Quantum Arithmetics and the Relationship between Real and p-Adic Physics

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Abstract

This chapter considers possible answers to the basic questions of the p-adicization program, which are following.

Some of the basic questions of the p-adicization program are following.

1. Is there some kind of duality between real and p-adic physics? What is its precise mathematic formulation? In particular, what is the concrete map of p-adic physics in long scales (in real sense) to real physics in short scales? Can one find a rigorous mathematical formulation of the canonical identification induced by the map $p \to 1/p$ in pinary expansion of p-adic number such that it is both continuous and respects symmetries or one must accept the finite measurement resolution.

   Few years after writing this the answer to this question is in terms of the notion of p-adic manifold. Canonical identification serving as its building brick however allows many variants and it seems that quantum arithmetics provides one further variant

2. What is the origin of the p-adic length scale hypothesis suggesting that primes near power of two are physically preferred? Why Mersenne primes seem to be especially important (p-adic mass calculations suggest this)?

   This chapter studies some ideas but does not provide a clearcut answer to these questions. The notion of quantum arithmetics obtained is central in this approach.

The starting point of quantum arithmetics is the map $n \to n_q$ taking integers to quantum integers: $n_q = (q^n - q^{-n})/(q - q^{-1})$. Here $q = exp(i\pi/n)$ is quantum phase defined as a root of unity. From TGD point of view prime roots $q = exp(i\pi/p)$ are of special interest. Also prime prime power roots $q = exp(i\pi/p^n)$ of unity are of interest. Quantum phase can be also generalized to complex number with modulus different from unity.

One can consider several variants of quantum arithmetics. One can regard finite integers as either real or p-adic. In the intersection of "real and p-adic worlds" finite integers can be regarded both p-adic and real.

1. If one regards the integer n real one can keep some information about the prime decomposition of n by dividing n to its prime factors and performing the mapping $p \to p_q$. The map takes prime first to finite field $G(p,1)$ and then maps it to quantum integer. Powers of p are mapped to zero unless one modifies the quantum map so that p is mapped to p or $1/p$ depending on whether one interprets the outcome as analog of p-adic number or real number. This map can be seen as a modification of p-adic norm to a map, which keeps some information about the prime factorization of the integer. Information about both real and p-adic structure of integer is kept.

2. For p-adic integers the decomposition into prime factors does not make sense. In this case it is natural to use pinary expansion of integer in powers of p and perform the quantum map for the coefficients without decomposition to products of primes $p_1 < p$. This map can be seen as a modification of canonical identification.

3. If one wants to interpret finite integers as both real and p-adic then one can imagine the definition of quantum integer obtained by de-compositing n to a product of primes, using pinary expansion and mapping coefficients to quantum integers looks natural. This map would keep information about both prime factorization and also a bout pinary series of factors. One can also decompose the coefficients to prime factors but it is not clear whether this really makes sense since in finite field $G(p,1)$ there are no primes.

One can distinguish between two basic options concerning the definition of quantum integers.

1. For option I the prime number decomposition of integer is mapped to its quantum counterpart by mapping the primes l to quantum primes $l_q = (q^l - q^{-l})/(q - q^{-1})$, $q = exp(i\pi/p)$ so that image of product is product of images. Sums are *not* mapped to sums as is easy to verify. p is mapped to zero for the standard definition of quantum integer. Now p is however mapped to itself or $1/p$ depending on whether one wants to interpret quantum integer as p-adic or real number. Quantum integers generate an algebra with respect to sum and product.

2. Option II one uses pinary expansion and maps the prime factors of coefficients to quantum primes. There seems to be no point in decomposing the pinary coefficients to their prime factors so that they are mapped to standard quantum integers smaller than p.

The quantum primes l_q act as generators of Kac-Moody type algebra defined by powers p^n such that sum is completely analogous to that for Kac-Moody algebra: $a + b = \sum_n a_n p^n + \sum b_n p^n = \sum_n (a_n + b_n)p^n$. For p-adic numbers this is not the case.

3. For both options it is natural to consider the variant for which one has expansion $n = \sum_k n_k p^{kr}$, $n_k < p^r$, $r = 1, 2....$ p^k would serve as cutoff.

The notion of quantum matrix group differing from ordinary quantum groups in that matrix elements are commuting numbers makes sense. This group forms a discrete counterpart of ordinary quantum group and its existence suggested by quantum classical correspondence. The existence of this group for matrices with unit determinant is guaranteed by mere ring property since the inverse matrix involves only arithmetic product and sum.

1. The quantum counterparts of special linear groups $SL(n, F)$ exists always. For the covering group $SL(2, C)$ of $SO(3, 1)$ this is the case so that 4-dimensional Minkowski space is in a very special position. For orthogonal, unitary, and orthogonal groups the quantum counterpart exists only if the number of powers of p for the generating elements of the quantum matrix group satisfies an upper bound characterizing the matrix group.
2. For the quantum counterparts of $SO(3)$ ($SU(2)$/ $SU(3)$) the orthogonality conditions state that at least some multiples of the prime characterizing quantum arithmetics is sum of three (four/six) squares. For $SO(3)$ this condition is strongest and satisfied for all integers, which are not of form $n = 2^{2r}(8k+7)$). The number $r_3(n)$ of representations as sum of squares is known and $r_3(n)$ is invariant under the scalings $n \rightarrow 2^{2r}n$. This means scaling by 2 for the integers appearing in the square sum representation.

The findings about quantum $SO(3)$ suggest a possible explanation for p-adic length scale hypothesis and preferred p-adic primes.

1. The idea to be studied is that the quantum matrix group which is discrete is in some sense very large for preferred p-adic primes. If cognitive representations correspond to the representations of quantum matrix group, the representational capacity of cognitive representations is high and this kind of primes are survivors in the algebraic evolution leading to algebraic extensions with increasing dimension. The simple estimates of this chapter restricting the consideration to finite fields ($O(p) = 0$ approximation) do not support this idea in the case of Mersenne primes.
2. An alternative idea is that number theoretic evolution leading to algebraic extensions of rationals with increasing dimension favors p-adic primes which do not split in the extensions to primes of the extension. There is also a nice argument that infinite primes which are in one-one correspondence with prime polynomials code for algebraic extensions. These primes code also for bound states of elementary particles. Therefore the stable bound states would define preferred p-adic primes as primes which do not split in the algebraic extension defined by infinite prime. This should select Mersenne primes as preferred ones.

1 Introduction

The construction of quantum counterparts for various mathematical structures of theoretical physics have been a fashion for decades. Quantum counterparts for groups, Lie algebras, coset spaces, etc... have been proposed often on purely formal grounds. In TGD framework quantum group like structures emerge via the hyper-finite factors of type II_1 (HFFs) about which WCW spinors represent a canonical example [K11]. The inclusions of HFFs provide a very attractive manner to realize mathematically the notion of finite measurement resolution.

In the following a proposal for what might be called quantum integers and quantum matrix groups is discussed. One can imagine two basic definitions of quantum integers n_q: option I and II. For option I the map $n \rightarrow n_q$ respects prime decomposition so that one obtains quantum variant of primeness. For option II ordinary primeness in the ordinary sense of word is lost as it is lost also for p-adic numbers (only p is prime for Q_p).

Also quantum rationals belonging to algebraic extension of rationals can be defined as well as their algebraic extensions. Quantum arithmetics differs from the usual one in that quantum sum is defined in such a manner that the map $n \rightarrow n_q$ commutes also with sum besides the product: $m_q +_q n_q = (m + n)_q$. Quantum matrix groups differ from their standard counterparts in that the matrix elements are not non-commutative. The matrix multiplication involving summation over products is however replaced with quantum summation.

The hope is that these new mathematical structures could allow a better understanding of the relationship between real and p-adic physics for various values of p-adic prime p, to be called l in the sequel because of its preferred physical nature resembling that of l-adic prime in l-adic cohomology. The correspondence with the ordinary quantum groups (see `http://tinyurl.com/3tors5`) [A15] is also considered and suggested to correspond to a discretization following as a correlate of finite measurement resolution.

1.1 Overall View About Variants Of Quantum Integers

The starting point of quantum arithmetics is the map $n \rightarrow n_q$ taking integers to quantum integers: $n_q = (q^n - q^{-n})/(q - q^{-1})$. Here $q = exp(i\pi/n)$ is quantum phase defined as a root of unity. From TGD point of view prime roots $q = exp(i\pi/p)$ are of special interest. Also prime prime power roots $q = exp(i\pi/p^n)$ of unity are of interest. Quantum phase can be also generalized to complex number with modulus different from unity.

One can consider several variants of quantum arithmetics. One can regard finite integers as either real or p-adic. In the intersection of "real and p-adic worlds" finite integers can be regarded both p-adic and real.

1. If one regards the integer n real one can keep some information about the prime decomposition of n by dividing n to its prime factors and performing the mapping $p \rightarrow p_q$. The map takes prime first to finite field $G(p, 1)$ and then maps it to quantum integer. Powers of p are mapped to zero unless one modifies the quantum map so that p is mapped to p or $1/p$ depending on whether one interprets the outcome as analog of p-adic number or real number. This map can be seen as a modification of p-adic norm to a map, which keeps some information about the prime factorization of the integer. Information about both real and p-adic structure of integer is kept.

2. For p-adic integers the decomposition into prime factors does not make sense. In this case it is natural to use pinary expansion of integer in powers of p and perform the quantum map for the coefficients without decomposition to products of primes $p_1 < p$. This map can be seen as a modification of canonical identification.

3. If one wants to interpret finite integers as both real and p-adic then one can imagine the definition of quantum integer obtained by de-compositing n to a product of primes, using pinary expansion and mapping coefficients to quantum integers looks natural. This map would keep information about both prime factorization and also a bout pinary series of factors. One can also decompose the coefficients to prime factors but it is not clear whether this really makes sense since in finite field $G(p, 1)$ there are no primes.

Clearly, many variants of quantum integers can be found and it is difficult to decide which of them - if any - has interesting from TGD point of view.

1. If one wants to really model something using quantum integers, the second options is perhaps the realistic one: the reason is that the decomposition into prime factors requires a lot of computation time.

2. A second fictive criterion would be whether the definition is maximally general. Does the definition makes sense for infinite primes? The simplest infinite primes at the first level of hierarchy have physical interpretation as many-particle states consisting of bosons and fermions, whose momentum values correspond to finite primes. The interpretation generalizes to higher levels of the hierarchy. A simple argument show that the option keeping information about prime factorization of the p-adic number allowing also infinite primes as factors makes sense only if prime factors are not expanded in series with respect to the prime p and if p does not correspond to a fermionic mode. The quantum map using prime root of unity therefore makes sense for all but fermionic primes. The presence of exceptional primes in number theory is basic phenomenon: typically they correspond to primes for which factorization is not unique in algebraic extension.

1.2 Motivations For Quantum Arithmetics

Quantum arithmetics has several motivations in TGD framework.

1.2.1 Model for Shnoll effect

The model for Shnoll effect [K1] suggests that this effect could be understand in terms of a deformation of probability distribution $f(n)$ (n non-negative integer) for random fluctuations. The deformation would replace the rational parameters characterizing the distribution with new ones obtained by mapping the parameters to new ones by using the analog of canonical identification respecting symmetries.

The idea of the model of Shnoll effect was to modify the quantum map $n \rightarrow n_q$ in such a manner that it is consistent with the prime decomposition of ordinary integers. This deformation would involve two parameters: quantum phase $q = exp(i\pi/m)$ and preferred prime l, which need not be independent however: $m = l$ is a highly suggestive restriction.

1.2.2 What could be the deeper mathematics behind dualities?

Dualities certainly represent one of the great ideas of theoretical physics of the last century. On could say that electric-magnetic duality due to Montonen and Olive [B2] is the mother of all dualities. Later a proliferation, one might say even inflation, of dualities has taken place. AdS/CFT correspondence (see `http://tinyurl.com/2zuek8`) [B3] is one example relating to each other perturbative QFT working in short scales and string theory working in long scales.

Also in TGD framework several dualities suggests itself. All of them seem to relate to dichotomies such as weak–strong, perturbative–non-perturbative, point like particle–string. Also number theory seems to be involved in an essential manner.

1. If $M^8 - -M^4 \times CP_2$ duality is true it is possible to regard space-times as surfaces in M^8 or $M^4 \times CP_2$ [K10]. The proper treatment of Minkowskian signature requires complexified version M^8_c of M^8 allowing identification as complexified octonions. One manner to interpret the duality would as the analog of q-p duality in wave mechanics. Surfaces in M^8 (or M^8_c) would be analogous to momentum space representation of the physical states: space-time surfaces in M^8 would represent in some sense the points for the tangent space of the "world of classical worlds" (WCW) just like tangent for a curve gives the first approximation for the curve near a given point.

   The argument supporting $M^8 - -M^4 \times CP_2$ duality involves the basic facts about classical number fields - in particular octonions and their complexification - and one can understand $M^4 \times CP_2$ in terms of number theory. The analog of the color group in M^8 picture would be the isometry group $SO(4)$ of E^4 which happens to be the symmetry group of the old fashioned hadron physics. Does this mean that $M^4 \times CP_2$ corresponds to short length scales and perturbative QCD whereas M^8 would correspond to long length scales and non-perturbative approach?

2. Second duality would relate partonic 2-surfaces and string world sheets playing a key role in the recent view about preferred extremals of Kähler action [L3]. Partonic 2-surfaces are magnetic monopoles and TGD counterparts of elementary particles, which in QFT approach are regarded as point like objects. The description in terms of partonic 2-surfaces forgetting that they are parts of bigger magnetically neutral structures would correspond to perturbative QFT. The description in terms of string like objects with vanishing magnetic charge is needed in longer length scales. Electroweak symmetry breaking and color confinement would be the natural applications. The essential point is that stringy description corresponds to long length scales (strong coupling) and partonic description to short length scales (weak coupling).

   Number theory seems to be involved also now: string world sheets could be seen as commutative (hyper-complex) 2-surfaces of space-time surface with hyper-quaternionic tangent space structure and partonic 2-surfaces as co-commutative (co-hyper-complex) 2-surfaces. To avoid inflation of clumsy "hyper-"s, the terms "associative"/"co-associative" and "commutative"/"co-commutative" will be used in the sequel.

The localization of the modes of induced spinor fields to string world sheets and partonic 2-surfaces could be seen as a physical realization this and is implied by the requirement that spinor modes are eigenstates of em charge operator [K12].

3. Space-time surface itself would decompose to associative and co-associative regions and a duality also at this level is suggestive [L1], [K2]. The most natural candidates for dual space-time regions are regions with Minkowskian and Euclidian signatures of the induced metric with latter representing the generalized Feynman graphs. Minkowskian regions would correspond to non-perturbative long length scale description and Euclidian regions to perturbative short length scale description. This duality should relate closely to quantum measurement theory and realize the assumption that the outcomes of quantum measurements are always macroscopic long length scale effects. Again number theory is in a key role.

Real and p-adic physics and their unification to a coherent whole represent the basic pieces of physics as generalized number theory program.

1. p-Adic physics in minimal sense would mean a discretization of real physics relying on effective p-adic topology. p-Adic physics could also mean genuine p-adic physics at p-adic space-time sheets identified as space-time correlates of cognition.

   Real continuity and smoothness is a powerful constraint on short distance physics. p-Adic continuity and smoothness pose similar constraints in short scales an therefore on real physics in long length scales if one accepts that real and space-time surfaces (partonic 2-surfaces for minimal option) intersect along rational points and possible common algebraics in preferred coordinates. p-Adic fractality implying short range chaos and long range correlations is the outcome. Therefore p-adic physics could allow to avoid the landscape problem of M-theory due to the fact that the IR limit is unpredictable although UV behavior is highly unique.

2. The recent argument [L3] suggesting that the areas for partonic 2-surfaces and string world sheets could characterize Kähler action leads to the proposal that the large N_c expansion (see `http://tinyurl.com/ya4xo926`) [B1] in terms of the number of colors defining non-perturbative stringy approach to strong coupling phase of gauge theories could have interpretation in terms of the expansion in powers of $1/\sqrt{p}$, p the p-adic prime. This expansion would converge extremely rapidly since N_c would be of the order of the ratio of the secondary and primary p-adic length scales and therefore of the order of $\sqrt{p}$: for electron one has $p = M_{127} = 2^{127} - 1$.

3. Could there exist a duality between genuinely p-adic physics and real physics? Could the mathematics used in p-adic mass calculations - in particular canonical identification $\sum_n x_n p^n \to \sum x_n p^{-n}$ - be extended to apply to quantum TGD itself and allow to understand the non-perturbative long length scale effects in terms of short distance physics dictated by continuity and smoothness but in different number field? Could a proper generalization of the canonical identification map allow to realize concretely the real–p-adic duality?

1.2.3 Could quantum arithmetics allow a variant of canonical identification respecting both symmetries and continuity?

One could argue that a generalization of the canonical identification [K7] and its variants is needed in order to solve the tension between algebra (symmetries) and topology: the correspondence via common rationals respects algebra and symmetries but is discontinuous. Canonical identification is continuous but does not respect algebra.

Concerning the correspondence between p-adics and reals the notion of p-adic manifolds seems to represent a real step of progress. The notion of p-adic manifold [K16] is based on simple idea. The chart maps of p-adic manifolds (now space-time surfaces) are to real manifolds (space-time surfaces) rather than p-adic counterpart of Euclidian space and realized in terms of some variant of canonical identification restricted to a discrete subset of rational points of manifold- now space-time surface- and preferred extremal property allows to find a space-time surface which contains these points. In accordance with finite measurement resolution, the correspondence is not unique.

The real image is interpreted as realization of intention represented as p-adic space-time surface. The reverse maps providing p-adic charges about real space-time surface are interpreted as cognitive representations. Building of cognitive representation and realization of intention as action could be time reversals of each other in the sense that quantum jump could lead from p-adic sector to real and vice versa: this requires zero energy ontology (ZEO) in order to make sense.

All forms of canonical identification break to some extent symmetries and continuity (this forces the restriction to a discrete subset of space-time points). One could accept this or ask whether a generalization of canonical identification resolving the tension between symmetries and continuity could exist.

It seems that this is not the case. The tension seems to be unresolvable and have interpretation in terms of finite measurement resolution. At best a given continuous symmetry group would be replaced by some of its discrete subgroups. Of course, both real and p-adic variants of symmetries are realized but the problem is that they are very different and canonical identification in its basic form does not give close connection between them.

This chapter was written before the emergence of the notion of p-adic manifold and in the hope that the symmetry respecting generalization of canonical identification might exist. In the new situation quantum variant of canonical identification provides a new variant of the map taking discretization of the p-adic space-time surfaces to its real counterpart.

1.2.4 Quantum integers and preferred extremals of Kähler action

One might hope that quantum integers have some deep function. Somehow the fact that the images of primes $1 < p_i < p$ are algebraic numbers might relate to this. Maybe their function might relate to the notion of p-adic manifold [K16]. The basic challenge is to continue the discrete canonical image of the p-adic space-time points to continuous and differentiable preferred extremal of Kähler action. O_c-real analytic functions (O_c denotes complexified octonions) [K17] defining four-surfaces in M_c^8 mappable to space-time surface in H by $M^8 - H$ correspondence might allow to code preferred extremals by real-valued analytic functions. A hierarchy of polynomials with rational or even algebraic arguments suggests itself.

Quantum integers might define discretization of real space-time surface by mapping p-adic integers (continuum) representing preferred imbedding space coordinates to a set of quantum integers n_q, $0 \leq n < p$.

The notion of deformation has played central role in attempts to generalize physics and one can see quantum physics as a deformation of classical physics. Suppose that p-adic preferred extremal is characterized by functions which are polynomials/ rational functions. Suppose that one can interpret these functions as functions in the ring of quantum integers. Since differentiability makes sense for the quantum ring one could hope that these functions could define preferred extremal in the ring of quantum integers and perhaps also in real imbedding space.

1.3 Correspondence Along Common Rationals And Canonical Identification: Two Manners To Relate Real And P-Adic Physics

The relationship between real and p-adic physics deserves a separate discussion.

1.3.1 Identification along common rationals

The first correspondence between reals and p-adics is based on the idea that rationals are common to all number fields implying that rational points are common to both real and p-adic worlds. This requires preferred coordinates. It also leads to a fusion of different number fields along rationals and common algebraics to a larger structure having a book like structure [K9, K7].

1. Quite generally, preferred space-time coordinates would correspond to a subset of preferred imbedding space coordinates, and the isometries of the imbedding space give rise to this kind of coordinates which are however not completely unique. This would give rise to a moduli space corresponding to different symmetry related coordinates interpreted in terms of different choices of causal diamonds (CDs: recall that CD is the intersection of future and past directed light-cones.

2. Cognitive representation in the rational (partly algebraic) intersection of real and p-adic worlds would necessarily select certain preferred coordinates and this would affect the physics

in a delicate manner. The selection of quantization axis would be basic example of this symmetry breaking. Finite measurement resolution would in turn reduce continuous symmetries to discrete ones. It deserves to be mentioned that for color color symmetries $SU(3)$ the space for the choices of quantization axes is flag-manifold $SU(3)/U(1) \times U(1)$ having interpretation as twistor space of CP_2: CP_2 is the only compact 4-manifold allowing twistor space with complex structure. M^4 twistors are assigned with light-like vectors defining plane $M^2 \subset M^4$ in turn defining quantization axis for spin.

3. Typically real and p-adic variants of given partonic 2-surface would have discrete and possibly finite set of rational points plus possible common algebraic points. The intersection of real and p-adic worlds would consist of discrete points. At more abstract level rational functions with rational coefficients used to define partonic 2-surfaces would correspond to common 2-surfaces in the intersection of real and p-adic WCW:s. As a matter fact, the quantum arithmetics would make most points algebraic numbers.

4. The correspondence along common rationals respects symmetries but not continuity: the graph for the p-adic norm of rational point is totally discontinuous. Most non-algebraic reals and p-adics do not correspond to each other. In particular, transcendental at both sides belong to different worlds with some exceptions like e^p which exists p-adically.

1.3.2 Canonical identification and its variants

There is however a totally different view about real–p-adic correspondence.

The predictions of p-adic mass calculations are mapped to real numbers via thecanonical identification applied to the p-adic value of mass squared [K7, K15]. One can imagine several forms of canonical identification but this affects very little the predictions since the convergence in powers of p for the mass squared thermal expectation is extremely fast.

As a matter fact, I proposed for more that 15 years ago that canonical identification could be essential element of cognition mapping external world to p-adic cognitive representations [K8] realized in short length scales and vice versa.

If so, then real–p-adic duality would be a cornerstone of cognition [K8]. Common rational points would relate to the intentionality which is second aspect of the p-adic real correspondence: the transformation of real to p-adic surfaces in quantum jump would be the correlate for the transformation of intention to action. The realization of intention would correspond to the correspondence along rationals and common algebraics (the more common points real and p-adic surface have, the more faithful the realization of intentional action) and the generation of cognitive representations to the canonical identification.

The already mentioned, notion of p-adic manifolds [K16] relies on this notion and provides a very promising approach to the description of space-time correlates of cognition. Various forms of canonical identification would define cognitive representations and their reverses.

Canonical identification is continuous but does does not respect symmetries: the action of the p-adic symmetry followed by a canonical identification to reals is not equal to the canonical identification map followed by the real symmetry.

1.3.3 Can one fuse the two views about real-p-adic correspondence

Could the two views about real-p-adic correspondence be fused if appropriately generalized canonical identification is interpreted as a concrete duality mapping short length scale physics and long length scale physics to each other? There are however hard technical problems involved.

1. Canonical identification is not consistent with general coordinate invariance unless one can identify some physicallyt preferred coordinate system. For imbedding spaces the isometries guarantee the existence of rather limited space of this kind of coordinate systems: linear coordinates for M^4 and complex coordinate systems relatd by color isometries for CP_2. This suggests taht canonical identification should be realized at the level of imbedding space.

2. Canonical identification would be locally continuous in both directions. Note that for the points with finite pinary expansion (ordinary integers) the map is two-valued. Note also that rationals can be expanded in infinite powers series with respect to p and one can ask whether

one should do this or map $q = m/n$ to $I(m)/I(n)$ (the representation of rational is unique if m and n have no common factors). Symmetries represented by matrix groups with rational matrix elements require the latter option.

One can map rationals by $m/n \rightarrow I(m)/I(n)$. One can also express m and n as power series of p^k as $x = \sum x_n p^{nk}$ and perform the map as $x \rightarrow \sum x_n p^{-nk}$. This allows to preserve symmetries in arbitrary good measurement resolution characterized by the power p^{-k} on real side. The reason would be that rationals m/n with $m < pk$ and $n < p^k$ would be mapped to themselves: algebra wins. If m or n or both are larger than p^k the behavior associated with canonical identification sets in: topology wins.

3. This compromize between algebra and topology looks nice but an additional problem emerges when one brings in more TGD. If one wants to map differentiable p-adic space-time surfaces (preferred extremals of Kähler action) to differentiable real surfaces (preferred extremals of Kähler action), canonical identification cannot work since it is not differentiable. Second pinary cutoff above which one simply throws out the pinary digits, is needed. p-Adic space-time sheets are discretized and mapped to a discrete subsets of the real space-time sheet. Completion to a preferred extremal is needed and assigning a preferred extremal to a discrete point set becomes the challenge. The p-adic manifold concept relies essentially on this idea about p-adic-real correspondence.

This chapter was originally written few years before the idea of p-adic manifold. The question was whether one could circumvent the tension between symmetries and continuity without approximations? After few years the answer is definitely "No!".

Despite this I have decided to keep this chapter since the quantum variant of canonical identification could also be involved with the definition of p-adic manifold. In particular, the fact that it maps p-adic numbers to algebraic numbers in the algebraic extension defined by p:th root of unity might have some deep meaning and relate to the connection between Galois group of maximal Abelian extension of rationals and adeles consisting of the Cartesian product of real and various p-adic number fields.

Could the canonical identification based on quantum integers provide a generalization of the notion of symmetry itself in order to circumvent ugly constructions? This is the question to be addressed in this chapter.

1.4 Brief Summary Of The General Vision

Some of the basic questions of the p-adicization program are following.

1. Is there a duality between real and p-adic physics? What is its precise mathematic formulation? In particular, what is the concrete map of p-adic physics in long scales (in real sense) to real physics in short scales? Can one find a rigorous mathematical formulation of the canonical identification induced by the map $p \rightarrow 1/p$ in pinary expansion of p-adic number such that it is both continuous and respects symmetries or one must accept the finite measurement resolution.

   Few years after writing this the answer to this question is in terms of the notion of p-adic manifold. Canonical identification serving as its building brick however allows many variants and it seems that quantum arithmetics provides one further variant. The physical interpretation could be in terms of inclusions of hyper-finite factors of type II_1 parametrized by quantum phases and allowing to interpret the action of the included algebra as having no effects on the state in the measurement resolution used [K11]. When quantum phase approaches unity one would obtained ordinary canonical identification.

2. What is the origin of the p-adic length scale hypothesis suggesting that primes near power of two are physically preferred? Why Mersenne primes seem to be especially important (p-adic mass calculations suggest this [K5])?

   This chapter studies some ideas but does not provide a clearcut answer to these questions.

1.4.1 Two options for quantum integers

In the sequel two options for definining quantum arithmetics are discussed: Options I and II. These are not the only one imaginable but represent kind of diametrical opposites. The two options are defined in the following manner.

1. For option I the prime number decomposition of integer is mapped to its quantum counterpart by mapping the primes l to $l\ modp$ (to guarantee positivity of the quantum integer) decomposed into primes $l < p$ and these in turn to quantum primes $l_q = (q^l - q^{-l})/(q - q^{-1})$, $q = exp(i\pi/p)$ so that image of the product is product of images. Sums are *not* mapped to sums as is easy to verify. p is mapped to zero for the standard definition of quantum integer. Now p is however mapped to itself or $1/p$ depending on whether one wants to interpret quantum integer as p-adic or real number. Quantum integers generate an algebra with respect to sum and product.

2. Option II one uses pinary expansion and maps the prime factors of coefficients to quantum primes. There seems to be no point in decomposing the pinary coefficients to their prime factors so that they are mapped to standard quantum integers smaller than p.

   The quantum primes l_q act as generators of Kac-Moody type algebra defined by powers p^n such that sum is completely analogous to that for Kac-Moody algebra: $a + b = \sum_n a_n p^n + \sum b_n p^n = \sum_n (a_n + b_n)p^n$. For p-adic numbers this is not the case.

3. For both options it is natural to consider the variant for which one has expansion $n = \sum_k n_k p^{kr}$, $n_k < p^r$, $r = 1, 2....$ p^k would serve as cutoff.

4. Non-negativity of quantum primes is important in the modelling of Shnoll effect by a deformation of probability distribution $P(n)$ by replacing the argument n by quantum integers and the parameters of the distribution by quantum rationals [K1]. One could also replace quantum prime by its square without losing the map of products to products.

5. At the limit when the quantum phase approaches to unit, ordinary quantum integers with p-adic norm 1 approach to ordinary integers in real sense and ordinary arithmetics results. Ordinary integers in real sense are obtained for option II when the coefficients of the pinary expansion of n are much smaller than p and p approaches infinity. Same is true for option I if the prime factors of the integer are much smaller than p.

The notion of quantum matrix group differing from ordinary quantum groups in that matrix elements are commuting numbers makes sense. This group forms a discrete counterpart of ordinary quantum group and its existence suggested by quantum classical correspondence. The existence of this group for matrices with unit determinant is guaranteed by mere ring property since the inverse matrix involves only arithmetic product and sum.

1.4.2 Quantum counterparts of classical groups

Quantum arithmetics inspires the notion of quantum matrix group as a counterpart of quantum group for which matrix elements are non-commuting numbers. Now the elements would be ordinary numbers. Quantum classical correspondence and the notion of finite measurement resolution realized at classical level in terms of discretization suggest that these two views about quantum groups are closely related. The preferred prime p defining the quantum matrix group is identified as p-adic prime or its power and the inversion $p \rightarrow 1/p$ is group homomorphism so that symmetries are respected.

Option I gives p-adic counterparts of classical groups. p-Adic numbers are replaced with the ring generated by the quantum images of p-adic numbers, which each correspond to some power of p: this extension gives powers series in p. By requiring the group conditions for a subgroup of special linear group to be satisfied in order $O(p) = 0$ one obtains classical groups for finite fields $G(p, 1)$ by simply requiring that group conditions are satisfied in order $O(p) = 0$. One can also have also classical groups associated with finite fields $G(p, n)$ having p^n elements.

Option II is more interesting and quantum counterparts could be seen as counterparts of classical groups obtained by replacing group elements with the elements of ring defined by Kac-Moody type algebra. The difference to Option I and its variants is that one does not map p-adic integer to $G(p,1)$ by $n \rightarrow n \mod p$ before quantum map but applies it to the entire p-adic integer.

1. The quantum counterparts of special linear groups $SL(n,F)$ exists always. For the covering group $SL(2,C)$ of $SO(3,1)$ this is the case so that 4-dimensional Minkowski space is in a very special position. For orthogonal, unitary, and orthogonal groups the quantum counterpart exists only if quantum arithmetics is characterized by a prime rather than general integer and when the number of powers of p for the generating elements of the quantum matrix group satisfies an upper bound characterizing the matrix group.

2. For the quantum counterparts of $SO(3)$ ($SU(2)$/ $SU(3)$) the orthogonality conditions state that at least some multiples of the prime characterizing quantum arithmetics is sum of three (four/six) squares. For $SO(3)$ this condition is strongest and satisfied for all integers, which are not of form $n = 2^{2r}(8k+7)$). The number $r_3(n)$ of representations as sum of squares is known and $r_3(n)$ is invariant under the scalings $n \rightarrow 2^{2r}n$. This means scaling by 2 for the integers appearing in the square sum representation.

3. $r_3(n)$ is proportional to the so called class number function $h(-n)$ telling how many non-equivalent decompositions algebraic integers have in the quadratic algebraic extension generated by $\sqrt{-n}$.

The findings about quantum $SO(3)$ encourages to consider a possible explanation for p-adic length scale hypothesis and preferred p-adic primes.

1. The idea to be studied is that the quantum matrix group which is discrete is in some sense very large for preferred p-adic primes. If cognitive representations correspond to the representations of quantum matrix group, the representational capacity of cognitive representations is high and this kind of primes are survivors in the algebraic evolution leading to algebraic extensions with increasing dimension. The simple estimates of this chapter restricting the consideration to finite fields ($O(p) = 0$ approximation) do not support this idea in the case of Mersenne primes.

2. An alternative idea discussed in [K14] is that number theoretic evolution leading to algebraic extensions of rationals with increasing dimension favors p-adic primes which do not split in the extensions to primes of the extension. There is also a nice argument that infinite primes which are in one-one correspondence with prime polynomials code for algebraic extensions. These primes code also for bound states of elementary particles. Therefore the stable bound states would define preferred p-adic primes as primes which do not split in the algebraic extension defined by infinite prime. This should select Mersenne primes as preferred ones.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L4].

2 Various options for Quantum Arithmetics

In this section the notion of quantum arithmetics as a deformation of p-adic number field to a ring is discussed. One can imagine several options for quantum arithmetics. Both for Option I and II p-adic integers are mapped to a subset of a ring of quantum integers and the sum operation for the ring has nothing to with that for p-adic numbers. In both cases the elements of ring makes sense as real numbers.

2.1 Comparing options I and II

The two options for defining quantum arithmetics are represented in the introduction so that it is no point writing the formulas again. It is interesting to compare these options.

Consider first what is common to these options.

1. For option I all integers are decomposed into products of primes mapped to their quantum counterparts by $p_1 \rightarrow p_1 \ mod \ p \rightarrow \prod p_i p_i^{k_i}$ followed by the mapping of p_i to its quantum counterpart. The modding operator for p_1 guarantees positivity of the outcome. Hence the information about prime decomposition is not lost. Also the information about p-adic norm is preserved if p is mapped to itself or $1/p$ (this depending on whether one speaks about p-adic or real variant of quantum integer). Quantum image of product is not however product of quantum images. The information about sum is lost.

   For option II the information about prime decomposition is lost.

   For both options it is also possible to decompose the coefficients of powers of p to prime factors. The information about pinary expansion is not lost. This option in turn respects continuity.

2. For both options the quantum image belongs to a ring larger than the image since for neither options the sum of two quantum integers need not be image of p-adic number. This makes possible to assign classical groups to this ring.

3. p-Adic–real duality can be identified as the analog of canonical identification induced by the map $p \rightarrow 1/p$ in the pinary expansion of quantum rational. This maps p-adic and real physics to each other and real long distances to short ones and vice versa. This map is especially interesting as a map for defining cognitive representations. The map $p^n \rightarrow p^{-n}$ is generalization of this map an maps p-adic integers $k < p^n$ to itself. Note that subgroups of $Gl(m, R)$ consisting of matrices with integer valued elements p^n are especially interesting p-adically since one avoid p-adic rationals for which canonical identification map allows several variants.

4. Quantum map $n \rightarrow n_q$ precedes canonical identification so that it could be interpreted as a modification for the chart map defined by canonical identification in the proposed definition for p-adic manifold already mentioned [K16]. My recent view is that this option is not promising. Canonical identification makes sense at the level of probability distributions and Lorentz invariants but not at space-time level since pinary expansion is not general coordinate invariant notion.

The differences between options I and II relate to how one treats integers $n > p$.

1. For option I one decomposes given integer to a product of primes and all primes are mapped to their quantum counterparts so that products go to products. Sums are not however mapped to sums. Quantum primes can be also negative. For $q = exp(i\pi/p)$ integers vanishing modulo p go to zero if one defines n_q by using the general formula for quantum integer. Also the extension of the map to rationals m/n meets with difficulties if n_q can vanish. It seems that p must be mapped to $1/p$ to avoid these problems and this is done in the proposal developed in the model for Shnoll effect [K1]. With this modification the image of integer is always product of quantum primes by some power of p and one does not obtain series in powers of p typical for p-adic numbers and canonical identification.

   If quantum map would respect both product and sum, the quantum counterparts of subgroups of classical matrix groups with elements elements smaller than p^n would exist. This condition cannot be satisfied. It is not clear whether subgroups of matrix groups exist for which their quantum counterparts defined by matrices with matrix elements smaller than p^n are groups too.

   This suggests that one must extend the image of p-adic integers (and its extension to that of p-adic rationals) to a ring defined by quantum sums and assign matrix group acting as symmetries to this ring. Matrix groups for which symmetries preserve volume the determinant of the matrix equals to unity so that the inverse exists always even when number field is replaced wit ring so that the existence of generalized matrix groups does not seem to be a problem.

2. For Option II one expands integer in powers p^k and maps the coefficients $n_k < p$ by quantum map just as for the first option. The quantum counterparts of p-adic integers generate a larger ring via products and sums.

One obtains the analog of Kac-Moody algebra with coefficients for a given power of p defining an algebra analogy to polynomial algebra. One can define also rationals and obtains a structure analogous to a function field. This field allows projection to p-adic numbers but is much larger than p-adic numbers. The construction works also for more general quantum phases q than those defined by primes and $q = exp(i\pi/p^n)$ is an especially interesting case. For this option the symmetries of quantum p-adics would be preserved in the canonical identification.

2.2 About the choice of the quantum parameter q

Some comments about the quantum parameter q are in order.

1. The basic formula for quantum integers in the case of quantum groups is

$$n_q = \frac{q^n - \overline{q}^n}{q - \overline{q}} \ . \tag{2.1}$$

Here q is *any* complex number. The generalization respective the notion of primeness is obtained by mapping only the primes p to their quantum counterparts and defining quantum integers as products of the quantum primes involved in their prime factorization.

$$\begin{aligned} p_q &= \frac{q^p - \overline{q}^p}{q - \overline{q}} \\ n_q &= \prod_p p_q^{n_p} \quad \text{for} \quad n = \prod_p p^{n_p} \ . \end{aligned} \tag{2.2}$$

2. In the general case quantum phase is complex number with magnitude different from unity:

$$q = exp(\eta)exp(i\pi/m) \ . \tag{2.3}$$

The quantum map is 1-1 for a non-vanishing value of η and the limit $m \to \infty$ gives ordinary integers. It seems that one must include the factor making the modulus of q different from unity if one wants 1-1 correspondence between ordinary and quantum integers guaranteeing a unique definition of quantum sum. In the p-adic context with $m = p$ the number $exp(\eta)$ exists as an ordinary p-adic number only for $\eta = np$. One can of course introduce a finite-dimensional extension of p-adic numbers generated by $e^{1/k}$.

3. The root of unity must correspond to an element of algebraic extension of p-adic numbers. Here Fermat's theorem $a^{p-1} \ mod \ p = 1$ poses constraints since $p-1$:th root of unity exists as ordinary p-adic number. Hence $m = p-1$:th root of unity is excluded. Also the modulus of q must exist either as a p-adic number or a number in the extension of p-adic numbers. .

4. $m = p$ the quantum counterparts of pinary digits are non-negative. The model of Sholl effect suggests that the most natural choice. One can however consider also expansions in powers of p^k and now $m = p^k$ is the most natural choice. For general value m it is natural to consider expansions in powers of m but now one loses number field property.

5. For p-adic rationals the quantum map reads as $m/n \to m_q/n_q$ by definition. But what about p-adic transcendentals such as e^p? There is no manner to decompose these numbers to finite primes and it seems that the only reasonable map is via the mapping of the coefficients x_n in $x = \sum x_n p^n$ to their quantum adic counterparts. It seems that one must expand all quantum transcendentals having as a signature non-periodic pinary expansion to quantum p-adics to achieve uniqueness. Second possibility is to restrict the consideration to rational p-adics. If one gives up the condition that products are mapped to products, one can map $n = n_k p^k$ to $n_q = \sum n_{kq} p^k$. Only the products of p-adic integers $n < p$ smaller than p would be mapped to products.

6. The index characterizing Jones inclusion [A19] [K4] is given by $[M:N] = 4cos^2(2\pi/n)$ and corresponds to quantum dimension of $2_q \times 2_q$ quantum matrices. TGD suggest that a series of more general quantum matrix dimensions identifiable as indices of inclusions and given by $[M:N] = l_q^2$, $l < p$ prime and $q = exp(i\pi/n)$, corresponding to prime Hilbert spaces and $q = n$-adicity. $l_q < l$ is in accordance with the idea about finite measurement resolution and for large values of p one would have $l_q \simeq l$.

To sum up, one can imagine several options and it is not clear which option is the correct one (if any). Certainly Option I for which the quantum map is only part of canonical identification is the simpler one- perhaps quite too simple. The model for Shnoll effect requires only Option I. The notion of quantum integer as defined for Opion II imbeds p-adic numbers to a much larger structure imbeddable to reals and therefore much more general than that proposed in the model of Shnoll effect [K1] but gives identical predictions when the parameters characterizing the probability distribution $P(n)$ correspond contain only single term in the p-adic power expansion. The mysterious dependence of nuclear decay rates on physics of solar system in the time scale of years reduces to similar dependence for the parameters characterizing $P(n)$. Could this dependence relate directly to the fact that canonical identification maps long length scale physics to short length scales physics. Could even microscopic systems such as atomic nuclei give rise to what might be called "cognitive representations" about the physics in astrophysical length scales?

2.3 Canonical identification for quantum rationals and symmetries

The fate of symmetries in canonical identification map is different for options I and II. Before continuing, one can of course ask why canonical identification should map p-adic symmetries to real symmetries. There is no obvious answer to the question.

1. For option II the prime p in the expansion $\sum x_n l^n$ is interpreted as a symbolic coordinate variable and the product of two quantum integers is analogous to the product of polynomials reducing to a convolution of the coefficient using quantum sum. The coefficient of a given power of p in the product would be just the convolution of the coefficients for factors using quantum sum. In the sum coefficients would be just the quantum sums of coefficients of summands.

2. Option I maps p-adic integers to their quantum counterparts by mapping the prime factors to their quantum counterparts defined by $q = exp(i\pi/p)$. The sums of the resulting quantum integers define a linear space consisting of sums $\sum k_n q^n$ of quantum phases with integer coefficients k_n subject to the condition that the sum $\sum_{0 \leq n < p} q^n$ vanishes. Given p-adic integer is mapped to single phase q^n. The map of all p-adic integers to p quantum phases means loss of information and generation or ring creates information not related to the p-adic numbers themselves.

   (a) One can also define quantum rationals by writing a given rational in unique manner as $r = p^k m/n$, expanding m and n as finite power series in p, and by replacing the coefficients with their quantum counterparts. The mapping of quantum rationals to their real counterparts would be by canonical identification $p \to 1/p$ in m_q/n_q. Also the completion of quantum rationals obtained by allowing infinite powers series for m and n makes sense and defines by canonical identification what might be called quantum reals.

   (b) Quantum arithmetics defined in this manner does not reflect faithfully the ordinary p-adic arithmetics and also leads to a problem with symmetries. In the product of ordinary p-adic integers the convolution for given power of p can lead to overflow and this leads to the emergence of modulo arithmetics. As a consequence, the canonical identification $\sum x_n l^n \to \sum x_n l^{-n}$ does not respect product and sum in general (simple example: $I((xl)^2) = x^2 l^{-2} \neq (I(xl))^2 = (x^2 modl)l^{-2} + (x^2 - x^2 modl)l^{-3}$ for $x > l/2$). Therefore canonical identification induced by $l \to 1/l$ does not respect symmetries represented affinely (as linear transformations and translations) although it is continuous.

(c) For quantum rationals defined as ratios m_q/n_q of quantum integers and mapped to $I(m_q)/I(n_q)$ the situation improves dramatically but is not cured completely. The breaking of symmetries could have a natural interpretation in terms of finite measurement resolution. For instance, one could argue that p-adic space-time sheets are extrema of Kähler action in algebraic sense and their real counterparts obtained by canonical identification are kind of smoothed out quantum average space-time surfaces, which do not satisfy real field equations and are not even differentiable. In this framework p-adicization would defined quantum average space-time as a p-adically smooth object which nice geometric properties.

Consider next Option II for quantum p-adics.

1. The original motivation for quantum rationals was to obtain correspondence between p-adics and reals respecting symmetries. For option II this dream can be achieved if the symmetries are defined for quantum rationals rather than p-adic numbers. Whether this means that quantum rationals are somehow deeper notion that p-adic number field is an interesting question. Since quantum rationals are obtained from quantum integers defining a Kac-Moody type algebra in powers of p^n symmetry conditions for quantum rational matrices reduce to conditions in terms of quantum integers and hold separately for each power of p. Therefore the value of p does not actually matter, and the replacement $p \rightarrow 1/p$ respects the symmetries.

   For instance, for the quantum counterpart of group $SL(2, Z)$ assuming that p^N is the largest power in the matrix elements the condition $det(A) = 1$ gives $2N + 1$ conditions for $4(N + 1)$ parameters leaving $2N + 3$ parameters. The matrix elements are integers so that actual conditions are more stringent.

2. Quantum integers generate a space in which the space of coefficients of p^n is the module generated by the sums $\sum k_n q^n$ of quantum phases with integer coefficients k_n subject to the condition that the sum $\sum_{0 \leq n < p} q^n$ vanishes. The huge extension of the original space is an obvious problem.

3. For this option non-uniqueness is a potential problem. One can have several quantum integers projecting to the same finite integer in powers of p. The number would be actually infinite when the coefficients of powers of p can occur with both signs. Does the non-uniqueness mean that quantum p-adics are more fundamental than p-adics?

4. The non-uniqueness inspires questions about the relationship between quantum field theory and number theory. Could the sum over different quantum representatives for p-adic integers define the analog of the functional integral in the ideal measurement resolution? Could loop corrections correspond number theoretically to the sum over all the alternatives allowed in a given measurement resolution defined by maximal number of powers of p in expansions of m and n in $r = m/n$? This would extend the vision about physics as generalized number theory considerably.

Note that quantum p-adic numbers are algebraic numbers so that quantum integers are algebraic numbers with prime p remaining ordinary integer.

2.4 More about the non-uniqueness of the correspondence between p-adic integers and their quantum counterparts

For both options the projection from quantum integers to p-adic numbers is many-to-one.

For option I p-adic integer is mapped to an integer proportional to a quantum integers proportional to power of p expressing its p-adic norm. Since the primes p_i in the decomposition of n are effectively replaced with $p_i \mod p$, a large number of integers with same p-adic norm is mapped to same quantum integer. A lot of information is lost.

For Option II p-adic number is mapped to a series in powers of p so that information is not lost. It is interesting to have some idea about how many quantum counterparts given p-adic integer has in this case and what might be their physical interpretation. If -1 is mapped to -1 rather than

$(p-1)_q(1+p+p^2+...)$ in quantum map and therefore also in canonical identification quantum p-adics form an analog of a function field. The number of quantum p-adics projected to same integer is infinite.

The number of quantum p-adics for which the coefficients of the polynomials of quantum primes $p_1 < p$ regarded as variables are positive is finite. These kind of quantum integers could be called strictly positive. It is easy to count the number of different strictly positive quantum counterparts of p-adic integer $n = n_0 + n_1 p + n_2 p^2 + ... + n_k p^k$ - that is elements of the ring of quantum integers projected to a given p-adic integer n.

1. For both options the number of quantum integers projected to a given integer n is simply the number of all partitions of to a sum of integers, whose number can vary from 1 to n and thus expressible as the sum $D(n) = \sum_{k=1}^{n} d(n,k)$ of numbers of partitions to k integers. Interestingly, the number of states with total conformal weight n constructible using at most k Virasoro generators equals to $d(n,k)$ and the total number of states with conformal weight n is just $D(n)$. This result follows if one does not assumes that different quantum representatives are really different. One cannot exclude the possibility that the condition $\sum_{n=0}^{p-1} q^n = 0$ for quantum phases implies this kind of dependencies.

   Similar situation occurs in the construction of tensor powers of group representations for any additive quantum number for which the basic unit is fixed. Could quantum classical correspondence be realized as a mapping of different states of a tensor product as different quantum p-adic space-time sheets?

2. The partition of n in all possible manners resembles combinatorially the insertion of loop corrections in all possible manners to a Feynman diagram containing corresponds up to p^{k-1}. Maybe the sum over quantum corrections could be reduced to the summation of amplitudes in which p-adic integer is mapped to its quantum counterpart in all possible manners. In zero energy ontology quantum corrections to generalized Feynman diagrams in a new p-adic length scaled defined by p^k indeed more or less reduces to the addition of zero energy states as a new tensor factor in all possible manners so that structurally the process would be like adding tensor factor.

To number of geometric objects to which one can assign quantum counterparts is rather limited. For the points of imbedding space with rational coordinates the number of quantum rational counterparts would be finite. If either of the integers appearing in the p-adic rational become infinite as a real integer, the number of quantum rationals becomes infinite and one obtains continuum in p-adic sense since p-adic integers form a continuum.

An infinite number of points of a $D > 0$-dimensional quantum counterpart of p-adic surface project to the same p-adic point. The restriction to a finite number of pinary digits makes sense only at the ends of braid strands at partonic 2-surfaces. This provides additional support for the effective 2-dimensionality and the braid representation for the finite measurement resolution. The selection of braid ends is strongly constrained by the condition that the number of pinary digits for the imbedding space coordinates is finite.

The interesting question is whether the summation over the infinite number of quantum copies of the p-adic partonic 2-surface could correspond to the functional integral over partonic 2-surfaces with braid ends fixed and thus having only one term in their pinary expansion. This kind of functional integral is indeed encountered in quantum TGD.

1. The summations in which the quantum positions of braid ends form a finite set would correspond to finite pinary cutoff. Second question is what the quantum summation for partonic 2-surfaces means: certainly there must be correlations between very nearby points if the summation is to make sense. The notion of finite measurement resolution suggests that summation reduces to that over the quantum positions of the braid ends.

2. Indeed, the reduction of the functional integral to a summation over quantum copies makes sense only if it can be carried out as a limit of a discrete sum analogous to Riemann sum and giving as a result what might be called quantum p-adic integral. This limit would mean inclusion of an increasing number of points of the partonic 2-surface to the quantum sum defined by the increasing pinary cutoff. One would also sum over the number of braid

strands. This approach could make sense physically if the collection of p-adic partonic 2-surfaces together with their tangent space data corresponds to a maximum of Kähler function. Quantum summation would correspond to a functional integral over small deformations with weight coming from the p-adic counterpart of vacuum functional mapped to its quantum counterpart. Canonical identification would give the real or complex counterpart of the integral.

2.5 The three basic options for Quantum Arithmetics

I have proposed two alternative definitions for quantum integers. In [K14] a third option is discussed.

1. For option I quantum counterparts of p-adic integers are identified as products of quantum counterparts for the primes dividing them. Powers of p are mapped to their inverses (straightforward quantum map would take them to zero). The quantum integers can be extended to ring (and algebra) by allowing sum operation. Field property is in general lost.

2. The approach adopted in the sequel is based on Option II based on the identification of quantum p-adics as an analog of Kac-Moody algebra with powers p^n in the same role as the powers z^n for Kac-Moody algebra. The two algebras have identical rules for sum and multiplication, and one does not require the arithmetics to be induced from the p-adic arithmetics (as assumed originally) since this would lead to a loss of associativity in the case of sum. Therefore the quantum counterparts of primes $l \neq p$ generate the algebra. One can also make the limitation $l < p^N$ to the generators. The counterparts of fixed integers in the map of integers to quantum integers are $0, 1, -1$ are $, 0, 1, -1$ as is easy to see. The number of quantum integers projecting to same p-adic integer is infinite.

3. One can consider also quantum m-adic option with expansion $l = \sum l_k m^k$ in powers of integer m with coefficients decomposable to products of primes $l < m$. This option is consistent with p-adic topology for primes p divisible by m and is suggested by the inclusion of hyper-finite factors [K4] characterized by quantum phases $q = exp(i\pi/m)$. Giving up the assumption that coefficients are smaller than m gives what could be called quantum covering of m-adic numbers. For this option all quantum primes l_q are non-vanishing. Phases $q = exp(i\pi/m)$ characterize Jones inclusions of hyper-finite factors of type II_1 assumed to characterize finite measurement resolution.

4. The definition of quantum p-adics discussed in [K14] replaces integers with Hilbert spaces of same dimension and $+$ and $\times$ with direct sum $\oplus$ and tensor product $\otimes$. Also co-product and co-sum must be introduced and assign to the arithmetics quantum dynamics, which leads to proposal that sequences of arithmetic operations can be interpreted arithmetic Feynman diagrams having direct TGD counterparts. This procedure leads to what might be called quantum mathematics or Hilbert mathematics since the replacement can be made for any structure such as rationals, algebraic numbers, reals, p-adic numbers, even quaternions and octonions. Even set theory has this kind of generalization. The replacement can be made also repeatedly so that one obtains a hierarchy of structures very similar to that obtained in the construction of infinite primes by a procedure analogous to repeated second quantization. One possible interpretation is in terms of a hierarchy of logics of various orders. Needless to say this definition is the really deep one and actually inspired by quantum TGD itself. In this picture the quantum p-adics as they are defined here would relate to the canonical identification map to reals and this map would apply also to Hilbert p-adics.

3 Could Lie groups possess quantum counterparts with commutative elements exist?

To begin with, it must be made clear that by commutativity it is meant that the matrix elements of the matrices representing the group elements are commutative numbers, not the matrices themselves.

The proposed definition of quantum rationals involves exceptional prime p expected to define what might be called p-adic prime. In p-adic mass calculations canonical identification is based on the map $p \rightarrow 1/p$ and has several variants but quite generally these variants fail to respect symmetries. Canonical identification for space-time coordinates fails also to be general coordinate invariant unless one has preferred coordinates. A possible interpretation could be that cognition affects physics: the choice of coordinate system to describe physics affects the physics.

The natural question is whether the proposed definition of quantum integers as series of powers of p-adic prime p with coefficients, which are arbitrary quantum rationals not divisible by p with product defined in terms of convolution for the coefficients of the series in powers of p using quantum sum for the summands in the convolution could change (should one say "save"?) the situation.

To see whether this is the case one must find whether the quantum analogues of classical matrix groups exist. To avoid confusion it should be emphasized that these quantum counterparts are distinct from the usual quantum groups having non-commutative matrix elements. Later a possible connection between these notions is discussed. In the recent case matrix elements commute but sum is replaced with quantum sum and the matrix element is interpreted as a powers series or polynomial in symbolic variable $x = p$ or $x = 1/p$, p prime such that coefficients are rationals not divisible by p.

The crucial points are the following ones.

1. All classical groups (see `http://tinyurl.com/y86oror3`) [A3] are subgroups of the special linear groups (see `http://tinyurl.com/3vpk8o8`) [A16] $SL_n(F)$, $F = R, C$, consisting of matrices with unit determinant. One can also replace F with the integers of the field F to get groups like $SL(2, Z)$. Classical groups are obtained by posing additional conditions on $SL_n(F)$ such as the orthonormality of the rows with respect to real, complex or quaternionic inner product. Determinant defines a homomorphism mapping the product of matrices to the product of determinants in the field F.

   Could one generalize rational special linear group (matrices with determinant 1) and its algebraic extensions by replacing the group elements by ratios of polynomials of a formal variable x, which has as its value the preferred prime p such that the coefficients of the polynomials are quantum integers not divisible by p? For Option I the situation one has just ratios of p-adic integers finite as real integers and for Option II the integers are polynomials $x = \sum x_n p^n$, where one has

   $$x_n = \sum_{\{n_i\}} N(\{n_i\}) \prod_i x_i^{n_i} \quad , \quad x_i = p_{i,q}, \;\; p_i < p \quad , \quad q = exp(i\pi/p) \quad .$$

   Here $N(\{n_i\})$ is integer. Could one perform this generalization in such a manner that the canonical identification $p \rightarrow 1/p$ maps this group to an isomorphic group? If quantum p-adic counterpart of the group is non-trivial, this seems to be the case since p plays the role of an argument of a polynomial with a specific values.

2. The identity $det(AB) = det(A)det(B)$ and the fact that the condition $det(A) = 1$ involves at the right hand side only the unit element common to all quantum integers suggests that this generalization could exist. If one has found a set of elements satisfying the condition $det_q(A) = 1$ all quantum products satisfy the same condition and subgroup of rational special linear group is generated.

3.1 Quantum counterparts of special linear groups

Special linear groups (see `http://tinyurl.com/3vpk8o8`) [A16] defined by matrices with determinant equal to 1 contain classical groups as subgroups and the conditions for their quantum counterparts are therefore the weakest possible. Special linear group makes sense also when one restricts the matrix elements to be integers of the field so that one has for instance $SL_n(Z)$. Opiton I reduces to that for ordinary p-adics. For Option II each power of p can be treated independently so that the situation is easier. The treatment of conditions in two cases differs only in that overflows in p are possible for Option I. The numbers of conditions are same.

Let us consider $SL_n(Z)$ first.

1. To see that the generalization exists in the case of special linear groups one just just writes the matrix elements a_{ij} in series in powers of p

$$a_{ij} = \sum_n a_{ij}(n)p^n \ . \tag{3.1}$$

   This expansion is very much analogous to that for the Kac-Moody algebra element and also the product and sum obey similar algebraic structure. p is treated as a symbolic variable in the conditions stating $det_q(A) = 1$. It is essential that $det_q(A) = 1$ holds true when p is treated as a formal symbol so that each power of p gives rise to separate conditions.

2. For SL_n the definition of determinant involves sum over products of n elements. Quantum sums of these elements are in question.

3. Consider now the number of conditions involved. The number of matrix elements is in real case $N^2(k+1)$, where k is the highest power of p involved. $det(A) = 1$ condition involves powers of p up to l^{Nk} and the total number of conditions is $kN+1$ - one for each power. For higher powers of p the conditions state the vanishing of the coefficients of p^m. This is achieved elegantly in the sense of modulo arithmetics if the quantum sum involved is proportional to l_q.

   The number of free parameters is

$$\# \ = (k+1)N^2 - kN - 1 = kN(N-1) + N^2 - 1 \ . \tag{3.2}$$

   For $N = 2, k = 0$ one obtains $\# = 3$ as expected for SL(2, R). For $N = 2, k = 1$ one obtains $\# = 5$. This can be verified by a direct calculation. Writing $a_{ij} = b_{ij} + c_{ij}p$ one obtains three conditions

$$det_q(B) = 1 \ , \ Tr_q(BC) = 0 \ , \ det_q(C) = 0 \ . \tag{3.3}$$

   for the 8 parameters leaving 5 integer parameters.

   Integer values of the parameters are indeed possible. Using the notation

$$b_{ij} = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \ , \ c_{ij} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \tag{3.4}$$

   one can write the solutions as

$$\begin{aligned} &(a_1, b_1) = k(c_1, d_1) \ , \quad (c_1, d_1) = l(a_0 - kc_0, b_0 - kd_0) \ , \\ &a_0 d_0 - b_0 c_0 = 1 \ . \end{aligned} \tag{3.5}$$

   Therefore 6 integers characterize the solution.

4. Complex case can be treated in similar manner. In this case the number of three parameters is $2(k+1)N^2$, the number of conditions is $2(kN+1)$ and the number of parameters is

$$\# = 2(k+1)N^2 - 2(kN+1) \ . \tag{3.6}$$

5. Since the conditions hold separately for each power of p, the formula $det_q(AB) = det_q(A)det_q(B)$ implies that the matrices satisfying the conditions generate a subgroup of SL_n.

One can generalize the argument to rational values of matrix elements in a simple manner. The matrix elements can be written in the form $A_{ij} = Z_{ij}/K$ and the only modification of the equations is that the zeroth order term in p gives $det(Z) = K^n$ for SL_n. One can expand K^n in powers of p and it gives inhomogenous term to in each power of p. For instance, if K is zeroth order in p, solutions to the conditions certainly exist.

The result means that rational subgroups of special linear groups $SL_n(R)$ and $SL(n,C)$ and also the real and complex counterparts of $SL(n,Z)$ quantum matrix groups characterized by prime p exist in both real and p-adic context and can be related by the map $p \rightarrow 1/p$ mapping short and length scales to each other.

It is remarkable that only the Lorentz groups $SO(2,1)$ and $SO(3,1)$ have covering groups are isomorphic to $SL(2,R)$ and $SL(2,C)$ allow these subgroups. All classical Lie groups involve additional conditions besides the condition that the determinant of the matrix equals to one and all these groups except symplectic groups fail to allow the generalization of this kind for arbitrary values of k. Therefore four-dimensional Minkowski space is in completely exceptional position.

3.2 Do classical lie groups allow quantum counterparts?

In the case of classical groups one has additional conditions stating orthonormality of the rows of the matrix in real, complex, or quaternionic number field. It is quite possible that the conditions might not be satisfied always and it turns out that for G_2 and probably also for other exceptional groups this is the case.

1. Non-exceptional classical groups

It is easy to see that all non-exceptional classical groups quantum counterparts in the proposed sense for sufficiently small values of k and in the case of symplectic groups quite generally. In this case one must assume rational values of group elements and one can transform the conditions to those involving integers by writing $A_{ij} = Z_{ij}/K$. The expansion of K gives for orthogonal groups the condition that the lengths of the integer rows defining Z_{ij} have length K^2 plus orthogonality conditions. $det(A) = 1$ condition holds true also now since a subgroup of special linear group is in question.

1. Consider first orthogonal groups $SO(N)$.
   (a) For $q = 1$ there are N^2 parameters. There are N conditions stating that the rows are unit vectors and $N(N-1)/2$ conditions stating that they are orthogonal. The total number of free parameters is $\# = N(N-1)/2$.
   (b) If the highest power of p is k there are $(k+1)N^2$ parameters and $(2k+1)[N+N(N-1)/2] = (2k+1)(N+1)/2$ conditions. The number of parameters is

$$\# = N^2(k+1) - \frac{N(N+1)(2k+1)}{2} = \frac{N(N-2k+1)}{2} \quad . \tag{3.7}$$

   This is negative for $k > (N+1)/2$. It is quite not clear how to interpret this result. Does it mean that when one forms products of group elements satisfying the conditions the powers higher than $k_{max} = [(N+1)/2]$ vanish by quantum modulo arithmetics. Or do the conditions separate to separate conditions for factors in AB: this indeed occurs in the unitarity conditions as is easy to verify. For $SO(3)$ and $SO(2,1)$ this would give $k_{max} = 2$. For $SO(3,1)$ one would have $k_{max} = 2$ too. Note that for the covering groups $SL(2,R)$ and $SL(2,C)$ there is no restrictions of this kind.
   (c) The normalization conditions for the coefficients of the highest power of a given row imply that the vector in question has vanishing length squared in quantum inner product. For $q = 1$ this implies that the coefficients vanish. The repeated application of this condition one would obtain that $k = 0$ is the only possible solution. For $q \neq 1$ the conditions can be satisfied if the quantum length squared is proportional to $l_q = 0$. It seems that this condition is absolutely essential and serves as a refined manner to realize p-adic cutoff and quantum group structure and p-adicity are extremely closely related to each other. This conclusion applies also in the case of unitary groups and symplectic groups.

(d) Complex forms of rotation groups can be treated similarly. Both the number of parameters and the number of conditions is doubled so that one obtains $\# = N^2(k+1) - N(N+1)(2k+1) = N(N-2k+1)$ which is negative for $k > (N+1)/2$.

2. Consider next the unitary groups $U(N)$. Similar argument leads to the expression

$$\# = 2N^2(k+1) - (2k+1)N^2 = N^2 \tag{3.8}$$

so that the number of three parameters would be N^2- same as for $U(N)$. The determinant has modulus one and the additional conditions requires that this phase is trivial. This is expected to give $k+1$ conditions since the fixed phase has l-adic expansion with $k+1$ powers. Hence the number of parameters for $SU(N)$ is

$$\# = N^2 - k + 1 \tag{3.9}$$

giving the condition $k_{max} < N^2 - 1$ which is the dimension of $SU(N)$.

3. Symplectic group can be regarded as a quaternionic unitary group. The number of parameters is $4N^2(k+1)$ and the number of conditions is $(2k+1)(N+2N(N-1)) = N(2N-1)(2k+1)$ so that the number of three parameters is $\# = 4N^2(k+1)-(2k+1)N(N-1) = (2k+3)N^2 + N(2k+1)$. Fixing single quaternionic phase gives 3(k+1) conditions so that the number of parameters reduces to

$$\# = (2k+3)N^2 + (2k+1)N - 3(k+1) = (k+1)(2N^2 + 2N - 3) + N(N-1) \ , \tag{3.10}$$

which is positive for all values of N and k so that also symplectic groups are in preferred position. This is rather interesting, since the infinite-dimensional variant of symplectic group associated with the $\delta M^4 \times CP_2$ is in the key role in quantum TGD and one expects that in finite measurement resolution its finite-dimensional counterparts should appear naturally.

2. Exceptional groups are exceptional

Also exceptional groups (see `http://tinyurl.com/y779ldyt`) [A7] [A7] related closely to octonions allow an analogous treatment once the nature of the conditions on matrix elements is known explicitly. The number of conditions can be deduced from the dimension of the ordinary variant of exceptional group in the defining matrix representation to deduce the number of conditions. The following argument allows to expect that exceptional groups are indeed exceptional in the sense that they do not allow non-trivial quantum counterparts.

The general reason for this is that exceptional groups are very low dimensional subgroups of matrix groups so that for the quantum counterparts of these groups the number N_{cond} of group conditions is too large since the number of parameters is $(k+1)N^2$ in the defining matrix representation (if such exists) and the number of conditions is at least $(2k+1)N_{class}$, where N_{class} is the number of condition for the classical counterpart of the exceptional group. Note that r-linear conditions the number of conditions is proportional to $rk+1$.

One can study the automorphism group G_2 (see `http://tinyurl.com/y9rrs7un`) [A8] of octonions as an example to demonstrate that the truth of the conjecture is plausible.

1. G_2 is a subgroup of $SO(7)$. One can consider 7-D real spinor representation so that a representation consists of real 7×7matrices so that one has $7^2 = 49$ parameters. One has $N(N+1)/2$ orthonormality conditions giving for $N = 7$ orthonormality conditions 28 conditions. This leaves 21 parameters. Besides this one has conditions stating that the 7-dimensional analogs of the 3-dimensional scalar-3-products $A \cdot (B \times C)$ for the rows are equal 1, -1, or 0. The number of these conditions is $N(N-1)(N-2)/3!$. For $N = 7$ this gives 35 conditions meaning that these conditions cannot be independent of orthonormalization conditions The number of parameters is $\# = 49 - 35 = 14$ - the dimension of G_2 - so that these conditions must imply orthonormality conditions.

2. Consider now the quantum counterpart of G_2. There are $(k+1)N^2 = 49(k+1)$ parameters altogether. The number of cross product conditions is $(3k+1)\times 35$ since the highest power of p in the scalar-3-product is l^{3k}. This would give

$$\# = -56k + 14 \ . \tag{3.11}$$

This number is negative for $k > 0$. Hence G_2 would not allow quantum variant. Could this be interpreted by saying that the breaking of G_2 to $SU(3)$ must take place and indeed occurs in quantum TGD as a consequence of associativity conditions for space-time surfaces.

3. The conjecture is that the situation is same for all exceptional groups.

The general results suggest that both the covering group of the Lorenz group of 4-D Minkowski space and the hierarchy symplectic groups have very special mathematical role and that the notions of finite measurement resolution and p-adic physics have tight connections to classical number fields, in particular to the non-associativity of octonions.

3.3 Questions

In the following some questions are introduced and discussed.

3.3.1 How to realize p-adic-real duality at the space-time level?

The concrete realization of p-adic–real duality would require a map from p-adic realm to real realm and vice-versa. The naive expectation is that it is induced by the map $p \rightarrow 1/p$ leading from p-adic number field to real number field or vice versa.

If possible, the realization of p-adic real duality at the space-time level should not pose additional conditions on the preferred extremals themselves. Together with effective 2-dimensionality this suggests that the map from p-adic realm to real realm maps partonic 2-surfaces to partonic 2-surfaces defining at least partially the boundary data for holography.

It turned out that the situation is not so simple. Or putting it correctly - so complex. The point is that the direct mapping of real space-time sheets to real ones requires discretization and length scale cutoff bringing in a lot of arbitrariness and the continuity of the map is in conflict with the preservation of symmetries.

A more realistic view is based on the idea that p-adic space-time sheets indeed define a theory about real space-time sheets. The interaction between real and p-adic number fields would mean that p-adic space-time surfaces define cognitive representations of real space-time surfaces (preferred extremals). One could also say that real space-time surface represents sensory aspects of conscious experience and p-adic space-time surfaces its cognitive aspects. Both real and p-adics rather than real or p-adics.

Strong form of holography implied by strong form of General Coordinate Invariance leads to the suggestion that partonic 2-surfaces and string world sheets at which the induced spinor fields are localized in order to have a well-defined em charge (this is only one of the reasons) and having having discrete set as intersection points with partonic 2-surfaces define what might called "space-time genes". Space-time surfaces would be obtained as preferred extremals satisfying certain boundary conditions at string world sheets. Space-time surfaces are defined only modulo transformations of super-symplectic algebra defining its sub-algebra and acting as conformal gauge transformations so that one can talk about conformal gauge equivalences classes of space-time surfaces.

The map assigning to real space-time surface a cognitive representation would be replaced by a correspondence assigning to the string world sheets preferred extremals of Kähler action in various number fields: string world sheets would be indeed like genes. String world sheets would be in the intersection of realities and p-adicities in the sense that the parameters characterizing them would be algebraic numbers associated with the algebraic extension of p-adic numbers in question. It is not clear whether the preferred extremal is possible for all p-adic primes but this would fit nicely with the vision that elementary particles are characterized by p-adic primes. It could be also that the classical non-determinism of Kähler action responsible for the conformal gauge symmetry corresponds to p-adic non-determinism for some particular prime so that the cognitive map is especially good for this prime.

3.3.2 How commutative quantum groups could relate to the ordinary quantum groups?

The interesting question is whether and how the commutative quantum groups relate to ordinary quantum groups.

This kind of question is also encountered when considers what finite measurement resolution means for second quantized induced spinor fields [K12]. Finite measurement resolution implies a cutoff on the number of the modes of the induced spinor fields on partonic 2-surfaces. As a consequence, the induced spinor fields at different points cannot ant-commute anymore. One can however require anti-commutativity at a discrete set of points with the number of points "more or less equal" to the number of modes. Discretization would follow naturally from finite measurement resolution in its quantum formulation.

The same line of thinking might apply to to quantum groups. The matrix elements of quantum group might be seen as quantum fields in the field of real or complex numbers or possibly p-adic number field or of its extension. Finite measurement resolution means a cutoff in the number of modes and commutativity of the matrix elements in a discrete set of points of the number field rather than for all points. Finite measurement resolution would apply already at the level of symmetry groups themselves. The condition that the commutative set of points defines a group would lead to the notion of commutative quantum group and imply p-adicity as an additional and completely universal outcome and select quantum phases $exp(i\pi/p)$ in a preferred position. Also the generalization of canonical identification so central for quantum TGD would emerge naturally.

One must of course remember that the above considerations probably generalize so that one should not take the details of the discussion too seriously.

3.3.3 How to define quantum counterparts of coset spaces?

The notion of commutative quantum group implies also a generalization of the notion of coset space G/H of two groups G and $H \subset G$. This allows to define the quantum counterparts of the proper time constant hyperboloid and $CP_2 = SU(3)/U(2)$ as discrete spaces consisting of quantum points identifiable as representatives of cosets of the coset space of discrete quantum groups. This approach is very similar but more precise than the earlier approach in which the points in discretization had angle coordinates corresponding to roots of unity and radial coordinates with discretization defined by p-adic prime.

The infinite-dimensional "world of classical worlds" (WCW) can be seen as a union of infinite-dimensional symmetric spaces (coset spaces) [K3] and the definition as a quantum coset group could make sense also now in finite measurement resolution. This kind of approach has been already suggested and might be made rigorous by constructing quantum counterparts for the coset spaces associated with the infinite-dimensional symplectic group associated with the boundary of causal diamond. The problem is that matrix group is not in question. There are however good hopes that the symplectic group could reduces to a finite-dimensional matrix group in finite measurement resolution. Maybe it is enough to achieve this reduction for matrix representations of the symplectic group.

3.4 Quantum P-Adic Deformations Of Space-Time Surfaces As A Representation Of Finite Measurement Resolution?

A mathematically fascinating question is whether one could use quantum arithmetics as a tool to build quantum deformations of partonic 2-surfaces or even of space-time surfaces and how could one achieve this. These quantum space-times would be commutative and therefore not like non-commutative geometries assigned with quantum groups. Perhaps one could see them as commutative semiclassical counterparts of non-commutative quantum geometries just as the commutative quantum groups discussed in [K13] could be seen commutative counterparts of quantum groups.

As one tries to develop a new mathematical notion and interpret it, one tends to forget the motivations for the notion. It is however extremely important to remember why the new notion is needed.

1. In the case of quantum arithmetics Shnoll effect is one excellent experimental motivation. The understanding of canonical identification and realization of number theoretical universality are also good motivations coming already from p-adic mass calculations. A further motivation

comes from a need to solve a mathematical problem: canonical identification for ordinary p-adic numbers does not commute with symmetries.

2. There are also good motivations for p-adic numbers. p-Adic numbers and quantum phases can be assigned to finite measurement resolution in length measurement and in angle measurement. This with a good reason since finite measurement resolution means the loss of ordering of points of real axis in short scales and this is certainly one outcome of a finite measurement resolution. This is also assumed to relate to the fact that cognition organizes the world to objects defined by clumps of matter and with the lumps ordering of points does not matter.

3. Why quantum deformations of partonic 2-surfaces (or more ambitiously: space-time surfaces) would be needed? Could they represent convenient representatives for partonic 2-surfaces (space-time surfaces) within finite measurement resolution?

   (a) If this is accepted, there is no compelling need to assume that this kind of space-time surfaces are preferred extremals of Kähler action.

   (b) The notion of quantum arithmetics and the interpretation of p-adic topology in terms of finite measurement resolution however suggest that they might obey field equations in preferred coordinates but not in the real differentiable structure but in what might be called quantum p-adic differentiable structure associated with prime p.

   (c) Canonical identification would map these quantum p-adic partonic (space-time surfaces) to their real counterparts in a unique continuous manner and the image would be real space-time surface in finite measurement resolution. It would be continuous but not differentiable and would not of course satisfy field equations for Kähler action anymore. What is nice is that the inverse of the canonical identification which is two-valued for finite number of pinary digits would not be needed in the correspondence.

   (d) This description might be relevant also to quantum field theories (QFTs). One usually assumes that minima obey partial differential equations although the local interactions in QFTs are highly singular so that the quantum average field configuration might not even possess differentiable structure in the ordinary sense! Therefore quantum p-adicity might be more appropriate for the minima of effective action.

   The cautious conclusion would be that commutative quantum deformations of space-time surfaces could have a useful function in TGD Universe.

Consider now in more detail the identification of the quantum deformations of space-time surfaces.

1. Rationals are in the intersection of real and p-adic number fields and the representation of numbers as rationals $r = m/n$ is the essence of quantum arithmetics. This means that m and n are expanded to series in powers of p and coefficients of the powers of p which are smaller than p are replaced by the quantum counterparts. They are quantum quantum counterparts of integers smaller than p. This restriction is essential for the uniqueness of the map assigning to a give rational quantum rationals.

2. One must get also quantum p-adics and the idea is simple: if the pinary expansions of m and n in positive powers of p are allowed o become infinite, one obtains a continuum very much analogous to that of ordinary p-adic integers with exactly the same arithmetics. This continuum can be mapped to reals by canonical identification. The possibility to work with numbers which are formally rationals is utmost importance for achieving the correct map to reals. It is possible to use the counterparts of ordinary pinary expansions in p-adic arithmetics.

3. One can defined quantum p-adic derivatives and the rules are familiar to anyone. Quantum p-adic variants of field equations for Kähler action make sense.

(a) One can take a solution of p-adic field equations and by the commutativity of the map $r = m/n \rightarrow r_q = m_q/n_q$ and of arithmetic operations replace p-adic rationals with their quantum counterparts in the expressions of quantum p-adic imbedding space coordinates h^k in terms of space-time coordinates x^α.

(b) After this one can map the quantum p-adic surface to a continuous real surface by using the replacement $p \rightarrow 1/p$ for every quantum rational. This space-time surface does not anymore satisfy the field equations since canonical identification is not even differentiable. This surface - or rather its quantum p-adic pre-image - would represent a space-time surface within measurement resolution. One can however map the induced metric and induced gauge fields to their real counterparts using canonical identification to get something which is continuous but non-differentiable.

4. This construction works nicely if in the preferred coordinates for imbedding space and partonic (space-time) surface itself the imbedding space coordinates are rational functions of space-time coordinates with rational coefficients of polynomials (also Taylor and Laurent series with rational coefficients could be considered as limits). This kind of assumption is very restrictive but in accordance with the fact that the measurement resolution is finite and that the representative for the space-time surface in finite measurement resolution is to some extent a convention. The use of rational coefficients for the polynomials involved implies that for polynomials of finite degree WCW reduces to a discrete set so that finite measurement resolution has been indeed realized quite concretely!

Consider now how the notion of finite measurement resolution allows to circumvent the objections against the construction.

1. Manifest GCI is lost because the expression for space-time coordinates as quantum rationals is not general coordinate invariant notion unless one restricts the consideration to rational maps and because the real counterpart of the quantum p-adic space-time surface depends on the choice of coordinates. The condition that the space-time surface is represented in terms of rational functions is a strong constraint but not enough to fix the choice of coordinates. Rational maps of both imbedding space and space-time produce new coordinates similar to these provided the coefficients are rational.

2. Different choices for imbedding space and space-time surface lead to different quantum p-adic space-time surface and its real counterpart. This is an outcome of finite measurement resolution. Since one cannot order the space-time points below the measurement resolution, one cannot fix uniquely the space-time surface nor uniquely fix the coordinates used. This implies the loss of manifest general coordinate invariance and also the non-uniqueness of quantum real space-time surface. The choice of coordinates is analogous to gauge choice and quantum real space-time surface preserves the information about the gauge.

4 Could one understand p-adic length scale hypothesis number theoretically?

p-Adic length scale hypothesis states that primes near powers of two are physically interesting. In particular, both real and Gaussian Mersenne primes seem to be fundamental and can be tentatively assigned to charged leptons and living matter in the length scales between cell membrane thickness and size of the cell nucleus. They can be also assigned to various scaled up variants of hadron physics and with lepto-hadron physics suggested by TGD.

4.1 Number theoretical evolution as a selector of the fittest p-adic primes?

How could one understand p-adic length scale hypothesis? The general explanation would be in terms of number theoretic evolution by quantum jumps selecting the primes that are the fittest. The vision discussed in [K14] d leads to the proposal that the fittest p-adic primes are those which do not split in the physically preferred algebraic extensions of rationals. Algebraic extensions are naturally characterized by infinite primes characterizing also stable bound states of particles.

Therefore these stable infinite primes or equivalently stable bound states would characterize also the p-adic primes which are fit. This explanation looks rather attractive.

p-Adic evolution would mean also a selection of preferred scales for CDs, instead of integer multiples of CP_2 scale only prime multiples or possibly prime power multiples would be favored and primes near powers of two were especially fit. A possible “biological” explanation is that for the preferred primes the number of quantum states is especially large making possible to build complex sensory and cognitive representations about external world.

The proposed vision about commutative quantum groups encourages to consider a number theoretic explanation for the p-adic length scale hypothesis consistent with the evolutionary explanation is that the quantum counterpart of symmetry groups are especially large for preferred primes. Large symmetries indeed imply large numbers of states related by symmetry transformations and high representational capacity provided by the p-adic–real duality. It is easy to make a rough test of the proposal for $G = SO(3)$, $SU(2)$ or $SU(3)$ associated with p-adic integers modulo p reducing to the counterpart of G for finite field might be especially large for physically preferred primes. Mersenne primes do not however seem to be special in this sense so that the following considerations can be taken as an exercise in the use of number theoretic functions and the reader can quite well skip the section.

4.2 Preferred p-adic primes as ramified primes?

As I wrote the first version of this chapter, I had not developed the vision about adelic physics. Adelic physics corresponds to a hierarchy of extensions of rationals inducing extensions of p-adic number fields and the proposal is that ramified primes of extension correspond to preferred p-adic primes.

1. Prime p of number field K can split in the extension L of K to primes P_i of L. Prime p is Galois invariant, which poses strong conditions on the decomposition. p need not split at all, or it splits to maximal number n of primes of extension, which is invariant under Galois group. In some exceptional cases the number of primes can be smaller than the dimension of extension and un this case there is product of primes of extension containing less than the maximal number of P_i. In this case speaks of ramification.

2. Adelic physics suggests that prime p and quite generally, all preferred p-adic primes, could correspond to ramified primes for the extension of rationals defining the adele. Ramified prime divides discriminant $D(P)$ of the irreducible polynomial P (monic polynomial with rational coefficients) defining the extension (see `http://tinyurl.com/oyumsnk`).

   Discriminant $D(P)$ of polynomial whose, roots give rise to extension of rationals, is essentially the resultant $Res(P, P')$ for P and its derivative P' defined as the determinant of so called Sylvester polynomial (see `http://tinyurl.com/p67rdgb`). $D(P)$ is proportional to the product of differences $r_i - r_j$, $i \neq j$ the roots of p and vanishes if there are two identical roots.

   Remark: For second order polynomials $P(x) = x^2 + bx + c$ one has $D = b^2 - 4c$.

3. Ramified primes divide D. Since the matrix defining $Res(P, P')$ is a polynomial of coefficients of p of order $2n - 1$, the size of ramified primes is bounded and their number is finite. The larger coefficients $P(x)$ has, the larger the value of ramified prime can be. Small discriminant means small ramified primes so that polynomials having nearly degenerate roots have also small ramifying primes. Galois ramification is of special interest: for them all primes of extension in the decomposition of p appear as same power. For instance, the polynomial $P(x) = x^2 + p$ has discriminant $D = -4p$ so that primes 2 and p are ramified primes. For Galois extensions one has $e_i = e$ and $f_i = f$ and n equals to the order of Galois group: in this case one has $p = (\prod_{i=1}^{g} P_i)^e$.

   Remark: All polynomials having pair of complex conjugate roots have $p = 2$ as ramified prime.

4. I try to formulate my poor man's understanding about the situation. The expression of the ramified prime p can be written as $p = \prod_{i=1}^{g} P_i^{e_i}$. $e_i > 1$ for some i and $\sum_{i=1}^{g} e_i < n$. The

interpretation is that the action of Galois group on each power $P_i^{e_i}$ is non-trivial and its orbit contains f_i points so that one has $\sum_{i=1}^g e_i f_i = n$. Although the numbers P^{e_i} are not invariant under Galois group, their product is. f_i can be identified as $f_i = [L/P_i^{e_i} : K/p]$. This says that $P/P_i^{e_i}$ consists of products of Galois images of P^{e_i} with integers $n < p$. The because the integers $n < p$ cannot decompose to a product of form $n = \prod_{i=1}^g P_i^{k_i}$ since they would divide p, which is impossible.

Since higher powers of P_i appear in the expression of ramified prime, one has $p \ mod \ P_i$=0 for $e_i > 1$. Why this can take place only for primes dividing D? Galois invariance of p must be involved. D is expressible as a product primes P_i of L and contains only higher powers P_i^k, $k > 1$. D is proportional to $\prod P_i^2$, where P_i are the primes dividing it. Why? Why the orbit consisting of f different integers of L contracts to single integer (this is just the criticality)?

5. What does ramification mean algebraically? The ring $\mathcal{O}(K)/(p)$ of integers of the extension K modulo $p = \pi_i^{e_i}$ can be written as product $\prod_i \mathcal{O}/\pi_i^{e_i}$ (see `http://tinyurl.com/y6yskkas`). If p is ramified, one has $e_i > 1$ for at least one i. Therefore there is at least one nilpotent element in $\mathcal{O}(K)/(p)$.

Could one interpret nilpotency quantum physically?

1. For Galois extensions one has $e_i = e > 1$ for ramified primes. e divides the dimension of extension. For the quadratic extensions ramified primes have $e = 2$. Quadratic extensions are fundamental extensions - kind of conserved genes -, whose further extensions give rise to physically relevant extensions.

   On the other hand, fermionic oscillator operators and Grassmann number used to describe fermions "classically" are nilpotent. Could they correspond to nilpotent elements of order $e_i = e = 2$ in $\mathcal{O}(K)/(p)$? Fermions are building bricks of all elementary particles in TGD. Could this number theoretic analogy for the fermionic statistics have a deeper meaning?

2. What about ramified primes with $e_i = e > 2$? Could they correspond to para-statistics (see `http://tinyurl.com/y4mq6j22`) or braid statistics (see `http://tinyurl.com/psuq45j`)?

   Both parabosonic and parafermionic fields of order n have the representation $\Psi = \sum_{i=1}^n \Psi_i$. For parafermion field one has $\{\Psi_i(x), \Psi_i(y)\} = 0$ and $[\Psi_i(x), \Psi_j(y)] = 0$, $i \neq j$, when x and y have space-like separation. For parabosons the roles of commutator and anti-commutator are changed.

   The states containing N identical parafermions are described by a representation of symmetric group S_N with N rows with at most n columns (anti-symmetrization). For states containing N identical parabosons one has N columns and at most n rows. For parafermions the wave function is symmetric in horizontal direction but the modes are different so that Bose-Einstein condensation is not possible.

   For parafermion of order n operator $\sum_{i=1}^n \Psi_i$ one has $(\sum_{i=1}^n \Psi_i)^n = \prod \Psi_1\Psi_2...\Psi_n$ and higher powers vanish so that one would have $n+1$-nilpotency. Therefore the interpretation for the nilpotent elements of order e in $\mathcal{O}(K)/(p)$ in terms of parafermion of order $n = e-1$ might make sense.

   It seems impossible to build a nilpotent operator from parabosonic field $\Psi = \sum_i \Psi_i$: the reason is that the powers Ψ_i^n are non-vanishing for arbitrarily high values of n.

3. Braid statistics differs from para-statistics and is assigned with quantum groups. It would naturally correspond to quantum phase $exp(i\pi/p)$ assignable to the exchange of particles by braid operation regarded as a homotopy permuting braid strands. Could ramified prime p would correspond to braid statistics and the index $e_i = e$ characterizing it to parafermion statistics of order $e-1$? This possibility cannot be excluded since this exotic physics would be associated in TGD framework to dark matter assigned to algebraic extensions of rationals whose dimension n equals to h_{eff}/h_0.

Why the primes, which do not split maximally in given extension would be physically special?

1. Do ramified primes possess exceptional evolutionary fitness and are ramified primes present for lower-dimensional extensions present also for higher-dimensional extensions? If higher extensions are formed as extensions of already existing extensions, this is the case. Hierarchy of polynomials of polynomials would to this kind of hierarchy with ramified primes of starting point polynomials analogous to conserved genes.

2. Quadratic extensions are the simplest ones and could serve as starting point extensions. Polynomials of form $x^2 - c$ are the simplest among them. Discriminant is now $D = -4c$.

3. Why $c = M_n = 2^n - 1$ allowing $p = 2$ and Mersenne prime $p = M_n$ as ramified primes would be favored? Extension of rationals defined by $x = 2^n$ is non-trivial for odd n and is equivalent with extension containing $\sqrt{2}$. $c = M_n = 2^n - 1$ as a small deformation of $c = 2^n$ gives an extension having both 2 as M_n as ramified primes.

   For $c = M_n$ the number of ramified primes is smallest possible and equal to 2: why minimal number of ramified primes would give rise to a fittest extension? Why smallest number of fermionic p-adic mass scales assignable to the ramified primes would be the fittest option?

   The p-adic length scale corresponding ro M_n would be maximal and mass scale minimal. Could one think that other quadratic extension are unstable against transforming to Mersenne extensions with smallest p-adic mass scale?

4.3 Could group theory select the preferred primes?

My recent view about the following considerations is that they are out-of-date. The notion of ramified prime so convincing that group theoretical considerations based on quantum-commutative generalization of Lie groups (matrix elements in commutative ring) look too tricky. I have not however had heart of throwing aways this piece of text yet.

In the following I consider *only the Option I*, which reduces to ordinary p-adic numbers effectively since quantum map induced by $p_i \to p_{iq}$ for $p_i < p$ can be combined with canonical identification. The arguments developed say nothing about option II. For option I the group transformations for which the conditions hold true modulo p make sense if matrix elements are integers satisfying $a_{ij} < p$. This makes sense for large values of p associated with elementary particles. This implies a reduction to finite field $G(p, 1)$. The original argument was more general and used same condition but involved an error.

1. For $SL(2, C)$ - the covering group of Lorentz group - one obtains no constraints and all quantum phases $exp(i\pi/n)$ are allowed: this would mean that all CDs are in the same position. The rational $SL(2, C)$ matrices whose determinant is zero modulo p form a group assignable to finite field andit might be that for some values of p this group is exceptionally large. $SL(2, C)$ defines also the covering group of conformal symmetries of sphere.

2. For orthogonal, unitary, and symplectic groups only $n = p$, p prime allows $k > 0$ and genuine p-adicity. Since $SO(3, 1)$, $SO(3)$, $SU(2)$ and $SU(3)$ should alow p-adicization this selects CDs with size scale characterized by prime p.

3. For orthogonal, unitary, and symplectic groups one obtains non-trivial solutions to the unitarity conditions only if the highest power of p corresponds quantum image of a vector with zero norm modulo p as follows from the basic properties of quantum arithmetics.

   (a) In the case of $SO(3)$ one has the condition

   $$\sum_{i=1}^{3} x_i^2 = 1 + k \times p \qquad (4.1)$$

   Note that this condition can degenerate to a condition stating that a sum of two squares is multiple of prime. As noticed the prime must be large and $x_i^2 < p$ holds true.

(b) For the covering group $SU(2)$ of $SO(3)$ one has the condition

$$\sum_{i=1}^{4} x_i^2 = 1 + k \times p \tag{4.2}$$

since two complex numbers for the row of SU(2) matrix correspond to four real numbers.

(c) For $SU(3)$ one has the condition

$$\sum_{i=1}^{6} x_i^2 = 1 + k \times p \tag{4.3}$$

corresponding to 3 complex numbers defining the row of SU(3) matrix.

What can one say about these conditions? The first thing to look is whether the conditions can be satisfied at all. Second thing to look is the number of solutions to the conditions.

4.4 Orthogonality conditions for $SO(3)$

The conditions for $SO(3)$ are certainly the strongest ones so that it is reasonable to study this case first.

1. One must remember that there are also integers -in particular primes- allowing representation as a sum of two squares. For instance, Fermat primes whose number is very small, allow representation $F_n = 2^{+}1$. More generally, Fermat's theorem on sums of two squares states that and odd prime is expressible as sum of two squares only if it satisfies $p \ mod \ 4 = 1$. The second possibility is $p \ mod \ 4 = 3$ so that roughly one half of primes satisfy the $p \ mod \ 4 = 1$ condition: Mersenne primes do not satisfy it.

   The more general condition giving sum proportional to prime is satisfied for all $n = k^2 l$, $k = 1, 2, ...$

2. For the sums of three non-vanishing squares one can use the well-known classical theorem stating that integer n can be represented as a sum of three squares (see `http://tinyurl.com/y6vkccv7`) [A11] only if it is it not of the form

$$n = 2^{2r}(8k + 7) \tag{4.4}$$

   For instance, squares of odd integers are of form $8k + 1$ and multiplied by any power of two satisfy the condition of being expressible as a sum of three squares.

   If n satisfies (does not satisfy) this condition then nm^2 satisfies (doe not satisfy) it for any m this since m^2 gives some power of 2 multiplied by a $8k + 1$ type factor so that one can say that square free odd integers for which the condition $n \neq 7 \pmod 8$ generate this set of integers. Note that the integers representable as sums of three non-vanishing squares do not allow a representation using two squares. The product of odd primes $p_1 = 8m_1 + k_1$ and $p_1 = 8m_2 + k_2$ fails to satisfy the condition only if one has $k_1 = 3$ and $k_2 = 5$. The product of n primes $p_i = 8m_i + k_i$ must satisfy the condition $\prod k_i \neq 7 \pmod 8$ in order to serve as a generating square free prime.

   In the recent case one must have $n \ mod \ p = 1$. For Mersenne primes $m = 1 + kM_n$ allows representation as a sum of three squares for most values of k. In particular, for $k = 1$ one obtains $m = 2^n$ allowing at least the representation $m = 2^{n-1} + 2^{n-1}$. One must also remember that all that is needed is that sufficiently small multiples of Mersenne primes correspond to large value of $r_3(n)$ if the proposed idea has any sense.

4.5 Number theoretic functions $r_k(n)$ for $k = 2, 4, 6$

The number theoretical functions $r_k(n)$ telling the number of vectors with length squared equal to a given integer n are well-known for $k = 2, 3, 4, 6$ and can be used to gain information about the constraints posed by the existence of quantum groups $SO(2)$, $SO(3)$, $SU(2)$ and $SU(3)$. In the following the easy cases corresponding to $k = 2, 4, 6$ are treated first and after than the more difficult case $k = 3$ is discussed. For the auxiliary function the reader can consult to the Appendix.

4.5.1 The behavior of $r_2(n)$

$r_2(n)$ gives information not only about quantum $SO(2)$ but also about $SO(3)$ since 2-D vectors define 3-D vectors in an obvious manner. The expression for $r_2(n)$ is given by

$$r_2(n) = \sum_{d|n} \chi(d) \ , \ \chi(d) = \left(\frac{-4}{d}\right) \ . \tag{4.5}$$

$\chi(d)$ is so called principal character defined in appendix. For $n = 1 + M_k = 2^k$ only powers of 2 and 1 divide n and for even numbers principal character vanishes so that one obtains $r_2(1 + M_k) = \chi(1) = 1$. This corresponds to the representation $2^k = 2^{k-1} + 2^{k-1}$.

4.5.2 The behavior of $r_4(n)$

The expression for $r_4(n)$ reads as

$$r_4(n) = \begin{cases} 8\sigma(n) & \text{if } n \text{ is odd} \ , \\ 24\sigma(m) & \text{if } n = 2^\nu m, \ m \text{ odd} \ . \end{cases} \ . \tag{4.6}$$

For $n = M_k + 1 = 2^k$ one has $r_4(n) = 24\sigma(1) = 24$.

The asymptotic behavior of σ function is known so that it is relatively easy to estimate the behavior of $r_4(n)$. The behavior involves random looking local fluctuation which can be understood as reflective the multiplicative character implying correlation between the values associated with multiples of a given prime.

4.5.3 The behavior of $r_6(n)$

The analytic expression for $r_6(n)$ is given by

$$\begin{aligned} r_6(n) &= \sum_{d|n} \left[16\chi(\frac{n}{d}) - 4\chi(d)\right] d^2 \ , \\ \chi(n) &= \left(\frac{-4}{n}\right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n = 1 \ (\text{mod } 4) \\ -1 & \text{if } n = 3 \ (\text{mod } 4) \end{cases} \end{aligned} \tag{4.7}$$

For $n = M_k + 1 = 2^k$ this gives $r_6(n) = 12 \times 2^{2k} - 4$ so that the number of representation is very large for large Mersenne primes.

4.6 What can one say about the behavior of r_3?

The proportionality of $r_3(D)$ to the order of $h(-D)$ (see `http://tinyurl.com/23sp45v`) [A1] of the ideal class group (see `http://tinyurl.com/cbxkhge`) [A10] for quadratic extensions of rationals [A1] inspires some conjectures.

1. The conjecture that preferred primes p correspond to large commutative quantum groups translates to a conjecture that the order of ideal class group is large for the algebraic extension generated by $\sqrt{-p-1}$ or more generally $\sqrt{-kp-1}$ - at least for some values of k. Could suitable integer multiples primes near power of 2 - in particular Mersenne primes - be such primes? Note that only integer multiple is required by the basic argument.

2. Also some kind of approximate fractal behavior $r_k(sp) \simeq r_k(p)f_k(s)$ for some values of s analogous to that encountered for $r_4(D)$ for all values of s might hold true since $k = 3$ is a critical transition dimension between $k = 2$ and $k = 3$. In particular, an approximate periodicity in octaves of primes might hold true: $r_k(2^s p) \simeq r_k(p)$: this would support p-adic length scale hypothesis and make the commutative quantum group large.

4.6.1 Expression of r_3 in terms of class number function

To proceed one must have an explicit expression for the class number function $h(D)$ and the expression of r_3 in terms of $h(D)$.

1. The expression for $h(D)$ discussed in the Appendix reads as gives

$$h(-D) \quad = \quad -\frac{1}{D}\sum_{1}^{D} r \times \left(\frac{-D}{r}\right) \quad . \tag{4.8}$$

The symbols$(\left(\frac{-D}{r}\right)$ are Dirichlet and Kronecker symbols defined in the Appendix. Note that for $D = M_k + 1 = 2^k$ the algebraic expansion in question reduces to that generated by $\sqrt{-2}$ so that the algebraic extension is definitely special.

2. One can express $r_3(|D|)$ in terms of $h(D)$ as

$$r_3(|D|) \quad = \quad 12(1 - (\frac{D}{2}))h(D) \quad . \tag{4.9}$$

Note that $(\frac{p}{2})$ refers to Kronecker symbol.

3. From Wolfram (see `http://tinyurl.com/ybl4hnp`) one finds the following expressions of $r_3(n)$ for square free integers.

$$\begin{array}{ll} r_3(n) = 24h(-n) & n = 3 \ (\mathrm{mod}\ 8) \ , \\ r_3(n) = 12h(-4n) & n = 1,2,5,6 \ (\mathrm{mod}\ 8) \ , \\ r_3(n) = 0 & n = 7 \ (\mathrm{mod}\ 8) \ . \end{array} \tag{4.10}$$

4. The generating function for r_3 (see `http://tinyurl.com/ybl4hnp`) [A17] is third power of theta function θ_3.

$$\sum_{n\geq 0} r_3(n)x^n \quad = \quad \theta_3^3(n) = 1 + 6x + 12x^2 + 8x^3 + 6x^4 + 24x^5 + 24x^6 + 12x^8 + 30x^9 + \tag{4.11}$$

This representation follows trivially from the definition of θ function as sum $\sum_{n=-\infty}^{\infty} x^{n^2}$.

The behavior of $h(-D)$ for large arguments is not easy to deduce without numerical calculations which probably get too heavy for primes of order M_{127}. The definition involves sum of p terms labeled by $r = 1, ..., p$, and each term is a product is product of terms expressible as a product over the prime factors of of r with over all term being a sign factor. "Interference" effects between terms of different sign are obviously possible in this kind of situation and one might hope that for large primes these effects imply wild fluctuations of $r_3(p)$.

4.6.2 Simplified formula for $r_3(D)$

Recall that the proportionality of $r_3(|D|)$ to the ideal class number $h(D)$ is for $D<-4$ given by

$$r_3(|D|) = 12[1-\left(\frac{D}{2}\right)]h(D) \ . \tag{4.12}$$

The expression for the Kronecker symbol appears in the formula as well as formulas to be discussed below and reads as

$$\left(\frac{D}{2}\right) = \begin{cases} 0 & \text{if } D \text{ is even} \ , \\ 1 & \text{if } D=-1 \ (\mathrm{mod}\ 8) \ , \\ -1 & \text{if } D=\pm 3 \ (\mathrm{mod}\ 8) \ . \end{cases} \tag{4.13}$$

The proportionality factor vanishes for $D=2^{2r}(8m+7)$ and equals to 12 for even values of D and to 24 for $D=\pm 3$ (mod 8).

To get more detailed information about r_3 one can begin from class number formula (see `http://tinyurl.com/yaopszpl`) [A2] for $D<-4$ reading as

$$h(D) = \frac{1}{|D|}\sum_{r=1}^{|D|} r\left(\frac{D}{r}\right) \ . \tag{4.14}$$

Each Jacobi symbol $\left(\frac{D}{r}\right)$ decomposes to a product of Legendre and Kronecker symbols $\left(\frac{D}{p_i}\right)$ in the decomposition of odd integer r to a product of primes p_i.

For $\left(\frac{D}{p_i}\right)=1$ p_i splits into a product of primes in quadratic extension generated by $\sqrt{D}$. If it vanishes p_i is square of prime in the quadratic extension. In the recent case neither of these options are possible for the primes involved as is easy to see by using the definition of algebraic integers. Hence one has $\left(\frac{D}{p_i}\right)=-1$ for all odd primes to transform the formula for $D<-4$ to the form

$$\begin{aligned} h(D) &= \frac{1}{|D|}\sum_{r=1}^{|D|} r[\left(\frac{D}{2}\right)]^{\nu_2(r)}(-1)^{\Omega(r)-\nu_2(r)} \\ &= \frac{1}{|D|}\sum_{r=1}^{|D|} r[-\left(\frac{D}{2}\right)]^{\nu_2(r)}(-1)^{\Omega(r))} \ . \\ & . \end{aligned} \tag{4.15}$$

Here $\nu_2(r)$ characterizes the power of 2 appearing in r and $\Omega(r)$ is the number of prime divisors of r with same divisor counted so many times as it appears. Hence the sign factor is same for all integers r which are obtained from the same square free integer by multiplying it by a product of even powers of primes.

Consider next various special cases.

1. For even values $D<-4$ (say $D=-1-M_n$) only odd integers r contribute to the sum since the Kronecker symbols vanish for even values of r.

$$\begin{aligned} h(D=2d) &= \frac{1}{|D|}\sum_{1\leq r<|D| \quad odd} r(-1)^{\Omega(r)} \\ & . \end{aligned} \tag{4.16}$$

2. For $D = \pm 1$ (mod 8), the factors $\left(\frac{D}{2}\right) = -1$ implies that one can forget the factors of 2 altogether in this case (note that for $D = -1$ (mod 8) $r_3(|D|)$ vanishes unlike $h(D)$).

$$h(D = \pm 1 (mod\ \ 8)) \quad = \quad \frac{1}{|D|}\sum_{r=1}^{|D|} r(-1)^{\Omega(r))} \quad . \tag{4.17}$$

3. For $D = \pm 3$ (mod 8), the factors $\left(\frac{D}{2}\right) = 1$ implies that one has

$$h(D = \pm 3 (mod\ \ 8)) \quad = \quad \frac{1}{|D|}\sum_{r=1}^{|D|} r(-1)^{\Omega(r)-\nu_2(r)} \quad . \tag{4.18}$$

The magnitudes of the terms in the sum increase linearly but the sign factor fluctuates wildly so that the value of $h(-D)$ varies chaotically but must be divisible by p and negative since $r_3(p)$ must be a positive integer.

4.6.3 Could thermodynamical analogy help?

For $D < -4$ $h(D)$ is expressible in terms of sign factors determined by the number of prime factors or odd prime factors modulo two for integers or odd integers $r < D$. This raises hopes that $h(D)$ could be calculated for even large values of D.

1. Consider first the case $D = \pm 1$ (mod 8)). The function $\lambda(r) = (-1)^{\Omega(r)}$ is known as Liouville function (see `http://tinyurl.com/y883uk5d`) [A12]. From the product expansion of zeta function in terms of "prime factors" it is easy to see that the generating function for $\lambda(r)$

$$\begin{aligned} \sum_n \lambda(n) n^{-s} &= \frac{\zeta(2s)}{\zeta(s)} = \frac{1}{\zeta_F(s)} \quad , \\ \zeta_(s) &= \prod_p (1-p^{-s})^{-1} \quad , \quad \zeta_F(s) = \prod_p (1+p^{-s}) \quad . \end{aligned} \tag{4.19}$$

Recall that $\zeta(s)$ *resp.* $\zeta_F(s)$ has a formal interpretation as partition functions for the thermodynamics of bosonic *resp.* fermionic system. This representation applies to $h(D = \pm 1 (mod 8))$.

2. For $D = 2d$ the representation is obtained just by dropping away the contribution of all even integers from Liouville function and this means division of $(1 + 2^{-s})$ from the fermionic partition function $\zeta_F(s)$. The generating function is therefore

$$\sum_{n\ \ odd} \lambda(n) n^{-s} \quad = \quad \prod_{p\ \ odd} (1+p^{-s})^{-1} = (1+2^{-s})\frac{1}{\zeta_F(s)} \quad . \tag{4.20}$$

3. For $h(D = \pm 3(\ \ mod\ \ 8))$. One most modify the Liouville function by replacing $\Omega(r)$ by the number of odd prime factors but allow also even integers r. The generating function is now

$$\sum_n \lambda(n)(-1)^{\nu_2(n)} n^{-s} \quad = \quad \frac{1}{1-2^{-s}} \prod_{p\ \ odd} (1+p^{-s})^{-1} = \frac{1}{1-2^{-s}}\frac{1}{\zeta_F(s)} \quad . \tag{4.21}$$

The generating functions raise the hope that it might be possible to estimate the values of the $h(D)$ numerically for large values of D using a thermodynamical analogy.

1. $h(D)$ is obtained as a kind of thermodynamical average $\langle r(-1)^{\Omega(r)}\rangle$ for particle number r weighted by a sign factor telling the number of divisors interpreted as particle number. s plays the role of the inverse of the temperature and infinite temperature limit $s = 0$ is considered. One can also interpret this number as difference of average particle number for states restricted to contain even *resp.* odd particle number identified as the number of prime divisors with 2 and even particle numbers possibly excluded.

2. The average is obtained at temperature corresponding to $s = 0$ so that $n^{-s} = 1$ holds true identically. The upper bound $r < D$ means cutoff in the partition sum and has interpretation as an upper bound on the energy $log(r)$ of many particle states defined by the prime decomposition. This means that one must replace Riemann zeta and its analogs with their cutoffs with $n \leq |D|$. Physically this is natural.

3. One must consider bosonic system all the cases considered. To get the required sign factor one must associated to the bosonic partition functions assigned with individual primes in $\zeta(s)$ the analog of chemical potential term $exp(-\mu/T)$ as the sign factor $exp(i\pi) = -1$ transforming ζ to $1/\zeta_F$ in the simplest case.

One might hope that one could calculate the partition function without explicitly constructing all the needed prime factorizations since only the number of prime factors modulo two is needed for $r \leq |D|$.

4.6.4 Expression of r_3 in terms of Dirichlet L-function

It is known [A13] that the function $r_3(D)$ is proportional to Dirichlet L-function (see `http://tinyurl.com/yatdk384`) $L(1, \chi(D))$ [A5]:

$$
\begin{aligned}
r_3(|D|) &= \frac{12\sqrt{D}}{\pi} L(1, \chi(D))) \ , \\
L(s, \chi) &= \sum_{n>0} \frac{\chi(n, D)}{n^s} \ ,
\end{aligned}
\tag{4.22}
$$

$\chi(n, D)$ is Dirichlet character (see `http://tinyurl.com/2fuudea`) [A4] which is periodic and multiplicative function - essentially a phase factor- satisfying the conditions

$$
\begin{array}{ll}
\chi(n, D) \neq 0 & \text{if n and D have no common divisors} > 1 \ , \\
\chi(n, D) = 0 & \text{if n and D have a common divisor} > 1 \ , \\
\chi(mn, D) = \chi(m, D)\chi(n, D) \ , & \chi(m + D, D) = \chi(m, D) \ , \\
\chi(1, D) = 1 \ . &
\end{array}
\tag{4.23}
$$

1. $L(1, \chi(D))$ varies in average sense slowly but fluctuates wildly between certain bounds (see `http://tinyurl.com/yc879v6e`). One can say that there is local chaos.

   The following estimates for the bounds are given in [A18]:

$$
c_1(D) \equiv k_1 log(log(D) < L_1(1, \chi(D)) < c_2(D) \equiv k_2 log(log(D)) \ . \tag{4.24}
$$

   Also other bounds are represented in the article.

4.6.5 Could preferred integers correspond to the maxima of Dirichlet L-function?

The maxima of Dirichlet L-function are excellent candidates for the local maxima of $r_3(D)$ since $\sqrt{D}$ is slowly varying function.

1. As already found, integers $n = 1 + M_k = 2^k$ cannot represent pronounced maxima of $r_3(n)$ since there are no representation as a sum of three squares and the proportionality constant vanishes. Note that in this case the representation reduces to a representation in terms of two integers. In this special case it does not matter whether L-function has a maximum or not.

   (a) Could just the fact that the representation for $n = 1 + M_k = 2^k$ in terms of three primes is not possible, select Mersenne primes $M_n > 3$ as preferred ones? For $SU(2)$, which is covering group of $SO(3)$ the representation as a sum of four squares is possible. Could it be that the spin 1/2 character of the fermionic building blocks of elementary particles means that a representation as sum of four squares is what matters. But why the non-existence of representation of n as a sum of three squares might make Mersenne primes so special?

2. Could also primes near power of 2 define maxima? Unfortunately, the calculations of [A18] involve averaging, minimum, and maximum over 10^6 integers in the ranges $n \times 10^6 < D < (n+1) \times 10^6$, so that they give very slowly varying maximum and minimum.

3. Could Dirichlet function have some kind of fractal structure such that for any prime one would have approximate factorization? The naivest guesses would be $L(1, \chi_{kl}) \simeq f_1(k)L(1, \chi_l)$ with $k = 2^s$. This would mean that the primes for which $D(1, \chi_p)$ is maximum would be of special importance.

4. p-Adic fractality and effective p-adic topology inspire the question whether L-function is p-adic fractal in the regions above certain primes defining effective p-adic topology $D(1, \chi_{p^k}) \simeq f_1(k)DK(1, \chi_p)$ for preferred primes.

4.6.6 Interference as a helpful physical analogy?

Could one use physical analog such as interference for the terms of varying sign appearing in L-function to gain some intuition about the situation?

1. One could interpret L-function as a number theoretic Fourier transform with D interpreted as a wave vector and one has an interference of infinite number of terms in position space whose points are labelled by positive integers defining a half -lattice with unit lattice length. The magnitude of n: th summand $1/n$ and its phase is periodic with period $D = kp$. The value of the Fourier component is finite except for $D = 0$ which corresponds to Riemann Zeta at $s = 1$. Could this means that the Fourier component behaves roughly like $1/D$ apart from an oscillating multiplicative factor.

2. The number theoretic counterparts of plane waves are special in that besides D-periodicity they are multiplicative making thema lso analogs of logarithmic waves. For ordinary Fourier components one additivity in the sense that $\Psi(k_1 + k_2) = \Psi(k_1)\Psi(k_2)$. Now one has $\Psi(k_1k_2) = \Psi(k_1)\Psi(k_2)$ so that $log(D)$ corresponds to ordinary wave vector. p-Adic fractality is an analog for periodicity in the sense of logarithmic waves so that powers rather than integer multiples of the basic scale define periodicity. Could the multiplicative nature of Dirichlet characters imply p-adic - or at least 2-adic - fractality, which also means logarithmic periodicity?

3. Could one say that for these special primes a constructive interference takes place in the sum defining the L-function. Certainly each prime represents the analog of fundamental wavelength whose multiples characterize the summands. In frequency space this would mean fundamental frequency and its sub-harmonics.

4.6.7 Period doubling as physical analogy?

1. For $k = 4$ all scales are present because of the multiplicative nature of σ function. Now only the Dirichlet characters are multiplicative which suggests that only few integers define preferred scales? Prime power multiples of the basic scale are certainly good candidates for preferred scales but amongst them must be some very special prime powers. $p = 2$ is the only even prime so that it is the first guess.

2. Could the system be chaotic or nearly chaotic in the sense of period doubling so that octaves of preferred primes interfere constructively? Why constructively? Could complete chaos -interpreted as randomness- correspond to a destructive interference and minimum of the L-function?

3. What about scalings by squares of a given prime? It seems that these scalings cannot be excluded by any simple argument. The point is that $r_3(n)$ contains also the factor $\sqrt{n}$ which must transform by integer in the scaling $n \to kn$. Therefore k must be power of square.

This leaves two extreme options. Both options are certainly testable by simple numerical calculations for small primes. For instance one can use generating function $\theta_3^3(x) = \sum r_3(n)x^n$ to kill the conjectures.

1. The first option corresponds to scalings by all integers that are squares. This option is also consistent with the condition $n \neq 2^k(8m+7)$ since both the scaling by a square of odd prime and by a square of 2 preserve this condition since one has $n^2 = 1 \pmod 8$ for odd integers. This is also consistent with the finding that $r_3(n) = 1$ holds true only for a finite number of integers. A simple numerical calculation for the sums of 3 squares of 16 first integers demonstrates that the conjecture is wrong.

2. The second option corresponds only to the scaling by even powers of two and is clearly the minimal option. This period quadrupling for n corresponds to period doubling for the components of 3-vector. A calculation of the sums of squares of the 16 first integers demonstrates that for $n = 3, 6, 9, 11, ..$ the conjecture the value of $r_3(n)$ is same so that the conjecture might hold true! If it holds true then Dirichlet L-function should suffer scaling by 2^{-r} in the scaling $n \to 2^{2r}n$. The integer solutions for n scaled by 2^r are certainly solutions for $2^{2r}n$. Quite generally, one has $r_3(m^2n) \geq r_3(n)$ for any integer m. The non-trivial question is whether some new solutions are possible when the scaling is by 2^{2r}.

   A simple argument demonstrates that there cannot be any other solutions to $\sum_{n_i=1}^{3} m_i^2 = 2^{2r}n$ than the scaled up solutions $m_i = 2n_i$ obtained from $\sum_{n_i=1}^{3} n_i^2 = n$. This is seen by noticing that non-scaled up solutions must contain $1, 2, or 3$ integers m_i, which are odd. For this kind of integers one has $m^2 = 1 \pmod 4$ so that the sum $(\sum_i m_i^2)$= 1, 2, or 3 (mod 4) whereas the right hand side vanishes mod 4.

3. If D is interpreted as wave vector, period quadrupling could be interpreted as a presence of logarithmic wave in wave-vector space with period $2log(2)$.

4.6.8 Does 2-adic quantum arithmetics prefer CD scales coming as powers of two?

For $p = 2$ quantum arithmetics looks singular at the first glance. This is actually not the case since odd quantum integers are equal to their ordinary counterparts in this case. This applies also to powers of two interpreted as 2-adic integers. The real counterparts of these are mapped to their inverses in canonical identification.

Clearly, odd 2-adic quantum quantum rationals are very special mathematically since they correspond to ordinary rationals. It is fair to call them "classical" rationals. This special role might relate to the fact that primes near powers of 2 are physically preferred. CDs with $n = 2^k$ would be in a unique position number theoretically. This would conform with the original - and as such wrong - hypothesis that only these time scales are possible for CDs. The preferred role of powers of two supports also p-adic length scale hypothesis.

The discussion of the role of quantum arithmetics in the construction of generalized Feynman diagrams in [K18] allows to understand how for a quantum arithmetics based on particular prime p

particle mass squared - equal to conformal weight in suitable mass units- divisible by p appears as an effective propagator pole for large values of p. In p-adic mass calculations real mass squared is obtained by canonical identification from the p-adic one. The construction of generalized Feynman diagrams allows to understand this strange sounding rule as a direct implication of the number theoretical universality realized in terms of quantum arithmetics.

5 How Quantum Arithmetics could affect basic TGD and TGD inspired view about life and consciousness?

The vision about real and p-adic physics as completions of rational physics or physics associated with extensions of rational numbers is central element of number theoretical universality. The physics in the extensions of rationals are assigned with the interaction of real and p-adic worlds.

1. At the level of the world of classical worlds (WCW) the points in the intersection of real and p-adic worlds are 2-surfaces defined by equations making sense both in real and p-adic sense. Rational functions with polynomials having rational (or algebraic coefficients in some extension of rationals) would define the partonic 2-surface. One can of course consider more stringent formulations obtained by replacing 2-surface with certain 3-surfaces or even by 4-surfaces.

2. At the space-time level the intersection of real and p-adic worlds corresponds to rational points common to real partonic 2-surface obeying same equations (the simplest assumption). This conforms with the vision that finite measurement resolution implies discretization at the level of partonic 2-surfaces and replaces light-like 3-surfaces and space-like 3-surfaces at the ends of causal diamonds with braids so that almost topological QFT is the outcome.

How does the replacement of rationals with quantum rationals modify quantum TGD and the TGD inspired vision about quantum biology and consciousness?

5.1 What happens to p-adic mass calculations and Quantum TGD?

The basic assumption behind the p-adic mass calculations and all applications is that one can assign to a given partonic 2-surface (or even light-like 3-surface) a preferred p-adic prime (or possibly several primes).

The replacement of rationals with quantum rationals in p-adic mass calculations implies effects, which are extremely small since the difference between rationals and quantum rationals is extremely small due to the fact that the primes assignable to elementary particles are so large ($M_{127} = 2^{127}-1$ for electron). The predictions of p-adic mass calculations remains almost as such in excellent accuracy. The bonus is the uniqueness of the canonical identification making the theory unique.

The problem of the original p-adic mass calculations is that the number of common rationals (plus possible algebraics in some extension of rationals) is same for all primes p. What is the additional criterion selecting the preferred prime assigned to the elementary particle?

Could the preferred prime correspond to the maximization of number theoretic negentropy for a quantum state involved and therefore for the partonic 2-surface by quantum classical correspondence? The solution ansatz for the Kähler-Dirac equation indeed allows this assignment [K12]: could this provide the first principle selecting the preferred p-adic prime? Here the replacement of rationals with quantum rationals improves the situation dramatically.

1. Quantum rationals are characterized by a quantum phase $q = exp(i\pi/p)$ and thus by prime p (in the most general but not so plausible case by an integer n). The set of points shared by real and p-adic partonic 2-surfaces would be discrete also now but consist of points in the algebraic extension defined by the quantum phase $q = exp(i\pi/p)$.

2. What is of crucial importance is that the number of common quantum rational points of partonic 2-surface and its p-adic counterpart would depend on the p-adic prime p. For some primes p would be large and in accordance with the original intuition this suggests that the interaction between p-adic and real partonic 2-surface is stronger. This kind of prime is

the natural candidate for the p-adic prime defining effective p-adic topology assignable to the partonic 2-surface and elementary particle. Quantum rationals would thus bring in the preferred prime and perhaps at the deepest possible level that one can imagine.

5.2 What happens to TGD inspired theory of consciousness and quantum biology?

The vision about rationals as common to reals and p-adics is central for TGD inspired theory of consciousness and the applications of TGD in biology.

1. One can say that life resides in the intersection of real and p-adic worlds. The basic motivation comes from the observation that number theoretical entanglement entropy can have negative values and has minimum for a unique prime [K6]. Negative entanglement entropy has a natural interpretation as a genuine information and this leads to a modification of Negentropy Maximization Principle (NMP) allowing quantum jumps generating negentropic entanglement. This tendency is something completely new: NMP for ordinary entanglement entropy would force always a state function reduction leading to unentangled states and the increase of ensemble entropy.

   What happens at the level of ensemble in TGD Universe is an interesting question. The pessimistic view (see `http://tinyurl.com/ybm6rxz3`) [K6], [L2] is that the generation of negentropic entanglement (see **Fig.** `http://tgdtheory.fi/appfigures/cat.jpg` or **Fig. ??** in the appendix of this book) is accompanied by entropic entanglement somewhere else guaranteeing that second law still holds true. Living matter would be bound to pollute its environment if the pessimistic view is correct. I cannot decide whether this is so: this seems like deciding whether Riemann hypothesis is true or not or perhaps unprovable.

2. Replacing rationals with quantum rationals however modifies somewhat the overall vision about what life is. It would be quantum rationals which would be common to real and p-adic variants of the partonic 2-surface. Also now an algebraic extension of rationals would be in question so that the proposal would be only more specific. The notion of number theoretic entropy still makes sense so that the basic vision about quantum biology survives the modification.

3. The large number of common points for some prime would mean that the quantum jump transforming p-adic partonic 2-surface to its real counterpart would take place with a large probability. Using the language of TGD inspired theory of consciousness one would say that the intentional powers are strong for the conscious entity involved. This applies also to the reverse transition generating a cognitive representation if p-adic-real duality induced by the canonical identification is true. This conclusion seems to apply even in the case of elementary particles. Could even elementary particles cognize and intend in some primitive sense? Intriguingly, the secondary p-adic time scale associated with electron defining the size of corresponding CD is.1 seconds defining the fundamental 10 Hz bio-rhythm. Just an accident or something very deep: a direct connection between elementary particle level and biology perhaps?

6 Appendix: Some Number Theoretical Functions

Explicit formulas for the number $r_k(n)$ of the solutions to the conditions $\sum_1^k x_k^2 = n$ are known and define standard number theoretical functions closely related to the quadratic algebraic extensions of rationals. The formulas for $r_k(n)$ require some knowledge about the basic number theoretical functions to be discussed first. Wikipedia contains a good overall summary about basic arithmetic functions (see `http://tinyurl.com/23sp45v`) [A1] including the most important multiplicative and additive arithmetic functions.

Included are character functions which are periodic and multiplicative: examples are symbols (m/n) assigned with the names of Legendre, Jacobi, and Kronecker as well as Dirichlet character.

6.1 Characters And Symbols

6.1.1 Principal character

Principal character (see http://tinyurl.com/23sp45v) [A1] $\chi(n)$ distinguishes between three situations: n is even, $n = 1 \pmod 4$, and $n = 3 \pmod 4$ and is defined as

$$\chi(n) = \left(\frac{-4}{n}\right) = \begin{cases} 0 & \text{if n=0 (mod 2)} \\ +1 & \text{if } n = 1 \pmod 4 \\ -1 & \text{if } n = 3 \pmod 4 \end{cases} \tag{6.1}$$

Principal character is multiplicative and periodic with period $k = 4$.

6.1.2 Legendre and Kronecker symbols

Legendre symbol $\left(\frac{n}{p}\right)$ characterizes what happens to ordinary primes in the quadratic extensions of rationals. Legendre symbol is defined for odd integers n and odd primes p as

$$\left(\frac{n}{p}\right) = \begin{cases} 0 & \text{if } n = 0 \text{ (mod p)} \ , \\ +1 & \text{if } n \neq 0 \text{ (mod p) and } n = x^2 \text{ (mod p)} \ , \\ -1 & \text{if there is no such } x \ . \end{cases} \tag{6.2}$$

When D is so called fundamental discriminant- that is discriminant $D = b^2 - 4c$ for the equation $x^2 - bx + c = 0$ with integer coefficients b, c, Legendre symbols tells what happens to ordinary primes in the extension:

1. $\left(\frac{D}{p}\right) = 0$ tells that the prime in question divides D and that p is expressible as a square in the quadratic extension of rationals defined by $\sqrt{D}$.
2. $\left(\frac{D}{p}\right) = 1$ tells that p splits into a product of two different primes in the quadratic extension.
3. For $\left(\frac{D}{p}\right) = -1$ the splitting of p does not occur.

This explains why Legendre symbols appear in the ideal class number $h(D)$ characterizing the number of different splittings of primes in quadratic extension.

Legendre symbol can be generalized to Kronecker symbol well-defined for also for even integers D. The multiplicative nature requires only the definition of $\left(\frac{n}{2}\right)$ for arbitrary n:

$$\left(\frac{n}{2}\right) = \begin{cases} 0 & \text{if } n \text{ is even} \ , \\ (-1)^{\frac{n^2-1}{8}} & \text{if n is odd} \ . \end{cases} \tag{6.3}$$

Kronecker symbol for $p = 2$ tells whether the integer is even, and if odd whether $n = \pm 1$ (mod 8) or $a = \pm 3$ (mod 8) holds true. Note that principal character $\chi(n)$ can be regarded as Dirichlet character $\left(\frac{-4}{n}\right)$.

For $D = p$ quadratic resiprocity (see http://tinyurl.com/yz2okpf) [A14] allows to transform the formula

$$\chi_p(n) = (-1)^{(p-1)/2}(-1)^{(n-1)/2}\left(\frac{p}{n}\right) = (-1)^{(p-1)/2}(-1)^{(n-1)/2}\prod_{p_i|n}\left(\frac{p}{p_i}\right) \ . \tag{6.4}$$

6.1.3 Dirichlet character

Dirichlet character (see http://tinyurl.com/2fuudea) [A4] $\left(\frac{a}{n}\right)$ is also a multiplicative function. Dirichlet character is defined for all values of a and odd values of n and is fixed completely by the conditions

$$\begin{aligned} &\chi_D(k) = \chi_D(k+D) \ , \qquad \chi_D(kl) = \chi_D(k)\chi_D(l) \ , \\ &\text{If } D|n \text{ then } \chi_D(n) = 0 \ , \quad \text{otherwise } \chi_D(n) \neq 0 \ . \end{aligned} \tag{6.5}$$

Dirichlet character associated with quadratic residues is real and can be expressed as

$$\chi_D(n) \ = \ \left(\frac{n}{D}\right) = \prod_{p_i|D} \left(\frac{n}{p_i}\right) \ . \tag{6.6}$$

Here $\left(\frac{n}{p_i}\right)$ is Legendre symbol described above. Note that the primes p_i are odd. $\left(\frac{n}{1}\right) = 1$ holds true by definition.

For prime values of D Dirichet character reduces to Legendre symbol. For odd integers Dirichlet character reduces to Jacobi symbol defined as a product of the Legendre symbols associated with the prime factors. For $n = p^k$ Dirichlet character reduces to $(\left(\frac{p}{n}\right))^k$ and is non-vanishing only for odd integers not divisible by p and containing only odd prime factors larger than p besides power of 2 factor.

6.2 Divisor Functions

Divisor functions (see http://tinyurl.com/2qyngq) [A6] $\sigma_k(n)$ are defined in terms of the divisors d of integer n with $d = 1$ and $d = n$ included and are also multiplicative functions. $\sigma_k(n)$ is defined as

$$\sigma_k(n) \ = \ \sum_{d|n} d^k \ , \tag{6.7}$$

and can be expressed in terms of prime factors of n as

$$\sigma_k(n) \ = \ \sum_i (p_i^k + p_i^{2k} + ... + p_i^{a_i k}) \ . \tag{6.8}$$

$\sigma_1 \equiv \sigma$ appears in the formula for $r_4(n)$.

The figures in Wikipedia (see http://tinyurl.com/y8vrrhx9) [A9] give an idea about the locally chaotic behavior of the sigma function.

6.3 Class Number Function And Dirichlet L-Function

In the most interesting $k = 3$ case the situation is more complicated and more refined number theoretic notions are needed. The function $r_3(D)$ is expressible in terms of so called class number function $h(n)$ characterizing the order of the ideal class group for a quadratic extension of rationals associated with D, which can be negative. In the recent case $D = -p$ is of special interest as also $D = -kp$, especially so for $k = 2^r$. $h(n)$ in turn is expressible in terms of Dirichlet L-function so that both functions are needed.

1. Dirichlet L-function (see http://tinyurl.com/yatdk384) [A5] can be regarded as a generalization of Riemann zeta and is also conjectured to satisfy Riemann hypothesis. Dirichlet L-function can be assigned to any Dirichlet character χ_D appearing in it as a function valued parameter and is defined as

$$L(s,\chi_D) = \sum_n \frac{\chi_D(n)}{n^s} \quad . \tag{6.9}$$

For $\chi_1 = 1$ one obtains Riemann Zeta. Also L-function has expression as product of terms associated with primes converging for $Re(s) > 1$, and must be analytically continued to get an analytic function in the entire complex plane. The value of L-function at $s = 1$ is needed and for Riemann zeta this corresponds to pole. For Dirichlet zeta the value is finite and $L(1,\chi_{-n})$ indeed appears in the formula for $r_3(n)$.

2. Consider next what class number function h means.

   (a) Class number function (see `http://tinyurl.com/yaopszpl`) [A2] characterizes quadratic extensions defined by $\sqrt{D}$ for both positive and negative values of D. For these algebraic extensions the prime factorization in the ring of algebraic integers need not be unique. Algebraic integers are complex algebraic numbers which are not solutions of a polynomial with coefficients in Z and with leading term with unit coefficient. What is important is that they are closed under addition and multiplication. One can also defined algebraic primes. For instance, for the quadratic extension generated by $\sqrt{\pm 5}$ algebraic integers are of form $m + n\sqrt{\pm 5}$ since $\sqrt{\pm 5}$ satisfies the polynomial equation $x^2 = \pm 5$.

   Given algebraic integer n can have several prime decompositions: $n = p_1p_2 = p_3p_4$, where p_i algebraic primes. In a more advance treatment primes correspond to ideals of the algebra involved: obviously algebra of algebraic integers multiplied by a prime is closed with respect to multiplication with any algebraic integer.

   A good example about non-unique prime decomposition is $6 = 2\times 3 = (1+\sqrt{-5})(\sqrt{1-\sqrt{-5}}$ in the quadratic extension generated by $\sqrt{-5}$.

   (b) Non-uniqueness means that one has what might be called fractional ideals: two ideals I and J are equivalent if one can write $(a)J = (b)I$ where (n) is the integer ideal consisting of algebraic integers divisible by algebraic integer n. This is the counterpart for the non-uniqueness of prime decomposition. These ideals form an Abelian group known as ideal class group (see `http://tinyurl.com/cbxkhge`) [A10]. For algebraic fields the ideal class group is always finite.

   (c) The order of elements of the ideal class group for the quadratic extension determined by integer D can be written as

$$h(D) = \frac{1}{D}\sum_1^{|D|} r \times \left(\frac{D}{r}\right) \quad , \quad D < -4 \quad . \tag{6.10}$$

   Here $\left(\frac{D}{r}\right)$ denotes the value of Dirichlet character. In the recent case D is negative.

3. It is perhaps not completely surprising that one can express $r_3(|D|)$ characterizing quadratic form in terms of $h(D)$ charactering quadratic algebraic extensions as

$$r_3(|D|) = 12(1 - \left(\frac{D}{2}\right))h(D) \quad , \quad D < -4 \quad . \tag{6.11}$$

Here $\left(\frac{D}{2}\right)$ denotes Kronecker symbol.

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