

# About the Identification of the Preferred extremals of Kähler Action

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### Abstract

Preferred extremal of Kähler action have remained one of the basic poorly defined notions of TGD. There are pressing motivations for understanding what the attribute “preferred” really means. Symmetries give a clue to the problem. The conformal invariance of string models naturally generalizes to 4-D invariance defined by quantum Yangian of quantum affine algebra (Kac-Moody type algebra) characterized by two complex coordinates and therefore explaining naturally the effective 2-dimensionality [?]. Preferred extremal property should rely on this symmetry.

In Zero Energy Ontology (ZEO) preferred extremals are space-time surfaces connecting two space-like 3-surfaces at the ends of space-time surfaces at boundaries of causal diamond (CD). A natural looking condition is that the symplectic Noether charges associated with a sub-algebra of symplectic algebra with conformal weights  $n$ -multiples of the weights of the entire algebra vanish for preferred extremals. These conditions would be classical counterparts the the condition that super-symplectic sub-algebra annihilates the physical states. This would give a hierarchy of super-symplectic symmetry breakings and quantum criticalities having interpretation in terms of hierarchy of Planck constants  $h_{eff} = n \times h$  identified as a hierarchy of dark matter.  $n$  could be interpreted as the number of space-time conformal gauge equivalence classes for space-time sheets connecting the 3-surfaces at the ends of space-time surface.

There are also many other proposals for what preferred extremal property could mean or imply. The weak form of electric-magnetic duality combined with the assumption that the contraction of the Kähler current with Kähler gauge potential vanishes for preferred extremals implies that Kähler action in Minkowskian space-time regions reduces to Chern-Simons terms at the light-like orbits of wormhole throats at which the signature of the induced metric changes its signature from Minkowskian to Euclidian. In regions with 4-D  $CP_2$  projection (wormhole contacts) also a 3-D contribution not assignable to the boundary of the region might be possible. These conditions pose strong physically feasible conditions on extremals and might be true for preferred extremals too.

Number theoretic vision leads to a proposal that either the tangent space or normal space of given point of space-time surface is associative and thus quaternionic. Also the formulation in terms of quaternion holomorphy and quaternion-Kähler property is an attractive possibility. So called  $M^8 - H$  duality is a variant of this vision and would mean that one can map associative/co-associative space-time surfaces from  $M^8$  to  $H$  and also iterate this mapping from  $H$  to  $H$  to generate entire category of preferred extremals. The signature of  $M^4$  is a general technical problem. For instance, the holomorphy in 2 complex variables could correspond to what I have called Hamilton-Jacobi property. Associativity/co-associativity of the tangent space makes sense also in Minkowskian signature.

In this chapter various views about preferred extremal property are discussed.

## 1 Introduction

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## 1.1 Preferred Extremals As Critical Extremals

The study of the Kähler-Dirac equation leads to a detailed view about criticality. Quantum criticality [D2] fixes the values of Kähler coupling strength as the analog of critical temperature. Quantum criticality implies that second variation of Kähler action vanishes for critical deformations and the existence of conserved current except in the case of Cartan algebra of isometries. Quantum criticality allows to fix the values of couplings appearing in the measurement interaction by using the condition  $K \rightarrow K + f + \bar{f}$ . p-Adic coupling constant evolution can be understood also and corresponds to scale hierarchy for the sizes of causal diamonds (CDs).

The discovery that the hierarchy of Planck constants realized in terms of singular covering spaces of  $CD \times CP_2$  can be understood in terms of the extremely non-linear dynamics of Kähler action implying 1-to-many correspondence between canonical momentum densities and time derivatives of the imbedding space coordinates led to a further very concrete understanding of the criticality at space-time level and its relationship to zero energy ontology [K10].

Criticality is accompanied by conformal invariance and this leads to the proposal that critical deformations correspond to Kac-Moody type conformal algebra respecting the light-likeness of the partonic orbits and acting trivially at partonic 2-surfaces. Sub-algebras of conformal algebras with conformal weights divisible by integer  $n$  would act as gauge symmetries and these algebras would form an inclusion hierarchy defining hierarchy of symmetry breakings.  $n$  would also characterize the value of Planck constant  $h_{eff} = n \times h$  assignable to various phases of dark matter.

## 1.2 Construction Of Preferred Extremals

There has been considerable progress in the understanding of both preferred extremals and Kähler-Dirac equation.

1. For preferred extremals the generalization of conformal invariance to 4-D situation is very attractive idea and leads to concrete conditions formally similar to those encountered in string model [K3]. In particular, Einstein's equations with cosmological constant would solve consistency conditions and field equations would reduce to a purely algebraic statements analogous to those appearing in equations for minimal surfaces if one assumes that space-time surface has Hermitian structure or its Minkowskian variant Hamilton-Jacobi structure (Appendix). The older approach based on basic heuristics for massless equations, on effective 3-dimensionality, weak form of electric magnetic duality, and Beltrami flows is also promising. An alternative approach is inspired by number theoretical considerations and identifies space-time surfaces as associative or co-associative sub-manifolds of octonionic imbedding space [K17].

The basic step of progress was the realization that the known extremals of Kähler action - certainly limiting cases of more general extremals - can be deformed to more general extremals having interpretation as preferred extremals.

- (a) The generalization boils down to the condition that field equations reduce to the condition that the traces  $Tr(TH^k)$  for the product of energy momentum tensor and second fundamental form vanish. In string models energy momentum tensor corresponds to metric and one obtains minimal surface equations. The equations reduce to purely algebraic conditions stating that  $T$  and  $H^k$  have no common components. Complex structure of string world sheet makes this possible.

Stringy conditions for metric stating  $g_{zz} = g_{\bar{z}\bar{z}} = 0$  generalize. The condition that field equations reduce to  $Tr(TH^k) = 0$  requires that the terms involving Kähler gauge current in field equations vanish. This is achieved if Einstein's equations hold true (one can consider also more general manners to satisfy the conditions). The conditions guaranteeing the vanishing of the trace in turn boil down to the existence of Hermitian structure in the case of Euclidian signature and to the existence of its analog - Hamilton-Jacobi structure - for Minkowskian signature (Appendix). These conditions state that certain components of the induced metric vanish in complex coordinates or Hamilton-Jacobi coordinates.

In string model the replacement of the imbedding space coordinate variables with quantized ones allows to interpret the conditions on metric as Virasoro conditions. In the recent case a generalization of classical Virasoro conditions to four-dimensional ones would be in question. An interesting question is whether quantization of these conditions could make sense also in TGD framework at least as a useful trick to deduce information about quantum states in WCW degrees of freedom.

The interpretation of the extended algebra as Yangian [A10] [B9] suggested previously [K28] to act as a generalization of conformal algebra in TGD Universe is attractive. There is also the conjecture that preferred extremals could be interpreted as quaternionic or co-quaternionic 4-surface of the octonionic imbedding space with octonionic representation of the gamma matrices defining the notion of tangent space quaternionicity.

## 2 Weak Form Electric-Magnetic Duality And Its Implications

The notion of electric-magnetic duality [B4] was proposed first by Olive and Montonen and is central in  $\mathcal{N} = 4$  supersymmetric gauge theories. It states that magnetic monopoles and ordinary particles are two different phases of theory and that the description in terms of monopoles can be applied at the limit when the running gauge coupling constant becomes very large and perturbation theory fails to converge. The notion of electric-magnetic self-duality is more natural since for  $CP_2$  geometry Kähler form is self-dual and Kähler magnetic monopoles are also Kähler electric monopoles and Kähler coupling strength is by quantum criticality renormalization group invariant rather than running coupling constant. The notion of electric-magnetic (self-)duality emerged already two decades ago in the attempts to formulate the Kähler geometric of world of classical worlds. Quite recently a considerable step of progress took place in the understanding of this notion [K7]. What seems to be essential is that one adopts a weaker form of the self-duality applying at partonic 2-surfaces. What this means will be discussed in the sequel.

Every new idea must be of course taken with a grain of salt but the good sign is that this concept leads to precise predictions. The point is that elementary particles do not generate monopole fields in macroscopic length scales: at least when one considers visible matter. The first question is whether elementary particles could have vanishing magnetic charges: this turns out to be impossible. The next question is how the screening of the magnetic charges could take place and leads to an identification of the physical particles as string like objects identified as pairs magnetic charged wormhole throats connected by magnetic flux tubes.

- (a) The first implication is a new view about electro-weak massivation reducing it to weak confinement in TGD framework. The second end of the string contains particle having electroweak isospin neutralizing that of elementary fermion and the size scale of the

string is electro-weak scale would be in question. Hence the screening of electro-weak force takes place via weak confinement realized in terms of magnetic confinement.

- (b) This picture generalizes to the case of color confinement. Also quarks correspond to pairs of magnetic monopoles but the charges need not vanish now. Rather, valence quarks would be connected by flux tubes of length of order hadron size such that magnetic charges sum up to zero. For instance, for baryonic valence quarks these charges could be  $(2, -1, -1)$  and could be proportional to color hyper charge.
- (c) The highly non-trivial prediction making more precise the earlier stringy vision is that elementary particles are string like objects: this could become manifest at LHC energies.
- (d) The weak form electric-magnetic duality together with Beltrami flow property of Kähler leads to the reduction of Kähler action to Chern-Simons action so that TGD reduces to almost topological QFT and that Kähler function is explicitly calculable. This has enormous impact concerning practical calculability of the theory.
- (e) One ends up also to a general solution ansatz for field equations from the condition that the theory reduces to almost topological QFT. The solution ansatz is inspired by the idea that all isometry currents are proportional to Kähler current which is integrable in the sense that the flow parameter associated with its flow lines defines a global coordinate. The proposed solution ansatz would describe a hydrodynamical flow with the property that isometry charges are conserved along the flow lines (Beltrami flow). A general ansatz satisfying the integrability conditions is found.

The strongest form of the solution ansatz states that various classical and quantum currents flow along flow lines of the Beltrami flow defined by Kähler current. Intuitively this picture is attractive. A more general ansatz would allow several Beltrami flows meaning multi-hydrodynamics. The integrability conditions boil down to two scalar functions: the first one satisfies massless d'Alembert equation in the induced metric and the gradients of the scalar functions are orthogonal. The interpretation in terms of momentum and polarization directions is natural.

## 2.1 Could A Weak Form Of Electric-Magnetic Duality Hold True?

Holography means that the initial data at the partonic 2-surfaces should fix the WCW metric. A weak form of this condition allows only the partonic 2-surfaces defined by the wormhole throats at which the signature of the induced metric changes. A stronger condition allows all partonic 2-surfaces in the slicing of space-time sheet to partonic 2-surfaces and string world sheets. Number theoretical vision suggests that hyper-quaternionicity *resp.* co-hyperquaternionicity constraint could be enough to fix the initial values of time derivatives of the imbedding space coordinates in the space-time regions with Minkowskian *resp.* Euclidian signature of the induced metric. This is a condition on modified gamma matrices and hyper-quaternionicity states that they span a hyper-quaternionic sub-space.

### 2.1.1 Definition of the weak form of electric-magnetic duality

One can also consider alternative conditions possibly equivalent with this condition. The argument goes as follows.

- (a) The expression of the matrix elements of the metric and Kähler form of WCW in terms of the Kähler fluxes weighted by Hamiltonians of  $\delta M_{\pm}^4$  at the partonic 2-surface  $X^2$  looks very attractive. These expressions however carry no information about the 4-D tangent space of the partonic 2-surfaces so that the theory would reduce to a genuinely 2-dimensional theory, which cannot hold true. One would like to code to the WCW metric also information about the electric part of the induced Kähler form assignable to the complement of the tangent space of  $X^2 \subset X^4$ .
- (b) Electric-magnetic duality of the theory looks a highly attractive symmetry. The trivial manner to get electric magnetic duality at the level of the full theory would be via the

identification of the flux Hamiltonians as sums of of the magnetic and electric fluxes. The presence of the induced metric is however troublesome since the presence of the induced metric means that the simple transformation properties of flux Hamiltonians under symplectic transformations -in particular color rotations- are lost.

- (c) A less trivial formulation of electric-magnetic duality would be as an initial condition which eliminates the induced metric from the electric flux. In the Euclidian version of 4-D YM theory this duality allows to solve field equations exactly in terms of instantons. This approach involves also quaternions. These arguments suggest that the duality in some form might work. The full electric magnetic duality is certainly too strong and implies that space-time surface at the partonic 2-surface corresponds to piece of  $CP_2$  type vacuum extremal and can hold only in the deep interior of the region with Euclidian signature. In the region surrounding wormhole throat at both sides the condition must be replaced with a weaker condition.
- (d) To formulate a weaker form of the condition let us introduce coordinates  $(x^0, x^3, x^1, x^2)$  such  $(x^1, x^2)$  define coordinates for the partonic 2-surface and  $(x^0, x^3)$  define coordinates labeling partonic 2-surfaces in the slicing of the space-time surface by partonic 2-surfaces and string world sheets making sense in the regions of space-time sheet with Minkowskian signature. The assumption about the slicing allows to preserve general coordinate invariance. The weakest condition is that the generalized Kähler electric fluxes are apart from constant proportional to Kähler magnetic fluxes. This requires the condition

$$J^{03}\sqrt{g_4} = KJ_{12} . \quad (2.1)$$

A more general form of this duality is suggested by the considerations of [K10] reducing the hierarchy of Planck constants to basic quantum TGD and also reducing Kähler function for preferred extremals to Chern-Simons terms [B1] at the boundaries of CD and at light-like wormhole throats. This form is following

$$J^{n\beta}\sqrt{g_4} = K\epsilon \times \epsilon^{n\beta\gamma\delta} J_{\gamma\delta}\sqrt{g_4} . \quad (2.2)$$

Here the index  $n$  refers to a normal coordinate for the space-like 3-surface at either boundary of CD or for light-like wormhole throat.  $\epsilon$  is a sign factor which is opposite for the two ends of CD. It could be also opposite of opposite at the opposite sides of the wormhole throat. Note that the dependence on induced metric disappears at the right hand side and this condition eliminates the potentials singularity due to the reduction of the rank of the induced metric at wormhole throat.

- (e) Information about the tangent space of the space-time surface can be coded to the WCW metric with loosing the nice transformation properties of the magnetic flux Hamiltonians if Kähler electric fluxes or sum of magnetic flux and electric flux satisfying this condition are used and  $K$  is symplectic invariant. Using the sum

$$J_e + J_m = (1 + K)J_{12} , \quad (2.3)$$

where  $J$  denotes the Kähler magnetic flux, , makes it possible to have a non-trivial WCW metric even for  $K = 0$ , which could correspond to the ends of a cosmic string like solution carrying only Kähler magnetic fields. This condition suggests that it can depend only on Kähler magnetic flux and other symplectic invariants. Whether local symplectic coordinate invariants are possible at all is far from obvious, If the slicing itself is symplectic invariant then  $K$  could be a non-constant function of  $X^2$  depending on string world sheet coordinates. The light-like radial coordinate of the light-cone boundary indeed defines a symplectically invariant slicing and this slicing could be shifted along the time axis defined by the tips of CD.

### 2.1.2 Electric-magnetic duality physically

What could the weak duality condition mean physically? For instance, what constraints are obtained if one assumes that the quantization of electro-weak charges reduces to this condition at classical level?

- (a) The first thing to notice is that the flux of  $J$  over the partonic 2-surface is analogous to magnetic flux

$$Q_m = \frac{e}{\hbar} \oint B dS = n .$$

$n$  is non-vanishing only if the surface is homologically non-trivial and gives the homology charge of the partonic 2-surface.

- (b) The expressions of classical electromagnetic and  $Z^0$  fields in terms of Kähler form [L1], [L1] read as

$$\begin{aligned} \gamma &= \frac{e F_{em}}{\hbar} = 3J - \sin^2(\theta_W) R_{03} , \\ Z^0 &= \frac{g_Z F_Z}{\hbar} = 2R_{03} . \end{aligned} \quad (2.4)$$

Here  $R_{03}$  is one of the components of the curvature tensor in vielbein representation and  $F_{em}$  and  $F_Z$  correspond to the standard field tensors. From this expression one can deduce

$$J = \frac{e}{3\hbar} F_{em} + \sin^2(\theta_W) \frac{g_Z}{6\hbar} F_Z . \quad (2.5)$$

- (c) The weak duality condition when integrated over  $X^2$  implies

$$\begin{aligned} \frac{e^2}{3\hbar} Q_{em} + \frac{g_Z^2 p}{6} Q_{Z,V} &= K \oint J = Kn , \\ Q_{Z,V} &= \frac{I_V^3}{2} - Q_{em} , \quad p = \sin^2(\theta_W) . \end{aligned} \quad (2.6)$$

Here the vectorial part of the  $Z^0$  charge rather than as full  $Z^0$  charge  $Q_Z = I_L^3 + \sin^2(\theta_W) Q_{em}$  appears. The reason is that only the vectorial isospin is same for left and right handed components of fermion which are in general mixed for the massive states. The coefficients are dimensionless and expressible in terms of the gauge coupling strengths and using  $\hbar = r\hbar_0$  one can write

$$\begin{aligned} \alpha_{em} Q_{em} + p \frac{\alpha_Z}{2} Q_{Z,V} &= \frac{3}{4\pi} \times rnK , \\ \alpha_{em} &= \frac{e^2}{4\pi\hbar_0} , \quad \alpha_Z = \frac{g_Z^2}{4\pi\hbar_0} = \frac{\alpha_{em}}{p(1-p)} . \end{aligned} \quad (2.7)$$

- (d) There is a great temptation to assume that the values of  $Q_{em}$  and  $Q_Z$  correspond to their quantized values and therefore depend on the quantum state assigned to the partonic 2-surface. The linear coupling of the Kähler-Dirac operator to conserved charges implies correlation between the geometry of space-time sheet and quantum numbers assigned to the partonic 2-surface. The assumption of standard quantized values for  $Q_{em}$  and  $Q_Z$  would be also seen as the identification of the fine structure constants  $\alpha_{em}$  and  $\alpha_Z$ . This however requires weak isospin invariance.

### 2.1.3 The value of $K$ from classical quantization of Kähler electric charge

The value of  $K$  can be deduced by requiring classical quantization of Kähler electric charge.

- (a) The condition that the flux of  $F^{03} = (\hbar/g_K)J^{03}$  defining the counterpart of Kähler electric field equals to the Kähler charge  $g_K$  would give the condition  $K = g_K^2/\hbar$ , where  $g_K$  is Kähler coupling constant which should be invariant under coupling constant evolution by quantum criticality. Within experimental uncertainties one has  $\alpha_K = g_K^2/4\pi\hbar_0 = \alpha_{em} \simeq 1/137$ , where  $\alpha_{em}$  is the fine structure constant in electron length scale and  $\hbar_0$  is the standard value of Planck constant.
- (b) The quantization of Planck constants makes the condition highly non-trivial. The most general quantization of  $r$  is as rationals but there are good arguments favoring the quantization as integers corresponding to the allowance of only singular coverings of  $CP_2$  and  $CP_2$ . The point is that in this case a given value of Planck constant corresponds to a finite number of pages of the “Big Book”. The quantization of the Planck constant implies a further quantization of  $K$  and would suggest that  $K$  scales as  $1/r$  unless the spectrum of values of  $Q_{em}$  and  $Q_Z$  allowed by the quantization condition scales as  $r$ . This is quite possible and the interpretation would be that each of the  $r$  sheets of the covering carries (possibly same) elementary charge. Kind of discrete variant of a full Fermi sphere would be in question. The interpretation in terms of anyonic phases [K14] supports this interpretation.
- (c) The identification of  $J$  as a counterpart of  $eB/\hbar$  means that Kähler action and thus also Kähler function is proportional to  $1/\alpha_K$  and therefore to  $\hbar$ . This implies that for large values of  $\hbar$  Kähler coupling strength  $g_K^2/4\pi$  becomes very small and large fluctuations are suppressed in the functional integral. The basic motivation for introducing the hierarchy of Planck constants was indeed that the scaling  $\alpha \rightarrow \alpha/r$  allows to achieve the convergence of perturbation theory: Nature itself would solve the problems of the theoretician. This of course does not mean that the physical states would remain as such and the replacement of single particles with anyonic states in order to satisfy the condition for  $K$  would realize this concretely.
- (d) The condition  $K = g_K^2/\hbar$  implies that the Kähler magnetic charge is always accompanied by Kähler electric charge. A more general condition would read as

$$K = n \times \frac{g_K^2}{\hbar}, n \in Z . \quad (2.8)$$

This would apply in the case of cosmic strings and would allow vanishing Kähler charge possible when the partonic 2-surface has opposite fermion and anti-fermion numbers (for both leptons and quarks) so that Kähler electric charge should vanish. For instance, for neutrinos the vanishing of electric charge strongly suggests  $n = 0$  besides the condition that abelian  $Z^0$  flux contributing to em charge vanishes.

It took a year to realize that this value of  $K$  is natural at the Minkowskian side of the wormhole throat. At the Euclidian side much more natural condition is

$$K = \frac{1}{\hbar} . \quad (2.9)$$

In fact, the self-duality of  $CP_2$  Kähler form favours this boundary condition at the Euclidian side of the wormhole throat. Also the fact that one cannot distinguish between electric and magnetic charges in Euclidian region since all charges are magnetic can be used to argue in favor of this form. The same constraint arises from the condition that the action for  $CP_2$  type vacuum extremal has the value required by the argument leading to a prediction for gravitational constant in terms of the square of  $CP_2$  radius and  $\alpha_K$  the effective replacement  $g_K^2 \rightarrow 1$  would spoil the argument.

The boundary condition  $J_E = J_B$  for the electric and magnetic parts of Kähler form at the Euclidian side of the wormhole throat inspires the question whether all Euclidian regions could be self-dual so that the density of Kähler action would be just the instanton density. Self-duality follows if the deformation of the metric induced by the deformation of the canonically imbedded  $CP_2$  is such that in  $CP_2$  coordinates for the Euclidian region the tensor  $(g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\nu}g^{\mu\beta})/\sqrt{g}$  remains invariant. This is certainly the case for  $CP_2$  type vacuum extremals since by the light-likeness of  $M^4$  projection the metric remains invariant. Also conformal scalings of the induced metric would satisfy this condition. Conformal scaling is not consistent with the degeneracy of the 4-metric at the wormhole.

#### 2.1.4 Reduction of the quantization of Kähler electric charge to that of electromagnetic charge

The best manner to learn more is to challenge the form of the weak electric-magnetic duality based on the induced Kähler form.

- (a) Physically it would seem more sensible to pose the duality on electromagnetic charge rather than Kähler charge. This would replace induced Kähler form with electromagnetic field, which is a linear combination of induced Kähler field and classical  $Z^0$  field

$$\begin{aligned}\gamma &= 3J - \sin^2\theta_W R_{03} \ , \\ Z^0 &= 2R_{03} \ .\end{aligned}\tag{2.10}$$

Here  $Z_0 = 2R_{03}$  is the appropriate component of  $CP_2$  curvature form [L1]. For a vanishing Weinberg angle the condition reduces to that for Kähler form.

- (b) For the Euclidian space-time regions having interpretation as lines of generalized Feynman diagrams Weinberg angle should be non-vanishing. In Minkowskian regions Weinberg angle could however vanish. If so, the condition guaranteeing that electromagnetic charge of the partonic 2-surfaces equals to the above condition stating that the em charge assignable to the fermion content of the partonic 2-surfaces reduces to the classical Kähler electric flux at the Minkowskian side of the wormhole throat. One can argue that Weinberg angle must increase smoothly from a vanishing value at both sides of wormhole throat to its value in the deep interior of the Euclidian region.
- (c) The vanishing of the Weinberg angle in Minkowskian regions conforms with the physical intuition. Above elementary particle length scales one sees only the classical electric field reducing to the induced Kähler form and classical  $Z^0$  fields and color gauge fields are effectively absent. Only in phases with a large value of Planck constant classical  $Z^0$  field and other classical weak fields and color gauge field could make themselves visible. Cell membrane could be one such system [K15]. This conforms with the general picture about color confinement and weak massivation.

The GRT limit of TGD suggests a further reason for why Weinberg angle should vanish in Minkowskian regions.

- (a) The value of the Kähler coupling strength must be very near to the value of the fine structure constant in electron length scale and these constants can be assumed to be equal.
- (b) GRT limit of TGD with space-time surfaces replaced with abstract 4-geometries would naturally correspond to Einstein-Maxwell theory with cosmological constant which is non-vanishing only in Euclidian regions of space-time so that both Reissner-Nordström metric and  $CP_2$  are allowed as simplest possible solutions of field equations [K18]. The extremely small value of the observed cosmological constant needed in GRT type cosmology could be equal to the large cosmological constant associated with  $CP_2$  metric multiplied with the 3-volume fraction of Euclidian regions.

- (c) Also at GRT limit quantum theory would reduce to almost topological QFT since Einstein-Maxwell action reduces to 3-D term by field equations implying the vanishing of the Maxwell current and of the curvature scalar in Minkowskian regions and curvature scalar + cosmological constant term in Euclidian regions. The weak form of electric-magnetic duality would guarantee also now the preferred extremal property and prevent the reduction to a mere topological QFT.
- (d) GRT limit would make sense only for a vanishing Weinberg angle in Minkowskian regions. A non-vanishing Weinberg angle would make sense in the deep interior of the Euclidian regions where the approximation as a small deformation of  $CP_2$  makes sense.

The weak form of electric-magnetic duality has surprisingly strong implications for the basic view about quantum TGD as following considerations show.

## 2.2 Magnetic Confinement, The Short Range Of Weak Forces, And Color Confinement

The weak form of electric-magnetic duality has surprisingly strong implications if one combines it with some very general empirical facts such as the non-existence of magnetic monopole fields in macroscopic length scales.

### 2.2.1 How can one avoid macroscopic magnetic monopole fields?

Monopole fields are experimentally absent in length scales above order weak boson length scale and one should have a mechanism neutralizing the monopole charge. How electroweak interactions become short ranged in TGD framework is still a poorly understood problem. What suggests itself is the neutralization of the weak isospin above the intermediate gauge boson Compton length by neutral Higgs bosons. Could the two neutralization mechanisms be combined to single one?

- (a) In the case of fermions and their super partners the opposite magnetic monopole would be a wormhole throat. If the magnetically charged wormhole contact is electromagnetically neutral but has vectorial weak isospin neutralizing the weak vectorial isospin of the fermion only the electromagnetic charge of the fermion is visible on longer length scales. The distance of this wormhole throat from the fermionic one should be of the order weak boson Compton length. An interpretation as a bound state of fermion and a wormhole throat state with the quantum numbers of a neutral Higgs boson would therefore make sense. The neutralizing throat would have quantum numbers of  $X_{-1/2} = \nu_L \bar{\nu}_R$  or  $X_{1/2} = \bar{\nu}_L \nu_R$ .  $\nu_L \bar{\nu}_R$  would not be neutral Higgs boson (which should correspond to a wormhole contact) but a super-partner of left-handed neutrino obtained by adding a right handed neutrino. This mechanism would apply separately to the fermionic and anti-fermionic throats of the gauge bosons and corresponding space-time sheets and leave only electromagnetic interaction as a long ranged interaction.
- (b) One can of course wonder what is the situation situation for the bosonic wormhole throats feeding gauge fluxes between space-time sheets. It would seem that these wormhole throats must always appear as pairs such that for the second member of the pair monopole charges and  $I_V^3$  cancel each other at both space-time sheets involved so that one obtains at both space-time sheets magnetic dipoles of size of weak boson Compton length. The proposed magnetic character of fundamental particles should become visible at TeV energies so that LHC might have surprises in store!

### 2.2.2 Well-definedness of electromagnetic charge implies stringiness

Well-definedness of electromagnetic charged at string world sheets carrying spinor modes is very natural constraint and not trivially satisfied because classical  $W$  boson fields are present.

As a matter fact, all weak fields should be effectively absent above weak scale. How this is possible classical weak fields identified as induced gauge fields are certainly present.

The condition that em charge is well defined for spinor modes implies that the space-time region in which spinor mode is non-vanishing has 2-D  $CP_2$  projection such that the induced  $W$  boson fields are vanishing. The vanishing of classical  $Z^0$  field can be poses as additional condition - at least in scales above weak scale. In the generic case this requires that the spinor mode is restricted to 2-D surface: string world sheet or possibly also partonic 2-surface. This implies that TGD reduces to string model in fermionic sector. Even for preferred extremals with 2-D projecting the modes are expected to allow restriction to 2-surfaces. This localization is possible only for Kähler-Dirac action.

A word of warning is however in order. The GRT limit or rather limit of TGD as Einstein Yang-Mills theory replaces the sheets of many-sheeted space-time with Minkowski space with effective metric obtained by summing to Minkowski metric the deviations of the induced metrics of space-time sheets from Minkowski metric. For gauge potentials a similar identification applies. YM-Einstein equations coupled with matter and with non-vanishing cosmological constant are expected on basis of Poincare invariance. One cannot exclude the possibility that the sums of weak gauge potentials from different space-time sheet tend to vanish above weak scale and that well-definedness of em charge at classical level follows from the effective absence of classical weak gauge fields.

### 2.2.3 Magnetic confinement and color confinement

Magnetic confinement generalizes also to the case of color interactions. One can consider also the situation in which the magnetic charges of quarks (more generally, of color excited leptons and quarks) do not vanish and they form color and magnetic singlets in the hadronic length scale. This would mean that magnetic charges of the state  $q_{\pm 1/2} - X_{\mp 1/2}$  representing the physical quark would not vanish and magnetic confinement would accompany also color confinement. This would explain why free quarks are not observed. To how degree then quark confinement corresponds to magnetic confinement is an interesting question.

For quark and antiquark of meson the magnetic charges of quark and antiquark would be opposite and meson would correspond to a Kähler magnetic flux so that a stringy view about meson emerges. For valence quarks of baryon the vanishing of the net magnetic charge takes place provided that the magnetic net charges are  $(\pm 2, \mp 1, \mp 1)$ . This brings in mind the spectrum of color hyper charges coming as  $(\pm 2, \mp 1, \mp 1)/3$  and one can indeed ask whether color hyper-charge correlates with the Kähler magnetic charge. The geometric picture would be three strings connected to single vertex. Amusingly, the idea that color hypercharge could be proportional to color hyper charge popped up during the first year of TGD when I had not yet discovered  $CP_2$  and believed on  $M^4 \times S^2$ .

p-Adic length scale hypothesis and hierarchy of Planck constants defining a hierarchy of dark variants of particles suggest the existence of scaled up copies of QCD type physics and weak physics. For p-adically scaled up variants the mass scales would be scaled by a power of  $\sqrt{2}$  in the most general case. The dark variants of the particle would have the same mass as the original one. In particular, Mersenne primes  $M_k = 2^k - 1$  and Gaussian Mersennes  $M_{G,k} = (1 + i)^k - 1$  has been proposed to define zoomed copies of these physics. At the level of magnetic confinement this would mean hierarchy of length scales for the magnetic confinement.

One particular proposal is that the Mersenne prime  $M_{89}$  should define a scaled up variant of the ordinary hadron physics with mass scaled up roughly by a factor  $2^{(107-89)/2} = 512$ . The size scale of color confinement for this physics would be same as the weal length scale. It would look more natural that the weak confinement for the quarks of  $M_{89}$  physics takes place in some shorter scale and  $M_{61}$  is the first Mersenne prime to be considered. The mass scale of  $M_{61}$  weak bosons would be by a factor  $2^{(89-61)/2} = 2^{14}$  higher and about  $1.6 \times 10^4$  TeV.  $M_{89}$  quarks would have virtually no weak interactions but would possess color interactions with weak confinement length scale reflecting themselves as new kind of jets at collisions above TeV energies.

In the biologically especially important length scale range 10 nm -2500 nm there are as many as four scaled up electron Compton lengths  $L_e(k) = \sqrt{5}L(k)$ : they are associated with Gaussian Mersennes  $M_{G,k}$ ,  $k = 151, 157, 163, 167$ . This would suggest that the existence of scaled up scales of magnetic-, weak- and color confinement. An especially interesting possibly testable prediction is the existence of magnetic monopole pairs with the size scale in this range. There are recent claims about experimental evidence for magnetic monopole pairs [D1] .

#### 2.2.4 Magnetic confinement and stringy picture in TGD sense

The connection between magnetic confinement and weak confinement is rather natural if one recalls that electric-magnetic duality in super-symmetric quantum field theories means that the descriptions in terms of particles and monopoles are in some sense dual descriptions. Fermions would be replaced by string like objects defined by the magnetic flux tubes and bosons as pairs of wormhole contacts would correspond to pairs of the flux tubes. Therefore the sharp distinction between gravitons and physical particles would disappear.

The reason why gravitons are necessarily stringy objects formed by a pair of wormhole contacts is that one cannot construct spin two objects using only single fermion states at wormhole throats. Of course, also super partners of these states with higher spin obtained by adding fermions and anti-fermions at the wormhole throat but these do not give rise to graviton like states [K9] . The upper and lower wormhole throat pairs would be quantum superpositions of fermion anti-fermion pairs with sum over all fermions. The reason is that otherwise one cannot realize graviton emission in terms of joining of the ends of light-like 3-surfaces together. Also now magnetic monopole charges are necessary but now there is no need to assign the entities  $X_{\pm}$  with gravitons.

Graviton string is characterized by some p-adic length scale and one can argue that below this length scale the charges of the fermions become visible. Mersenne hypothesis suggests that some Mersenne prime is in question. One proposal is that gravitonic size scale is given by electronic Mersenne prime  $M_{127}$ . It is however difficult to test whether graviton has a structure visible below this length scale.

What happens to the generalized Feynman diagrams is an interesting question. It is not at all clear how closely they relate to ordinary Feynman diagrams. All depends on what one is ready to assume about what happens in the vertices. One could of course hope that zero energy ontology could allow some very simple description allowing perhaps to get rid of the problematic aspects of Feynman diagrams.

- (a) Consider first the recent view about generalized Feynman diagrams which relies ZEO. A highly attractive assumption is that the particles appearing at wormhole throats are on mass shell particles. For incoming and outgoing elementary bosons and their super partners they would be positive it resp. negative energy states with parallel on mass shell momenta. For virtual bosons they the wormhole throats would have opposite sign of energy and the sum of on mass shell states would give virtual net momenta. This would make possible twistor description of virtual particles allowing only massless particles (in 4-D sense usually and in 8-D sense in TGD framework). The notion of virtual fermion makes sense only if one assumes in the interaction region a topological condensation creating another wormhole throat having no fermionic quantum numbers.
- (b) The addition of the particles  $X^{\pm}$  replaces generalized Feynman diagrams with the analogs of stringy diagrams with lines replaced by pairs of lines corresponding to fermion and  $X_{\pm 1/2}$ . The members of these pairs would correspond to 3-D light-like surfaces glued together at the vertices of generalized Feynman diagrams. The analog of 3-vertex would not be splitting of the string to form shorter strings but the replication of the entire string to form two strings with same length or fusion of two strings to single string along all their points rather than along ends to form a longer string. It is not clear whether the duality symmetry of stringy diagrams can hold true for the TGD variants of stringy diagrams.

- (c) How should one describe the bound state formed by the fermion and  $X^\pm$ ? Should one describe the state as superposition of non-parallel on mass shell states so that the composite state would be automatically massive? The description as superposition of on mass shell states does not conform with the idea that bound state formation requires binding energy. In TGD framework the notion of negentropic entanglement has been suggested to make possible the analogs of bound states consisting of on mass shell states so that the binding energy is zero [K12]. If this kind of states are in question the description of virtual states in terms of on mass shell states is not lost. Of course, one cannot exclude the possibility that there is infinite number of this kind of states serving as analogs for the excitations of string like object.
- (d) What happens to the states formed by fermions and  $X_{\pm 1/2}$  in the internal lines of the Feynman diagram? Twistor philosophy suggests that only the higher on mass shell excitations are possible. If this picture is correct, the situation would not change in an essential manner from the earlier one.

The highly non-trivial prediction of the magnetic confinement is that elementary particles should have stringy character in electro-weak length scales and could behaving to become manifest at LHC energies. This adds one further item to the list of non-trivial predictions of TGD about physics at LHC energies [K13].

### 2.3 Could Quantum TGD Reduce To Almost Topological QFT?

There seems to be a profound connection with the earlier unrealistic proposal that TGD reduces to almost topological quantum theory in the sense that the counterpart of Chern-Simons action assigned with the wormhole throats somehow dictates the dynamics. This proposal can be formulated also for the Kähler-Dirac action action. I gave up this proposal but the following argument shows that Kähler action with weak form of electric-magnetic duality effectively reduces to Chern-Simons action plus Coulomb term.

- (a) Kähler action density can be written as a 4-dimensional integral of the Coulomb term  $j_K^\alpha A_\alpha$  plus and integral of the boundary term  $J^{n\beta} A_\beta \sqrt{g_4}$  over the wormhole throats and of the quantity  $J^{0\beta} A_\beta \sqrt{g_4}$  over the ends of the 3-surface.
- (b) If the self-duality conditions generalize to  $J^{n\beta} = 4\pi\alpha_K \epsilon^{n\beta\gamma\delta} J_{\gamma\delta}$  at throats and to  $J^{0\beta} = 4\pi\alpha_K \epsilon^{0\beta\gamma\delta} J_{\gamma\delta}$  at the ends, the Kähler function reduces to the counterpart of Chern-Simons action evaluated at the ends and throats. It would have same value for each branch and the replacement  $\hbar \rightarrow n \times \hbar$  would effectively describe this. Boundary conditions would however give  $1/n$  factor so that  $\hbar$  would disappear from the Kähler function! It is somewhat surprising that Kähler action gives Chern-Simons action in the vacuum sector defined as sector for which Kähler current is light-like or vanishes.

Holography encourages to ask whether also the Coulomb interaction terms could vanish. This kind of dimensional reduction would mean an enormous simplification since TGD would reduce to an almost topological QFT. The attribute “almost” would come from the fact that one has non-vanishing classical Noether charges defined by Kähler action and non-trivial quantum dynamics in  $M^4$  degrees of freedom. One could also assign to space-time surfaces conserved four-momenta which is not possible in topological QFTs. For this reason the conditions guaranteeing the vanishing of Coulomb interaction term deserve a detailed analysis.

- (a) For the known extremals  $j_K^\alpha$  either vanishes or is light-like (“massless extremals” for which weak self-duality condition does not make sense [K3]) so that the Coulomb term vanishes identically in the gauge used. The addition of a gradient to  $A$  induces terms located at the ends and wormhole throats of the space-time surface but this term must be cancelled by the other boundary terms by gauge invariance of Kähler action. This implies that the  $M^4$  part of WCW metric vanishes in this case. Therefore massless extremals as such are not physically realistic: wormhole throats representing particles are needed.

- (b) The original naive conclusion was that since Chern-Simons action depends on  $CP_2$  coordinates only, its variation with respect to Minkowski coordinates must vanish so that the WCW metric would be trivial in  $M^4$  degrees of freedom. This conclusion is in conflict with quantum classical correspondence and was indeed too hasty. The point is that the allowed variations of Kähler function must respect the weak electro-magnetic duality which relates Kähler electric field depending on the induced 4-metric at 3-surface to the Kähler magnetic field. Therefore the dependence on  $M^4$  coordinates creeps via a Lagrange multiplier term

$$\int \Lambda_\alpha (J^{n\alpha} - K \epsilon^{n\alpha\beta\gamma} J_\beta \text{ gamma}) \sqrt{g_4} d^3x . \quad (2.11)$$

The (1,1) part of second variation contributing to  $M^4$  metric comes from this term.

- (c) This erratic conclusion about the vanishing of  $M^4$  part WCW metric raised the question about how to achieve a non-trivial metric in  $M^4$  degrees of freedom. The proposal was a modification of the weak form of electric-magnetic duality. Besides  $CP_2$  Kähler form there would be the Kähler form assignable to the light-cone boundary reducing to that for  $r_M = \text{constant}$  sphere - call it  $J^1$ . The generalization of the weak form of self-duality would be  $J^{n\beta} = \epsilon^{n\beta\gamma\delta} K (J_{\gamma\delta} + \epsilon J_{\gamma\delta}^1)$ . This form implies that the boundary term gives a non-trivial contribution to the  $M^4$  part of the WCW metric even without the constraint from electric-magnetic duality. Kähler charge is not affected unless the partonic 2-surface contains the tip of CD in its interior. In this case the value of Kähler charge is shifted by a topological contribution. Whether this term can survive depends on whether the resulting vacuum extremals are consistent with the basic facts about classical gravitation.
- (d) The Coulombic interaction term is not invariant under gauge transformations. The good news is that this might allow to find a gauge in which the Coulomb term vanishes. The vanishing condition fixing the gauge transformation  $\phi$  is

$$j_K^\alpha \partial_\alpha \phi = -j^\alpha A_\alpha . \quad (2.12)$$

This differential equation can be reduced to an ordinary differential equation along the flow lines  $j_K$  by using  $dx^\alpha/dt = j_K^\alpha$ . Global solution is obtained only if one can combine the flow parameter  $t$  with three other coordinates- say those at the either end of CD to form space-time coordinates. The condition is that the parameter defining the coordinate differential is proportional to the covariant form of Kähler current:  $dt = \phi j_K$ . This condition in turn implies  $d^2t = d(\phi j_K) = d\phi \wedge j_K + \phi dj_K = 0$  implying  $j_K \wedge dj_K = 0$  or more concretely,

$$\epsilon^{\alpha\beta\gamma\delta} j_\beta^K \partial_\gamma j_{\text{delta}}^K = 0 . \quad (2.13)$$

$j_K$  is a four-dimensional counterpart of Beltrami field [B8] and could be called generalized Beltrami field.

The integrability conditions follow also from the construction of the extremals of Kähler action [K3]. The conjecture was that for the extremals the 4-dimensional Lorentz force vanishes (no dissipation): this requires  $j_K \wedge J = 0$ . One manner to guarantee this is the topologization of the Kähler current meaning that it is proportional to the instanton current:  $j_K = \phi j_I$ , where  $j_I = *(J \wedge A)$  is the instanton current, which is not conserved for 4-D  $CP_2$  projection. The conservation of  $j_K$  implies the condition  $j_I^\alpha \partial_\alpha \phi = \partial_\alpha j^\alpha \phi$  and from this  $\phi$  can be integrated if the integrability condition  $j_I \wedge dj_I = 0$  holds true implying the same condition for  $j_K$ . By introducing at least 3 or  $CP_2$  coordinates as space-time coordinates, one finds that the contravariant form of  $j_I$  is purely topological so that the integrability condition fixes the dependence on  $M^4$  coordinates and this selection is coded into the scalar function  $\phi$ . These functions define families of conserved currents  $j_K^\alpha \phi$  and  $j_I^\alpha \phi$  and could be also interpreted as conserved currents associated with the critical deformations of the space-time surface.

- (e) There are gauge transformations respecting the vanishing of the Coulomb term. The vanishing condition for the Coulomb term is gauge invariant only under the gauge transformations  $A \rightarrow A + \nabla\phi$  for which the scalar function the integral  $\int j_K^\alpha \partial_\alpha \phi$  reduces to a total divergence a giving an integral over various 3-surfaces at the ends of CD and at throats vanishes. This is satisfied if the allowed gauge transformations define conserved currents

$$D_\alpha(j^\alpha\phi) = 0 . \quad (2.14)$$

As a consequence Coulomb term reduces to a difference of the conserved charges  $Q_\phi^e = \int j^0 \phi \sqrt{g_4} d^3x$  at the ends of the CD vanishing identically. The change of the Chern-Simons type term is trivial if the total weighted Kähler magnetic flux  $Q_\phi^m = \sum \int J \phi dA$  over wormhole throats is conserved. The existence of an infinite number of conserved weighted magnetic fluxes is in accordance with the electric-magnetic duality. How these fluxes relate to the flux Hamiltonians central for WCW geometry is not quite clear.

- (f) The gauge transformations respecting the reduction to almost topological QFT should have some special physical meaning. The measurement interaction term in the Kähler-Dirac interaction corresponds to a critical deformation of the space-time sheet and is realized as an addition of a gauge part to the Kähler gauge potential of  $CP_2$ . It would be natural to identify this gauge transformation giving rise to a conserved charge so that the conserved charges would provide a representation for the charges associated with the infinitesimal critical deformations not affecting Kähler action. The gauge transformed Kähler gauge potential couples to the Kähler-Dirac equation and its effect could be visible in the value of Kähler function and therefore also in the properties of the preferred extremal. The effect on WCW metric would however vanish since  $K$  would transform only by an addition of a real part of a holomorphic function.
- (g) A first guess for the explicit realization of the quantum classical correspondence between quantum numbers and space-time geometry is that the deformation of the preferred extremal due to the addition of the measurement interaction term is induced by a  $U(1)$  gauge transformation induced by a transformation of  $\delta CD \times CP_2$  generating the gauge transformation represented by  $\phi$ . This interpretation makes sense if the fluxes defined by  $Q_\phi^m$  and corresponding Hamiltonians affect only zero modes rather than quantum fluctuating degrees of freedom.
- (h) Later a simpler proposal assuming Kähler action with Chern-Simons term at partonic orbits and Kähler-Dirac action with Chern-Simons Dirac term at partonic orbits emerged. Measurement interaction terms would correspond to Lagrange multiplier terms at the ends of space-time surface fixing the values of classical conserved charges to their quantum values. Super-symmetry requires the assignment of this kind of term also to Kähler-Dirac action as boundary term.

Kähler-Dirac equation gives rise to a boundary condition at space-like ends of the space-time surface stating that the action of the Kähler-Dirac gamma matrix in normal direction annihilates the spinor modes. The normal vector would be light-like and the value of the incoming on mass shell four-momentum would be coded to the geometry of the space-time surface and string world sheet.

One can assign to partonic orbits Chern-Simons Dirac action and now the condition would be that the action of C-S-D operator equals to that of massless  $M^4$  Dirac operator. C-S-D Dirac action would give rise to massless Dirac propagator. Twistor Grassmann approach suggests that also the virtual fermions reduce effectively to massless on-shell states but have non-physical helicity.

To sum up, one could understand the basic properties of WCW metric in this framework. Effective 2-dimensionality would result from the existence of an infinite number of conserved charges in two different time directions (genuine conservation laws plus gauge fixing). The infinite-dimensional symmetric space for given values of zero modes corresponds to the Cartesian product of the WCWs associated with the partonic 2-surfaces at both ends of CD and

the generalized Chern-Simons term decomposes into a sum of terms from the ends giving single particle Kähler functions and to the terms from light-like wormhole throats giving interaction term between positive and negative energy parts of the state. Hence Kähler function could be calculated without any knowledge about the interior of the space-time sheets and TGD would reduce to almost topological QFT as speculated earlier. Needless to say this would have immense boost to the program of constructing WCW Kähler geometry.

### 3 An attempt to understand preferred extremals of Kähler action

Preferred extremal of Kähler action is one of the basic poorly defined notions of TGD. There are pressing motivations for understanding what "preferred" really means. For instance, the conformal invariance of string models naturally generalizes to 4-D invariance defined by quantum Yangian of quantum affine algebra (Kac-Moody type algebra) characterized by two complex coordinates and therefore explaining naturally the effective 2-dimensionality [K30]. The problem is however how to assign a complex coordinate with the string world sheet having Minkowskian signature of metric. One can hope that the understanding of preferred extremals could allow to identify two preferred complex coordinates whose existence is also suggested by number theoretical vision giving preferred role for the rational points of partonic 2-surfaces in preferred coordinates. The best one could hope is a general solution of field equations in accordance with the hints that TGD is integrable quantum theory.

#### 3.1 What "preferred" could mean?

The first question is what preferred extremal could mean.

- (a) In positive energy ontology preferred extremal would be a space-time surface assignable to given 3-surface and unique in the ideal situation: since one cannot pose conditions to the normal derivatives of imbedding space coordinates at 3-surface, there is infinity of extremals. Some additional conditions are required and space-time surface would be analogous to Bohr orbit: hence the attribute "preferred". The problem would be to understand what "preferred" could mean. The non-determinism of Kähler action however destroyed this dream in its original form and led to zero energy ontology (ZEO).
- (b) In ZEO one considers extremals as space-time surfaces connecting two space-like 3-surfaces at the boundaries. One might hope that these 4-surfaces are unique. The non-determinism of Kähler action suggests that this is not the case. At least there is conformal invariance respecting the light-likeness of the 3-D parton orbits at which the signature of the induced metric changes: the conformal transformations would leave the space-like 3-D ends or at least partonic 2-surfaces invariant. This non-determinism would correspond to quantum criticality.
- (c) Effective 2-dimensionality follows from strong form of general coordinate invariance (GCI) stating that light-like partonic orbits and space-like 3-surfaces at the ends of space-time surface are equivalent physically: partonic 2-surfaces and their 4-D tangent space data would determine everything. One can however worry about how effective 2-dimensionality relates to the fact that the modes of the induced spinor field are localized at string world sheets and partonic 2-surface. Are the tangent space data equivalent with the data characterizing string world sheets as surfaces carrying vanishing electroweak fields?

There is however a problem: the hierarchy of Planck constants (dark matter) requires that the conformal equivalence classes of light-like surfaces must be counted as physical degrees of freedom so that either space-like or light-like surfaces do not seem to be quite enough.

Should one then include also the light-like partonic orbits to the what one calls 3-surface? The resulting connected 3-surfaces would define analogs of Wilson loops. Could the

conformal equivalence class of the preferred extremal be unique without any additional conditions? If so, one could get rid of the attribute “preferred”. The fractal character of the many-sheeted space-time however suggests that one can have this kind of uniqueness only in given length scale resolution and that “radiative corrections” due to the non-determinism are always present.

These considerations show that the notion of preferred extremal is still far from being precisely defined and it is not even clear whether the attribute “preferred” is needed. If not then the question is what are the extremals of Kähler action.

### 3.2 What is known about extremals?

A lot is known about properties of extremals and just by trying to integrate all this understanding, one might gain new visions. The problem is that all these arguments are heuristic and rely heavily on physical intuition. The following considerations relate to the space-time regions having Minkowskian signature of the induced metric. The attempt to generalize the construction also to Euclidian regions could be very rewarding. Only a humble attempt to combine various ideas to a more coherent picture is in question.

The core observations and visions are following.

- (a) Hamilton-Jacobi coordinates for  $M^4$  (discussed in this chapter) define natural preferred coordinates for Minkowskian space-time sheet and might allow to identify string world sheets for  $X^4$  as those for  $M^4$ . Hamilton-Jacobi coordinates consist of light-like coordinate  $m$  and its dual defining local 2-plane  $M^2 \subset M^4$  and complex transversal complex coordinates  $(w, \bar{w})$  for a plane  $E_x^2$  orthogonal to  $M_x^2$  at each point of  $M^4$ . Clearly, hyper-complex analyticity and complex analyticity are in question.
- (b) Space-time sheets allow a slicing by string world sheets (partonic 2-surfaces) labelled by partonic 2-surfaces (string world sheets).
- (c) The quaternionic planes of octonion space containing preferred hyper-complex plane are labelled by  $CP_2$ , which might be called  $CP_2^{mod}$  [K17]. The identification  $CP_2 = CP_2^{mod}$  motivates the notion of  $M^8 - M^4 \times CP_2$  duality [K6]. It also inspires a concrete solution ansatz assuming the equivalence of two different identifications of the quaternionic tangent space of the space-time sheet and implying that string world sheets can be regarded as strings in the 6-D coset space  $G_2/SU(3)$ . The group  $G_2$  of octonion automorphisms has already earlier appeared in TGD framework.
- (d) The duality between partonic 2-surfaces and string world sheets in turn suggests that the  $CP_2 = CP_2^{mod}$  conditions reduce to string model for partonic 2-surfaces in  $CP_2 = SU(3)/U(2)$ . String model in both cases could mean just hypercomplex/complex analyticity for the coordinates of the coset space as functions of hyper-complex/complex coordinate of string world sheet/partonic 2-surface.

The considerations of this section lead to a revival of an old very ambitious and very romantic number theoretic idea.

- (a) To begin with express octonions in the form  $o = q_1 + Iq_2$ , where  $q_i$  is quaternion and  $I$  is an octonionic imaginary unit in the complement of fixed a quaternionic sub-space of octonions. Map preferred coordinates of  $H = M^4 \times CP_2$  to octonionic coordinate, form an arbitrary octonion analytic function having expansion with real Taylor or Laurent coefficients to avoid problems due to non-commutativity and non-associativity. Map the outcome to a point of  $H$  to get a map  $H \rightarrow H$ . This procedure is nothing but a generalization of Wick rotation to get an 8-D generalization of analytic map.
- (b) Identify the preferred extremals of Kähler action as surfaces obtained by requiring the vanishing of the imaginary part of an octonion analytic function. Partonic 2-surfaces and string world sheets would correspond to commutative sub-manifolds of the space-time surface and of imbedding space and would emerge naturally. The ends of braid strands

at partonic 2-surface would naturally correspond to the poles of the octonion analytic functions. This would mean a huge generalization of conformal invariance of string models to octonionic conformal invariance and an exact solution of the field equations of TGD and presumably of quantum TGD itself.

### 3.3 Basic ideas about preferred extremals

#### 3.3.1 The slicing of the space-time sheet by partonic 2-surfaces and string world sheets

The basic vision is that space-time sheets are sliced by partonic 2-surfaces and string world sheets. The challenge is to formulate this more precisely at the level of the preferred extremals of Kähler action.

- (a) Almost topological QFT property means that the Kähler action reduces to Chern-Simons terms assignable to 3-surfaces. This is guaranteed by the vanishing of the Coulomb term in the action density implied automatically if conserved Kähler current is proportional to the instanton current with proportionality coefficient some scalar function.
- (b) The field equations reduce to the conservation of isometry currents. An attractive ansatz is that the flow lines of these currents define global coordinates. This means that these currents are Beltrami flows [B8] so that corresponding 1-forms  $J$  satisfy the condition  $J \wedge dJ = 0$ . These conditions are satisfied if

$$J = \Phi \nabla \Psi$$

hold true for conserved currents. From this one obtains that  $\Psi$  defines global coordinate varying along flow lines of  $J$ .

- (c) A possible interpretation is in terms of local polarization and momentum directions defined by the scalar functions involved and natural additional conditions are that the gradients of  $\Psi$  and  $\Phi$  are orthogonal:

$$\nabla \Phi \cdot \nabla \Psi = 0 \quad ,$$

and that the  $\Psi$  satisfies massless d'Alembert equation

$$\nabla^2 \Psi = 0$$

as a consequence of current conservation. If  $\Psi$  defines a light-like vector field - in other words

$$\nabla \Psi \cdot \nabla \Psi = 0 \quad ,$$

the light-like dual of  $\Phi$  -call it  $\Phi_c$ - defines a light-like like coordinate and  $\Phi$  and  $\Phi_c$  defines a light-like plane at each point of space-time sheet.

If also  $\Phi$  satisfies d'Alembert equation

$$\nabla^2 \Phi = 0 \quad ,$$

also the current

$$K = \Psi \nabla \Phi$$

is conserved and its flow lines define a global coordinate in the polarization plane orthogonal to time-like plane defined by local light-like momentum direction.

If  $\Phi$  allows a continuation to an analytic function of the transversal complex coordinate, one obtains a coordinatization of space-time surface by  $\Psi$  and its dual (defining

hyper-complex coordinate) and  $w, \bar{w}$ . Complex analyticity and its hyper-complex variant would allow to provide space-time surface with four coordinates very much analogous with Hamilton-Jacobi coordinates of  $M^4$ .

This would mean a decomposition of the tangent space of space-time surface to orthogonal planes defined by light-like momentum and plane orthogonal to it. If the flow lines of  $J$  defined Beltrami flow it seems that the distribution of momentum planes is integrable.

- (d) General arguments suggest that the space-time sheets allow a slicing by string world sheets parametrized by partonic 2-surfaces or vice versa. This would mean a intimate connection with the mathematics of string models. The two complex coordinates assignable to the Yangian of affine algebra would naturally relate to string world sheets and partonic 2-surfaces and the highly non-trivial challenge is to identify them appropriately.

### 3.3.2 Hamilton-Jacobi coordinates for $M^4$

The earlier attempts to construct preferred extremals [K3] led to the realization that so called Hamilton-Jacobi coordinates  $(m, w)$  for  $M^4$  define its slicing by string world sheets parametrized by partonic 2-surfaces.  $m$  would be pair of light-like conjugate coordinates associated with an integrable distribution of planes  $M^2$  and  $w$  would define a complex coordinate for the integrable distribution of 2-planes  $E^2$  orthogonal to  $M^2$ . There is a great temptation to assume that these coordinates define preferred coordinates for  $M^4$ .

- (a) The slicing is very much analogous to that for space-time sheets and the natural question is how these slicings relate. What is of special interest is that the momentum plane  $M^2$  can be defined by massless momentum. The scaling of this vector does not matter so that these planes are labelled by points  $z$  of sphere  $S^2$  telling the direction of the line  $M^2 \cap E^3$ , when one assigns rest frame and therefore  $S^2$  with the preferred time coordinate defined by the line connecting the tips of CD. This direction vector can be mapped to a twistor consisting of a spinor and its conjugate. The complex scalings of the twistor  $(u, \bar{u}) \rightarrow \lambda u, \bar{u}/\lambda$  define the same plane. Projective twistor like entities defining  $CP_1$  having only one complex component instead of three are in question. This complex number defines with certain prerequisites a local coordinate for space-time sheet and together with the complex coordinate of  $E^2$  could serve as a pair of complex coordinates  $(z, w)$  for space-time sheet. This brings strongly in mind the two complex coordinates appearing in the expansion of the generators of quantum Yangian of quantum affine algebra [K30].
- (b) The coordinate  $\Psi$  appearing in Beltrami flow defines the light-like vector field defining  $M^2$  distribution. Its hyper-complex conjugate would define  $\Psi_c$  and conjugate light-like direction. An attractive possibility is that  $\Psi$  allows analytic continuation to a holomorphic function of  $w$ . In this manner one would have four coordinates for  $M^4$  also for space-time sheet.
- (c) The general vision is that at each point of space-time surface one can decompose the tangent space to  $M^2(x) \subset M^4 = M_x^2 \times E_x^2$  representing momentum plane and polarization plane  $E^2 \subset E_x^2 \times T(CP_2)$ . The moduli space of planes  $E^2 \subset E^6$  is 8-dimensional and parametrized by  $SO(6)/SO(2) \times SO(4)$  for a given  $E_x^2$ . How can one achieve this selection and what conditions it must satisfy? Certainly the choice must be integrable but this is not the only condition.

### 3.3.3 Space-time surfaces as associative/co-associative surfaces

The idea that number theory determines classical dynamics in terms of associativity condition means that space-time surfaces are in some sense quaternionic surfaces of an octonionic space-time. It took several trials before the recent form of this hypothesis was achieved.

- (a) Octonionic structure is defined in terms of the octonionic representation of gamma matrices of the imbedding space existing only in dimension  $D = 8$  since octonion units are in one-one correspondence with tangent vectors of the tangent space. Octonionic real unit corresponds to a preferred time axes (and rest frame) identified naturally as that connecting the tips of CD. What modified gamma matrices mean depends on variational principle for space-time surface. For volume action one would obtain induced gamma matrices. For Kähler action one obtains something different. In particular, the modified gamma matrices do not define vector basis identical with tangent vector basis of space-time surface.
- (b) Quaternionicity means that the modified gamma matrices defined as contractions of gamma matrices of  $H$  with canonical momentum densities for Kähler action span quaternionic sub-space of the octonionic tangent space [K19, K23]. A further condition is that each quaternionic space defined in this manner contains a preferred hyper-complex sub-space of octonions.
- (c) The sub-space defined by the modified gamma matrices does not co-incide with the tangent space of space-time surface in general so that the interpretation of this condition is far from obvious. The canonical momentum densities need not define four independent vectors at given point. For instance, for massless extremals these densities are proportional to light-like vector so that the situation is degenerate and the space in question reduces to 2-D hyper-complex sub-space since light-like vector defines plane  $M^2$ .

The obvious questions are following.

- (a) Does the analog of tangent space defined by the octonionic modified gammas contain the local tangent space  $M^2 \subset M^4$  for preferred extremals? For massless extremals [K3] this condition would be true. The orthogonal decomposition  $T(X^4) = M^2 \oplus_{\perp} E^2$  can be defined at each point if this is true. For massless extremals also the functions  $\Psi$  and  $\Phi$  can be identified.
- (b) One should answer also the following delicate question. Can  $M^2$  really depend on point  $x$  of space-time?  $CP_2$  as a moduli space of quaternionic planes emerges naturally if  $M^2$  is *same* everywhere. It however seems that one should allow an integrable distribution of  $M_x^2$  such that  $M_x^2$  is same for all points of a given partonic 2-surface. How could one speak about fixed  $CP_2$  (the imbedding space) at the entire space-time sheet even when  $M_x^2$  varies?
  - i. Note first that  $G_2$  (see <http://tinyurl.com/y9rrs7un>) defines the Lie group of octonionic automorphisms and  $G_2$  action is needed to change the preferred hyper-octonionic sub-space. Various  $SU(3)$  subgroups of  $G_2$  are related by  $G_2$  automorphism. Clearly, one must assign to each point of a string world sheet in the slicing parameterizing the partonic 2-surfaces an element of  $G_2$ . One would have Minkowskian string model with  $G_2$  as a target space. As a matter fact, this string model is defined in the target space  $G_2/SU(3)$  having dimension  $D = 6$  since  $SU(3)$  automorphisms leave given  $SU(3)$  invariant.
  - ii. This would allow to identify at each point of the string world sheet standard quaternionic basis - say in terms of complexified basis vectors consisting of two hyper-complex units and octonionic unit  $q_1$  with "color isospin"  $I_3 = 1/2$  and "color hypercharge"  $Y = -1/3$  and its conjugate  $\bar{q}_1$  with opposite color isospin and hypercharge.
  - iii. The  $CP_2$  point assigned with the quaternionic basis would correspond to the  $SU(3)$  rotation needed to rotate the standard basis to this basis and would actually correspond to the first row of  $SU(3)$  rotation matrix. Hyper-complex analyticity is the basic property of the solutions of the field equations representing Minkowskian string world sheets. Also now the same assumption is highly natural. In the case of string models in Minkowski space, the reduction of the induced metric to standard form implies Virasoro conditions and similar conditions are expected also now. There is no need to introduce action principle -just the hyper-complex analyticity is enough-since Kähler action already defines it.

- (c) The WZW model (see <http://tinyurl.com/ydxcvfhv>) inspired approach to the situation would be following. The parameterization corresponds to a map  $g : X^2 \rightarrow G_2$  for which  $g$  defines a flat  $G_2$  connection at string world sheet. WZW type action would give rise to this kind of situation. The transition  $G_2 \rightarrow G_2/SU(3)$  would require that one gauges  $SU(3)$  degrees of freedom by bringing in  $SU(3)$  connection. Similar procedure for  $CP_2 = SU(3)/U(2)$  would bring in  $SU(3)$  valued chiral field and  $U(2)$  gauge field. Instead of introducing these connections one can simply introduce  $G_2/SU(3)$  and  $SU(3)/U(2)$  valued chiral fields. What this observation suggests that this ansatz indeed predicts gluons and electroweak gauge bosons assignable to string like objects so that the mathematical picture would be consistent with physical intuition.

### 3.3.4 The two interpretations of $CP_2$

An old observation very relevant for what I have called  $M^8 - H$  duality [K6] is that the moduli space of quaternionic sub-spaces of octonionic space (identifiable as  $M^8$ ) containing preferred hyper-complex plane is  $CP_2$ . Or equivalently, the space of two planes whose addition extends hyper-complex plane to some quaternionic subspace can be parametrized by  $CP_2$ . This  $CP_2$  can be called it  $CP_2^{mod}$  to avoid confusion. In the recent case this would mean that the space  $E^2(x) \subset E_x^2 \times T(CP_2)$  is represented by a point of  $CP_2^{mod}$ . On the other hand, the imbedding of space-time surface to  $H$  defines a point of "real"  $CP_2$ . This gives two different  $CP_2$ s.

- (a) The highly suggestive idea is that the identification  $CP_2^{mod} = CP_2$  (apart from isometry) is crucial for the construction of preferred extremals. Indeed, the projection of the space-time point to  $CP_2$  would fix the local polarization plane completely. This condition for  $E^2(x)$  would be purely local and depend on the values of  $CP_2$  coordinates only. Second condition for  $E^2(x)$  would involve the gradients of imbedding space coordinates including those of  $CP_2$  coordinates.
- (b) The conditions that the planes  $M_x^2$  form an integrable distribution at space-like level and that  $M_x^2$  is determined by the modified gamma matrices. The integrability of this distribution for  $M^4$  could imply the integrability for  $X^2$ .  $X^4$  would differ from  $M^4$  only by a deformation in degrees of freedom transversal to the string world sheets defined by the distribution of  $M^2$ s.

Does this mean that one can begin from vacuum extremal with constant values of  $CP_2$  coordinates and makes them non-constant but allows to depend only on transversal degrees of freedom? This condition is too strong even for simplest massless extremals for which  $CP_2$  coordinates depend on transversal coordinates defined by  $\epsilon \cdot m$  and  $\epsilon \cdot k$ . One could however allow dependence of  $CP_2$  coordinates on light-like  $M^4$  coordinate since the modification of the induced metric is light-like so that light-like coordinate remains light-like coordinate in this modification of the metric.

Therefore, if one generalizes directly what is known about massless extremals, the most general dependence of  $CP_2$  points on the light-like coordinates assignable to the distribution of  $M_x^2$  would be dependence on either of the light-like coordinates of Hamilton-Jacobi coordinates but not both.

### 3.4 What could be the construction recipe for the preferred extremals assuming $CP_2 = CP_2^{mod}$ identification?

The crucial condition is that the planes  $E^2(x)$  determined by the point of  $CP_2 = CP_2^{mod}$  identification and by the tangent space of  $E_x^2 \times CP_2$  are same. The challenge is to transform this condition to an explicit form.  $CP_2 = CP_2^{mod}$  identification should be general coordinate invariant. This requires that also the representation of  $E^2$  as  $(e^2, e^3)$  plane is general coordinate invariant suggesting that the use of preferred  $CP_2$  coordinates - presumably complex Eguchi-Hanson coordinates - could make life easy. Preferred coordinates are also suggested by number theoretical vision. A careful consideration of the situation would be required.

The modified gamma matrices define a quaternionic sub-space analogous to tangent space of  $X^4$  but not in general identical with the tangent space: this would be the case only if the action were 4-volume. I will use the notation  $T_x^m(X^4)$  about the modified tangent space and call the vectors of  $T_x^m(X^4)$  modified tangent vectors. I hope that this would not cause confusion.

### 3.4.1 $CP_2 = CP_2^{mod}$ condition

Quaternionic property of the counterpart of  $T_x^m(X^4)$  allows an explicit formulation using the tangent vectors of  $T_x^m(X^4)$ .

- (a) The unit vector pair  $(e_2, e_3)$  should correspond to a unique tangent vector of  $H$  defined by the coordinate differentials  $dh^k$  in some natural coordinates used. Complex Eguchi-Hanson coordinates [L1] are a natural candidate for  $CP_2$  and require complexified octonionic imaginary units. If octonionic units correspond to the tangent vector basis of  $H$  uniquely, this is possible.
- (b) The pair  $(e_2, e_3)$  as also its complexification  $(q_1 = e_2 + ie_3, \bar{q}_1 = e_2 - ie_3)$  is expressible as a linear combination of octonionic units  $I_2, \dots, I_7$  should be mapped to a point of  $CP_2^{mod} = CP_2$  in canonical manner. This mapping is what should be expressed explicitly. One should express given  $(e_2, e_3)$  in terms of  $SU(3)$  rotation applied to a standard vector. After that one should define the corresponding  $CP_2$  point by the bundle projection  $SU(3) \rightarrow CP_2$ .
- (c) The tangent vector pair

$$(\partial_w h^k, \partial_{\bar{w}} h^k)$$

defines second representation of the tangent space of  $E^2(x)$ . This pair should be equivalent with the pair  $(q_1, \bar{q}_1)$ . Here one must be however very cautious with the choice of coordinates. If the choice of  $w$  is unique apart from constant the gradients should be unique. One can use also real coordinates  $(x, y)$  instead of  $(w = x + iy, \bar{w} = x - iy)$  and the pair  $(e_2, e_3)$ . One can project the tangent vector pair to the standard vielbein basis which must correspond to the octonionic basis

$$(\partial_x h^k, \partial_y h^k) \rightarrow (\partial_x h^k e_k^A e_A, \partial_y h^k e_k^A e_A) \leftrightarrow (e_2, e_3) ,$$

where the  $e_A$  denote the octonion units in 1-1 correspondence with vielbein vectors. This expression can be compared to the expression of  $(e_2, e_3)$  derived from the knowledge of  $CP_2$  projection.

### 3.4.2 Formulation of quaternionicity condition in terms of octonionic structure constants

One can consider also a formulation of the quaternionic tangent planes in terms of  $(e_2, e_3)$  expressed in terms of octonionic units deducible from the condition that unit vectors obey quaternionic algebra. The expressions for octonionic (see <http://tinyurl.com/5m5lqr>) resp. quaternionic (see <http://tinyurl.com/3rr79p9>) structure constants can be found at [A5] resp. [A6].

- (a) The ansatz is

$$\begin{aligned} \{E_k\} &= \{1, I_1, E_2, E_3\} , \\ E_2 &= E_{2k} e^k \equiv \sum_{k=2}^7 E_{2k} e^k , \quad E_3 = E_{3k} e^k \equiv \sum_{k=2}^7 E_{3k} e^k , \\ |E_2| &= 1 , \quad |E_3| = 1 . \end{aligned} \tag{3.1}$$

- (b) The multiplication table for octonionic units expressible in terms of octonionic triangle (see <http://tinyurl.com/5m51qr>) [A5] gives

$$f^{1kl} E_{2k} = E_{3l} \ , \ f^{1kl} E_{3k} = -E_{2l} \ , \ f^{klr} E_{2k} E_{3l} = \delta_1^r \ . \quad (3.2)$$

Here the indices are raised by unit metric so that there is no difference between lower and upper indices. Summation convention is assumed. Also the contribution of the real unit is present in the structure constants of third equation but this contribution must vanish.

- (c) The conditions are linear and quadratic in the coefficients  $E_{2k}$  and  $E_{3k}$  and are expected to allow an explicit solution. The first two conditions define homogenous equations which must allow solution. The coefficient matrix acting on  $(E_2, E_3)$  is of the form

$$\begin{pmatrix} f_1 & 1 \\ -1 & f_1 \end{pmatrix} \ ,$$

where 1 denotes unit matrix. The vanishing of the determinant of this matrix should be due to the highly symmetric properties of the structure constants. In fact the equations can be written as eigen conditions

$$f_1 \circ (E_2 \pm iE_3) = \mp i(E_2 \pm iE_3) \ ,$$

and one can say that the structure constants are eigenstates of the hermitian operator defined by  $I_1$  analogous to color hyper charge. Both values of color hyper charged are obtained.

### 3.4.3 Explicit expression for the $CP_2 = CP_2^{mod}$ conditions

The symmetry under  $SU(3)$  allows to construct the solutions of the above equations directly.

- (a) One can introduce complexified basis of octonion units transforming like  $(1, 1, 3, \bar{3})$  under  $SU(3)$ . Note the analogy of triplet with color triplet of quarks. One can write complexified basis as  $(1, e_1, (q_1, q_2, q_3), (\bar{q}_1, \bar{q}_2, \bar{q}_3))$ . The expressions for complexified basis elements are

$$(q_1, q_2, q_3) = \frac{1}{\sqrt{2}}(e_2 + ie_3, e_4 + ie_5, e_6 + ie_7) \ .$$

These options can be seen to be possible by studying octonionic triangle in which all lines containing 3 units defined associative triple: any pair of octonion units at this kind of line can be used to form pair of complexified unit and its conjugate. In the tangent space of  $M^4 \times CP_2$  the basis vectors  $q_1$ , and  $q_2$  are mixtures of  $E_x^2$  and  $CP_2$  tangent vectors.  $q_3$  involves only  $CP_2$  tangent vectors and there is a temptation to interpret it as the analog of the quark having no color isospin.

- (b) The quaternionic basis is real and must transform like  $(1, 1, q_1, \bar{q}_1)$ , where  $q_1$  is any quark in the triplet and  $\bar{q}_1$  its conjugate in antitriplet. Having fixed some basis one can perform  $SU(3)$  rotations to get a new basis. The action of the rotation is by  $3 \times 3$  special unitary matrix. The over all phases of its rows do not matter since they induce only a rotation in  $(e_2, e_3)$  plane not affecting the plane itself. The action of  $SU(3)$  on  $q_1$  is simply the action of its first row on  $(q_1, q_2, q_3)$  triplet:

$$\begin{aligned} q_1 &\rightarrow (Uq)_1 = U_{11}q_1 + U_{12}q_2 + U_{13}q_3 \equiv z_1q_1 + z_2q_2 + z_3q_3 \\ &= z_1(e_2 + ie_3) + z_2(e_4 + ie_5) + z_3(e_6 + ie_7) \ . \end{aligned} \quad (3.3)$$

The triplets  $(z_1, z_2, z_3)$  defining a complex unit vector and point of  $S^5$ . Since overall phase does not matter a point of  $CP_2$  is in question. The new real octonion units are given by the formulas

$$\begin{aligned}
e_2 &\rightarrow \operatorname{Re}(z_1)e_2 + \operatorname{Re}(z_2)e_4 + \operatorname{Re}(z_3)e_6 - \operatorname{Im}(z_1)e_3 - \operatorname{Im}(z_2)e_5 - \operatorname{Im}(z_3)e_7 \ , \\
e_3 &\rightarrow \operatorname{Im}(z_1)e_2 + \operatorname{Im}(z_2)e_4 + \operatorname{Im}(z_3)e_6 + \operatorname{Re}(z_1)e_3 + \operatorname{Re}(z_2)e_5 + \operatorname{Re}(z_3)e_7 \ .
\end{aligned}
\tag{3.4}$$

For instance the  $CP_2$  coordinates corresponding to the coordinate patch  $(z_1, z_2, z_3)$  with  $z_3 \neq 0$  are obtained as  $(\xi_1, \xi_2) = (z_1/z_3, z_2/z_3)$ .

Using these expressions the equations expressing the conjecture  $CP_2 = CP_2^{mod}$  equivalence can be expressed explicitly as first order differential equations. The conditions state the equivalence

$$(e_2, e_3) \leftrightarrow (\partial_x h^k e_k^A e_A, \partial_y h^k e_k^A e_A) \ , \tag{3.5}$$

where  $e_A$  denote octonion units. The comparison of two pairs of vectors requires normalization of the tangent vectors on the right hand side to unit vectors so that one takes unit vector in the direction of the tangent vector. After this the vectors can be equated. This allows to express the contractions of the partial derivatives with vielbein vectors with the 6 components of  $e_2$  and  $e_3$ . Each condition gives 6+6 first order partial differential equations which are non-linear by the presence of the overall normalization factor for the right hand side. The equations are invariant under scalings of  $(x, y)$ . The very special form of these equations suggests that some symmetry is involved.

It must be emphasized that these equations make sense only in preferred coordinates: ordinary Minkowski coordinates and Hamilton-Jacobi coordinates for  $M^4$  and Eguchi-Hanson complex coordinates in which  $SU(2) \times U(1)$  is represented linearly for  $CP_2$ . These coordinates are preferred because they carry deep physical meaning.

#### 3.4.4 Does TGD boil down to two string models?

It is good to look what have we obtained. Besides Hamilton-Jacobi conditions, and  $CP_2 = CP_2^{mod}$  conditions one has what one might call string model with 6-dimensional  $G_2/SU(3)$  as target space. The orbit of string in  $G_2/SU(3)$  allows to deduce the  $G_2$  rotation identifiable as a point of  $G_2/SU(3)$  defining what one means with standard quaternionic plane at given point of string world sheet. The hypothesis is that hyper-complex analyticity solves these equations.

The conjectured electric-magnetic duality implies duality between string world sheet and partonic 2-surfaces central for the proposed mathematical applications of TGD [K20, K11, K16, K21]. This duality suggests that the solutions to the  $CP_2 = CP_2^{mod}$  conditions could reduce to holomorphy with respect to the coordinate  $w$  for partonic 2-surface plus the analogs of Virasoro conditions. The dependence on light-like coordinate would appear as a parametric dependence.

If this were the case, TGD would reduce at least partially to what might be regarded as dual string models in  $G_2/SU(3)$  and  $SU(3)/U(2)$  and also to string model in  $M^4$  and  $X^4$ ! In the previous arguments one ends up to string models in moduli spaces of string world sheets and partonic 2-surfaces. TGD seems to yield an inflation of string models! This not actually surprising since the slicing of space-time sheets by string world sheets and partonic 2-surfaces implies automatically various kinds of maps having interpretation in terms of string orbits.

## 4 In What Sense TGD Could Be An Integrable Theory?

During years evidence supporting the idea that TGD could be an integrable theory in some sense has accumulated. The challenge is to show that various ideas about what integrability

means form pieces of a bigger coherent picture. Of course, some of the ideas are doomed to be only partially correct or simply wrong. Since it is not possible to know beforehand what ideas are wrong and what are right the situation is very much like in experimental physics and it is easy to claim (and has been and will be claimed) that all this argumentation is useless speculation. This is the price that must be paid for real thinking.

Integrable theories allow to solve nonlinear classical dynamics in terms of scattering data for a linear system. In TGD framework this translates to quantum classical correspondence. The solutions of Kähler-Dirac equation define the scattering data. This data should define a real analytic function whose octonionic extension defines the space-time surface as a surface for which its imaginary part in the representation as bi-quaternion vanishes. There are excellent hopes about this thanks to the reduction of the Kähler-Dirac equation to geometric optics.

In the following I will first discuss briefly what integrability means in (quantum) field theories, list some bits of evidence for integrability in TGD framework, discuss once again the question whether the different pieces of evidence are consistent with other and what one really means with various notions. As an outcome I represent what I regard as a more coherent view about integrability of TGD. The notion of octonion analyticity developed in the previous section is essential for the for what follows.

## 4.1 What Integrable Theories Are?

The following is an attempt to get some bird's eye of view about the landscape of integrable theories.

### 4.1.1 Examples of integrable theories

Integrable theories are typically non-linear 1+1-dimensional (quantum) field theories. Solitons and various other particle like structures are the characteristic phenomenon in these theories. Scattering matrix is trivial in the sense that the particles go through each other in the scattering and suffer only a phase change. In particular, momenta are conserved. Korteweg-de Vries equation (see <http://tinyurl.com/3cyt8hk>) [B2] was motivated by the attempt to explain the experimentally discovered shallow water wave preserving its shape and moving with a constant velocity. Sine-Gordon equation (see <http://tinyurl.com/yaf1243x>) [B6] describes geometrically constant curvature surfaces and defines a Lorentz invariant non-linear field theory in 1+1-dimensional space-time, which can be applied to Josephson junctions (in TGD inspired quantum biology it is encountered in the model of nerve pulse [K15]). Non-linear Schrödinger equation (see <http://tinyurl.com/y88efb07>) [B5] having applications to optics and water waves represents a further example. All these equations have various variants.

From TGD point of view conformal field theories represent an especially interesting example of integrable theories. (Super-)conformal invariance is the basic underlying symmetry and by its infinite-dimensional character implies infinite number of conserved quantities. The construction of the theory reduces to the construction of the representations of (super-)conformal algebra. One can solve 2-point functions exactly and characterize them in terms of (possibly anomalous) scaling dimensions of conformal fields involved and the coefficients appearing in 3-point functions can be solved in terms of fusion rules leading to an associative algebra for conformal fields. The basic applications are to 2-dimensional critical thermodynamical systems whose scaling invariance generalizes to conformal invariance. String models represent second application in which a collection of super-conformal field theories associated with various genera of 2-surface is needed to describe loop corrections to the scattering amplitudes. Also moduli spaces of conformal equivalence classes become important.

Topological quantum field theories (see <http://tinyurl.com/lsvx7g3>) are also examples of integrable theories. Because of its independence on the metric Chern-Simons action (see <http://tinyurl.com/ydgsqm2c>) is in 3-D case the unique action defining a topological quantum field theory. The calculations of knot invariants (for TGD approach see [K20]), topological invariants of 3-manifolds and 4-manifolds, and topological quantum computation

(see <http://tinyurl.com/dkpo4y>) (for a model of DNA as topological quantum computer see [K8] ) represent applications of this approach. TGD as almost topological QFT means that the Kähler action for preferred extremals reduces to a surface term by the vanishing of Coulomb term in action and by the weak form of electric-magnetic duality reduces to Chern-Simons action. Both Euclidian and Minkowskian regions give this kind of contribution.

$\mathcal{N} = 4$  SYM is the a four-dimensional and very nearly realistic candidate for an integral quantum field theory. The observation that twistor amplitudes allow also a dual of the 4-D conformal symmetry motivates the extension of this symmetry to its infinite-dimensional Yangian variant [A10]. Also the enormous progress in the construction of scattering amplitudes suggests integrability. In TGD framework Yangian symmetry would emerge naturally by extending the symplectic variant of Kac-Moody algebra from light-cone boundary to the interior of causal diamond and the Kac-Moody algebra from light-like 3-surface representing wormhole throats at which the signature of the induced metric changes to the space-time interior [K28].

#### 4.1.2 About mathematical methods

The mathematical methods used in integrable theories are rather refined and have contributed to the development of the modern mathematical physics. Mention only quantum groups, conformal algebras, and Yangian algebras.

The basic element of integrability is the possibility to transform the non-linear classical problem for which the interaction is characterized by a potential function or its analog to a linear scattering problem depending on time. For instance, for the ordinary Schrödinger function one can solve potential once single solution of the equation is known. This does not work in practice. One can however gather information about the asymptotic states in scattering to deduce the potential. One cannot do without information about bound state energies too.

In TGD framework asymptotic states correspond to partonic 2-surfaces at the two light-like boundaries of CD (more precisely: the largest CD involved and defining the IR resolution for momenta). From the scattering data coding information about scattering for various values of energy of the incoming particle one deduced the potential function or its analog.

- (a) The basic tool is inverse scattering transform known as Gelfand-Marchenko-Levitan (GML) transform (see <http://tinyurl.com/y9f7ybln>) described in simple terms in [B7].
  - i. In 1+1 dimensional case the S-matrix characterizing scattering is very simple since the only thing that can take place in scattering is reflection or transmission. Therefore the S-matrix elements describe either of these processes and by unitarity the sum of corresponding probabilities equals to 1. The particle can arrive to the potential either from left or right and is characterized by a momentum. The transmission coefficient can have a pole meaning complex (imaginary in the simplest case) wave vector serving as a signal for the formation of a bound state or resonance. The scattering data are represented by the reflection and transmission coefficients as function of time.
  - ii. One can deduce an integral equation for a propagator like function  $K(t, x)$  describing how delta pulse moving with light velocity is scattered from the potential and is expressible in terms of time integral over scattering data with contributions from both scattering states and bound states. The derivation of GML transform [B7] uses time reversal and time translational invariance and causality defined in terms of light velocity. After some tricks one obtains the integral equation as well as an expression for the time independent potential as  $V(x) = K(x, x)$ . The argument can be generalized to more complex problems to deduce the GML transform.
- (b) The so called Lax pair (see <http://tinyurl.com/yc93nw53>) is one manner to describe integrable systems [B3]. Lax pair consists of two operators  $L$  and  $M$ . One studies what might be identified as “energy” eigenstates satisfying  $L(x, t)\Psi = \lambda\Psi$ .  $\lambda$  does not depend

on time and one can say that the dynamics is associated with  $x$  coordinate whereas as  $t$  is time coordinate parametrizing different variants of eigenvalue problem with the same spectrum for  $L$ . The operator  $M(t)$  does not depend on  $x$  at all and the independence of  $\lambda$  on time implies the condition

$$\partial_t L = [L, M] .$$

This equation is analogous to a quantum mechanical evolution equation for an operator induced by time dependent ‘‘Hamiltonian’’  $M$  and gives the non-linear classical evolution equation when the commutator on the right hand side is a multiplicative operator (so that it does not involve differential operators acting on the coordinate  $x$ ). Non-linear classical dynamics for the time dependent potential emerges as an integrability condition.

One could say that  $M(t)$  introduces the time evolution of  $L(t, x)$  as an automorphism which depends on time and therefore does not affect the spectrum. One has  $L(t, x) = U(t)L(0, x)U^{-1}(t)$  with  $dU(t)/dt = M(t)U(t)$ . The time evolution of the analog of the quantum state is given by a similar equation.

- (c) A more refined view about Lax pair is based on the observation that the above equation can be generalized so that  $M$  depends also on  $x$ . The generalization of the basic equation for  $M(x, t)$  reads as

$$\partial_t L - \partial_x M - [L, M] = 0 .$$

The condition has interpretation as a vanishing of the curvature of a gauge potential having components  $A_x = L, A_t = M$ . This generalization allows a beautiful geometric formulation of the integrability conditions and extends the applicability of the inverse scattering transform. The monodromy of the flat connection becomes important in this approach. Flat connections in moduli spaces are indeed important in topological quantum field theories and in conformal field theories.

- (d) There is also a connection with the so called Riemann-Hilbert problem (see <http://tinyurl.com/ybay4qjg>) [A8]. The monodromies of the flat connection define monodromy group and Riemann-Hilbert problem concerns the existence of linear differential equations having a given monodromy group. Monodromy group emerges in the analytic continuation of an analytic function and the action of the element of the monodromy group tells what happens for the resulting many-valued analytic function as one turns around a singularity once (‘‘mono-’’). The linear equations obviously relate to the linear scattering problem. The flat connection  $(M, L)$  in turn defines the monodromy group. What is needed is that the functions involved are analytic functions of  $(t, x)$  replaced with a complex or hyper-complex variable. Again Wick rotation is involved. Similar approach generalizes also to higher dimensional moduli spaces with complex structures. In TGD framework the effective 2-dimensionality raises the hope that this kind of mathematical apparatus could be used. An interesting possibility is that finite measurement resolution could be realized in terms of a gauge group or Kac-Moody type group represented by trivial gauge potential defining a monodromy group for  $n$ -point functions. Monodromy invariance would hold for the full  $n$ -point functions constructed in terms of analytic  $n$ -point functions and their conjugates. The ends of braid strands are natural candidates for the singularities around which monodromies are defined.

## 4.2 Why TGD Could Be Integrable Theory In Some Sense?

There are many indications that TGD could be an integrable theory in some sense. The challenge is to see which ideas are consistent with each other and to build a coherent picture where everything finds its own place.

- (a) 2-dimensionality or at least effective 2-dimensionality seems to be a prerequisite for integrability. Effective 2-dimensionality is suggested by the strong form of General

Coordinate Invariance implying also holography and generalized conformal invariance predicting infinite number of conservation laws. The dual roles of partonic 2-surfaces and string world sheets supports a four-dimensional generalization of conformal invariance. Twistor considerations [K28, K26] indeed suggest that Yangian invariance and Kac-Moody invariances combine to a 4-D analog of conformal invariance induced by 2-dimensional one by algebraic continuation.

- (b) Octonionic representation of imbedding space Clifford algebra and the identification of the space-time surfaces as quaternionic space-time surfaces would define a number theoretically natural generalization of conformal invariance. The reason for using gamma matrix representation is that vector field representation for octonionic units does not exist. The problem concerns the precise meaning of the octonionic representation of gamma matrices.

Space-time surfaces could be quaternionic also in the sense that conformal invariance is analytically continued from string curve to 8-D space by octonion real-analyticity. The question is whether the Clifford algebra based notion of tangent space quaternionicity is equivalent with octonionic real-analyticity based notion of quaternionicity.

The notions of co-associativity and co-quaternionicity make also sense and one must consider seriously the possibility that associativity-co-associativity dichotomy corresponds to Minkowskian-Euclidian dichotomy.

- (c) Field equations define hydrodynamic Beltrami flows satisfying integrability conditions of form  $J \wedge dJ = 0$ .

- i. One can assign local momentum and polarization directions to the preferred extremals and this gives a decomposition of Minkowskian space-time regions to massless quanta analogous to the 1+1-dimensional decomposition to solitons. The linear superposition of modes with 4-momenta with different directions possible for free Maxwell action does not look plausible for the preferred extremals of Kähler action. This rather quantal and solitonic character is in accordance with the quantum classical correspondence giving very concrete connection between quantal and classical particle pictures. For 4-D volume action one does not obtain this kind of decomposition. In 2-D case volume action gives superposition of solutions with different polarization directions so that the situation is nearer to that for free Maxwell action and is not like soliton decomposition.

- ii. Beltrami property in strong sense allows to identify 4 preferred coordinates for the space-time surface in terms of corresponding Beltrami flows. This is possible also in Euclidian regions using two complex coordinates instead of hyper-complex coordinate and complex coordinate. The assumption that isometry currents are parallel to the same light-like Beltrami flow implies hydrodynamic character of the field equations in the sense that one can say that each flow line is analogous to particle carrying some quantum numbers. This property is not true for all extremals (say cosmic strings).

- iii. The tangent bundle theoretic view about integrability is that one can find a Lie algebra of vector fields in some manifold spanning the tangent space of a lower-dimensional manifolds and is expressed in terms of Frobenius theorem (see <http://tinyurl.com/of6vfz5>) [A2]. The gradients of scalar functions defining Beltrami flows appearing in the ansatz for preferred extremals would define these vector fields and the slicing. Partonic 2-surfaces would correspond to two complex conjugate vector fields (local polarization direction) and string world sheets to light-like vector field and its dual (light-like momentum directions). This slicing generalizes to the Euclidian regions.

- (d) Infinite number of conservation laws is the signature of integrability. Classical field equations follow from the condition that the vector field defined by Kähler-Dirac gamma matrices has vanishing divergence and can be identified an integrability condition for the Kähler-Dirac equation guaranteeing also the conservation of super currents so that one obtains an infinite number of conserved charges.

- (e) Quantum criticality is a further signal of integrability. 2-D conformal field theories describe critical systems so that the natural guess is that quantum criticality in TGD framework relates to the generalization of conformal invariance and to integrability. Quantum criticality implies that Kähler coupling strength is analogous to critical temperature. This condition does affect classical field equations only via boundary conditions expressed as weak form of electric magnetic duality at the wormhole throats at which the signature of the metric changes.

For finite-dimensional systems the vanishing of the determinant of the matrix defined by the second derivatives of potential is similar signature and applies in catastrophe theory. Therefore the existence of vanishing second variations of Kähler action should characterize criticality and define a property of preferred extremals. The vanishing of second variations indeed leads to an infinite number of conserved currents [K3] following the conditions that the deformation of Kähler-Dirac gamma matrix is also divergenceless and that the Kähler-Dirac equation associated with it is satisfied.

### 4.3 Could TGD Be An Integrable Theory?

Consider first the abstraction of integrability in TGD framework. Quantum classical correspondence could be seen as a correspondence between linear quantum dynamics and non-linear classical dynamics. Integrability would realize this correspondence. In integrable models such as Sine-Gordon equation particle interactions are described by potential in 1+1 dimensions. This too primitive for the purposes of TGD. The vertices of generalized Feynman diagrams take care of this. At lines one has free particle dynamics so that the situation could be much simpler than in integrable models if one restricts the considerations to the lines or Minkowskian space-time regions surrounding them.

The non-linear dynamics for the space-time sheets representing incoming lines of generalized Feynman diagram should be obtainable from the linear dynamics for the induced spinor fields defined by Kähler-Dirac operator. There are two options.

- (a) Strong form of the quantum classical correspondence states that each solution for the linear dynamics of spinor fields corresponds to space-time sheet. This is analogous to solving the potential function in terms of a single solution of Schrödinger equation. Coupling of space-time geometry to quantum numbers via measurement interaction term is a proposal for realizing this option. It is however the quantum numbers of positive/negative energy parts of zero energy state which would be visible in the classical dynamics rather than those of induced spinor field modes.
- (b) Only overall dynamics characterized by scattering data- the counterpart of  $S$ -matrix for the Kähler-Dirac operator- is mapped to the geometry of the space-time sheet. This is much more abstract realization of quantum classical correspondence.
- (c) Can these two approaches be equivalent? This might be the case since quantum numbers of the state are not those of the modes of induced spinor fields.

What the scattering data could be for the induced spinor field satisfying Kähler-Dirac equation?

- (a) If the solution of field equation has hydrodynamic character, the solutions of the Kähler-Dirac equation can be localized to light-like Beltrami flow lines of hydrodynamic flow. These correspond to basic solutions and the general solution is a superposition of these. There is no dispersion and the dynamics is that of geometric optics at the basic level. This means geometric optics like character of the spinor dynamics.

Solutions of the Kähler-Dirac equation are completely analogous to the pulse solutions defining the fundamental solution for the wave equation in the argument leading from wave equation with external time independent potential to Marchenko-Gelfand-Levitan equation allowing to identify potential in terms of scattering data. There is however no potential present now since the interactions are described by the vertices of Feynman

diagram where the particle lines meet. Note that particle like regions are Euclidian and that this picture applies only to the Minkowskian exteriors of particles.

- (b) Partonic 2-surfaces at the ends of the line of generalized Feynman diagram are connected by flow lines. Partonic 2-surfaces at which the signature of the induced metric changes are in a special position. Only the imaginary part of the bi-quaternionic value of the octonion valued map is non-vanishing at these surfaces which can be said to be co-complex 2-surfaces. By geometric optics behavior the scattering data correspond to a diffeomorphism mapping initial partonic 2-surface to the final one in some preferred complex coordinates common to both ends of the line.
- (c) What could be these preferred coordinates? Complex coordinates for  $S^2$  at light-cone boundary define natural complex coordinates for the partonic 2-surface. With these coordinates the diffeomorphism defining scattering data is diffeomorphism of  $S^2$ . Suppose that this map is real analytic so that maps “real axis” of  $S^2$  to itself. This map would be same as the map defining the octonionic real analyticity as algebraic extension of the complex real analytic map. By octonionic analyticity one can make large number of alternative choices for the coordinates of partonic 2-surface.
- (d) There can be non-uniqueness due to the possibility of  $G_2/SU(3)$  valued map characterizing the local octonionic units. The proposal is that the choice of octonionic imaginary units can depend on the point of string like orbit: this would give string model in  $G_2/SU(3)$ . Conformal invariance for this string model would imply analyticity and helps considerably but would not probably fix the situation completely since the element of the coset space would constant at the partonic 2-surfaces at the ends of CD. One can of course ask whether the  $G_2/SU(3)$  element could be constant for each propagator line and would change only at the 2-D vertices?

This would be the inverse scattering problem formulated in the spirit of TGD. There could be also dependence of space-time surface on quantum numbers of quantum states but not on individual solution for the induced spinor field since the scattering data of this solution would be purely geometric.

## 5 Do Geometric Invariants Of Preferred Extremals Define Topological Invariants Of Space-time Surface And Code For Quantumphysics?

The recent progress in the understanding of preferred extremals [K3] led to a reduction of the field equations to conditions stating for Euclidian signature the existence of Kähler metric. The resulting conditions are a direct generalization of corresponding conditions emerging for the string world sheet and stating that the 2-metric has only non-diagonal components in complex/hypercomplex coordinates. Also energy momentum of Kähler action and has this characteristic (1, 1) tensor structure. In Minkowskian signature one obtains the analog of 4-D complex structure combining hyper-complex structure and 2-D complex structure.

The construction lead also to the understanding of how Einstein’s equations with cosmological term follow as a consistency condition guaranteeing that the covariant divergence of the Maxwell’s energy momentum tensor assignable to Kähler action vanishes. This gives  $T = kG + \Lambda g$ . By taking trace a further condition follows from the vanishing trace of  $T$ :

$$R = \frac{4\Lambda}{k} . \tag{5.1}$$

That any preferred extremal should have a constant Ricci scalar proportional to cosmological constant is very strong prediction. Note that the accelerating expansion of the Universe would support positive value of  $\Lambda$ . Note however that both  $\Lambda$  and  $k \propto 1/G$  are both parameters characterizing one particular preferred extremal. One could of course argue that the dynamics

allowing only constant curvature space-times is too simple. The point is however that particle can topologically condense on several space-time sheets meaning effective superposition of various classical fields defined by induced metric and spinor connection.

The following considerations demonstrate that preferred extremals can be seen as canonical representatives for the constant curvature manifolds playing central role in Thurston's geometrization theorem (see <http://tinyurl.com/y8bbz1nr>) [A9] known also as hyperbolization theorem implying that geometric invariants of space-time surfaces transform to topological invariants. The generalization of the notion of Ricci flow to Maxwell flow in the space of metrics and further to Kähler flow for preferred extremals in turn gives a rather detailed vision about how preferred extremals organize to one-parameter orbits. It is quite possible that Kähler flow is actually discrete. The natural interpretation is in terms of dissipation and self organization.

Quantum classical correspondence suggests that this line of thought could be continued even further: could the geometric invariants of the preferred extremals code not only for space-time topology but also for quantum physics? How to calculate the correlation functions and coupling constant evolution has remained a basic unresolved challenge of quantum TGD. Could the correlation functions be reduced to statistical geometric invariants of preferred extremals? The latest (means the end of 2012) and perhaps the most powerful idea hitherto about coupling constant evolution is quantum classical correspondence in statistical sense stating that the statistical properties of a preferred extremal in quantum superposition of them are same as those of the zero energy state in question. This principle would be quantum generalization of ergodic theorem stating that the time evolution of a single member of ensemble represents the ensemble statistically. This principle would allow to deduce correlation functions and S-matrix from the statistical properties of single preferred extremal alone using classical intuition. Also coupling constant evolution would be coded by the statistical properties of the representative preferred extremal.

## 5.1 Preferred Extremals Of Kähler Action As Manifolds With Constant Ricci Scalar Whose Geometric Invariants Are Topological Invariants

An old conjecture inspired by the preferred extremal property is that the geometric invariants of space-time surface serve as topological invariants. The reduction of Kähler action to 3-D Chern-Simons terms (see <http://tinyurl.com/ybp86sho>) [K3] gives support for this conjecture as a classical counterpart for the view about TGD as almost topological QFT. The following arguments give a more precise content to this conjecture in terms of existing mathematics.

- (a) It is not possible to represent the scaling of the induced metric as a deformation of the space-time surface preserving the preferred extremal property since the scale of  $CP_2$  breaks scale invariance. Therefore the curvature scalar cannot be chosen to be equal to one numerically. Therefore also the parameter  $R = 4\Lambda/k$  and also  $\Lambda$  and  $k$  separately characterize the equivalence class of preferred extremals as is also physically clear.

Also the volume of the space-time sheet closed inside causal diamond CD remains constant along the orbits of the flow and thus characterizes the space-time surface.  $\Lambda$  and even  $k \propto 1/G$  can indeed depend on space-time sheet and p-adic length scale hypothesis suggests a discrete spectrum for  $\Lambda/k$  expressible in terms of p-adic length scales:  $\Lambda/k \propto 1/L_p^2$  with  $p \simeq 2^k$  favored by p-adic length scale hypothesis. During cosmic evolution the p-adic length scale would increase gradually. This would resolve the problem posed by cosmological constant in GRT based theories.

- (b) One could also see the preferred extremals as 4-D counterparts of constant curvature 3-manifolds in the topology of 3-manifolds. An interesting possibility raised by the observed negative value of  $\Lambda$  is that most 4-surfaces are constant negative curvature 4-manifolds. By a general theorem coset spaces (see <http://tinyurl.com/y8d3udpr>)  $H^4/\Gamma$ , where  $H^4 = SO(1,4)/SO(4)$  is hyperboloid of  $M^5$  and  $\Gamma$  a torsion free discrete

subgroup of  $SO(1,4)$  [A3]. It is not clear to me, whether the constant value of Ricci scalar implies constant sectional curvatures and therefore hyperbolic space property. It could happen that the space of spaces with constant Ricci curvature contain a hyperbolic manifold as an especially symmetric representative. In any case, the geometric invariants of hyperbolic metric are topological invariants.

By Mostow rigidity theorem (see <http://tinyurl.com/yacbu8sk>) [A4] finite-volume hyperbolic manifold is unique for  $D > 2$  and determined by the fundamental group of the manifold. Since the orbits under the Kähler flow preserve the curvature scalar the manifolds at the orbit must represent different imbeddings of one and hyperbolic 4-manifold. In 2-D case the moduli space for hyperbolic metric for a given genus  $g > 0$  is defined by Teichmueller parameters and has dimension  $6(g - 1)$ . Obviously the exceptional character of  $D = 2$  case relates to conformal invariance. Note that the moduli space in question (see <http://tinyurl.com/ybowqm5v>) plays a key role in p-adic mass calculations [K5].

In the recent case Mostow rigidity theorem could hold true for the Euclidian regions and maybe generalize also to Minkowskian regions. If so then both “topological” and “geometro” in “Topological Geometrodynamics” would be fully justified. The fact that geometric invariants become topological invariants also conforms with “TGD as almost topological QFT” and allows the notion of scale to find its place in topology. Also the dream about exact solvability of the theory would be realized in rather convincing manner.

These conjectures are the main result independent of whether the generalization of the Ricci flow discussed in the sequel exists as a continuous flow or possibly discrete sequence of iterates in the space of preferred extremals of Kähler action. My sincere hope is that the reader could grasp how far reaching these result really are.

## 5.2 Is There A Connection Between Preferred Extremals And AdS<sub>4</sub>/CFT Correspondence?

The preferred extremals satisfy Einstein Maxwell equations with a cosmological constant and have negative scalar curvature for negative value of  $\Lambda$ . 4-D space-times with hyperbolic metric provide canonical representation for a large class of four-manifolds and an interesting question is whether these spaces are obtained as preferred extremals and/or vacuum extremals.

4-D hyperbolic space with Minkowski signature is locally isometric with AdS<sub>4</sub>. This suggests at connection with AdS<sub>4</sub>/CFT correspondence of M-theory. The boundary of AdS would be now replaced with 3-D light-like orbit of partonic 2-surface at which the signature of the induced metric changes. The metric 2-dimensionality of the light-like surface makes possible generalization of 2-D conformal invariance with the light-like coordinate taking the role of complex coordinate at light-like boundary. AdS could represent a special case of a more general family of space-time surfaces with constant Ricci scalar satisfying Einstein-Maxwell equations and generalizing the AdS<sub>4</sub>/CFT correspondence. There is however a strong objection from cosmology: the accelerated expansion of the Universe requires positive value of  $\Lambda$  and favors De Sitter Space  $dS_4$  instead of  $AdS_4$ .

These observations provide motivations for finding whether AdS<sub>4</sub> and/or  $dS_4$  allows an imbedding as a vacuum extremal to  $M^4 \times S^2 \subset M^4 \times CP_2$ , where  $S^2$  is a homologically trivial geodesic sphere of  $CP_2$ . It is easy to guess the general form of the imbedding by writing the line elements of,  $M^4$ ,  $S^2$ , and AdS<sub>4</sub>.

- (a) The line element of  $M^4$  in spherical Minkowski coordinates  $(m, r_M, \theta, \phi)$  reads as

$$ds^2 = dm^2 - dr_M^2 - r_M^2 d\Omega^2 . \quad (5.2)$$

- (b) Also the line element of  $S^2$  is familiar:

$$ds^2 = -R^2(d\Theta^2 + \sin^2(\theta)d\Phi^2) . \quad (5.3)$$

- (c) By visiting in Wikipedia (see <http://tinyurl.com/y9hw95q1>) one learns that in spherical coordinate the line element of AdS<sub>4</sub>/dS<sub>4</sub> is given by

$$\begin{aligned} ds^2 &= A(r)dt^2 - \frac{1}{A(r)}dr^2 - r^2d\Omega^2 , \\ A(r) &= 1 + \epsilon y^2 , \quad y = \frac{r}{r_0} , \\ \epsilon &= 1 \text{ for } AdS_4 , \quad \epsilon = -1 \text{ for } dS_4 . \end{aligned} \quad (5.4)$$

- (d) From these formulas it is easy to see that the ansatz is of the same general form as for the imbedding of Schwarzschild-Nordstöm metric:

$$\begin{aligned} m &= \Lambda t + h(y) , \quad r_M = r , \\ \Theta &= s(y) , \quad \Phi = \omega(t + f(y)) . \end{aligned} \quad (5.5)$$

The non-trivial conditions on the components of the induced metric are given by

$$\begin{aligned} g_{tt} &= \Lambda^2 - x^2 \sin^2(\Theta) = A(r) , \\ g_{tr} &= \frac{1}{r_0} \left[ \Lambda \frac{dh}{dy} - x^2 \sin^2(\theta) \frac{df}{dr} \right] = 0 , \\ g_{rr} &= \frac{1}{r_0^2} \left[ \left( \frac{dh}{dy} \right)^2 - 1 - x^2 \sin^2(\theta) \left( \frac{df}{dy} \right)^2 - R^2 \left( \frac{d\Theta}{dy} \right)^2 \right] = -\frac{1}{A(r)} , \\ x &= R\omega . \end{aligned} \quad (5.6)$$

By some simple algebraic manipulations one can derive expressions for  $\sin(\Theta)$ ,  $df/dr$  and  $dh/dr$ .

- (a) For  $\Theta(r)$  the equation for  $g_{tt}$  gives the expression

$$\begin{aligned} \sin(\Theta) &= \pm \frac{P^{1/2}}{x} , \\ P &= \Lambda^2 - A = \Lambda^2 - 1 - \epsilon y^2 . \end{aligned} \quad (5.7)$$

The condition  $0 \leq \sin^2(\Theta) \leq 1$  gives the conditions

$$\begin{aligned} (\Lambda^2 - x^2 - 1)^{1/2} \leq y \leq (\Lambda^2 - 1)^{1/2} & \quad \text{for } \epsilon = 1 \text{ (} AdS_4 \text{)} , \\ (-\Lambda^2 + 1)^{1/2} \leq y \leq (x^2 + 1 - \Lambda^2)^{1/2} & \quad \text{for } \epsilon = -1 \text{ (} dS_4 \text{)} . \end{aligned} \quad (5.8)$$

Only a spherical shell is possible in both cases. The model for the final state of star considered in [K18] predicted similar layer layer like structure and inspired the proposal that stars quite generally have an onion-like structure with radii of various shells characterize by p-adic length scale hypothesis and thus coming in some powers of  $\sqrt{2}$ . This brings in mind also Titius-Bode law.

- (b) From the vanishing of  $g_{tr}$  one obtains

$$\frac{dh}{dy} = \frac{P}{\Lambda} \frac{df}{dy} . \quad (5.9)$$

(c) The condition for  $g_{rr}$  gives

$$\left(\frac{df}{dy}\right)^2 = \frac{r_0^2}{AP} [A^{-1} - R^2 \left(\frac{d\Theta}{dy}\right)^2] . \quad (5.10)$$

Clearly, the right-hand side is positive if  $P \geq 0$  holds true and  $Rd\Theta/dy$  is small. One can express  $d\Theta/dy$  using chain rule as

$$\left(\frac{d\Theta}{dy}\right)^2 = \frac{x^2 y^2}{P(P-x^2)} . \quad (5.11)$$

One obtains

$$\left(\frac{df}{dy}\right)^2 = \Lambda r_0^2 \frac{y^2}{AP} \left[ \frac{1}{1+y^2} - x^2 \left(\frac{R}{r_0}\right)^2 \frac{1}{P(P-x^2)} \right] . \quad (5.12)$$

The right hand side of this equation is non-negative for certain range of parameters and variable  $y$ . Note that for  $r_0 \gg R$  the second term on the right hand side can be neglected. In this case it is easy to integrate  $f(y)$ .

The conclusion is that both  $AdS_4$  and  $dS^4$  allow a local imbedding as a vacuum extremal. Whether also an imbedding as a non-vacuum preferred extremal to  $M^4 \times S^2$ ,  $S^2$  a homologically non-trivial geodesic sphere is possible, is an interesting question.

### 5.3 Generalizing Ricci Flow To Maxwell Flow For 4-Geometries And Kähler Flow For Space-Time Surfaces

The notion of Ricci flow has played a key part in the geometrization of topological invariants of Riemann manifolds. I certainly did not have this in mind when I choose to call my unification attempt “Topological Geometro-dynamics” but this title strongly suggests that a suitable generalization of Ricci flow could play a key role in the understanding of also TGD.

#### 5.3.1 Ricci flow and Maxwell flow for 4-geometries

The observation about constancy of 4-D curvature scalar for preferred extremals inspires a generalization of the well-known volume preserving Ricci flow (see <http://tinyurl.com/2cw1zh91>) [A7] introduced by Richard Hamilton. Ricci flow is defined in the space of Riemann metrics as

$$\frac{dg_{\alpha\beta}}{dt} = -2R_{\alpha\beta} + 2\frac{R_{avg}}{D}g_{\alpha\beta} . \quad (5.13)$$

Here  $R_{avg}$  denotes the average of the scalar curvature, and  $D$  is the dimension of the Riemann manifold. The flow is volume preserving in average sense as one easily checks ( $\langle g^{\alpha\beta} dg_{\alpha\beta}/dt \rangle = 0$ ). The volume preserving property of this flow allows to intuitively understand that the volume of a 3-manifold in the asymptotic metric defined by the Ricci flow is topological invariant. The fixed points of the flow serve as canonical representatives for the topological equivalence classes of 3-manifolds. These 3-manifolds (for instance hyperbolic 3-manifolds with constant sectional curvatures) are highly symmetric. This is easy to understand since the flow is dissipative and destroys all details from the metric.

What happens in the recent case? The first thing to do is to consider what might be called Maxwell flow in the space of all 4-D Riemann manifolds allowing Maxwell field.

- (a) First of all, the vanishing of the trace of Maxwell's energy momentum tensor codes for the volume preserving character of the flow defined as

$$\frac{dg_{\alpha\beta}}{dt} = T_{\alpha\beta} . \quad (5.14)$$

Taking covariant divergence on both sides and assuming that  $d/dt$  and  $D_\alpha$  commute, one obtains that  $T^{\alpha\beta}$  is divergenceless.

This is true if one assumes Einstein's equations with cosmological term. This gives

$$\frac{dg_{\alpha\beta}}{dt} = kG_{\alpha\beta} + \Lambda g_{\alpha\beta} = kR_{\alpha\beta} + \left(-\frac{kR}{2} + \Lambda\right)g_{\alpha\beta} . \quad (5.15)$$

The trace of this equation gives that the curvature scalar is constant. Note that the value of the Kähler coupling strength plays a highly non-trivial role in these equations and it is quite possible that solutions exist only for some critical values of  $\alpha_K$ . Quantum criticality should fix the allow value triplets  $(G, \Lambda, \alpha_K)$  apart from overall scaling

$$(G, \Lambda, \alpha_K) \rightarrow (xG, \Lambda/x, x\alpha_K) .$$

Fixing the value of  $G$  fixes the values remaining parameters at critical points. The rescaling of the parameter  $t$  induces a scaling by  $x$ .

- (b) By taking trace one obtains the already mentioned condition fixing the curvature to be constant, and one can write

$$\frac{dg_{\alpha\beta}}{dt} = kR_{\alpha\beta} - \Lambda g_{\alpha\beta} . \quad (5.16)$$

Note that in the recent case  $R_{avg} = R$  holds true since curvature scalar is constant. The fixed points of the flow would be Einstein manifolds (see <http://tinyurl.com/ybrnakuu>) [A1, A11] satisfying

$$R_{\alpha\beta} = \frac{\Lambda}{k}g_{\alpha\beta} \quad (5.17)$$

- (c) It is by no means obvious that continuous flow is possible. The condition that Einstein-Maxwell equations are satisfied might pick up from a completely general Maxwell flow a discrete subset as solutions of Einstein-Maxwell equations with a cosmological term. If so, one could assign to this subset a sequence of values  $t_n$  of the flow parameter  $t$ .
- (d) I do not know whether 3-dimensionality is somehow absolutely essential for getting the topological classification of closed 3-manifolds using Ricci flow. This ignorance allows me to pose some innocent questions. Could one have a canonical representation of 4-geometries as spaces with constant Ricci scalar? Could one select one particular Einstein space in the class four-metrics and could the ratio  $\Lambda/k$  represent topological invariant if one normalizes metric or curvature scalar suitably. In the 3-dimensional case curvature scalar is normalized to unity. In the recent case this normalization would give  $k = 4\Lambda$  in turn giving  $R_{\alpha\beta} = g_{\alpha\beta}/4$ . Does this mean that there is only single fixed point in local sense, analogous to black hole toward which all geometries are driven by the Maxwell flow? Does this imply that only the 4-volume of the original space would serve as a topological invariant?

### 5.3.2 Maxwell flow for space-time surfaces

One can consider Maxwell flow for space-time surfaces too. In this case Kähler flow would be the appropriate term and provides families of preferred extremals. Since space-time surfaces inside CD are the basic physical objects in TGD framework, a possible interpretation of these families would be as flows describing physical dissipation as a four-dimensional phenomenon polishing details from the space-time surface interpreted as an analog of Bohr orbit.

- (a) The flow is now induced by a vector field  $j^k(x, t)$  of the space-time surface having values in the tangent bundle of imbedding space  $M^4 \times CP_2$ . In the most general case one has Kähler flow without the Einstein equations. This flow would be defined in the space of all space-time surfaces or possibly in the space of all extremals. The flow equations reduce to

$$h_{kl}D_\alpha j^k(x, t)D_\beta h^l = \frac{1}{2}T_{\alpha\beta} . \quad (5.18)$$

The left hand side is the projection of the covariant gradient  $D_\alpha j^k(x, t)$  of the flow vector field  $j^k(x, t)$  to the tangent space of the space-time surface.  $D_{alpha}$  is covariant derivative taking into account that  $j^k$  is imbedding space vector field. For a fixed point space-time surface this projection must vanish assuming that this space-time surface is reachable. A good guess for the asymptotia is that the divergence of Maxwell energy momentum tensor vanishes and that Einstein's equations with cosmological constant are well-defined.

Asymptotia corresponds to vacuum extremals. In Euclidian regions  $CP_2$  type vacuum extremals and in Minkowskian regions to any space-time surface in any 6-D sub-manifold  $M^4 \times Y^2$ , where  $Y^2$  is Lagrangian sub-manifold of  $CP_2$  having therefore vanishing induced Kähler form. Symplectic transformations of  $CP_2$  combined with diffeomorphisms of  $M^4$  give new Lagrangian manifolds. One would expect that vacuum extremals are approached but never reached at second extreme for the flow.

If one assumes Einstein's equations with a cosmological term, allowed vacuum extremals must be Einstein manifolds. For  $CP_2$  type vacuum extremals this is the case. It is quite possible that these fixed points do not actually exist in Minkowskian sector, and could be replaced with more complex asymptotic behavior such as limit, chaos, or strange attractor.

- (b) The flow could be also restricted to the space of preferred extremals. Assuming that Einstein Maxwell equations indeed hold true, the flow equations reduce to

$$h_{kl}D_\alpha j^k(x, t)\partial_\beta h^l = \frac{1}{2}(kR_{\alpha\beta} - \Lambda g_{\alpha\beta}) . \quad (5.19)$$

Preferred extremals would correspond to a fixed sub-manifold of the general flow in the space of all 4-surfaces.

- (c) One can also consider a situation in which  $j^k(x, t)$  is replaced with  $j^k(h, t)$  defining a flow in the entire imbedding space. This assumption is probably too restrictive. In this case the equations reduce to

$$(D_r j_l(x, t) + D_l j_r)\partial_\alpha h^r \partial_\beta h^l = kR_{\alpha\beta} - \Lambda g_{\alpha\beta} . \quad (5.20)$$

Here  $D_r$  denotes covariant derivative. Asymptotia is achieved if the tensor  $D_k j_l + D_l j_k$  becomes orthogonal to the space-time surface. Note for that Killing vector fields of  $H$  the left hand side vanishes identically. Killing vector fields are indeed symmetries of also asymptotic states.

It must be made clear that the existence of a continuous flow in the space of preferred extremals might be too strong a condition. Already the restriction of the general Maxwell flow in the space of metrics to solutions of Einstein-Maxwell equations with cosmological term might lead to discretization, and the assumption about representability as 4-surface in  $M^4 \times CP_2$  would give a further condition reducing the number of solutions. On the other hand, one might consider a possibility of a continuous flow in the space of constant Ricci scalar metrics with a fixed 4-volume and having hyperbolic spaces as the most symmetric representative.

### 5.3.3 Dissipation, self organization, transition to chaos, and coupling constant evolution

A beautiful connection with concepts like dissipation, self-organization, transition to chaos, and coupling constant evolution suggests itself.

- (a) It is not at all clear whether the vacuum extremal limits of the preferred extremals can correspond to Einstein spaces except in special cases such as  $CP_2$  type vacuum extremals isometric with  $CP_2$ . The imbeddability condition however defines a constraint force which might well force asymptotically more complex situations such as limit cycles and strange attractors. In ordinary dissipative dynamics an external energy feed is essential prerequisite for this kind of non-trivial self-organization patterns.

In the recent case the external energy feed could be replaced by the constraint forces due to the imbeddability condition. It is not too difficult to imagine that the flow (if it exists!) could define something analogous to a transition to chaos taking place in a stepwise manner for critical values of the parameter  $t$ . Alternatively, these discrete values could correspond to those values of  $t$  for which the preferred extremal property holds true for a general Maxwell flow in the space of 4-metrics. Therefore the preferred extremals of Kähler action could emerge as one-parameter (possibly discrete) families describing dissipation and self-organization at the level of space-time dynamics.

- (b) For instance, one can consider the possibility that in some situations Einstein's equations split into two mutually consistent equations of which only the first one is independent

$$\begin{aligned} xJ^\alpha{}_\nu J^{\nu\beta} &= R^{\alpha\beta} , \\ L_K &= xJ^\alpha{}_\nu J^{\nu\beta} = 4\Lambda , \\ x &= \frac{1}{16\pi\alpha_K} . \end{aligned} \tag{5.21}$$

Note that the first equation indeed gives the second one by tracing. This happens for  $CP_2$  type vacuum extremals.

Kähler action density would reduce to cosmological constant which should have a continuous spectrum if this happens always. A more plausible alternative is that this holds true only asymptotically. In this case the flow equation could not lead arbitrary near to vacuum extremal, and one can think of situation in which  $L_K = 4\Lambda$  defines an analog of limiting cycle or perhaps even strange attractor. In any case, the assumption would allow to deduce the asymptotic value of the action density which is of utmost importance from calculational point of view: action would be simply  $S_K = 4\Lambda V_4$  and one could also say that one has minimal surface with  $\Lambda$  taking the role of string tension.

- (c) One of the key ideas of TGD is quantum criticality implying that Kähler coupling strength is analogous to critical temperature. Second key idea is that p-adic coupling constant evolution represents discretized version of continuous coupling constant evolution so that each p-adic prime would correspond a fixed point of ordinary coupling constant evolution in the sense that the 4-volume characterized by the p-adic length scale remains constant. The invariance of the geometric and thus geometric parameters of hyperbolic 4-manifold under the Kähler flow would conform with the interpretation as a flow preserving scale assignable to a given p-adic prime. The continuous evolution

in question (if possible at all!) might correspond to a fixed p-adic prime. Also the hierarchy of Planck constants relates to this picture naturally. Planck constant  $\hbar_{eff} = n\hbar$  corresponds to a multi-furcation generating n-sheeted structure and certainly affecting the fundamental group.

- (d) One can of course question the assumption that a continuous flow exists. The property of being a solution of Einstein-Maxwell equations, imbeddability property, and preferred extremal property might allow only discrete sequences of space-time surfaces perhaps interpretable as orbit of an iterated map leading gradually to a fractal limit. This kind of discrete sequence might be also be selected as preferred extremals from the orbit of Maxwell flow without assuming Einstein-Maxwell equations. Perhaps the discrete p-adic coupling constant evolution could be seen in this manner and be regarded as an iteration so that the connection with fractality would become obvious too.

### 5.3.4 Does a 4-D counterpart of thermodynamics make sense?

The interpretation of the Kähler flow in terms of dissipation, the constancy of  $R$ , and almost constancy of  $L_K$  suggest an interpretation in terms of 4-D variant of thermodynamics natural in zero energy ontology (ZEO), where physical states are analogs for pairs of initial and final states of quantum event are quantum superpositions of classical time evolutions. Quantum theory becomes a “square root” of thermodynamics so that 4-D analog of thermodynamics might even replace ordinary thermodynamics as a fundamental description. If so this 4-D thermodynamics should be qualitatively consistent with the ordinary 3-D thermodynamics.

- (a) The first naive guess would be the interpretation of the action density  $L_K$  as an analog of energy density  $e = E/V_3$  and that of  $R$  as the analog to entropy density  $s = S/V_3$ . The asymptotic states would be analogs of thermodynamical equilibria having constant values of  $L_K$  and  $R$ .
- (b) Apart from an overall sign factor  $\epsilon$  to be discussed, the analog of the first law  $de = Tds - pdV/V$  would be

$$dL_K = kdR + \Lambda \frac{dV_4}{V_4} .$$

One would have the correspondences  $S \rightarrow \epsilon RV_4$ ,  $e \rightarrow \epsilon L_K$  and  $k \rightarrow T$ ,  $p \rightarrow -\Lambda$ .  $k \propto 1/G$  indeed appears formally in the role of temperature in Einstein’s action defining a formal partition function via its exponent. The analog of second law would state the increase of the magnitude of  $\epsilon RV_4$  during the Kähler flow.

- (c) One must be very careful with the signs and discuss Euclidian and Minkowskian regions separately. Concerning purely thermodynamic aspects at the level of vacuum functional Euclidian regions are those which matter.
  - i. For  $CP_2$  type vacuum extremals  $L_K \propto E^2 + B^2$ ,  $R = \Lambda/k$ , and  $\Lambda$  are positive. In thermodynamical analogy for  $\epsilon = 1$  this would mean that pressure is negative.
  - ii. In Minkowskian regions the value of  $R = \Lambda/k$  is negative for  $\Lambda < 0$  suggested by the large abundance of 4-manifolds allowing hyperbolic metric and also by cosmological considerations. The asymptotic formula  $L_K = 4\Lambda$  considered above suggests that also Kähler action is negative in Minkowskian regions for magnetic flux tubes dominating in TGD inspired cosmology: the reason is that the magnetic contribution to the action density  $L_K \propto E^2 - B^2$  dominates.

Consider now in more detail the 4-D thermodynamics interpretation in Euclidian and Minkowskian regions assuming that the evolution by quantum jumps has Kähler flow as a space-time correlate.

- (a) In Euclidian regions the choice  $\epsilon = 1$  seems to be more reasonable one. In Euclidian regions  $-\Lambda$  as the analog of pressure would be negative, and asymptotically (that is for

$CP_2$  type vacuum extremals) its value would be proportional to  $\Lambda \propto 1/GR^2$ , where  $R$  denotes  $CP_2$  radius defined by the length of its geodesic circle.

A possible interpretation for negative pressure is in terms of string tension effectively inducing negative pressure (note that the solutions of the Kähler-Dirac equation indeed assign a string to the wormhole contact). The analog of the second law would require the increase of  $RV_4$  in quantum jumps. The magnitudes of  $L_K$ ,  $R$ ,  $V_4$  and  $\Lambda$  would be reduced and approach their asymptotic values. In particular,  $V_4$  would approach asymptotically the volume of  $CP_2$ .

- (b) In Minkowskian regions Kähler action contributes to the vacuum functional a phase factor analogous to an imaginary exponent of action serving in the role of Morse function so that thermodynamics interpretation can be questioned. Despite this one can check whether thermodynamic interpretation can be considered. The choice  $\epsilon = -1$  seems to be the correct choice now.  $-\Lambda$  would be analogous to a negative pressure whose gradually decreases. In 3-D thermodynamics it is natural to assign negative pressure to the magnetic flux tube like structures as their effective string tension defined by the density of magnetic energy per unit length.  $-R \geq 0$  would entropy and  $-L_K \geq 0$  would be the analog of energy density.

$R = \Lambda/k$  and the reduction of  $\Lambda$  during cosmic evolution by quantum jumps suggests that the larger the volume of CD and thus of (at least) Minkowskian space-time sheet the smaller the negative value of  $\Lambda$ .

Assume the recent view about state function reduction explaining how the arrow of geometric time is induced by the quantum jump sequence defining experienced time [K2]. According to this view zero energy states are quantum superpositions over CDs of various size scales but with common tip, which can correspond to either the upper or lower light-like boundary of CD. The sequence of quantum jumps the gradual increase of the average size of CD in the quantum superposition and therefore that of average value of  $V_4$ . On the other hand, a gradual decrease of both  $-L_K$  and  $-R$  looks physically very natural. If Kähler flow describes the effect of dissipation by quantum jumps in ZEO then the space-time surfaces would gradually approach nearly vacuum extremals with constant value of entropy density  $-R$  but gradually increasing 4-volume so that the analog of second law stating the increase of  $-RV_4$  would hold true.

- (c) The interpretation of  $-R > 0$  as negentropy density assignable to entanglement is also possible and is consistent with the interpretation in terms of second law. This interpretation would only change the sign factor  $\epsilon$  in the proposed formula. Otherwise the above arguments would remain as such.

## 5.4 Could Correlation Functions, S-Matrix, And Coupling Constant Evolution Be Coded The Statistical Properties Of Preferred Extremals?

How to calculate the correlation functions and coupling constant evolution has remained a basic unresolved challenge. Generalized Feynman diagrams provide a powerful vision which however does not help in practical calculations. Some big idea has been lacking.

Quantum classical correspondence states that all aspects of quantum states should have correlates in the geometry of preferred extremals. In particular, various elementary particle propagators should have a representation as properties of preferred extremals. This would allow to realize the old dream about being able to say something interesting about coupling constant evolution although it is not yet possible to calculate the M-matrices and U-matrix. The general structure of U-matrix is however understood [K22]. Hitherto everything that has been said about coupling constant evolution has been rather speculative arguments except for the general vision that it reduces to a discrete evolution defined by p-adic length scales. General first principle definitions are however much more valuable than ad hoc guesses even if the latter give rise to explicit formulas.

In quantum TGD and also at its QFT limit various correlation functions in given quantum state should code for its properties. By quantum classical correspondence these correlation

functions should have counterparts in the geometry of preferred extremals. Even more: these classical counterparts for a given preferred extremal ought to be identical with the quantum correlation functions for the superposition of preferred extremals. This correspondence could be called quantum ergodicity by its analogy with ordinary ergodicity stating that the member of ensemble becomes representative of ensemble.

This principle would be a quantum generalization of ergodic theorem stating that the time evolution of a single member of ensemble represents the ensemble statistically. This symmetry principle analogous to holography might allow to fix S-matrix uniquely even in the case that the hermitian square root of the density matrix appearing in the M-matrix would lead to a breaking of quantum ergodicity as also 4-D spin glass degeneracy suggests.

This principle would allow to deduce correlation functions from the statistical properties of single preferred extremal alone using just classical intuition. Also coupling constant evolution would be coded by the statistical properties of preferred extremals. Quantum ergodicity would mean an enormous simplification since one could avoid the horrible conceptual complexities involved with the functional integrals over WCW .

This might of course be too optimistic guess. If a sub-algebra of symplectic algebra acts as gauge symmetries of the preferred extremals in the sense that corresponding Noether charges vanish, it can quite well be that correlations functions correspond to averages for extremals belonging to single conformal equivalence class.

- (a) The marvellous implication of quantum ergodicity would be that one could calculate everything solely classically using the classical intuition - the only intuition that we have. Quantum ergodicity would also solve the paradox raised by the quantum classical correspondence for momentum eigenstates. Any preferred extremal in their superposition defining momentum eigenstate should code for the momentum characterizing the superposition itself. This is indeed possible if every extremal in the superposition codes the momentum to the properties of classical correlation functions which are identical for all of them.
- (b) The only manner to possibly achieve quantum ergodicity is in terms of the statistical properties of the preferred extremals. It should be possible to generalize the ergodic theorem stating that the properties of statistical ensemble are represented by single space-time evolution in the ensemble of time evolutions. Quantum superposition of classical worlds would effectively reduce to single classical world as far as classical correlation functions are considered. The notion of finite measurement resolution suggests that one must state this more precisely by adding that classical correlation functions are calculated in a given UV and IR resolutions meaning UV cutoff defined by the smallest CD and IR cutoff defined by the largest CD present.
- (c) The skeptic inside me immediately argues that TGD Universe is 4-D spin glass so that this quantum ergodic theorem must be broken. In the case of the ordinary spin classes one has not only statistical average for a fixed Hamiltonian but a statistical average over Hamiltonians. There is a probability distribution over the coupling parameters appearing in the Hamiltonian. Maybe the quantum counterpart of this is needed to predict the physically measurable correlation functions.

Could this average be an ordinary classical statistical average over quantum states with different classical correlation functions? This kind of average is indeed taken in density matrix formalism. Or could it be that the square root of thermodynamics defined by ZEO actually gives automatically rise to this average? The eigenvalues of the “hermitian square root” of the density matrix would code for components of the state characterized by different classical correlation functions. One could assign these contributions to different “phases”.

- (d) Quantum classical correspondence in statistical sense would be very much like holography (now individual classical state represents the entire quantum state). Quantum ergodicity would pose a rather strong constraint on quantum states. This symmetry principle could actually fix the spectrum of zero energy states to a high degree and fix therefore the M-matrices given by the product of hermitian square root of density

matrix and unitary S-matrix and unitary U-matrix constructible as inner products of M-matrices associated with CDs with various size scales [K22].

- (e) In TGD inspired theory of consciousness the counterpart of quantum ergodicity is the postulate that the space-time geometry provides a symbolic representation for the quantum states and also for the contents of consciousness assignable to quantum jumps between quantum states. Quantum ergodicity would realize this strongly self-referential looking condition. The positive and negative energy parts of zero energy state would be analogous to the initial and final states of quantum jump and the classical correlation functions would code for the contents of consciousness like written formulas code for the thoughts of mathematician and provide a sensory feedback.

How classical correlation functions should be defined?

- (a) General Coordinate Invariance and Lorentz invariance are the basic constraints on the definition. These are achieved for the space-time regions with Minkowskian signature and 4-D  $M^4$  projection if linear Minkowski coordinates are used. This is equivalent with the contraction of the indices of tensor fields with the space-time projections of  $M^4$  Killing vector fields representing translations. Accepting this generalization, there is no need to restrict oneself to 4-D  $M^4$  projection and one can also consider also Euclidian regions identifiable as lines of generalized Feynman diagrams.

Quantum ergodicity very probably however forces to restrict the consideration to Minkowskian and Euclidian space-time regions and various phases associated with them. Also  $CP_2$  Killing vector fields can be projected to space-time surface and give a representation for classical gluon fields. These in turn can be contracted with  $M^4$  Killing vectors giving rise to gluon fields as analogs of graviton fields but with second polarization index replaced with color index.

- (b) The standard definition for the correlation functions associated with classical time evolution is the appropriate starting point. The correlation function  $G_{XY}(\tau)$  for two dynamical variables  $X(t)$  and  $Y(t)$  is defined as the average  $G_{XY}(\tau) = \int_T X(t)Y(t + \tau)dt/T$  over an interval of length  $T$ , and one can also consider the limit  $T \rightarrow \infty$ . In the recent case one would replace  $\tau$  with the difference  $m_1 - m_2 = m$  of  $M^4$  coordinates of two points at the preferred extremal and integrate over the points of the extremal to get the average. The finite time interval  $T$  is replaced with the volume of causal diamond in a given length scale. Zero energy state with given quantum numbers for positive and negative energy parts of the state defines the initial and final states between which the fields appearing in the correlation functions are defined.
- (c) What correlation functions should be considered? Certainly one could calculate correlation functions for the induced spinor connection given electro-weak propagators and correlation functions for  $CP_2$  Killing vector fields giving correlation functions for gluon fields using the description in terms of Killing vector fields. If one can uniquely separate from the Fourier transform uniquely a term of form  $Z/(p^2 - m^2)$  by its momentum dependence, the coefficient  $Z$  can be identified as coupling constant squared for the corresponding gauge potential component and one can in principle deduce coupling constant evolution purely classically. One can imagine of calculating spinorial propagators for string world sheets in the same manner. Note that also the dependence on color quantum numbers would be present so that in principle all that is needed could be calculated for a single preferred extremal without the need to construct QFT limit and to introduce color quantum numbers of fermions as spin like quantum numbers (color quantum numbers corresponds to  $CP_2$  partial wave for the tip of the CD assigned with the particle).

Many detailed speculations about coupling constant evolution to be discussed in the sections below must be taken as innovative guesses doomed to have the eventual fate of guesses. The notion of quantum ergodicity could however be one of the really deep ideas about coupling constant evolution comparable to the notion of p-adic coupling constant evolution. Quantum Ergodicity (briefly QE) would also state something extremely non-trivial also about the

construction of correlation functions and S-matrix. Because this principle is so new, the rest of the chapter does not yet contain any applications of QE. This should not lead the reader to under-estimate the potential power of QE.

## 6 About Deformations Of Known Extremals Of Kähler Action

I have done a considerable amount of speculative guesswork to identify what I have used to call preferred extremals of Kähler action. The difficulty is that the mathematical problem at hand is extremely non-linear and that I do not know about existing mathematical literature relevant to the situation. One must proceed by trying to guess the general constraints on the preferred extremals which look physically and mathematically plausible. The hope is that this net of constraints could eventually chrysalize to Eureka! Certainly the recent speculative picture involves also wrong guesses. The need to find explicit ansatz for the deformations of known extremals based on some common principles has become pressing. The following considerations represent an attempt to combine the existing information to achieve this.

### 6.1 What Might Be The Common Features Of The Deformations Of Known Extremals

The dream is to discover the deformations of all known extremals by guessing what is common to all of them. One might hope that the following list summarizes at least some common features.

#### 6.1.1 Effective three-dimensionality at the level of action

- (a) Holography realized as effective 3-dimensionality also at the level of action requires that it reduces to 3-dimensional effective boundary terms. This is achieved if the contraction  $j^\alpha A_\alpha$  vanishes. This is true if  $j^\alpha$  vanishes or is light-like, or if it is proportional to instanton current in which case current conservation requires that  $CP_2$  projection of the space-time surface is 3-dimensional. The first two options for  $j$  have a realization for known extremals. The status of the third option - proportionality to instanton current - has remained unclear.
- (b) As I started to work again with the problem, I realized that instanton current could be replaced with a more general current  $j = *B \wedge J$  or concretely:  $j^\alpha = \epsilon^{\alpha\beta\gamma\delta} B_\beta J_{\gamma\delta}$ , where  $B$  is vector field and  $CP_2$  projection is 3-dimensional, which it must be in any case. The contractions of  $j$  appearing in field equations vanish automatically with this ansatz.
- (c) Almost topological QFT property in turn requires the reduction of effective boundary terms to Chern-Simons terms: this is achieved by boundary conditions expressing weak form of electric magnetic duality. If one generalizes the weak form of electric-magnetic duality to  $J = \Phi * J$  one has  $B = d\Phi$  and  $j$  has a vanishing divergence for 3-D  $CP_2$  projection. This is clearly a more general solution ansatz than the one based on proportionality of  $j$  with instanton current and would reduce the field equations in concise notation to  $Tr(TH^k) = 0$ .
- (d) Any of the alternative properties of the Kähler current implies that the field equations reduce to  $Tr(TH^k) = 0$ , where  $T$  and  $H^k$  are shorthands for Maxwellian energy momentum tensor and second fundamental form and the product of tensors is obvious generalization of matrix product involving index contraction.

### 6.1.2 Could Einstein's equations emerge dynamically?

For  $j^\alpha$  satisfying one of the three conditions, the field equations have the same form as the equations for minimal surfaces except that the metric  $g$  is replaced with Maxwell energy momentum tensor  $T$ .

- (a) This raises the question about dynamical generation of small cosmological constant  $\Lambda$ :  $T = \Lambda g$  would reduce equations to those for minimal surfaces. For  $T = \Lambda g$  Kähler-Dirac gamma matrices would reduce to induced gamma matrices and the Kähler-Dirac operator would be proportional to ordinary Dirac operator defined by the induced gamma matrices. One can also consider weak form for  $T = \Lambda g$  obtained by restricting the consideration to a sub-space of tangent space so that space-time surface is only "partially" minimal surface but this option is not so elegant although necessary for other than  $CP_2$  type vacuum extremals.
- (b) What is remarkable is that  $T = \Lambda g$  implies that the divergence of  $T$  which in the general case equals to  $j^\beta J_\beta^\alpha$  vanishes. This is guaranteed by one of the conditions for the Kähler current. Since also Einstein tensor has a vanishing divergence, one can ask whether the condition to  $T = \kappa G + \Lambda g$  could be the general condition. This would give Einstein's equations with cosmological term besides the generalization of the minimal surface equations. GRT would emerge dynamically from the non-linear Maxwell's theory although in slightly different sense as conjectured [K18] ! Note that the expression for  $G$  involves also second derivatives of the imbedding space coordinates so that actually a partial differential equation is in question. If field equations reduce to purely algebraic ones, as the basic conjecture states, it is possible to have  $Tr(GH^k) = 0$  and  $Tr(gH^k) = 0$  separately so that also minimal surface equations would hold true.

What is amusing that the first guess for the action of TGD was curvature scalar. It gave analogs of Einstein's equations as a definition of conserved four-momentum currents. The recent proposal would give the analog of ordinary Einstein equations as a dynamical constraint relating Maxwellian energy momentum tensor to Einstein tensor and metric.

- (c) Minimal surface property is physically extremely nice since field equations can be interpreted as a non-linear generalization of massless wave equation: something very natural for non-linear variant of Maxwell action. The theory would be also very "stringy" although the fundamental action would not be space-time volume. This can however hold true only for Euclidian signature. Note that for  $CP_2$  type vacuum extremals Einstein tensor is proportional to metric so that for them the two options are equivalent. For their small deformations situation changes and it might happen that the presence of  $G$  is necessary. The GRT limit of TGD discussed in [K18] [L2] indeed suggests that  $CP_2$  type solutions satisfy Einstein's equations with large cosmological constant and that the small observed value of the cosmological constant is due to averaging and small volume fraction of regions of Euclidian signature (lines of generalized Feynman diagrams).
- (d) For massless extremals and their deformations  $T = \Lambda g$  cannot hold true. The reason is that for massless extremals energy momentum tensor has component  $T^{vv}$  which actually quite essential for field equations since one has  $H_{vv}^k = 0$ . Hence for massless extremals and their deformations  $T = \Lambda g$  cannot hold true if the induced metric has Hamilton-Jacobi structure meaning that  $g^{uu}$  and  $g^{vv}$  vanish. A more general relationship of form  $T = \kappa G + \Lambda G$  can however be consistent with non-vanishing  $T^{vv}$  but require that deformation has at most 3-D  $CP_2$  projection ( $CP_2$  coordinates do not depend on  $v$ ).
- (e) The non-determinism of vacuum extremals suggest for their non-vacuum deformations a conflict with the conservation laws. In, also massless extremals are characterized by a non-determinism with respect to the light-like coordinate but like-likeness saves the situation. This suggests that the transformation of a properly chosen time coordinate of vacuum extremal to a light-like coordinate in the induced metric combined with Einstein's equations in the induced metric of the deformation could allow to handle the non-determinism.

**6.1.3 Are complex structure of  $CP_2$  and Hamilton-Jacobi structure of  $M^4$  respected by the deformations?**

The complex structure of  $CP_2$  and Hamilton-Jacobi structure of  $M^4$  could be central for the understanding of the preferred extremal property algebraically.

- (a) There are reasons to believe that the Hermitian structure of the induced metric ((1, 1) structure in complex coordinates) for the deformations of  $CP_2$  type vacuum extremals could be crucial property of the preferred extremals. Also the presence of light-like direction is also an essential elements and 3-dimensionality of  $M^4$  projection could be essential. Hence a good guess is that allowed deformations of  $CP_2$  type vacuum extremals are such that (2, 0) and (0, 2) components the induced metric and/or of the energy momentum tensor vanish. This gives rise to the conditions implying Virasoro conditions in string models in quantization:

$$g_{\xi^i \xi^j} = 0 \quad , \quad g_{\bar{\xi}^i \bar{\xi}^j} = 0 \quad , \quad i, j = 1, 2 \quad . \quad (6.1)$$

Holomorphisms of  $CP_2$  preserve the complex structure and Virasoro conditions are expected to generalize to 4-dimensional conditions involving two complex coordinates. This means that the generators have two integer valued indices but otherwise obey an algebra very similar to the Virasoro algebra. Also the super-conformal variant of this algebra is expected to make sense.

These Virasoro conditions apply in the coordinate space for  $CP_2$  type vacuum extremals. One expects similar conditions hold true also in field space, that is for  $M^4$  coordinates.

- (b) The integrable decomposition  $M^4(m) = M^2(m) + E^2(m)$  of  $M^4$  tangent space to longitudinal and transversal parts (non-physical and physical polarizations) - Hamilton-Jacobi structure- could be a very general property of preferred extremals and very natural since non-linear Maxwellian electrodynamics is in question. This decomposition led rather early to the introduction of the analog of complex structure in terms of what I called Hamilton-Jacobi coordinates  $(u, v, w, \bar{w})$  for  $M^4$ .  $(u, v)$  defines a pair of light-like coordinates for the local longitudinal space  $M^2(m)$  and  $(w, \bar{w})$  complex coordinates for  $E^2(m)$ . The metric would not contain any cross terms between  $M^2(m)$  and  $E^2(m)$ :  $g_{uw} = g_{vw} = g_{u\bar{w}} = g_{v\bar{w}} = 0$ .

A good guess is that the deformations of massless extremals respect this structure. This condition gives rise to the analog of the constraints leading to Virasoro conditions stating the vanishing of the non-allowed components of the induced metric.  $g_{uu} = g_{vv} = g_{ww} = g_{\bar{w}\bar{w}} = g_{uw} = g_{vw} = g_{u\bar{w}} = g_{v\bar{w}} = 0$ . Again the generators of the algebra would involve two integers and the structure is that of Virasoro algebra and also generalization to super algebra is expected to make sense. The moduli space of Hamilton-Jacobi structures would be part of the moduli space of the preferred extremals and analogous to the space of all possible choices of complex coordinates. The analogs of infinitesimal holomorphic transformations would preserve the modular parameters and give rise to a 4-dimensional Minkowskian analog of Virasoro algebra. The conformal algebra acting on  $CP_2$  coordinates acts in field degrees of freedom for Minkowskian signature.

**6.1.4 Field equations as purely algebraic conditions**

If the proposed picture is correct, field equations would reduce basically to purely algebraically conditions stating that the Maxwellian energy momentum tensor has no common index pairs with the second fundamental form. For the deformations of  $CP_2$  type vacuum extremals  $T$  is a complex tensor of type (1, 1) and second fundamental form  $H^k$  a tensor of type (2, 0) and (0, 2) so that  $Tr(TH^k) = 0$  is true. This requires that second light-like coordinate of  $M^4$  is constant so that the  $M^4$  projection is 3-dimensional. For Minkowskian signature of the induced metric Hamilton-Jacobi structure replaces conformal structure. Here the dependence of  $CP_2$  coordinates on second light-like coordinate of  $M^2(m)$  only plays a fundamental role. Note that now  $T^{vv}$  is non-vanishing (and light-like). This picture generalizes to the deformations of cosmic strings and even to the case of vacuum extremals.

## 6.2 What Small Deformations Of $CP_2$ Type Vacuum Extremals Could Be?

I was led to these arguments when I tried find preferred extremals of Kähler action, which would have 4-D  $CP_2$  and  $M^4$  projections - the Maxwell phase analogous to the solutions of Maxwell's equations that I conjectured long time ago. It however turned out that the dimensions of the projections can be ( $D_{M^4} \leq 3, D_{CP_2} = 4$ ) or ( $D_{M^4} = 4, D_{CP_2} \leq 3$ ). What happens is essentially breakdown of linear superposition so that locally one can have superposition of modes which have 4-D wave vectors in the same direction. This is actually very much like quantization of radiation field to photons now represented as separate space-time sheets and one can say that Maxwellian superposition corresponds to union of separate photonic space-time sheets in TGD.

Approximate linear superposition of fields is fundamental in standard physics framework and is replaced in TGD with a linear superposition of effects of classical fields on a test particle topologically condensed simultaneously to several space-time sheets. One can say that linear superposition is replaced with a disjoint union of space-time sheets. In the following I shall restrict the consideration to the deformations of  $CP_2$  type vacuum extremals.

### 6.2.1 Solution ansatz

I proceed by the following arguments to the ansatz.

- (a) Effective 3-dimensionality for action (holography) requires that action decomposes to vanishing  $j^\alpha A_\alpha$  term + total divergence giving 3-D "boundary" terms. The first term certainly vanishes (giving effective 3-dimensionality) for

$$D_\beta J^{\alpha\beta} = j^\alpha = 0 .$$

Empty space Maxwell equations, something extremely natural. Also for the proposed GRT limit these equations are true.

- (b) How to obtain empty space Maxwell equations  $j^\alpha = 0$ ? The answer is simple: assume self duality or its slight modification:

$$J = *J$$

holding for  $CP_2$  type vacuum extremals or a more general condition

$$J = k * J ,$$

In the simplest situation  $k$  is some constant not far from unity.  $*$  is Hodge dual involving 4-D permutation symbol.  $k = \text{constant}$  requires that the determinant of the induced metric is apart from constant equal to that of  $CP_2$  metric. It does not require that the induced metric is proportional to the  $CP_2$  metric, which is not possible since  $M^4$  contribution to metric has Minkowskian signature and cannot be therefore proportional to  $CP_2$  metric.

One can consider also a more general situation in which  $k$  is scalar function as a generalization of the weak electric-magnetic duality. In this case the Kähler current is non-vanishing but divergenceless. This also guarantees the reduction to  $Tr(TH^k) = 0$ . In this case however the proportionality of the metric determinant to that for  $CP_2$  metric is not needed. This solution ansatz becomes therefore more general.

- (c) Field equations reduce with these assumptions to equations differing from minimal surfaces equations only in that metric  $g$  is replaced by Maxwellian energy momentum tensor  $T$ . Schematically:

$$Tr(TH^k) = 0 ,$$

where  $T$  is the Maxwellian energy momentum tensor and  $H^k$  is the second fundamental form - asymmetric 2-tensor defined by covariant derivative of gradients of imbedding space coordinates.

### 6.2.2 How to satisfy the condition $Tr(TH^k) = 0$ ?

It would be nice to have minimal surface equations since they are the non-linear generalization of massless wave equations. It would be also nice to have the vanishing of the terms involving Kähler current in field equations as a consequence of this condition. Indeed,  $T = \kappa G + \Lambda g$  implies this. In the case of  $CP_2$  vacuum extremals one cannot distinguish between these options since  $CP_2$  itself is constant curvature space with  $G \propto g$ . Furthermore, if  $G$  and  $g$  have similar tensor structure the algebraic field equations for  $G$  and  $g$  are satisfied separately so that one obtains minimal surface property also now. In the following minimal surface option is considered.

- (a) The first option is achieved if one has

$$T = \Lambda g \ .$$

Maxwell energy momentum tensor would be proportional to the metric! One would have dynamically generated cosmological constant! This begins to look really interesting since it appeared also at the proposed GRT limit of TGD [L2] (see <http://tinyurl.com/hzkldnb>). Note that here also non-constant value of  $\Lambda$  can be considered and would correspond to a situation in which  $k$  is scalar function: in this case the the determinant condition can be dropped and one obtains just the minimal surface equations.

- (b) Very schematically and forgetting indices and being sloppy with signs, the expression for  $T$  reads as

$$T = JJ - g/4Tr(JJ) \ .$$

Note that the product of tensors is obtained by generalizing matrix product. This should be proportional to metric.

Self duality implies that  $Tr(JJ)$  is just the instanton density and does not depend on metric and is constant.

For  $CP_2$  type vacuum extremals one obtains

$$T = -g + g = 0 \ .$$

Cosmological constant would vanish in this case.

- (c) Could it happen that for deformations a small value of cosmological constant is generated?

The condition would reduce to

$$JJ = (\Lambda - 1)g \ .$$

$\Lambda$  must relate to the value of parameter  $k$  appearing in the generalized self-duality condition. For the most general ansatz  $\Lambda$  would not be constant anymore.

This would generalize the defining condition for Kähler form

$$JJ = -g \ (i^2 = -1 \text{ geometrically})$$

stating that the square of Kähler form is the negative of metric. The only modification would be that index raising is carried out by using the induced metric containing also  $M^4$  contribution rather than  $CP_2$  metric.

(d) Explicitly:

$$J_{\alpha\mu}J^{\mu}_{\beta} = (\Lambda - 1)g_{\alpha\beta} .$$

Cosmological constant would measure the breaking of Kähler structure. By writing  $g = s + m$  and defining index raising of tensors using  $CP_2$  metric and their product accordingly, this condition can be also written as

$$Jm = (\Lambda - 1)mJ .$$

If the parameter  $k$  is constant, the determinant of the induced metric must be proportional to the  $CP_2$  metric. If  $k$  is scalar function, this condition can be dropped. Cosmological constant would not be constant anymore but the dependence on  $k$  would drop out from the field equations and one would hope of obtaining minimal surface equations also now. It however seems that the dimension of  $M^4$  projection cannot be four. For 4-D  $M^4$  projection the contribution of the  $M^2$  part of the  $M^4$  metric gives a non-holomorphic contribution to  $CP_2$  metric and this spoils the field equations.

For  $T = \kappa G + \Lambda g$  option the value of the cosmological constant is large - just as it is for the proposed GRT limit of TGD [K18] [L2]. The interpretation in this case is that the average value of cosmological constant is small since the portion of space-time volume containing generalized Feynman diagrams is very small.

### 6.2.3 More detailed ansatz for the deformations of $CP_2$ type vacuum extremals

One can develop the ansatz to a more detailed form. The most obvious guess is that the induced metric is apart from constant conformal factor the metric of  $CP_2$ . This would guarantee self-duality apart from constant factor and  $j^{\alpha} = 0$ . Metric would be in complex  $CP_2$  coordinates tensor of type (1, 1) whereas  $CP_2$  Riemann connection would have only purely holomorphic or anti-holomorphic indices. Therefore  $CP_2$  contributions in  $Tr(TH^k)$  would vanish identically.  $M^4$  degrees of freedom however bring in difficulty. The  $M^4$  contribution to the induced metric should be proportional to  $CP_2$  metric and this is impossible due to the different signatures. The  $M^4$  contribution to the induced metric breaks its Kähler property but would preserve Hermitian structure.

A more realistic guess based on the attempt to construct deformations of  $CP_2$  type vacuum extremals is following.

- (a) Physical intuition suggests that  $M^4$  coordinates can be chosen so that one has integrable decomposition to longitudinal degrees of freedom parametrized by two light-like coordinates  $u$  and  $v$  and to transversal polarization degrees of freedom parametrized by complex coordinate  $w$  and its conjugate.  $M^4$  metric would reduce in these coordinates to a direct sum of longitudinal and transverse parts. I have called these coordinates Hamilton-Jacobi coordinates.
- (b)  $w$  would be holomorphic function of  $CP_2$  coordinates and therefore satisfy the analog of massless wave equation. This would give hopes about rather general solution ansatz.  $u$  and  $v$  cannot be holomorphic functions of  $CP_2$  coordinates. Unless wither  $u$  or  $v$  is constant, the induced metric would receive contributions of type (2, 0) and (0, 2) coming from  $u$  and  $v$  which would break Kähler structure and complex structure. These contributions would give no-vanishing contribution to all minimal surface equations. Therefore either  $u$  or  $v$  is constant: the coordinate line for non-constant coordinate -say  $u$ - would be analogous to the  $M^4$  projection of  $CP_2$  type vacuum extremal.
- (c) With these assumptions the induced metric would remain (1,1) tensor and one might hope that  $Tr(TH^k)$  contractions vanishes for all variables except  $u$  because there are no common index pairs (this if non-vanishing Christoffel symbols for  $H$  involve only holomorphic or anti-holomorphic indices in  $CP_2$  coordinates). For  $u$  one would obtain massless wave equation expressing the minimal surface property.

- (d) If the value of  $k$  is constant the determinant of the induced metric must be proportional to the determinant of  $CP_2$  metric. The induced metric would contain only the contribution from the transversal degrees of freedom besides  $CP_2$  contribution. Minkowski contribution has however rank 2 as  $CP_2$  tensor and cannot be proportional to  $CP_2$  metric. It is however enough that its determinant is proportional to the determinant of  $CP_2$  metric with constant proportionality coefficient. This condition gives an additional non-linear condition to the solution. One would have wave equation for  $u$  (also  $w$  and its conjugate satisfy massless wave equation) and determinant condition as an additional condition.

The determinant condition reduces by the linearity of determinant with respect to its rows to sum of conditions involved 0, 1, 2 rows replaced by the transversal  $M^4$  contribution to metric given if  $M^4$  metric decomposes to direct sum of longitudinal and transversal parts. Derivatives with respect to derivative with respect to particular  $CP_2$  complex coordinate appear linearly in this expression they can depend on  $u$  via the dependence of transversal metric components on  $u$ . The challenge is to show that this equation has (or does not have) non-trivial solutions.

- (e) If the value of  $k$  is scalar function the situation changes and one has only the minimal surface equations and Virasoro conditions.

What makes the ansatz attractive is that special solutions of Maxwell empty space equations are in question, equations reduces to non-linear generalizations of Euclidian massless wave equations, and possibly space-time dependent cosmological constant pops up dynamically. These properties are true also for the GRT limit of TGD [L2] (see <http://tinyurl.com/hzkldnb>).

### 6.3 Hamilton-Jacobi Conditions In Minkowskian Signature

The maximally optimistic guess is that the basic properties of the deformations of  $CP_2$  type vacuum extremals generalize to the deformations of other known extremals such as massless extremals, vacuum extremals with 2-D  $CP_2$  projection which is Lagrangian manifold, and cosmic strings characterized by Minkowskian signature of the induced metric. These properties would be following.

- (a) The recomposition of  $M^4$  tangent space to longitudinal and transversal parts giving Hamilton-Jacobi structure. The longitudinal part has hypercomplex structure but the second light-like coordinate is constant: this plays a crucial role in guaranteeing the vanishing of contractions in  $Tr(TH^k)$ . It is the algebraic properties of  $g$  and  $T$  which are crucial.  $T$  can however have light-like component  $T^{vv}$ . For the deformations of  $CP_2$  type vacuum extremals (1, 1) structure is enough and is guaranteed if second light-like coordinate of  $M^4$  is constant whereas  $w$  is holomorphic function of  $CP_2$  coordinates.
- (b) What could happen in the case of massless extremals? Now one has 2-D  $CP_2$  projection in the initial situation and  $CP_2$  coordinates depend on light-like coordinate  $u$  and single real transversal coordinate. The generalization would be obvious: dependence on single light-like coordinate  $u$  and holomorphic dependence on  $w$  for complex  $CP_2$  coordinates. The constraint is  $T = \Lambda g$  cannot hold true since  $T^{vv}$  is non-vanishing (and light-like). This property restricted to transversal degrees of freedom could reduce the field equations to minimal surface equations in transversal degrees of freedom. The transversal part of energy momentum tensor would be proportional to metric and hence covariantly constant. Gauge current would remain light-like but would not be given by  $j = *d\phi \wedge J$ .  $T = \kappa G + \Lambda g$  seems to define the attractive option.

It therefore seems that the essential ingredient could be the condition

$$T = \kappa G + \lambda g \quad ,$$

which has structure (1, 1) in both  $M^2(m)$  and  $E^2(m)$  degrees of freedom apart from the presence of  $T^{vv}$  component with deformations having no dependence on  $v$ . If the second

fundamental form has  $(2, 0)+(0, 2)$  structure, the minimal surface equations are satisfied provided Kähler current satisfies on of the proposed three conditions and if  $G$  and  $g$  have similar tensor structure.

One can actually pose the conditions of metric as complete analogs of stringy constraints leading to Virasoro conditions in quantization to give

$$g_{uu} = 0 \ , \ g_{vv} = 0 \ , \ g_{ww} = 0 \ , \ g_{\bar{w}\bar{w}} = 0 \ . \quad (6.2)$$

This brings in mind the generalization of Virasoro algebra to four-dimensional algebra for which an identification in terms of non-local Yangian symmetry [A10] [B11, B9, B10] has been proposed [K28]. The number of conditions is four and the same as the number of independent field equations. One can consider similar conditions also for the energy momentum tensor  $T$  but allowing non-vanishing component  $T^{vv}$  if deformations has no  $v$ -dependence. This would solve the field equations if the gauge current vanishes or is light-like. On this case the number of equations is 8. First order differential equations are in question and they can be also interpreted as conditions fixing the coordinates used since there is infinite number of manners to choose the Hamilton-Jacobi coordinates.

One can can try to apply the physical intuition about general solutions of field equations in the linear case by writing the solution as a superposition of left and right propagating solutions:

$$\xi^k = f_+^k(u, w) + f_-^k(v, w) \ . \quad (6.3)$$

This could guarantee that second fundamental form is of form  $(2, 0)+(0, 2)$  in both  $M^2$  and  $E^2$  part of the tangent space and these terms if  $Tr(TH^k)$  vanish identically. The remaining terms involve contractions of  $T^{uw}$ ,  $T^{u\bar{w}}$  and  $T^{vw}$ ,  $T^{v\bar{w}}$  with second fundamental form. Also these terms should sum up to zero or vanish separately. Second fundamental form has components coming from  $f_+^k$  and  $f_-^k$

Second fundamental form  $H^k$  has as basic building bricks terms  $\hat{H}^k$  given by

$$\hat{H}_{\alpha\beta}^k = \partial_\alpha \partial_\beta h^k + \binom{k}{l\ m} \partial_\alpha h^l \partial_\beta h^m \ . \quad (6.4)$$

For the proposed ansatz the first terms give vanishing contribution to  $H_{uv}^k$ . The terms containing Christoffel symbols however give a non-vanishing contribution and one can allow only  $f_+^k$  or  $f_-^k$  as in the case of massless extremals. This reduces the dimension of  $CP_2$  projection to  $D = 3$ .

What about the condition for Kähler current? Kähler form has components of type  $J_{w\bar{w}}$  whose contravariant counterpart gives rise to space-like current component.  $J_{uw}$  and  $J_{u\bar{w}}$  give rise to light-like currents components. The condition would state that the  $J^{w\bar{w}}$  is covariantly constant. Solutions would be characterized by a constant Kähler magnetic field. Also electric field is represent. The interpretation both radiation and magnetic flux tube makes sense.

## 6.4 Deformations Of Cosmic Strings

In the physical applications it has been assumed that the thickening of cosmic strings to Kähler magnetic flux tubes takes place. One indeed expects that the proposed construction generalizes also to the case of cosmic strings having the decomposition  $X^4 = X^2 \times Y^2 \subset M^4 \times CP_2$ , where  $X^2$  is minimal surface and  $Y^2$  a complex homologically non-trivial submanifold of  $CP_2$ . Now the starting point structure is Hamilton-Jacobi structure for  $M_m^2 \times Y^2$  defining the coordinate space.

- (a) The deformation should increase the dimension of either  $CP_2$  or  $M^4$  projection or both. How this thickening could take place? What comes in mind that the string orbits  $X^2$  can be interpreted as a distribution of longitudinal spaces  $M^2(x)$  so that for the deformation  $w$  coordinate becomes a holomorphic function of the natural  $Y^2$  complex coordinate so that  $M^4$  projection becomes 4-D but  $CP_2$  projection remains 2-D. The new contribution to the  $X^2$  part of the induced metric is vanishing and the contribution to the  $Y^2$  part is of type (1, 1) and the ansatz  $T = \kappa G + \Lambda g$  might be needed as a generalization of the minimal surface equations. The ratio of  $\kappa$  and  $G$  would be determined from the form of the Maxwellian energy momentum tensor and be fixed at the limit of undeformed cosmic string to  $T = (ag(Y^2) - bg(Y^2))$ . The value of cosmological constant is now large, and overall consistency suggests that  $T = \kappa G + \Lambda g$  is the correct option also for the  $CP_2$  type vacuum extremals.
- (b) One could also imagine that remaining  $CP_2$  coordinates could depend on the complex coordinate of  $Y^2$  so that also  $CP_2$  projection would become 4-dimensional. The induced metric would receive holomorphic contributions in  $Y^2$  part. As a matter fact, this option is already implied by the assumption that  $Y^2$  is a complex surface of  $CP_2$ .

## 6.5 Deformations Of Vacuum Extremals?

What about the deformations of vacuum extremals representable as maps from  $M^4$  to  $CP_2$ ?

- (a) The basic challenge is the non-determinism of the vacuum extremals. One should perform the deformation so that conservation laws are satisfied. For massless extremals there is also non-determinism but it is associated with the light-like coordinate so that there are no problems with the conservation laws. This would suggest that a properly chosen time coordinate consistent with Hamilton-Jacobi decomposition becomes light-like coordinate in the induced metric. This poses a conditions on the induced metric.
- (b) Physical intuition suggests that one cannot require  $T = \Lambda g$  since this would mean that the rank of  $T$  is maximal whereas the original situation corresponds to the vanishing of  $T$ . For small deformations rank two for  $T$  looks more natural and one could think that  $T$  is proportional to a projection of metric to a 2-D subspace. The vision about the long length scale limit of TGD is that Einstein's equations are satisfied and this would suggest  $T = kG$  or  $T = \kappa G + \Lambda g$ . The rank of  $T$  could be smaller than four for this ansatz and this conditions binds together the values of  $\kappa$  and  $G$ .
- (c) These extremals have  $CP_2$  projection which in the generic case is 2-D Lagrangian sub-manifold  $Y^2$ . Again one could assume Hamilton-Jacobi coordinates for  $X^4$ . For  $CP_2$  one could assume Darboux coordinates  $(P_i, Q_i)$ ,  $i = 1, 2$ , in which one has  $A = P_i dQ^i$ , and that  $Y^2 \subset CP_2$  corresponds to  $Q_i = \text{constant}$ . In principle  $P_i$  would depend on arbitrary manner on  $M^4$  coordinates. It might be more convenient to use as coordinates  $(u, v)$  for  $M^2$  and  $(P_1, P_2)$  for  $Y^2$ . This covers also the situation when  $M^4$  projection is not 4-D. By its 2-dimensionality  $Y^2$  allows always a complex structure defined by its induced metric: this complex structure is not consistent with the complex structure of  $CP_2$  ( $Y^2$  is not complex sub-manifold).

Using Hamilton-Jacobi coordinates the pre-image of a given point of  $Y^2$  is a 2-dimensional sub-manifold  $X^2$  of  $X^4$  and defines also 2-D sub-manifold of  $M^4$ . The following picture suggests itself. The projection of  $X^2$  to  $M^4$  can be seen for a suitable choice of Hamilton-Jacobi coordinates as an analog of Lagrangian sub-manifold in  $M^4$  that is as surface for which  $v$  and  $Im(w)$  vary and  $u$  and  $Re(w)$  are constant.  $X^2$  would be obtained by allowing  $u$  and  $Re(w)$  to vary: as a matter fact,  $(P_1, P_2)$  and  $(u, Re(w))$  would be related to each other. The induced metric should be consistent with this picture. This would requires  $g_{uRe(w)} = 0$ .

For the deformations  $Q_1$  and  $Q_2$  would become non-constant and they should depend on the second light-like coordinate  $v$  only so that only  $g_{uu}$  and  $g_{uv}$  and  $g_{u\bar{v}}$   $g_{v,w}$  and  $g_{\bar{v},\bar{w}}$  receive contributions which vanish. This would give rise to the analogs of Virasoro conditions guaranteeing that  $T$  is a tensor of form (1, 1) in both  $M^2$  and  $E^2$  indices and

that there are no cross components in the induced metric. A more general formulation states that energy momentum tensor satisfies these conditions. The conditions on  $T$  might be equivalent with the conditions for  $g$  and  $G$  separately.

- (d) Einstein's equations provide an attractive manner to achieve the vanishing of effective 3-dimensionality of the action. Einstein equations would be second order differential equations and the idea that a deformation of vacuum extremal is in question suggests that the dynamics associated with them is in directions transversal to  $Y^2$  so that only the deformation is dictated partially by Einstein's equations.
- (e) Lagrangian manifolds do not involve complex structure in any obvious manner. One could however ask whether the deformations could involve complex structure in a natural manner in  $CP_2$  degrees of freedom so that the vanishing of  $g_{ww}$  would be guaranteed by holomorphy of  $CP_2$  complex coordinate as function of  $w$ .

One should get the complex structure in some natural manner: in other words, the complex structure should relate to the geometry of  $CP_2$  somehow. The complex coordinate defined by say  $z = P_1 + iQ_1$  for the deformation suggests itself. This would suggest that at the limit when one puts  $Q_1 = 0$  one obtains  $P_1 = P_1(Re(w))$  for the vacuum extremals and the deformation could be seen as an analytic continuation of real function to region of complex plane. This is in spirit with the algebraic approach. The vanishing of Kähler current requires that the Kähler magnetic field is covariantly constant:  $D_z J^{z\bar{z}} = 0$  and  $D_{\bar{z}} J^{z\bar{z}} = 0$  .

- (f) One could consider the possibility that the resulting 3-D sub-manifold of  $CP_2$  can be regarded as contact manifold with induced Kähler form non-vanishing in 2-D section with natural complex coordinates. The third coordinate variable- call it  $s$ - of the contact manifold and second coordinate of its transversal section would depend on time space-time coordinates for vacuum extremals. The coordinate associated with the transversal section would be continued to a complex coordinate which is holomorphic function of  $w$  and  $u$ .
- (g) The resulting thickened magnetic flux tubes could be seen as another representation of Kähler magnetic flux tubes: at this time as deformations of vacuum flux tubes rather than cosmic strings. For this ansatz it is however difficult to imagine deformations carrying Kähler electric field.

## 6.6 About The Interpretation Of The Generalized Conformal Algebras

The long-standing challenge has been finding of the direct connection between the super-conformal symmetries assumed in the construction of the geometry of the "world of classical worlds" ( WCW ) and possible conformal symmetries of field equations. 4-dimensionality and Minkowskian signature have been the basic problems. The recent construction provides new insights to this problem.

- (a) In the case of string models the quantization of the Fourier coefficients of coordinate variables of the target space gives rise to Kac-Moody type algebra and Virasoro algebra generators are quadratic in these. Also now Kac-Moody type algebra is expected. If one were to perform a quantization of the coefficients in Laurents series for complex  $CP_2$  coordinates, one would obtain interpretation in terms of  $su(3) = u(2) + t$  decomposition, where  $t$  corresponds to  $CP_3$ : the oscillator operators would correspond to generators in  $t$  and their commutator would give generators in  $u(2)$ .  $SU(3)/SU(2)$  coset representation for Kac-Moody algebra would be in question. Kac-Moody algebra would be associated with the generators in both  $M^4$  and  $CP_2$  degrees of freedom. This kind of Kac-Moody algebra appears in quantum TGD.
- (b) The constraints on induced metric imply a very close resemblance with string models and a generalization of Virasoro algebra emerges. An interesting question is how the two algebras acting on coordinate and field degrees of freedom relate to the super-conformal algebras defined by the symplectic group of  $\delta M_+^4 \times CP_2$  acting on space-like 3-surfaces

at boundaries of CD and to the Kac-Moody algebras acting on light-like 3-surfaces. It has been conjectured that these algebras allow a continuation to the interior of space-time surface made possible by its slicing by 2-surfaces parametrized by 2-surfaces. The proposed construction indeed provides this kind of slicings in both  $M^4$  and  $CP_2$  factor.

- (c) In the recent case, the algebras defined by the Fourier coefficients of field variables would be Kac-Moody algebras. Virasoro algebra acting on preferred coordinates would be expressed in terms of the Kac-Moody algebra in the standard Sugawara construction applied in string models. The algebra acting on field space would be analogous to the conformal algebra assignable to the symplectic algebra so that also symplectic algebra is present. Stringy pragmatist could imagine quantization of symplectic algebra by replacing  $CP_2$  coordinates in the expressions of Hamiltonians with oscillator operators. This description would be counterpart for the construction of spinor harmonics in WCW and might provide some useful insights.
- (d) For given type of space-time surface either  $CP_2$  or  $M^4$  corresponds to Kac-Moody algebra but not both. From the point of view of quantum TGD it looks as that something were missing. An analogous problem was encountered at GRT limit of TGD [L2]. When Euclidian space-time regions are allowed Einstein-Maxwell action is able to mimic standard model with a surprising accuracy but there is a problem: one obtains either color charges or  $M^4$  charges but not both. Perhaps it is not enough to consider either  $CP_2$  type vacuum extremal or its exterior but both to describe particle: this would give the direct product of the Minkowskian and Euclidian algebras acting on tensor product. This does not however seem to be consistent with the idea that the two descriptions are duality related (the analog of T-duality).

## 7 About TGD counterparts of classical field configurations in Maxwell's theory

Classical physics is an exact part of TGD so that the study of extremals of dimensionally reduces 6-D Kähler action can provide a lot of intuition about quantum TGD and see how quantum-classical correspondence is realized. In the following I will try to develop further understanding about TGD counterparts of the simplest field configurations in Maxwell's theory.

In the sequel  $CP_2$  type extremals will be considered from the point of view of quantum criticality and the view about string world sheets, their lightlike boundaries as carriers of fermion number, and the ends as point like particles as singularities acting as sources for minimal surfaces satisfying non-linear generalization of d'Alembert equation.

I will also discuss the delicacies associated with  $M^4$  Kähler structure and its connection with what I call Hamilton-Jacobi structure and with  $M^8$  approach based on classical number fields. I will argue that the breaking of CP symmetry associated with  $M^4$  Kähler structure is small without any additional assumptions: this is in contrast with the earlier view.

The difference between TGD and Maxwell's theory and consider the TGD counterparts of simple em field configurations will be also discussed. Topological field quantization provides a geometric view about formation of atoms as bound states based on flux tubes as correlates for binding, and allows to identify space-time correlates for second quantization. These considerations force to take seriously the possibility that preferred extremals besides being minimal surfaces also possess generalized holomorphy reducing field equations to purely algebraic conditions and that minimal surfaces without this property are not preferred extremals. If so, at microscopic level only  $CP_2$  type extremals, massless extremals, and string like objects and their deformations would exist as preferred extremals and serve as building bricks for the counterparts of Maxwellian field configurations and the counterparts of Maxwellian field configurations such as Coulomb potential would emerge only at the QFT limit.

## 7.1 About differences between Maxwell's ED and TGD

TGD differs from Maxwell's theory in several important aspects.

- (a) The TGD counterparts of classical electroweak gauge potentials are induced from component of spinor connection of  $CP_2$ . Classical color gauge potentials corresponds to the projections of Killing vector fields of color isometries.
- (b) Also  $M^4$  has Kähler potential, which is induced to space-time surface and gives rise to an additional  $U(1)$  force. The couplings of  $M^4$  gauge potential to quarks and leptons are of same sign whereas the couplings of  $CP_2$  Kähler potential to B and L are of opposite sign so that the contributions to 6-D Kähler action reduce to separate terms without interference term.

Coupling to induced  $M^4$  Kähler potential implies CP breaking. This could explain the small CP breaking in hadronic systems and also matter antimatter asymmetry in which there are opposite matter-antimatter asymmetries inside cosmic strings and their exteriors respectively. A priori it is however not obvious that the CP breaking is small.

- (c) General coordinate invariance implies that there are only 4 local field like degrees of freedom so that for extremals with 4-D  $M^4$  projection corresponding to GRT space-time both metric, electroweak and color gauge potentials can be expressed in terms four  $CP_2$  coordinates and their gradients. Preferred extremal property realized as minimal surface condition means that field equations are satisfied separately for the 4-D Kähler and volume action reduces the degrees of freedom further.

If the  $CP_2$  part of Kähler form is non-vanishing, minimal surface conditions can be guaranteed by a generalization of holomorphy realizing quantum criticality (satisfied by known extremals). One can say that there is no dependence on coupling parameters. If  $CP_2$  part of Kähler form vanishes identically, the minimal surface condition need not be guaranteed by holomorphy. It is not at all clear whether quantum criticality and preferred extremal property allow this kind of extremals.

- (d) Supersymplectic symmetries act as isometries of "world of classical worlds" (WCW). In a well-defined sense supersymplectic symmetry generalizes 2-D conformal invariance to 4-D context. The key observation here is that light-like 3-surfaces are metrically 2-D and therefore allow extended conformal invariance.

Preferred extremal property realizing quantum criticality boils down to a condition that sub-algebra of SSA and its commutator with SSA annihilate physical states and that corresponding Noether charges vanish. These conditions could be equivalent with minimal surface property. This implies that the set of possible field patterns is extremely restricted and one might talk about "archetypal" field patterns analogous to partial waves or plane waves in Maxwell's theory.

- (e) Linear superposition of the archetypal field patterns is not possible. TGD however implies the notion of many-sheeted space-time and each sheet can carry its own field pattern. A test particle which is space-time surface itself touches all these sheets and experiences the sum of the effects caused by fields at various sheets. Effects are superposed rather than fields and this is enough. This means weakening of the superposition principle of Maxwell's theory and the linear superposition of fields at same space-time sheet is replaced with set theoretic union of space-time sheets carrying the field patterns whose effects superpose.

This observation is also essential in the construction of QFT limit of TGD. The gauge potentials in standard model and gravitational field in general relativity are superpositions of those associated with space-time sheets idealized with slightly curved piece of Minkowski space  $M^4$ .

- (f) An important implication is that each system has field identity - field body or magnetic body (MB). In Maxwell's theory superposition of fields coming from different sources leads to a loss of information since one does not anymore now which part of field came from particular source. In TGD this information loss does not happen and this is essential for TGD inspired quantum biology.

**Remark:** An interesting algebraic analog is the notion of co-algebra. Co-product is analogous to reversal of product  $AB=C$  in the sense that it assigns to  $C$  and a linear combination of products  $\sum A_i \otimes B_i$  such that  $A_i B_i = C$ . Quantum groups and co-algebras are indeed important in TGD and it might be that there is a relationship. In TGD inspired quantum biology magnetic body plays a key role as an intentional agent receiving sensory data from biological body and using it as motor instrument.

- (g) I have already earlier considered a space-time correlate for second quantization in terms of sheets of covering for  $h_{eff} = nh_0$ . In [L5] it is proposed that  $n$  factorizes as  $n = n_1 n_2$  such that  $n_1$  ( $n_2$ ) is the number sheets for space-time surface as covering of  $CP_2$  ( $M^4$ ). One could have quantum mechanical linear superposition of space-time sheets, each with a particular field pattern. This kind state would correspond to single particle state created by quantum field in QFT limit. For instance, one could have spherical harmonic for orientations of magnetic flux tube or electric flux tube.

One could also have superposition of configurations containing several space-time sheets simultaneously as analogs of many-boson states. Many-sheeted space-time would correspond to this kind many-boson states. Second quantization in quantum field theory (QFT) could be seen as an algebraic description of many-sheetedness having no obvious classical correlate in classical QFT.

- (h) Flux tubes should be somehow different for gravitational fields, em fields, and also weak and color gauge fields. The value of  $n = n_1 n_2$  [L5] for gravitational flux tubes is very large by Nottale formula  $\hbar_{eff} = \hbar_{gr} = GMm/v_0$ . The value of  $n_2$  for gravitational flux tubes is  $n_2 \sim 10^7$  if one accepts the formula  $G = R^2/n_2 \hbar$ . For em fields much smaller values of  $n$  and therefore of  $n_2$  are suggestive. There the value of  $n$  measuring in adelic physics algebraic complexity and evolutionary level would distinguish between gravitational and em flux tubes.

Large value of  $n$  would mean quantum coherence in long scales. For gravitation this makes sense since screening is absent unlike for gauge interactions. Note that the large value of  $h_{eff} = h_{gr}$  implies that  $\alpha_{em} = e^2/4\pi\hbar_{eff}$  is extremely small for gravitational flux tubes so that they would indeed be gravitational in an excellent approximation.

$n$  would be the dimension of extension of rationals involved and  $n_2$  would be the number space-time sheets as covering of  $M^4$ . If this picture is correct, gravitation would correspond to much larger algebraic complexity and much larger value of Planck constant. This conforms with the intuition that gravitation plays essential role in the quantum physics of living matter.

There are also other number theoretic characteristics such as ramified primes of the extension identifiable as preferred p-adic primes in turn characterizing elementary particle. Also flux tubes mediating weak and strong interactions should allow characterization in terms of number theoretic parameters. There are arguments that in atomic physics one has  $h = 6h_0$ . Since the quantum coherence scale of hadrons is smaller than atomic scale, one can ask whether one could have  $h_{eff} < h$ .

## 7.2 $CP_2$ type extremals as ultimate sources of fields and singularities

$CP_2$  type extremals have Euclidian signature of induced metric and therefore represent the most radical deviation from Maxwell's ED, gauge theories, and GRT.  $CP_2$  type extremal with light-like geodesic as  $M^4$  projection represents a model for wormhole contact. The light-like orbit of partonic 2-surface correspond to boundary between wormhole contact and Minkowskian region and is associated with both throats of wormhole contact. The throats of wormhole contact can carry part of a boundary of string world sheet connecting the partonic orbits associated with different particles. These light-like lines can carry fermion number and would correspond to lines of TGD counterparts of twistor diagrams.

These world lines would correspond to singularities for the minimal surface equations analogous to sources of massless vector fields carrying charge [L4, L7]. These singularities would serve as ultimate sources of classical em fields. Various currents would consist of wormhole throat pairs representing elementary particle and carrying charges at the partonic orbits.

Two-sheetedness is essential and could be interpreted in terms of a double covering formed by space-time sheet glued along their common boundary. This necessary since space-time sheet has a finite size being not continuable beyond certain minimal size as preferred extremal since some of the real coordinates would become complex.

### 7.2.1 Quantum criticality for $CP_2$ type extremals

TGD predicts a hierarchy of quantum criticalities. The increase in criticality means that some space-time sheets for space-time surface regarded as a covering with sheets related by Galois group of extension of rationals degenerate to single sheet. The action of Galois group would reduce to that for its subgroup.

This is analogous to the degeneration of some roots of polynomial to single root and in  $M^8$  representation space-time sheets are indeed quite concretely roots of octonionic polynomial defined by vanishing of real or imaginary part in the decomposition  $o = q_1 + iq_2$  of octonion to a sum quaternionic real and imaginary parts.

The hierarchy of criticalities is closely related to the hierarchy of Planck constants  $h_{eff}/h_0 = n = n_1 n_2$ , where  $n_1$  corresponds to number of sheets as covering over  $CP_2$  and  $n_2$  as covering over  $M^4$ . One can also consider special cases in which  $M^4$  projection has dimension  $D < 4$ . The proposal is that  $n$  corresponds to the dimension of Galois group for extension of rationals defining the level of dark matter hierarchy. If  $n$  is prime, one has either  $n_1 = 1$  or  $n_2 = 1$ .

It seems that the range of  $n_2$  is rather limited since the expression for Newton's constant as  $G = R^2/n_2 \hbar$  varies in rather narrow range. If the covering has symmetries assignable to some discrete subgroup of  $SU(3)$  acting as isometries of  $CP_2$  this could be understood. The increase of criticality could mean that  $n_1$  or  $n_2$  or both are reduced.

What is the position of  $CP_2$  type extremals in the hierarchies of Planck constants and quantum criticalities?

- (a) Consider first  $n_2$ .  $CP_2$  type extremal have 1-D geodesic line as  $M^4$  projection. The light-like geodesic as 1-D structure could be interpreted as covering for which two geodesic lines along the orbits of opposite throats of wormhole contact form a kind of time loop. In this case one would have  $n_2 = 2$  and one could have  $n = 2p$ ,  $p$  prime.

In this sense  $CP_2$  type extremal or at least its core would be maximally critical. Deformations replacing the light-like geodesic as projection with higher-D region of  $M^4$  presumably reduce criticality and one has  $n_2 > 2$  is obtained. Whether this is possible inside wormhole contact is not clear. One can imagine that as one approaches partonic 2-surface, the criticality and degeneration increase in  $CP_2$  degrees of freedom step by step and reach maximum in its core. This would be like realization of Thom's catastrophe involving parts with various degrees of criticalities.

At the flux tubes mediating gravitational interaction  $n_2 \sim 10^7$  would hold true in the exterior of associated  $CP_2$  type extremals. This would suggest that  $CP_2$  type extremals have maximal criticality in  $M^4$  degrees of freedom and  $M^4$  covering reduces to 2-fold covering for wormhole contacts.

- (b) What about criticality as  $n_1$ -fold covering of  $CP_2$ . This covering corresponds to a situation in which  $CP_2$  coordinates as field in  $M^4$  have given values of  $CP_2$  coordinates  $n_1$  times. A lattice like structure formed by  $n_1$  wormhole contacts is suggestive.  $n_1$  can be arbitrary large in principle and the gravitational Planck constant  $h_{gr}/h_0 = n_1 n_2$  would correspond to this situation. Singularities would now correspond to a degeneration of some wormhole contacts to single wormhole contact and could have interpretation in terms of fusion of particles to single particle. One might perhaps interpret elementary particle reaction vertices as catastrophes.

Wormhole contacts can be regarded as  $CP_2$  type extremals having two holes corresponding to the 3-D orbits of wormhole contacts. Mathematician would probably speak of a blow up.  $CP_2$  type extremals is glued to surrounding Minkowskian space-time sheets at the 3-D

boundaries of these holes. At the orbit of partonic 2-surface the induced 4-metric degenerates to 3-D metric and 4-D tangent space becomes metrically 3-D. Light-likeness of the  $M^4$  projection would correspond to this. For  $CP_2$  type extremal 3 space-like  $M^4$  directions of Minkowskian region would transmute to  $CP_2$  directions at the light-like geodesic and time direction would become light-like. This is like graph of function for which tangent becomes vertical. For deformations of  $CP_2$  type extremals this process could take place in several steps, one dimension in given step. This process could take place inside  $CP_2$  or outside it depending on which order the transmutation of dimensions takes place.

### 7.3 Delicacies associated with $M^4$ Kähler structure

Twistor lift forces to assume that also  $M^4$  possesses the analog of Kähler form, and Minkowskian signature does not prevent this [K29].  $M^4$  Kähler structure breaks CP symmetry and provides a very attractive manner to break CP symmetry and explain generation of matter antimatter symmetry and CP breaking in hadron physics. The CP breaking is very small characterized by a dimensionless number of order  $10^{-9}$  identifiable as photon/baryon ratio. Can one understand the smallness of CP breaking in TGD framework?

#### 7.3.1 Hamilton-Jacobi structure

Hamilton-Jacobi structure [K25] can be seen as a generalization of complex structure and involves a local but integrable selection of subspaces of various dimension for the tangent space of  $M^4$ . Integrability means that the selected subspaces are tangent spaces of a submanifold of  $M^4$ .  $M^8 - H$  duality allows to interpret this selection as being induced by a global selection of a hierarchy of real, complex, and quaternionic subspaces associated with octonionic structure mapped to  $M^4$  in such a manner that this global selection becomes local at the level of  $H$ .

- (a) The 4-D analog of conformal invariance is due to very special conformal properties of light-like 3-surfaces and light-cone boundary of  $M^4$ . This raises hopes about construction of general solution families by utilizing the generalized form of conformal invariance. Massless extremals (MEs) in fact define extremely general solution family of this kind and involve light-like direction vector  $k$  and polarization vector  $\epsilon$  orthogonal to it defining decomposition  $M^4 = M^2 \times E^2$ . I have proposed that this decomposition generalizes to local but integrable decomposition so that the distributions for  $M^2$  and  $E^2$  integrate to string world sheets and partonic 2-surfaces.
- (b) One can have decomposition  $M^4 = M^2 \times E^2$  such that one has Minkowskian analog of conformal symmetry in  $M^2$ . This decomposition is defined by the vectors  $k$  and  $\epsilon$ . An unproven conjecture is that these vectors can depend on point and the proposed Hamilton-Jacobi structure would mean a *local* decomposition of tangent space of  $M^4$ , which is integrable meaning that local  $M^2$ s integrate to string world sheet in  $M^4$  and local  $E^2$ s integrate to closed 2-surface as special case corresponds to partonic 2-surface. Generalizing the terminology, one could talk about family of partonic surfaces. These decompositions could define families of extremals.  
An integrable decomposition of  $M^4$  to string world sheets and partonic 2-surfaces would characterize the preferred extremals with 4-D  $M^4$  projection. Integrable distribution would mean assignment of partonic 2-surface to each point of string world sheet and vice versa.
- (c)  $M^4$  Kähler form defines unique decomposition  $M^2 \times E^2$ . This is however not consistent Lorentz invariance. To cure this problem one must allow moduli space for  $M^4$  Kähler forms such that one can assign to each Hamilton-Jacobi structure  $M^4$  Kähler form defining the corresponding integrable surfaces in terms of light-like vector and polarization vector whose directions depend on point of  $M^4$ .

This looks strange since the very idea is that the imbedding space is unique. However, this local decomposition could be secondary being associated only with  $H = M^4 \times CP_2$

and emerge in  $M^8 - H$  duality mapping of space-time surfaces  $X^4 \subset M^8$  to surfaces in  $M^4 \times CP_2$ . There is a moduli space for octonion structures in  $M^8$  defined as a choice of preferred time axis  $M^1$  (rest system), preferred  $M^2$  defining hypercomplex plane and preferred direction (light-like vector), and quaternionic plane  $M^2 \times E^2$  (also polarization direction is included). Lorentz boosts mixing the real and imaginary octonion coordinates and changing the direction of time axis give rise to octonion structures not equivalent with the original one.

Thus the choice  $M^1 \subset M^2 \subset M^4 = M^2 \times E^2 \subset M^8$  is involved with the definition of octonion structure and quaternionion structure. The image of this decomposition under  $M^8 - H$  duality mapping quaternionic tangent space of  $X^4 \subset M^8$  containing  $M^1$  and  $M^2$  as sub-spaces would be such that the image of  $M^1 \subset M^2 \subset M^2 \times E^2$  depends on point of  $M^4 \subset H$  in integrable manner so that Hamilton-Jacobi structure in  $H$  is obtained.

Also  $CP_2$  allows the analog of Hamilton-Jacobi structure as a local decomposition integrating to a family of geodesic spheres  $S_I^2$  as analog of partonic 2-surfaces with complex structure and having at each point as a fiber different  $S_I^2$  - these spheres necessary intersect at single point. This decomposition could correspond to the 4-D complex structure of  $CP_2$  and complex coordinates of  $CP_2$  would serve as coordinates for the two geodesic spheres.

Could one imagine decompositions in which fiber is 2-D Lagrangian manifold - say  $S_{II}^2$  - with vanishing induced Kähler form and not possessing induced complex structure?  $S_{II}^2$  does not have complex structure as induced complex structure and is therefore analogous to  $M^2$ .  $S_{II}^2$  coordinates would be functions of string world sheet coordinates (in special as analytic in hypercomplex sense and describing wave propagating with light-velocity).  $S_I^2$  coordinates would be analytic functions of complex coordinates of partonic 2-surface.

### 7.3.2 CP breaking and $M^4$ Kähler structure

The CP breaking induce by  $M^4$  Kähler structure should be small. Is this automatically true or must one make some assumptions to achieve this.

Could one guarantee this by brute force by assuming  $M^4$  and  $CP_2$  parts of Kähler action to have different normalizations. The proposal for the length scale evolution of cosmological constant however relies on almost cancellation  $M^4$  induced Kähler forms of  $M^4$  and  $CP_2$  parts due to the fact that the induced forms differ from each other by a rotation of the twistor sphere  $S^2$ . The  $S^2$  part  $M^4 \times S^2$  Kähler form can have opposite with respect to  $T(CP_2) = SU(3)/U(1) \times U(1)$  Kähler so that for trivial rotation the forms cancel completely. If the normalizations of Kähler actions differ this cannot happen at the level of 4-D Kähler action.

To make progress, it is useful to look at the situation more concretely.

- (a) Kähler action is dimensionless. The square of Kähler form is metric so that  $J_{kl}J^{kl}$  is dimensionless. One must include to the 4-D Kähler action a dimensional factor  $1/L^4$  to make it dimensionless. The natural choice for  $L$  is as the radius  $R$  of  $CP_2$  geodesic sphere to radius of twistor spheres for  $M^4$  and  $CP_2$ . Note however that there is numerical constant involved and if it is changed there must be a compensating change of Kähler coupling strength. Therefore  $M^4$  contribution to action is proportional to the volume of  $M^4$  region using  $R^4$  as unit. This contribution is very large for macroscopic regions of  $M^4$  unless self-duality of  $M^4$  Kähler form would not cause cancellation ( $E^2 - B^2 = 0$ ).
- (b) What about energy density? The naive expectation based on Maxwell's theory is that the energy density assignable to  $M^4$  Kähler form is by self-duality proportional to  $E^2 + B^2 = 2E^2$  and non-vanishing. By naive order of magnitude estimate using Maxwellian formula for the energy of this kind extremal is proportional to  $Vol_3/R^4$  and very large. Does this exclude these extremals or should one assume that they have very small volume? For macroscopic lengths of one should assume extremely thin MEs with thickness smaller than  $R$ . Could one have 2-fold covering formed by gluing to copies of very thin MEs together along their boundaries. This does not look feasible.

Luckily, the Maxwellian intuition fails in TGD framework. The Noether currents associated in presence of  $M^4$  Kähler action involve also a term coming from the variation of the induced  $M^4$  Kähler form. This term guarantees that canonical momentum currents as  $H$ -vector fields are orthogonal to the space-time surface. In the case of  $CP_2$  type extremals this causes the cancellation of the canonical momentum currents associated with Kähler action and corresponding contributions to conserved charges. The complete symmetry between  $M^4$  and  $CP_2$  and also physical intuition demanding that canonically imbedded  $M^4$  os vacuum require that cancellation takes place also for  $M^4$  part so that only the term corresponding to cosmological constant remains.

### 7.3.3 $M^4$ Kähler form and CP breaking for various kinds of extremals

I have considered already earlier the proposal that CP breaking is due to  $M^4$  Kähler form [K29]. CP breaking is very small and the proposal inspired by the Cartesian product structure of the imbedding space and its twistor bundle and also by the similar decomposition of  $T(M^4) = M^4 \times S^2$  was that the coefficient of  $M^4$  part of Kähler action can be chosen to be much smaller than the coefficient of  $CP_2$  part. The proposed mechanism giving rise to p-adic length scale evolution of cosmological constant however requires that the coefficients of are identical. Luckily, the CP breaking term is automatically very small as the following arguments based on the examination of various kinds of extremals demonstrate.

- (a) For  $CP_2$  type extremals with light-like  $M^4$  geodesics as  $M^4$  projection the induced  $M^4$  Kähler form vanishes so that there is no CP breaking. For small deformations  $CP_2$  type extremals thickening the  $M^4$  projection the induced  $M^4$  Kähler form is non-vanishing. An attractive hypothesis is that the small CP breaking parameter quantifies the order of magnitude of the induced  $M^4$  Kähler form. This picture could allow to understand CP breaking of hadrons.
- (b) Canonically imbedded  $M^4$  is a minimal surface. A small breaking of CP symmetry is generated in small deformations of  $M^4$ . In particular, for massless extremals (MEs) having 4-D  $M^4$  projection the action associated with  $M^4$  part of Kähler action vanishes at the  $M^4$  limit when the local polarization vector characterizing ME approaches zero. The small CP breaking is characterized by the size of the polarization vector  $\epsilon$  giving a contribution to the induced metric. This conforms with the perturbative CP breaking.
- (c) String like objects of type  $X^4 = X^2 \times Y^2 \subset M^4 \times CP_2$ , where  $X^2$  is minimal surface and  $Y^2$  is 2-surface in  $CP_2$ . The  $M^4$  projection contains only electric part but no magnetic part. The  $M^4$  part of action is proportional to the volume  $Y^2$  and therefore very small. This in turn guarantees smallness of CP breaking effects.
  - i. If  $Y^2$  is homologically non-trivial (magnetic flux tube carries monopole flux),  $CP_2$  part of action is large since action density is proportional  $1/\sqrt{\det(g_2)}$  for  $Y^2$  and therefore large. The thickening of the flux tube however reduces the value of the action by flux conservation as discussed already earlier.  
 $M^4$  and  $CP_2$  contributions to the actions are of opposite sign but  $M^4$  contribution os however very small as compared to  $CP_2$  contribution. One can look the situation in  $M^2 \times S^2$  coordinates. The transverse deformation would correspond to the dependence of  $E^2$  coordinates on  $S^2$  coordinates. The induced Kähler form would give a contribution to the  $S^2$  part of induced Kähler form whose size would characterize CP breaking.
  - ii.  $Y^2$  can be also homologically trivial. In particular, for  $Y^2 = S^2_{II}$  the  $CP_2$  contribution to the total Kähler action vanishes and only the small  $M^4$  contribution proportional to the area of  $Y^2$  remains.

## 7.4 About TGD counterparts for the simplest classical field patterns

What could be the TGD counterparts of typical configurations of classical fields? Since minimal surface equation is a nonlinear generalization of massless field equations, one can

hope that the simplest solutions of Maxwell's equations have TGD analogs. The strong non-linearity poses a strong constraint, which can be solved if the extremal allows generalization of holomorphic structure so that field equations are trivially true since they involve in complex coordinates a contraction of tensors of type (1,1) with tensors of type (2,0) or (0,2). It is not clear whether minimal surface property reducing to holomorphy is equivalent with preferred extremal property.

Can one have the basic field patterns such as multipoles as structures with 4-D  $M^4$  projection or could it be that flux tube picture based on spherical harmonics for the orientation of flux tube is all that one can have? Same question can be made for radiation fields having MEs as archetypal representatives in TGD framework. What about the possible consistency problems produced by  $M^4$  Kähler form breaking Lorentz invariance?

I have considered these questions already earlier. The following approach is just making questions and guesses possibly helping to develop general ideas about the correspondence.

- (a) In QFT approach one expresses fields as superpositions of partial waves, which are indeed very simple field patterns and the coefficients in the superposition become oscillator operators. What could be the analogs of partial waves in TGD? Simultaneous extremals of Kähler action and volume strongly suggest themselves as carriers of field archetypes but the non-linearity of field equations does not support the idea that partial waves could be realized at classical level as extremals with 4-D  $M^4$  projection. A more plausible option is that they correspond to spherical harmonics for the orientation of flux tube carrying say electric flux. Could the flux tubes of various kinds serve as building of all classical fields?
- (b) String-like objects  $X^2 \times Y^2 \subset M^4 \times CP_2$ , where string world sheet  $X^2$  is minimal surface and  $Y^2$  is sub-manifold of  $CP_2$  and their deformations in  $M^4$  degrees of freedom transversal to  $X^2$  and depending on the coordinates  $Y^2$  are certainly good candidates for archetypal field configurations.  
 $Y^2$  can be homologically trivial and could correspond to Lagrangian sub-manifold.  $Y^2$  can also carry homology charge  $n$  identifiable as Kähler magnetic charge and correspond to complex sub-manifold of  $CP_2$  with complex structure induced from that of  $CP_2$ .

The simplest option corresponds to geodesic sphere  $Y^2 = S^2$ . There are two geodesic spheres in  $CP_2$  and they correspond to simplest string like objects.

- (a)  $S_I^2$  has Kähler magnetic charge of one unit and the cosmic and its deformations carry monopole flux. These field configurations are not possible in Maxwell's electrodynamics and the proposal is that they appear in all length scales. The model for the formation of galaxies solving also the problem of galactic dark matter relies on long cosmic strings. They are proposed to appear also in biology.
- (b)  $S_{II}^2$  is homologically trivial so that magnetic flux over it vanishes although magnetic field is non-vanishing. Note that although the Kähler magnetic field is vanishing, the electromagnetic ordinary magnetic field is non-vanishing because em field is a combination of Kähler form and component of  $CP_2$  curvature form with vanishing weak isospin. The total flux of ordinary magnetic field over  $S_{II}^2$  vanishes whereas electric flux can be non-vanishing.

#### 7.4.1 Coulomb fields

By the vanishing of magnetic flux flux tubes for  $S_{II}^2$  cannot represent ordinary magnetic field. They can however serve as radial flux tubes carrying electromagnetic flux. Magnetic flux tubes indeed allow time dependent deformations for which the phase angles of  $CP_2$  coordinates depend linearly of  $M^4$  time coordinate. This would give rise to an archetypal flux tube representation of the electric field created by point charge. Also gravitational flux tubes should correspond to this kind flux tubes emanating radially from the source.

Charge quantization suggests that these flux tubes carry unit charge. In the case of charged elementary particle there would be only single flux tubes but there would be wave function

for its orientation having no angular dependence. In principle, this wave function can any spherical harmonic.

Does the orientation angle dependence of flux distribution have any counterpart in Maxwell's theory. One would have the analog of  $1/r$  Coulomb potential with the modulus squared of spherical harmonic  $Y_{lm}$  modulating it. Could one consider the possibility that in atoms the spherical harmonics for excited states correspond to this kind of distribution for the electric flux coming from nucleus. The probability amplitude for electrons touching the flux tube would inherit this distribution.

For many particle system with large em charge there would be large number of radial flux tubes and the approximation of electric field with Coulomb field becomes natural. In the case of atoms this limit is achieved for large enough nuclear charges. This does not exclude the possibility of having space-time surfaces carrying Coulomb potential in Maxwellian sense: in this case however the field equations cannot be solved by holomorphy and quantum criticality might exclude these configurations.

What about gravitation? The notion of gravitational Planck constant requires that Planck mass replaced in TGD framework by  $CP_2$  mass defining the unit of gravitational flux -  $h_{gr}0GMm/v_0$  cannot be smaller than  $h_0$ . What happens in systems possessing mass smaller than  $CP_2$  mass? Are gravitational flux tubes absent. Is gravitational interaction absent in this kind of systems or is its description analogous to string model description meaning that  $h_{gr} = h_0$  for masses smaller than  $CP_2$  mass?

#### 7.4.2 Magnetic fields

As such  $S_{II}^2$  flux tubes cannot serve as counterparts of ordinary magnetic fields. The flux tubes have now boundary and the current at boundary creates the magnetic field inside the tube. This would mean cutting of a disk  $D^2$  from  $S_{II}^2$  so that the net magnetic flux becomes non-vanishing.

The assumption has been that genuine boundaries are not possible since conservation laws very probably prevent them (the normal components of canonical momentum currents should vanish at boundaries but this is not possible). This requires that this flux tube must be glued along the boundary of  $D^2 \times D^1$  to surrounding space-time surface  $X^4$ , which has a similar hole. At the boundary of this hole the space-time surface must turn to the direction of  $CP_2$  meaning that the dimension of  $M^4$  projection is reduced to  $D = 2$ . Algebraic geometer would talk about blow-up.

Ordinary multipole magnetic field could correspond to spherical harmonic for the orientation of this kind flux tubes. They could also carry electric flux but the em charge could be fractionized. These flux tubes might relate to anyons carrying fractional em charge. Also the fractional charges of quarks could classically correspond to flux tubes mediating both color magnetic field and em flux. The spherical harmonic in question corresponds to that associated with electron in atoms.

#### 7.4.3 Magnetic and electric fields associated with straight current wire

Magnetic and electric fields associated with straight current wire need not allow representation as archetypes since they are obviously macroscopic entities.

- (a) Is the magnetic field associated with straight current wire representable in terms of extremal with 4-D  $M^4$  projection. The magnetic field lines rotate around the current and it does not seem natural to model it the field in terms of flux tubes. Forget the presence of  $M^4$  Kähler form. One can imbed this kind of magnetic field as a surface with 4-D  $M^4$  projection and possessing cylindrical symmetry. Line current would correspond to a source of the magnetic field and could be realized as a flux tube carrying em current and topologically condensed to the space-time sheet in question.

The imbedding however fails at certain critical radius and the assumption is that no boundaries are allowed by conservation laws. Should one glue the structure to the

surrounding space-time surface at this radius. In Maxwell's theory one would have surface current in direction opposite to the source cancelling the magnetic field outside. Could this current have interpretation as a return current?

One can also imagine glueing its copy to it along the boundary at critical radius. It would seem that the magnetic fields must have same direction at the boundary and therefore also in interior.

- (b) What about current ring? Separation of variables is essential for the simplest imbeddings implying a reduction of partial differential equations to differential equation. There is rather small number of coordinates system in  $E^3$  in which Laplacian allows separation of variables. The metric is diagonal in these coordinates. One example is toroidal coordinates assignable with a current ring having toroidal geometry. This would allow a construction of minimal surface solution in some finite volume. Minimal surface property would *not* reduce to complex analyticity for these extremals and they would be naturally associated  $M^4 \times S^2_{II}$ .

**Remark:** This kind of extremals are not holomorphic and could be excluded by quantum criticality and preferred extremal property. GRT space-time would be idealization making sense only at the QFT limit of TGD.

#### 7.4.4 Time dependent fields

What about time dependent fields such as the field created by oscillating dipole and radiation fields? One can imagine quantal and classical option.

- (a) The simplest possibility is reduction to quantum description at single particle level. The dipole current corresponds to a wave function for the source particle system consisting of systems with opposite total charge.

Spherical harmonics representing multipoles would induce wave function for the orientations of MEs (topological light ray) carrying radial wave. This is certainly the most natural options as far radiation field at large distances from sources is considered. One can also have second quantization in the proposed sense giving rise to multi-photon states and one can also define coherent states.

One should also understand time dependent fields near sources having also non-radiative part. This requires a model for source such as oscillating dipole. The simplest possibility is that in the case of dipole there are charges of opposite sign with oscillating distance creating Coulomb fields represented in the proposed manner. It is however not obvious that preferred extremals of this kind exist.

- (b) One can consider also classical description. The model of elementary particle as consisting of two wormhole contacts, whose throats effectively serve as end of monopole flux tubes at the two sheets involved suggests a possible model. If the wormhole contacts carry opposite em charges realized in terms of fermion and antifermions an oscillating dipole could correspond to flux tube whose length oscillates. This means generation of radiation and for elementary particles this would suggest instability against decay. One can however consider excitation which decay to ground states - say for hadrons. For scaled up variants of this structure this would not mean instability although energy is lost and the system must end up to non-oscillating state.

One possibility is that there are two charges at different space-time sheets connected by wormhole contacts and oscillating by their mutual interaction in harmonic oscillator state. Ground state would be stable and have not dipole moment.

#### 7.4.5 Effectively 2-D systems

In classical electrodynamics effectively 2-D systems are very special in that they allow conformal invariance assignable to 2-D Laplacian.

- (a) Since minimal surface equation is generalization of massless d'Alembertian and since field equations are trivially true for analytic solutions, one can hope that the basic solutions of 4-D d'Alembertian generalize in TGD framework. This would conform with the universality of quantum criticality meaning that coupling parameters disappear from field equations. Conformal invariance or its generalization would mean huge variety of field patterns. This suggests that effectively 2-D systems serve as basic building bricks of more complex field configurations. Flux tubes of various kinds would represent basic examples of this kind of surfaces. Also the magnetic and electric fields associated with straight current wire would serve as an example.
- (b) Are there preferred extremals analogous to the solutions of field equations of general relativity in faraway regions, where they become simple and might allow an analog in TGD framework? If our mathematical models reflect the preferred extremals as archetypal structures, this could be the case.

Forget for a moment the technicalities related to  $M^4$  Kähler form. One can construct a spherically symmetric ansatz in  $M^4 \times S^2_{II}$  as a minimal surface for which  $\Phi$  depends linearly on time  $t$  and  $u$  is function of  $r$ . The ansatz reduces to a highly non-linear differential equation for  $u$ . In this case hyper-complex analyticity is obviously not satisfied. This ansatz could give the analog of Schwarzschild metric giving also the electric field of point charge appearing as source of the non-linear variant of d'Alembertian. It is however far from clear whether this kind extremals is allowed as preferred extremals.

Under which conditions spherically symmetric ansatz is consistent with  $M^4$  Kähler form? Obviously, the  $M^4$  Kähler form must be spherically symmetric as also the Hamilton-Jacobi structure it. Suppose local Hamilton-Jacobi structures for which  $M^2$ s integrate to  $t, r$  coordinate planes and  $E^2$ s integrate to  $(\theta, \phi)$  sphere are allowed and that  $M^4$  Kähler form defines this decomposition. In this case there are hopes that consistency conditions can be satisfied. Note however that  $M^4$  Kähler form defines in this case orthogonal magnetic and electric monopole fields defining an analog of instanton. Can one really allow this or should one exclude the time line with  $r = 0$ ?

Similar  $M^4$  Kähler structure can be associated with cylindrical coordinates and other separable coordinates system.  $M^4$  Kähler structure would define Hamilton-Jacobi structure.

## 8 Minimal surfaces and TGD

The twistor lift of TGD [K28, K33, L6] meant a revolution in the understanding of TGD and led to a new view about what preferred extremal property means physically and why it is needed.

- (a) The construction of twistor lift of TGD replaces space-time surfaces with 6-D surfaces but requires that they are dynamically effectively 4-D as the analogs of twistor space having the structure of  $S^2$  bundle with space-time surface as the base. This requires dimensional reduction making  $S^2$  fiber of the twistor space non-dynamical. One can say that twistor structure is induced from that for 12-D product of the geometric 6-D twistor spaces of  $M^4$  and  $CP_2$ . The condition that 6-D Kähler action exists requires that the twistor spaces of  $M^4$  and  $CP_2$  have Kähler structure. This condition allows only  $H = M^4 \times CP_2$  [A13]. The condition that one obtains standard model symmetries leads to the same conclusion.
- (b) The dimensionally reduced Kähler action decomposes to a sum of 4-D Kähler action and volume term. The interaction is as analog of Maxwell action plus action of point-like particle replaced with 3-D surface. The coefficient of the volume term has an interpretation as cosmological constant having a discrete spectrum [L7]. The natural proposal is that it depends on p-adic length scale approaching zero in long length scales. This solves the cosmological constant problem.
- (c) I had actually known for decades that all non-vacuum extremals of 4-D Kähler action are minimal surfaces thus minimizing the space-time volume in the induced metric.

This is because the field equations for Kähler action for known non-vacuum extremals were reduced essentially to algebraic conditions realizing holomorphy. Also so called  $CP_2$  type vacuum extremals of 4-D Kähler action are minimal surfaces. This finding conforms with the fact that in  $M^8 - H$  duality [K31] one has regard field equations as purely algebraic conditions at  $M^8$  side of the duality.

This inspired the proposal that preferred extremal property of space-time surface is realized by requiring that space-time surfaces as base spaces of these 6-D twistor spaces are quite generally minimal surfaces, and therefore represent a non-linear geometrization for the notion of massless field in accordance with conformal invariance forced by quantum criticality.

Also a more general proposal that space-time contains regions inside which there is an exchange of canonical momenta between Kähler action and volume term was considered. Minimal surface regions would correspond to incoming particles and non-minimal ones to interaction regions.

Later this proposal was simplified by requiring that interaction regions are 2-D string world sheets as singularities: this implied that string world sheets required by general considerations [K19] indeed emerge from 4-D action. This could happen also at the 1-D boundaries of string world sheets at 3-D light-like boundaries between Minkowskian and Euclidian regions behaving like ordinary point-like particles and carrying fermion number, and in the most general case also at these 3-D light-like 3-surfaces.

## 8.1 Space-time surfaces as singular minimal surfaces

From the physics point this is not surprising since minimal surface equations are the geometric analog for massless field equations.

- (a) The boundary value problem in TGD is analogous to that defining soap films spanned by frames: space-time surface is thus like a 4-D soap film. Space-time surface has 3-D ends at the opposite boundaries of causal diamond of  $M^4$  with points replaced with  $CP_2$ : I call this 8-D object just causal diamond (CD). Geometrically CD brings in mind big-bang followed by big crunch.

These 3-D ends are like the frame of a soap film. This and the Minkowskian signature guarantees the existence of minimal surface extremals. Otherwise one would expect that the non-compactness does not allow minimal surfaces as non-self-intersecting surfaces.

- (b) Space-time is a 4-surface in 8-D  $H = M^4 \times CP_2$  and is a minimal surface, which can have 2-D or 1-D singularities identifiable as string world sheets having 1-D singularities as light-like orbits - they could be geodesics of space-time surface.

**Remark:** I considered in [L3] the possibility that the minimal surface property could fail only at the reaction vertices associated with partonic 2-surfaces defining the ends of string world sheet boundaries. This condition however seems to be too strong. It is essential that the singular surface defines a sub-manifold giving deltafunction like contribution to the action density and that one can assign conserved quantities to this surface. This requires that the singular contributions to energy momentum tensor and canonical momentum currents as spacetime vectors are parallel to the singular surface. Singular points do not satisfy this condition.

String boundaries represent orbits of fundamental point-like fermions located at 3-D light-like surfaces which represent orbits of partonic 2-surfaces. String world sheets are minimal surfaces and correspond to stringy objects associated with say hadrons. There are also degrees of freedom associated with space-time interior. One have objects of various dimension which all are minimal surfaces. Modified Dirac equation extends the field equations to supersymmetric system and assigns fermionic degrees of freedom to these minimal surfaces of varying dimension.

From the physics point of view, the singular surfaces are analogous to carriers of currents acting as point- and string-like sources of massless field equations.

- (c) Geometrically string world sheets are analogous to folds of paper sheet. Space-time surfaces are extremals of an action which is sum of volume term having interpretation in terms of cosmological constant and what I call Kähler action - analogous to Maxwell action. Outside singularities one has minimal surfaces stationary with respect to variations of both volume term and Kähler action - note the analogy with free massless field. At singularities there is an exchange of conserved quantities between volume and Kähler degrees of freedom analogous to the interaction of charged particle with electromagnetic field. One can see TGD as a generalization of a dynamics of point-like particle coupled to Maxwell field by making particle 3-D surface.
- (d) The condition that the exchange of conserved charges such as four-momentum is restricted to lower-D surfaces realizes preferred extremal property as a consequence of quantum criticality demanding a universal dynamics independent of coupling parameters [L7]. Indeed, outside the singularities the minimal surfaces dynamics has no explicit dependence on coupling constants provided local minimal surface property guarantees also the local stationarity of Kähler action.

Preferred extremal property has also other formulations. What is essential is the generalization of super-conformal symmetry playing key role in super string models and in the theory of 2-D critical systems so that field equations reduce to purely algebraic conditions just like for analytic functions in 2-D space providing solutions of Laplace equations.

- (e) TGD provides a large number of specific examples about closed minimal surfaces [K25]. Cosmic strings are objects, which are Cartesian products of minimal surfaces (string world sheets) in  $M^4$  and of complex algebraic curves (2-D surfaces). Both are minimal surfaces and extremize also Kähler action. These algebraic surfaces are non-contractible and characterized by homology charge having interpretation as Kähler magnetic charge. These surfaces are genuine minima just like the geodesics at torus.

$CP_2$  contains two kinds of geodesic spheres, which are trivially minimal surfaces. The reason is that the second fundamental form defining as its trace the analogs of external curvatures in the normal space of the surfaces vanishes identically. The geodesic sphere of the first kind is non-contractible minimal surface and absolute minimum. Geodesic spheres of second kind is contractible and one has Minimax type situation.

These geodesic spheres are analogous to 2-planes in flat 3-space with vanishing external curvatures. For a generic minimal surface in 3-space the principal curvatures are non-vanishing and sum up to zero. This implies that minimal surfaces look locally like saddles. For 2-plane the curvatures vanish identically so that saddle is not formed.

## 8.2 Kähler action as Morse function in the space of minimal 4-surfaces

It was found that surface volume could define a Morse function in the space of surfaces. What about the situation in TGD, where volume is replaced with action which is sum of volume term and Kähler action [L6, ?, L7]?

Morse function interpretation could appear in two manners. The first possibility is that the action defines an analog of Morse function in the space of 4-surfaces connecting given 3-surfaces at the boundaries of CD. Could it be that there is large number of preferred extremals connecting given 3-surfaces at the boundaries of CD? This would serve as analogy for the existence of infinite number of closed surfaces in the case of compact imbedding space. The fact that preferred extremals extremize almost everywhere two different actions suggests that this is not the case but one must consider also this option.

- (a) The simplest realization of general coordinate invariance would allow only single preferred extremal but I have considered also the option for which one has several preferred extremals. In this case one encounters problem with the definition of Kähler function which would become many-valued unless one is ready to replace 3-surfaces with its covering so that each preferred extremal associated with the given 3-surface gives rise to its

own 3-surface in the covering space. Note that analogy with the definition of covering space of say circle by replacing points with the set of homologically equivalence classes of closed paths at given point (rotating arbitrary number of times around circle).

- (b) Number theoretic vision [K24, K27] suggests that these possibly existing different preferred extremals are analogous to same algebraic computation but performed in different manners or theorem proved in different manners. There is always the shortest manner to do the computation and an attractive idea is that the physical predictions of TGD do not depend on what preferred extremal is chosen.
- (c) An interesting question is whether the “drum theorem” could generalize to TGD framework. If there exists infinite series of preferred extremals which are singular minimal surfaces, the volume of space-time surface for surfaces in the series would depend only on the volume of the CD containing it. The analogy with the high frequencies and drum suggests that the surfaces in the series have more and more local details. In number theoretic vision this would correspond to emergence of more and more un-necessary pieces to the computation. One cannot exclude the possibility that these details are analogs for what is called loop corrections in quantum field theory.
- (d) If the action defines Morse action, the preferred extremals give information about its topology. Note however that the requirement that one has extremum of both volume term and Kähler action almost everywhere is an extremely strong additional condition and corresponds physically to quantum criticality.

*Remark:* The original assumption was that the space-time surface decomposes to critical regions which are minimal surfaces locally and to non-critical regions inside which there is flow of canonical momentum currents between volume and Kähler degrees of freedom. The stronger hypothesis is that this flow occurs at 2-D and 1-D surfaces only.

### 8.3 Kähler function as Morse function in the space of 3-surfaces

The notion of Morse function can make sense also in the space of 3-surfaces - the world of classical worlds which in zero energy ontology consists of pairs of 3-surfaces at opposite boundaries of CD connected by preferred extremal of Kähler action [K7, K23, L6, L4]. Kähler action for the preferred extremal is assumed to define Kähler function defining Kähler metric of WCW via its second derivatives  $\partial_K \partial_{\bar{L}} K$ . Could Kähler function define a Morse function?

- (a) First of all, Morse function must be a genuine function. For general Kähler metric this is not the case. Rather, Kähler function  $K$  is a section in a  $U(1)$  bundle consisting of patches transforming by real part of a complex gradient as one moves between the patches of the bundle. A good example is  $CP_2$ , which has non-trivial topology, and which decomposes to 3 coordinate patches such that Kähler functions in overlapping patches are related by the analog of  $U(1)$  gauge transformation.

Kähler action for preferred extremal associated with given 3-surface is however uniquely defined unless one includes Chern-Simons term which changes in  $U(1)$  gauge transformation for Kähler gauge potential of  $CP_2$ .

- (b) What could one conclude about the topology of WCW if the action for preferred extremal defines a Morse function as a functional of 3-surface? This function cannot have saddle points: in a region of WCW around saddle point the WCW metric depending on the second derivatives of Morse function would not be positive definite, and this is excluded by the positivity of Hilbert space inner product defined by the Kähler metric essential for the unitarity of the theory. This would suggest that the space of 3-surfaces has very simple topology if Kähler function.

This is too hasty conclusion! WCW metric is expected to depend also on zero modes, which do not contribute to the WCW line element. What suggests itself is bundle structure. Zero modes define the base space and dynamical degrees of freedom contributing to WCW line element as fiber. The space of zero modes can be topologically complex.

There is a fascinating open problem related to the metric of WCW.

- (a) The conjecture is that WCW metric possess the symplectic symmetries of  $\Delta M_+^4 \times CP_2$  as isometries. In infinite dimensional case the existence of Riemann/Kähler geometry is not at all obvious as the work of Dan Freed demonstrated in the case of loops spaces [A12], and the maximal group of isometries would guarantee the existence of WCW Kähler geometry. Geometry would be determined by symmetries alone and all points of the space would be metrically equivalent. WCW would be an infinite-dimensional analog of symmetric space.
- (b) Isometry group property does not require that symplectic symmetries leave Kähler action, and even less volume term for preferred extremal, invariant. Just the opposite: if the action would remain invariant, Kähler function and Kähler metric would be trivial!
- (c) The condition for the existence of symplectic isometries must fix the ratio of the coefficients of Kähler action and volume term highly uniquely. The physical interpretation is in terms of quantum criticality realized mathematically in terms of the symplectic symmetry serving as analog of ordinary conformal symmetry characterizing 2-D critical systems. Note that at classical level quantum criticality realized as minimal surface property says nothing outside singular surfaces since the field equations in this regions are algebraic. At singularities the situation changes. Note also that the minimal surface property is a geometric analog of masslessness which in turn is a correlate of criticality.
- (d) Twistor lift of TGD [?] leads to a proposal for the spectra of Kähler coupling strength and cosmological constant allowed by quantum criticality [L4]. What is surprising that cosmological constant identified as the coefficient of the volume term takes the role of cutoff mass in coupling constant evolution in TGD framework. Coupling constant evolution discretizes in accordance with quantum criticality which must give rise to infinite-D group of WCW isometries. There is also a connection with number theoretic vision in which coupling constant evolution has interpretation in terms of extensions of rationals [K24, K32, K31].

#### 8.4 Kähler calibrations: an idea before its time?

While updating book introductions I was surprised to find that I had talked about so called calibrations of sub-manifolds as something potentially important for TGD and later forgotten the whole idea! A closer examination however demonstrated that I had ended up with the analog of this notion completely independently later as the idea that preferred extremals are minimal surfaces apart from 2-D singular surfaces, where there would be exchange of Noether charges between Kähler and volume degrees of freedom.

- (a) The original idea that I forgot too soon was that the notion of calibration (see <http://tinyurl.com/y31yead3>) generalizes and could be relevant for TGD. A calibration in Riemann manifold  $M$  means the existence of a  $k$ -form  $\phi$  in  $M$  such that for any orientable  $k$ -D sub-manifold the integral of  $\phi$  over  $M$  equals to its  $k$ -volume in the induced metric. One can say that metric  $k$ -volume reduces to homological  $k$ -volume. Calibrated  $k$ -manifolds are minimal surfaces in their homology class, in other words their volume is minimal. Kähler calibration is induced by the  $k^{th}$  power of Kähler form and defines calibrated sub-manifold of real dimension  $2k$ . Calibrated sub-manifolds are in this case precisely the complex sub-manifolds. In the case of  $CP_2$  they would be complex curves (2-surfaces) as has become clear.
- (b) By the Minkowskian signature of  $M^4$  metric, the generalization of calibrated sub-manifold so that it would apply in  $M^4 \times CP_2$  is non-trivial. Twistor lift of TGD however forces to introduce the generalization of Kähler form in  $M^4$  (responsible for CP breaking and matter antimatter asymmetry) and calibrated manifolds in this case would be naturally analogs of string world sheets and partonic 2-surfaces as minimal surfaces. Cosmic strings are Cartesian products of string world sheets and complex curves of  $CP_2$ . Calibrated manifolds, which do not reduce to Cartesian products of string world sheets and complex surfaces of  $CP_2$  should also exist and are minimal surfaces.

One can also have 2-D calibrated surfaces and they could correspond to string world sheets and partonic 2-surfaces which also play key role in TGD. Even discrete points assignable to partonic 2-surfaces and representing fundamental fermions play a key role and would trivially correspond to calibrated surfaces.

- (c) Much later I ended up with the identification of preferred extremals as minimal surfaces by totally different route without realizing the possible connection with the generalized calibrations. Twistor lift and the notion of quantum criticality led to the proposal that preferred extremals for the twistor lift of Kähler action containing also volume term are minimal surfaces. Preferred extremals would be separately minimal surfaces and extrema of Kähler action and generalization of complex structure to what I called Hamilton-Jacobi structure would be an essential element. Quantum criticality outside singular surfaces would be realized as decoupling of the two parts of the action. May be all preferred extremals be regarded as calibrated in generalized sense.

If so, the dynamics of preferred extremals would define a homology theory in the sense that each homology class would contain single preferred extremal. TGD would define a generalized topological quantum field theory with conserved Noether charges (in particular rest energy) serving as generalized topological invariants having extremum in the set of topologically equivalent 3-surfaces.

It is interesting to recall that the original proposal for the preferred extremals as absolute minima of Kähler action has transformed during years to a proposal that they are absolute minima of volume action within given homology class and having fixed ends at the boundaries of CD.

- (d) The experience with  $CP_2$  would suggest that the Kähler structure of  $M^4$  defining the counterpart of form  $\phi$  is unique. There is however infinite number of different closed self-dual Kähler forms of  $M^4$  defining what I have called Hamilton-Jacobi structures. These forms can have subgroups of Poincare group as symmetries. For instance, magnetic flux tubes correspond to given cylindrically symmetry Kähler form. The problem disappears as one realizes that Kähler structures characterize families of preferred extremals rather than  $M^4$  itself.

If the notion of calibration indeed generalizes, one ends up with the same outcome - preferred extremals as minimal surfaces with 2-D string world sheets and partonic 2-surfaces as singularities - from many different directions.

- (a) Quantum criticality requires that dynamics does not depend on coupling parameters so that extremals must be separately extremals of both volume term and Kähler action and therefore minimal surfaces for which these degrees of freedom decouple except at singular 2-surfaces, where the necessary transfer of Noether charges between two degrees of freedom takes place at these. One ends up with string picture but strings alone are of course not enough. For instance, the dynamical string tension is determined by the dynamics for the twistor lift.
- (b) Almost topological QFT picture implies the same outcome: topological QFT property fails only at the string world sheets.
- (c) Discrete coupling constant evolution, vanishing of loop corrections, and number theoretical condition that scattering amplitudes make sense also in p-adic number fields, requires a representation of scattering amplitudes as sum over resonances realized in terms of string world sheets.
- (d) In the standard QFT picture about scattering incoming states are solutions of free massless field equations and interaction regions the fields have currents as sources. This picture is realized by the twistor lift of TGD in which the volume action corresponds to geodesic length and Kähler action to Maxwell action and coupling corresponds to a transfer of Noether charges between volume and Kähler degrees of freedom. Massless modes are represented by minimal surfaces arriving inside causal diamond (CD) and minimal surface property fails in the scattering region consisting of string world sheets.

- (e) Twistor lift forces  $M^4$  to have generalize Kähler form and this in turn strongly suggests a generalization of the notion of calibration. At physics side the implication is the understanding of CP breaking and matter anti-matter asymmetry.
- (f)  $M^8-H$  duality requires that the dynamics of space-time surfaces in  $H$  is equivalent with the algebraic dynamics in  $M^8$ . The effective reduction to almost topological dynamics implied by the minimal surface property implies this. String world sheets (partonic 2-surfaces) in  $H$  would be images of complex (co-complex sub-manifolds) of  $X^4 \subset M^8$  in  $H$ . This should allow to understand why the partial derivatives of imbedding space coordinates can be discontinuous at these edges/folds but there is no flow between interior and singular surface implying that string world sheets are minimal surfaces (so that one has conformal invariance).

The analogy with foams in 3-D space deserves to be noticed.

- (a) Foams can be modelled as 2-D minimal surfaces with edges meeting at vertices. TGD space-time could be seen as a dynamically generated foam in 4-D many-sheeted space-time consisting of 2-D minimal surfaces such that also the 4-D complement is a minimal surface. The counterparts for vertices would be light-like curves at light like orbits of partonic 2-surfaces from which several string world sheets can emanate.
- (b) Can one imagine something more analogous to the usual 3-D foam? Could the light-like orbits of partonic 2-surfaces define an analog of ordinary foam? Could also partonic 2-surfaces have edges consisting of 2-D minimal surfaces joined along edges representing strings connecting fermions inside partonic 2-surface?

For years ago I proposed what I called as symplectic QFT (SQFT) as an analog of conformal QFT and as part of quantum TGD [K4]. SQFT would have symplectic transformations as symmetries, and provide a description for the symplectic dynamics of partonic 2-surfaces. SQFT involves an analog of triangulation at partonic 2-surfaces and Kähler magnetic fluxes associated with them serve as observables. The problem was how to fix this kind of network. Partonic foam could serve as a concrete physical realization for the symplectic network and have fundamental fermions at vertices. The edges at partonic 2-surfaces would be space-like geodesics. The outcome would be a calibration involving objects of all dimensions  $0 \leq D \leq 4$  - a physical analog of homology theory.

## 9 Appendix: Hamilton-Jacobi Structure

In the following the definition of Hamilton-Jacobi structure is discussed in detail.

### 9.1 Hermitian And Hyper-Hermitian Structures

The starting point is the observation that besides the complex numbers forming a number field there are hyper-complex numbers. Imaginary unit  $i$  is replaced with  $e$  satisfying  $e^2 = 1$ . One obtains an algebra but not a number field since the norm is Minkowskian norm  $x^2 - y^2$ , which vanishes at light-cone  $x = y$  so that light-like hypercomplex numbers  $x \pm e$  do not have inverse. One has “almost” number field.

Hyper-complex numbers appear naturally in 2-D Minkowski space since the solutions of a massless field equation can be written as  $f = g(u = t - ex) + h(v = t + ex)$  with  $e^2 = 1$  realized by putting  $e = 1$ . Therefore Wick rotation relates sums of holomorphic and antiholomorphic functions to sums of hyper-holomorphic and anti-hyper-holomorphic functions. Note that  $u$  and  $v$  are hyper-complex conjugates of each other.

Complex  $n$ -dimensional spaces allow Hermitian structure. This means that the metric has in complex coordinates  $(z_1, \dots, z_n)$  the form in which the matrix elements of metric are non-vanishing only between  $z_i$  and complex conjugate of  $z_j$ . In 2-D case one obtains just  $ds^2 = g_{z\bar{z}} dz d\bar{z}$ . Note that in this case metric is conformally flat since line element is proportional

to the line element  $ds^2 = dzd\bar{z}$  of plane. This form is always possible locally. For complex n-D case one obtains  $ds^2 = g_{i\bar{j}}dz^i d\bar{z}^j$ .  $g_{i\bar{j}} = \overline{g_{j\bar{i}}}$  guaranteeing the reality of  $ds^2$ . In 2-D case this condition gives  $g_{z\bar{z}} = \overline{g_{z\bar{z}}}$ .

How could one generalize this line element to hyper-complex n-dimensional case. In 2-D case Minkowski space  $M^2$  one has  $ds^2 = g_{uv}dudv$ ,  $g_{uv} = 1$ . The obvious generalization would be the replacement  $ds^2 = g_{u_i v_j} du^i dv^j$ . Also now the analogs of reality conditions must hold with respect to  $u_i \leftrightarrow v_i$ .

## 9.2 Hamilton-Jacobi Structure

Consider next the path leading to Hamilton-Jacobi structure.

4-D Minkowski space  $M^4 = M^2 \times E^2$  is Cartesian product of hyper-complex  $M^2$  with complex plane  $E^2$ , and one has  $ds^2 = dudv + dzd\bar{z}$  in standard Minkowski coordinates. One can also consider more general integrable decompositions of  $M^4$  for which the tangent space  $TM^4 = M^4$  at each point is decomposed to  $M^2(x) \times E^2(x)$ . The physical analogy would be a position dependent decomposition of the degrees of freedom of massless particle to longitudinal ones ( $M^2(x)$ : light-like momentum is in this plane) and transversal ones ( $E^2(x)$ : polarization vector is in this plane). Cylindrical and spherical variants of Minkowski coordinates define two examples of this kind of coordinates (it is perhaps a good exercise to think what kind of decomposition of tangent space is in question in these examples). An interesting mathematical problem highly relevant for TGD is to identify all possible decompositions of this kind for empty Minkowski space.

The integrability of the decomposition means that the planes  $M^2(x)$  are tangent planes for 2-D surfaces of  $M^4$  analogous to Euclidian string world sheet. This gives slicing of  $M^4$  to Minkowskian string world sheets parametrized by euclidian string world sheets. The question is whether the sheets are stringy in a strong sense: that is minimal surfaces. This is not the case: for spherical coordinates the Euclidian string world sheets would be spheres which are not minimal surfaces. For cylindrical and spherical coordinates however  $M^2(x)$  integrate to plane  $M^2$ , which is minimal surface.

Integrability means in the case of  $M^2(x)$  the existence of light-like vector field  $J$  whose flow lines define a global coordinate. Its existence implies also the existence of its conjugate and together these vector fields give rise to  $M^2(x)$  at each point. This means that one has  $J = \Psi \nabla \Phi$ :  $\Phi$  indeed defines the global coordinate along flow lines. In the case of  $M^2$  either the coordinate  $u$  or  $v$  would be the coordinate in question. This kind of flows are called Beltrami flows. Obviously the same holds for the transversal planes  $E^2$ .

One can generalize this metric to the case of general 4-D space with Minkowski signature of metric. At least the elements  $g_{uv}$  and  $g_{z\bar{z}}$  are non-vanishing and can depend on both  $u, v$  and  $z, \bar{z}$ . They must satisfy the reality conditions  $g_{z\bar{z}} = \overline{g_{z\bar{z}}}$  and  $g_{uv} = \overline{g_{vu}}$  where complex conjugation in the argument involves also  $u \leftrightarrow v$  besides  $z \leftrightarrow \bar{z}$ .

The question is whether the components  $g_{uz}$ ,  $g_{vz}$ , and their complex conjugates are non-vanishing if they satisfy some conditions. They can. The direct generalization from complex 2-D space would be that one treats  $u$  and  $v$  as complex conjugates and therefore requires a direct generalization of the hermiticity condition

$$g_{uz} = \overline{g_{v\bar{z}}} \quad , \quad g_{vz} = \overline{g_{u\bar{z}}} \quad .$$

This would give complete symmetry with the complex 2-D (4-D in real sense) spaces. This would allow the algebraic continuation of hermitian structures to Hamilton-Jacobi structures by just replacing  $i$  with  $e$  for some complex coordinates.

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