness of the perturbation theory, the suggested manner to fix the direction of the projections of p_1 and p_2 in $P(p)$ in terms of the preferred polarization plane $M^2 \subset M^4$ characterizing a given line of Feynman diagram, and the formulation of super-symmetric QFT limit of TGD [K7] be consistent with each other?

1. There are good arguments that the generalized SUSY based on bosonic emergence and the generalization of super field concept guarantees the cancelation of divergences associated with particles and their super-partners. The new view about off mass shell particles encourages a dream about the finiteness of the individual diagrams justifying the motivations for the primitive model of [K9] .

2. The description of bosons and their superpartners as wormhole throats requires at the fundamental level the introduction of new degrees of freedom associated with $p = p_1 - p_2$ decomposition. On mass shell property is possible and would realize twistorial dreams. If one keeps the original view about virtual fermions and their super-partners as single throated objects, there is no need to describe virtual fermions as wormhole contacts.

3. Quantum classical correspondence suggests that the projections of p_1 and p_2 into $P(p)$ lie in the intersection $M^2 \cap P(p)$, where M^2 characterizes the line of the generalized Feynman diagram. If so, then the new degrees of freedom mean integral over the planes M^2 labeled by the points of $s \in S^2$. If also virtual fermions correspond to wormhole contacts, BFF-vertices would contain an amplitude $f(\alpha, s_1, s_2, s_3)$ with s_i characterizing the lines. The parameters α would code information about the momenta of virtual particles, about the masses of on mass shell particles comprising the virtual particles, and also about the dynamics of Kähler action involving exponent of Kähler function for the extremal in question. If virtual fermions are single throated, one has $f(\alpha, s)$ with s characterizing the bosonic line. The generalization would require a characterization of the form factor $f(\alpha, s_1, s_2, s_3)$ or $f(\alpha, s)$ in principle predicted by TGD proper but probably only modelable at QFT limit. The view about preferred extremals allows the possibility that s_i is not conserved along line. If the values of s_i at the ends of the line are not correlated, the integral over s_i gives a form factor $F(\alpha)$.

4. The propagators for the generalized chiral super-field describing fermions would not be affected, and the effects of f would be only seen at the level of propagators and vertices for bosons and their super-parterns. f could in principle guarantee the finiteness of individual contributions to both fermionic and bosonic loops without the need for Wick rotation.

6.6 Trying to sum up

The proposed replacement of virtual particles as a convenient mathematical abstraction with something very real suggests that the black box of the loop integrals could be opened and one might even construct concrete models for off mass shell particles using twistorial formulation. The conservative approach would interpret the non-uniqueness of the decomposition of the loop momenta to on mass shell momenta in terms of gauge invariance. A more radical approach would assign two additional degrees of freedom to each line of generalized Feynman diagram and allow vertices to depend on the decomposition. This would give even hopes about the smoothing out of the standard divergences. As a matter fact, this idea was followed already in the chapter about bosonic emergence [K9] , where it was proposed that natural physical cutoffs on mass squared and hyperbolic angle characterizing the energy of virtual particle could guarantee the finiteness of fermionic loops. The construction of the super-symmetric QFT limit of TGD [K7] however suggests that the cancelation of infinities takes place by super-symmetry even without cutoffs. One interpretation is that this cancelation justifies the neglect of the physical cutoff as an excellent approximation. An interesting question is whether the loop integrals could make sense even without Wick rotation.

7 The first attempt to formulate twistorial description

This section summarizes a further vision about how twistors might emerge from quantum TGD. It is only loosely related to the other visions and is certainly the simplest one and also very closely related to the recent picture about generalized Feynman diagrams. Of course, it is bound to be speculative just like all other considerations of this chapter and one cannot take the details of the proposal too seriously.

7.1 The simplest vision about how twistors might emerge from TGD

The vision involves the notions of bosonic emergence, the identification of virtual states as pairs of on mass shell states assignable to wormhole throats inspired by zero energy ontology and associated realization of Cutkosky rules in terms of manifestly finite Feynman diagrammatics, and as the latest piece the weak form of electric-magnetic duality and the notion of pseudo-momentum emerging from the generalized eigenstates of the Chern-Simons Dirac operator.

There must be a correlation between pseudo-momenta and real momenta. One can imagine two identifications.

1. Chern-Simons action emerges as boundary term in variation of Kähler Dirac action [K6] and variation gives Chern-Simons Dirac equation with an additional contribution to modified gamma

matrices from the 3-D Lagrange multiplier term expressing electric-magnetic duality. The Chern-Simons Dirac propagator is very difficult to handle mathematically. One also wants a correlation between incoming quantum numbers and also those assignable to fermionic lines and action of Chern-Simons Dirac operator. This is achieved by adding a 3-D measurement interaction term having same form as massless Dirac interaction in M^4 but obtained by replacing the derivatives with components of light-like four-momentum [K6]. The outcome is that Chern-Simons Dirac operator effective acts as the massless Dirac operator $p^k\gamma_k$ of M^4.

What is important is that $p^k\gamma_k$ does not annihilate fermions with unphysical helicities so that on virtual fermions can be taken as on mass shell massless fermions with unphysical helicity. Indeed, the integration over loop momentum interpreted as residue integral reduces for fermion propagator to D with 3-D integral over light-like four-momenta. A result highly analogous to that obtained for gauge bosons in twistor approach.

2. Second possibility is that the eigenvalues of C-S Dirac operator (this identification does not follow from action principle) are identified as analogs of region momenta whose differences define the incoming massless momenta in twistor diagram. This does not give direct connection with the quantum numbers of the fermions appearing in incoming lines and brings in additional complications. As one might expect, I chose just this option when I wrote this chapter for the first time!

In the following the basic arguments supporting this still speculative picture are described.

7.2 Generalized eigen modes for the modified Chern-Simons Dirac equation and hydrodynamical picture

Hydrodynamical picture and the reduction of TGD to almost topological QFT discussed in detail in [K6] helps to understand also the construction of generalized eigen modes of 3-D Chern-Simons Dirac equation.

7.3 Generalized Feynman diagrams at fermionic and momentum space level

Negative energy ontology has already led to the idea of interpreting the virtual particles as pairs of positive and negative energy wormhole throats. Hitherto I have taken it as granted that ordinary Feynman diagrammatics generalizes more or less as such. It is however far from clear what really happens in the verties of the generalized Feynmann diagrams. The safest approach relies on the requirement that unitarity realized in terms of Cutkosky rules in ordinary Feynman diagrammatics allows a generalization. This requires loop diagrams. In particular, photon-photon scattering can take place only via a fermionic square loop so that it seems that loops must be present at least in the topological sense.

One must be however ready for the possibility that something unexpectedly simple might emerge. For instance, the vision about algebraic physics allows naturally only finite sums for diagrams and does not favor infinite perturbative expansions. Hence the true believer on algebraic physics might dream about finite number of diagrams for a given reaction type. For simplicity generalized Feynman diagrams without the complications brought by the magnetic confinement since by the previous arguments the generalization need not bring in anything essentially new.

The basic idea of duality in early hadronic models was that the lines of the dual diagram representing particles are only re-arranged in the vertices. This however does not allow to get rid of off mass shell momenta. Zero energy ontology encourages to consider a stronger form of this principle in the sense that the virtual momenta of particles could correspond to pairs of on mass shell momenta of particles. If also interacting fermions are pairs of positive and negative energy throats in the interaction region the idea about reducing the construction of Feynman diagrams to some kind of lego rules might work.

7.3.1 Virtual particles as pairs of on mass shell particles in ZEO

The first thing is to try to define more precisely what generalized Feynman diagrams are. The direct generalization of Feynman diagrams implies that both wormhole throats and wormhole contacts join at vertices.

1. A simple intuitive picture about what happens is provided by diagrams obtained by replacing the points of Feynman diagrams (wormhole contacts) with short lines and imagining that the throats correspond to the ends of the line. At vertices where the lines meet the incoming on mass shell quantum numbers would sum up to zero. This approach leads to a straightforward generalization of Feynman diagrams with virtual particles replaced with pairs of on mass shell throat states of type $++$, $--$, and $+-$. Incoming lines correspond to $++$ type lines and outgoing ones to $--$ type lines. The first two line pairs allow only time like net momenta whereas $+-$ line pairs allow also space-like virtual momenta. The sign assigned to a given throat is dictated by the the sign of the on mass shell momentum on the line. The condition that Cutkosky rules generalize as such requires $++$ and $--$ type virtual lines since the cut of the diagram in Cutkosky rules corresponds to on mass shell outgoing or incoming states and must therefore correspond to $++$ or $--$ type lines.

2. The basic difference as compared to the ordinary Feynman diagrammatics is that loop integrals reduce to integrals over over light-like mass shell momenta for fermions with unphysical helicity by interpreting 4-D momentum integral as residue integral. This if all wormhole throats are assumed to carry light-like on mass shell momenta. In each vertex of the loop mass incoming on mass shell momenta must sum up to on mass shell momentum.

   The original hope was that these constraints improve the behavior of loop integrals dramatically and give rise to finiteness.

   It does not however seem that only a finite number of diagrams contribute to the scattering amplitude besides tree diagrams. The point is that if a the reactions $N_1 \rightarrow N_2$ and $N_2 \rightarrow N_3$,, where N_i denote particle numbers, are possible in a common kinematical region for N_2-particle states then also the diagrams $N_1 \rightarrow N_2 \rightarrow N_2 \rightarrow N_3$ are possible. The virtual states N_2 include all all states in the intersection of kinematically allow regions for $N_1 \rightarrow N_2$ and $N_2 \rightarrow N_3$. Hence the dream about finite number possible diagrams is not fulfilled if one allows massless particles. If all particles are massive then the particle number N_2 for given N_1 is limited from above and the dream is realized.

   It has also tgurned out that finiteness is too much to hope. To achieve finiteness one must replace Feynman diagrams with stringy diagrams and these indeed emerge naturally in TGD framework as became clear after writing the first draft of this chapter.

3. The original argument suggesting finiteness went as follows. For instance, in gauge theories loops are not possible in the massless case or are highly singular (bringing in mind twistor diagrams) since the conservation laws at vertices imply that the momenta are parallel. In the massive case and allowing mass spectrum the situation is not so simple. As a first example one can consider a loop with three vertices and thus three internal lines. Three on mass shell conditions are present so that the four-momentum can vary in 1-D subspace only. For a loop involving four vertices there are four internal lines and four mass shell conditions so that loop integrals would reduce to discrete sums. Loops involving more than four vertices are expected to be impossible.

   There is however an important distinction between TGD and gauge theories. At microcopic level one has four-fermion vertices although wormhole contacts with fermion and antifermion at throats behave effectively as virtual bosons. This means that momentum conservation for massless momenta does not force them to be parallel.

4. The proposed replacement of the elementary fermions with bound states of elementary fermions and monopoles $X_\pm$ brings in the analog of stringy diagrammatics. The 2-particle wave functions in the momentum degrees of freedom of fermiona and $X_\pm$ migh allow more flexibility and allow more loops. Note however that there are excellent hopes about the finiteness of the theory also in this case.

7.3.2 Are loop integrals manifestly finite?

One can make also more detailed observations about loops.

1. The simplest situation is obtained if only 3-vertices are allowed. In this case conservation of momentum however allows only collinear momenta although the signs of energy need not be the same. Particle creation and annihilation is possible and momentum exchange is possible but is always light-like in the massless case. The scattering matrices of supersymmetric YM theories would suggest something less trivial and this raises the question whether something is missing. Magnetic monopoles are an essential element of also these theories as also massivation and symmetry breaking and this encourages to think that the formation of massive states as fermion $X_{\pm}$ pairs is needed. Of course, in TGD framework one has also high mass excitations of the massless states making the scattering matrix non-trivial.

2. In YM theories on mass shell lines would be singular. In TGD framework this is not the case since the propagator is defined as the inverse of the 3-D dimensional reduction of the modified Dirac operator D containing also coupling to four-momentum (this is required by quantum classical correspondence and guarantees stringy propagators),

$$\begin{aligned} D &= i\hat{\Gamma}^{\alpha}p_{\alpha} + \hat{\Gamma}^{\alpha}D_{\alpha} \ , \\ p_{\alpha} &= p_k\partial_{\alpha}h^k \ . \end{aligned} \tag{7.1}$$

   The propagator does not diverge for on mass shell massless momenta and the propagator lines are well-defined. This is of course of essential importance also in general case. Only for the incoming lines one can consider the possibility that 3-D Dirac operator annihilates the induced spinor fields. All lines correspond to generalized eigenstates of the propagator in the sense that one has $D_3\Psi = \lambda\gamma\Psi$, where γ is modified gamma matrix in the direction of the stringy coordinate emanating from light-like surface and D_3 is the 3-dimensional dimensional reduction of the 4-D modified Dirac operator. The eigenvalue λ is analogous to energy. Note that the eigenvalue spectrum depends on 4-momentum as a parameter.

3. Massless incoming momenta can decay to massless momenta with both signs of energy. The integration measure $d^2k/2E$ reduces to dx/x where $x \geq 0$ is the scaling factor of massless momentum. Only light-like momentum exchanges are however possible and scattering matrix is essentially trivial. The loop integrals are finite apart from the possible delicacies related to poles since the loop integrands for given massless wormhole contact are proportional to dx/x^3 for large values of x.

4. Irrrespective of whether the particles are massless or not, the divergences are obtained only if one allows too high vertices as self energy loops for which the number of momentum degrees of freedom is $3N-4$ for N-vertex. The construction of SUSY limit of TGD in [K7] led to the conclusion that the parallelly propagating N fermions for given wormhole throat correspond to a product of N fermion propagators with same four-momentum so that for fermions and ordinary bosons one has the standard behavior but for $N>2$ non-standard so that these excitations are not seen as ordinary particles. Higher vertices are finite only if the total number N_F of fermions propagating in the loop satisfies $N_F > 3N-4$. For instance, a 4-vertex from which $N=2$ states emanate is finite.

There is however a heavy objection against this line of thought [K10]. In TGD framework the vertices are four-fermion vertices with on fermion-antifermion pair forming virtual boson. This effectively gives BFF vertices.

1. One can generalize the idea about residue integration over virtual four-momenta for bosons allowing to reduce everything to on mass shell particles. For bosons the massless particles have complex momenta. For fermions the situation is simpled: momentum integration gives the inverse of the propagator as $D = p^k\gamma_k$. For non-physical helicities this does not annihilate the spinor at the end of the line. All particles could be regarded as massless on mass shell particles but virtual ones would have unphysical helicity.

2. There is however cold shower waiting: the study of the behavior of diagrams discussed in [K10] suggests that the resulting diagrams involving 3-D integrals over mass shell and need not give finite results! One cannot avoid logarithmic divergences, which should be the standard divergences of massless theories.

3. It seems that the only manner to escape difficulties is to start from stringy diagrams [K10], which are indeed forced by the fact that the modes of induced spinor fields are localized at string world sheets with one exception: right-handed neutrino. These diagrams are manifestly finite and right handed neutrino cannot spoil the situation since it has no electroweak couplings.

   When D is slashed between fermionic stringy propagator and its hermitian conjugate one obtains well-defined propagator lines although fermionic stringy propagator identified as super Virasoro generator G carries quark or lepton number so that the old problem caused by non-Majorana property of fermions in TGD framework disappears.

In the light of after wisdom the observation that stringy diagrammatics is necessary to achieve finiteness does not give rise to a hot news! The paths to the truth are sometimes very tortuous.

7.3.3 Taking into account magnetic confinement

What has been said above is not quite enough. The weak form of electric-magnetic duality [B2] leads to the picture about elementary particles as pairs of magnetic monopoles inspiring the notions of weak confinement based on magnetic monopole force. Also color confinement would have magnetic counterpart. This means that elementary particles would behave like string like objects in weak boson length scale. Therefore one must also consider the stringy case with wormhole throats replaced with fermion-$X_\pm$ pairs ($X_\pm$ is electromagnetically neutral and $\pm$ refers to the sign of the weak isospin opposite to that of fermion) and their super partners.

1. The simplest assumption in the stringy case is that fermion-$X_\pm$ pairs behave as coherent objects, that is scatter elastically. In more general case only their higher excitations identifiable in terms of stringy degrees of freedom would be created in vertices. The massivation of these states makes possible non-collinear vertices. An open question is how the massivation fermion-$X_\pm$ pairs relates to the existing TGD based description of massivation in terms of Higgs mechanism and modified Dirac operator.

2. Mass renormalization could come from self energy loops with negative energy lines as also vertex normalization. By very general arguments supersymmetry implies the cancellation of the self energy loops but would allow non-trivial vertex renormalization [K7] .

3. If only 3-vertices are allowed, the loops containing only positive energy lines are possible if on mass shell fermion-$X_\pm$ pair (or its superpartner) can decay to a pair of positive energy pair particles of same kind. Whether this is possible depends on the masses involved. For ordinary particles these decays are not kinematically possible below intermediate boson mass scale (the decays $F_1 \rightarrow F_2 + \gamma$ are forbidden kinematically or by the absence of flavor changing neutral currents whereas intermediate gauge bosons can decay to on mass shell fermion-antifermion pair).

4. The introduction of IR cutoff for 3-momentum in the rest system associated with the largest CD (causal diamond) looks natural as scale parameter of coupling constant evolution and p-adic length scale hypothesis favors the inverse of the size scale of CD coming in powers of two. This parameter would define the momentum resolution as a discrete parameter of the p-adic coupling constant evolution. This scale does not have any counterpart in standard physics. For electron, d quark, and u quark the proper time distance between the tips of CD corresponds to frequency of 10 Hz, 1280 Hz, and 160 Hz: all these frequencies define fundamental bio-rhythms [K5] .

These considerations have left completely untouched one important aspect of generalized Feynman diagrams: the necessity to perform a functional integral over the deformations of the partonic 2-surfaces at the ends of the lines- that is integration over WCW. Number theoretical universality requires that WCW and these integrals make sense also p-adically and in the following these aspects of generalized Feynman diagrams are discussed.

REFERENCES

Mathematics

[A1] Hyper-Kähler manifold. `http://en.wikipedia.org/wiki/Hyper-Khler_manifold`.

[A2] Octonions. http://en.wikipedia.org/wiki/Octonions.

[A3] Yangian symmetry. http://en.wikipedia.org/wiki/Yangian.

[A4] E. Witten. Fivebranes and Knots. http://arxiv.org/pdf/1101.3216v1, 2010.

Theoretical Physics

[B1] Chern-Simons theory. http://en.wikipedia.org/wiki/ChernSimons_theory.

[B2] Montonen Olive Duality. http://en.wikipedia.org/wiki/Montonen-Olive_duality.

[B3] N. Arkani-Hamed et al. A duality for the S-matrix. http://arxiv.org/abs/0907.5418, 2009.

[B4] Nima Arkani-Hamed et al. The All-Loop Integrand For Scattering Amplitudes in Planar N=4 SYM. http://arxiv.org/find/hep-th/1/au:+Bourjaily_J/0/1/0/all/0/1, 2010.

[B5] R. Penrose. The Central Programme of Twistor Theory. *Chaos, Solitons and Fractals*, 10, 1999.

[B6] E. Witten. Perturbative Gauge Theory As a String Theory In Twistor Space. http://arxiv.org/abs/hep-th/0312171, 2003.

Cosmology and Astro-Physics

[E1] M. Grusenick. Extended Michelson-Morley interferometer experiment. http://www.youtube.com/watch?v=7T0d7o8X2-E, 2009.

Books related to TGD

[K1] M. Pitkänen. Basic Properties of CP_2 and Elementary Facts about p-Adic Numbers. In *Towards M-matrix*. Onlinebook. http://tgdtheory.fi/public_html/pdfpool/append.pdf, 2006.

[K2] M. Pitkänen. Basic Extremals of Kähler Action. In *Physics in Many-Sheeted Space-Time*. Onlinebook. http://tgdtheory.fi/public_html/tgdclass/tgdclass.html#class, 2006.

[K3] M. Pitkänen. Category Theory and Quantum TGD. In *Towards M-Matrix*. Onlinebook. http://tgdtheory.fi/public_html/tgdquant/tgdquantum.html#categorynew, 2006.

[K4] M. Pitkänen. Configuration Space Spinor Structure. In *Quantum Physics as Infinite-Dimensional Geometry*. Onlinebook. http://tgdtheory.fi/public_html/tgdgeom/tgdgeom.html#cspin, 2006.

[K5] M. Pitkänen. Dark Matter Hierarchy and Hierarchy of EEGs. In *TGD and EEG*. Onlinebook. http://tgdtheory.fi/public_html/tgdeeg/tgdeeg.html#eegdark, 2006.

[K6] M. Pitkänen. Does the Modified Dirac Equation Define the Fundamental Action Principle? In *Quantum Physics as Infinite-Dimensional Geometry*. Onlinebook. http://tgdtheory.fi/public_html/tgdgeom/tgdgeom.html#Dirac, 2006.

[K7] M. Pitkänen. Does the QFT Limit of TGD Have Space-Time Super-Symmetry? In *Towards M-Matrix*. Onlinebook. http://tgdtheory.fi/public_html/tgdquant/tgdquantum.html#susy, 2006.

[K8] M. Pitkänen. Knots and TGD. In *Quantum Physics as Infinite-Dimensional Geometry*. Onlinebook. http://tgdtheory.fi/public_html/tgdgeom/tgdgeom.html#knotstgd, 2006.

[K9] M. Pitkänen. Quantum Field Theory Limit of TGD from Bosonic Emergence. In *Towards M-Matrix*. Onlinebook. http://tgdtheory.fi/public_html/tgdquant/tgdquantum.html#emergence, 2006.

[K10] M. Pitkänen. Some Fresh Ideas about Twistorialization of TGD. In *Towards M-Matrix*. Onlinebook. http://tgdtheory.fi/public_html/tgdquant/tgdquantum.html#twistornotices, 2006.

[K11] M. Pitkänen. TGD as a Generalized Number Theory: Infinite Primes. In *TGD as a Generalized Number Theory*. Onlinebook. http://tgdtheory.fi/public_html/tgdnumber/tgdnumber.html#visionc, 2006.

[K12] M. Pitkänen. The Relationship Between TGD and GRT. In *Physics in Many-Sheeted Space-Time*. Onlinebook. http://tgdtheory.fi/public_html/tgdclass/tgdclass.html#tgdgrt, 2006.

[K13] M. Pitkänen. Twistors, N=4 Super-Conformal Symmetry, and Quantum TGD. In *Towards M-Matrix*. Onlinebook. http://tgdtheory.fi/public_html/tgdquant/tgdquantum.html#twistor, 2006.

[K14] M. Pitkänen. Yangian Symmetry, Twistors, and TGD. In *Towards M-Matrix*. Onlinebook. http://tgdtheory.fi/public_html/tgdquant/tgdquantum.html#Yangian, 2006.

Articles about TGD

[L1] M. Pitkänen. Basic Properties of CP_2 and Elementary Facts about p-Adic Numbers. http://tgdtheory.fi/pdfpool/append.pdf, 2006.