Could Categories, Tensor Networks, and Yangians Provide the Tools for Handling the Complexity of TGD?

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Contents

Abstract

TGD Universe is extremely simple locally but the presence of various hierarchies make it to look extremely complex globally. Category theory and quantum groups, in particular Yangian or its TGD generalization are most promising tools to handle this complexity. The arguments developed in the sequel suggest the following overall view.

1. Positive and negative energy parts of zero energy states can be regarded as tensor networks identifiable as categories. The new element is that one does not have only particles (objects) replaced with partonic 2-surfaces but also strings connecting them (morphisms). Morphisms and functors provide a completely new element not present in standard model. For instance, S-matrix would be a functor between categories. Various hierarchies of of TGD would in turn translate to hierarchies of categories.
2. TGD view about generalized Feynman diagrams relies on two general ideas. First, the twistor lift of TGD replaces space-time surfaces with their twistor-spaces getting their twistor structure as induced twistor structure from the product of twistor spaces of M^4 and CP_2. Secondly, topological scattering diagrams are analogous to computations and can be reduced to tree diagrams with braiding. This picture fits very nicely with the picture suggested by fusion categories. At fermionic level the basic interaction is 2+2 scattering of fermions occurring at the vertices identifiable as partonic 2-surface and re-distributes the fermion lines between partonic 2-surfaces. This interaction is highly analogous to what happens in braiding interaction but vertices expressed in terms of twistors depend on momenta of fermions.
3. Braiding transformations take place inside the light-like orbits of partonic 2-surfaces defining boundaries of space-time regions with Minkowskian and Euclidian signature of induced metric respectively permuting two braid strands. R-matrix satisfying Yang-Baxter equation characterizes this operation algebraically.
4. Reconnections of fermionic strings connecting partonic 2-surfaces are possible and suggest interpretation in terms of 2-braiding generalizing ordinary braiding: string world sheets get knotted in 4-D space-time forming 2-knots and strings form 1-knots in 3-D space. Reconnection induces an exchange of braid strands defined by the boundaries of the string world sheet and therefore exchange of fermion lines defining boundaries of string world sheets. A generalization of quantum algebras to include also algebraic representation for reconnection is needed. Also reconnection might reduce to a braiding type operation.

Yangians look especially natural quantum algebras from TGD point of view. They are bi-algebras with co-product Δ. This makes the algebra multi-local raising hopes about the understanding of bound states. Δ-iterates of single particle system would give many-particle systems with non-trivial interactions reducing to kinematics.

One should assign Yangian to various Kac-Moody algebras (SKMAs) involved and even with super-conformal algebra (SSA), which however reduces effectively to SKMA for finite-dimensional Lie group if the proposed gauge conditions meaning vanishing of Noether charges for some sub-algebra H of SSA isomorphic to it and for its commutator $[SSA, H]$ with the entire SSA. Strong form of holography (SH) implying almost 2-dimensionality motivates these gauge conditions. Each SKMA would define a direct summand with its own parameter defining coupling constant for the interaction in question.

1 Introduction

The dynamics of TGD is extremely simple locally: space-times are surfaces of 8-D embedding space so that only four field-like dynamical variables are present and preferred extremals satisfy strong form of holography (SH) meaning that almost 2-D data determine them. TGD Universe looks however also extremely complex. There is a hierarchy of space-times sheets, hierarchy of p-adic length scales, hierarchy of dark matters labelled by the values of Planck constant $h_{eff}/h = n$, hierarchy of extensions of rationals defining hierarchy of adeles in adelic physics view about TGD, hierarchy of infinite primes (and rationals), and also the hierarchy of conscious entities (quantum measurement theory in zero energy ontology can be seen as theory of consciousness [L9]).

During years it has become gradually clear that category theory could be the mathematical language of quantum TGD [K5, K4, K3]. Only category theory gives hopes about unifying various hierarchies making TGD Universe to look so horribly complex. Hierarchy formed by categories,

categories of categories, ... could be the mathematics needed to keep book about this complexity and provide also otherwise unexpected constraints.

The arguments developed in the sequel suggest the following overall view.

1. Positive and negative energy parts of zero energy states can be regarded as tensor networks [L3] identifiable as categories. The new element is that one does not have only particles (objects) replaced with partonic 2-surfaces but also strings connecting them (morphisms). Morphisms and functors provide a completely new element not present in the standard model. For instance, S-matrix would be a functor between categories. Various hierarchies of of TGD would in turn translate to hierarchies of categories.

2. The recent view about generalized Feynman diagrams [K8, K2, L8] is inspired by two general ideas. First, the twistor lift of TGD replaces space-time surfaces with their twistor-spaces getting their twistor structure as induced twistor structure from the product of twistor spaces of M^4 and CP_2. Secondly, topological scattering diagrams are analogous to computations and can be reduced to minimal diagrams, which are tree diagrams with braiding. This picture fits very nicely with the picture provided by fusion categories. At fermionic level the basic interaction is 2+2 scattering of fermions occurring at the vertices identifiable as partonic 2-surface and re-distributes the fermion lines between partonic 2-surfaces. This interaction is highly analogous to what happens in braiding interaction defining basic gate in topological quantum computation [K1] but vertices expressed in terms of twistors depend on momenta of fermions.

3. Braiding transformations for fermionic lines identified as boundaries of string world sheets can take place inside the light-like orbits of partonic 2-surfaces defining boundaries of space-time regions with Minkowskian and Euclidian signature of induced metric respectively. Braiding transformation is essentially a permutation for two braid strands mapping tensor product $A \otimes B$ to $B \otimes A$. R-matrix satisfying Yang-Baxter equation [B6] characterizes this operation algebraically.

4. Reconnections of fermionic strings connecting partonic 2-surfaces are possible and suggest interpretation in terms of 2-braiding generalizing ordinary braiding. I have2-braiding in [K11]: string world sheets get knotted in 4-D space-time forming 2-knots and strings form 1-knots in 3-D space. I do not actually know whether my intuitive believe that 2-braiding reduces to reconnections is correct. Reconnection induces an exchange of braid strands defined by boundaries of the string world sheet and therefore exchange of fermion lines defining boundaries string world sheets. This requires a generalization of quantum algebras to include also algebraic representation for reconnection: this representation could reduce to a representation in terms of an analog of R-matrix.

Yangians [B2] seem to be especially natural quantum algebras from TGD point of view [K21, L8]. Quantum algebras are bi-algebras having co-product Δ, which in well-defined sense is the inverse of the product. This makes the algebra multi-local: this feature is very attractive as far as understanding of bound states is considered. Δ-iterates of single particle system would give many-particle systems with non-trivial interactions reducing to kinematics.

One should assign Yangian to various Super-Kac-Moody algebras (SKMAs) involved and even with super-symplectic algebra (SSA) [K7, K24, K17], which however reduces effectively to SKMA for finite-dimensional Lie group if the proposed gauge conditions meaning vanishing of Noether charges for some sub-algebra H of SSA isomorphic to it and for its commutator $[SSA, H]$ with the entire SSA. Strong form of holography (SH) implying almost 2-dimensionality motivates these gauge conditions. Each SKMA would define a direct summand with its own parameter defining coupling constant for the interaction in question. There is also extended SKMA associated with the light-like orbits of partonic 2-surfaces and it seems natural to identify appropriate sub-algebras of these two algebras as duals in Yangian sense.

There is also partonic super-Kac-Moody algebra (PSKMA) associated with partonic 2-surfaces extending ordinary SKMA. On old conjecture is that SSA and PSKMA are physically dual in the same sense as the conformal algebra and its dual in twistor Grassmannian approach and that this generalizes equivalence principle (EP) to all conserved charges.

The plan of the article is following.

1. The basic notions and ideas about tensor networks as categories and about Yangians as multi-local symmetries and fundamental description of interactions are described.
2. The questions related to the Yangianization in TGD framework are considered. Yangianization of four-momentum and mass squared operator are discussed as examples.
3. The next section is devoted to category theory as tool of TGD: braided categories and fusion categories are briefly described and the notion of category with reconnection is considered.
4. The last section tries to represent the "great vision" in more detail.

2 Basic vision

The existing vision about TGD is summarized first and followed by a proposal about tensor networks as categories and Yangians as a multi-local generalization of symmetries with partonic surfaces replacing point like particles.

2.1 Very concise summary about basic notions and ideas of TGD

Let us briefly summarize the basic notions and ideas of TGD.

1. Space-times are regarded as 4-surfaces in $H = M^4 \times CP_2$, which is fixed uniquely by the condition that the factors of $H = M^4 \times S$ allow twistor space with Kähler structure [A6]. The twistor spaces of dynamically allowed space-time surfaces are assumed to be representable as 6-D surfaces in twistor space $T(H) = T(M^4) \times T(CP_2)$ getting their twistor structure by induction from that of $T(H)$. $T(M^4)$ is identified as its purely geometric variant $T(M^4) = M^4 \times CP_1$. At the level of momentum space the usual identification is more appropriate. It is also assumed that these space-time surfaces are obtained as extremals of 6-D Kähler action [K21, K2, L8]. At space-time level this gives rise to dimensionally reduced Kähler action equal to the sum of volume term and 4-D Kähler action. Either the entire action or volume term would correspond to vacuum energy parameterized by cosmological constant in standard cosmology. Planck length corresponds to the radius of twistor sphere of M^4.
2. Strong form of holography (SH) implied by strong form of general coordinate invariance (SGCI) stating that light-like 3-surfaces defined by parton orbits and 3-D space-like ends of space-time surface at boundaries of CD separately code 3-D holography. SH states that 2-D data at string world sheets plus condition fixing the points of space-time surface with H-coordinates in extension of rationals fix the real space-time surface.

   (a) SH strongly suggests that the preferred extremals of the dimensionally reduced action satisfy gauge conditions (vanishing Noether charges) for a subalgebra H of super-symplectic algebras (SSA) isomorphic to it and its commutator $[H, SSA]$ with SSA: this effectively reduces SSA to a finite-dimensional Kac-Moody algebra.

   (b) Similar dimensional reduction would take place in fermionic degrees of freedom, where super-conformal symmetry fixes 4-D Dirac action, when bosonic action is known [K24, K17]. This involves the new notion of modified gamma matrices determined in terms of canonical momentum currents associated with the action.

   Quantum classical correspondence (QCC) states that classical Cartan charges for SSA are equal to the eigenvalues of corresponding fermionic charges. This gives a correlation between space-time dynamics and quantum numbers of positive (negative) parts of zero energy states.

   (c) SH implies that fermions are effectively localized at string world sheets: in other words, the induced spinor fields Ψ_{int} in space-time interior are determined their values Ψ_{string} at string world sheets. There are two options: Ψ_{int} is either continuation of Ψ_{string} or Ψ_{string} serves as the source of Ψ_{int} [L4].

3. At space-time level the dynamics is extremely simple locally since by general coordinate invariance (GCI) only 4 field-like variables are dynamical, and one has also SH by SGCI. Topologically the situation is rather complex: one has many-sheeted space-time having hierarchical structure. The GRT limit of TGD [K22] is obtained in long length scales by mapping the many-sheeted structure to a slightly curved piece of M^4 by demanding that the deformation of M^4 metric is sum of the deformation of he induced metrics of space-time surface from M^4 metric. Similar description implies to gauge potentials in terms of induced gauge potentials. The many-sheetedness is visible as anomalies of GRT and plays central role in quantum biology [K16].

4. Zero energy ontology (ZEO) means that one consider space-time surfaces inside causal diamonds (CDs defined as intersections of future and past directed light-cones with points replaced with CP_2) forming a scale hierarchy. Zero energy states are tensor products of positive and negative energy parts at opposite boundaries of CD. Zero energy property means that the total conserved quantum numbers are opposite at the opposite boundaries of CD so that one has consistency with ordinary positive energy ontology. Zero energy states are analogous to physical events in the usual ontology but is much more flexible since given zero energy energy states is in principle creatable from vacuum.

5. The "world of classical worlds" (WCW) [K10, K7, K17] generalizes the superspace of Wheeler. WCW decomposes to sub-WCWs assignable to CDs forming a scale hierarchy. Note that 3-surface in ZEO corresponds to a pair of disjoint collections 3-surfaces at opposite boundaries of CD- initial and final state in standard ontology. Super-symplectic symmetries (SCA) act as isometries of WCW. Zero energy states correspond to WCW spinor fields and the gamma matrices of WCW are expressible as linear combinations of fermionic oscillator operators for induced spinor fields. Besides SCA there is partonic super-Kac-Moody algebra (PSCA) acting on light-like orbits of partonic 2-surfaces and these algebras are suggested to be dual physically (generalized EP).

6. One ends up with an extension of real physics to adelic physics [L6]. p-Adic physics for various primes are introduced as physical correlates of cognition and imagination: the original motivation come from p-adic mass calculations [K12]. p-Adic non-determinism (pseudo constants) [K14, K20] strongly suggests that one can always assign to 2-D holographic data a p-adic variant of space-time surface as a preferred extremal. In real case this need not be the case so that the space-time surface realized as preferred extremal is imaginable but not necessarily realizable.

   p-Adic physics and real physics are fused to adelic physics: space-time surface isa book-like structure with pages labelled by real number field and p-adic number fields in an extension induced by some extension of rationals. Planck constants $h_{eff} = n \times h$ corresponds to the dimension of the extension dividing the order of its Galois group and favored p-adic primes correspond to ramified primes for favored extensions. Evolution corresponds to increasing complexity of extension of rationals and favored extensions are the survivors in fight for number theoretic survival.

7. Twistor lift of TGD leads to a proposal for the construction of scattering amplitudes assuming Yangian symmetry assignable to Kac-Moody algebras for embedding space isometries, with electroweak gauge group, and for finite-D Lie dynamically generated Lie group selected by conditions on SSA algebra. 2+2 fermion vertex analogous to braiding interaction serves as the basic vertex in the formulation of [L8].

2.2 Tensor networks as categories

The challenge has been the identification of relevant categories and physical realization of them. One can imagine endless number of identifications but the identification of absolutely convincing candidate has been difficult. Quite recently an astonishingly simple proposal emerged.

1. The notion of tensor network [B5] has emerged in condensed matter physics to describe strongly entangled systems and complexity associated with them. Holography is in an essential role in this framework. In TGD framework tensor network is realized physically at

the level of the topology and geometry of many-sheeted space-time [L3]. Nodes would correspond to objects and links between them to morphisms. This structure would be realized as partonic 2-surfaces - objects - connected by fermionic strings - morphisms - assignable to magnetic flux tubes. Morphisms would be realized as Hilbert space isometries defined by entanglement. Physical state would be category or set of them!

Functors are morphisms of categories mapping objects to objects and morphisms to morphisms and respecting the composition of morphisms so that the structure of the category is preserved. For instance, in zero energy ontology (ZEO) S-matrix for given space-time surface could be a unitary functor assigning to an initial category final category: they would be represented as quantum states at the opposite boundaries of causal diamond (CD). Also quantum states could be categories of categories of in accordance with various hierarchies.

2. Skeptic could argue as follows. The passive part of zero energy states for which active part evolves by unitary time evolutions following by state function reductions inducing time localization in moduli space of CDs, could be category. But isn't the active path more naturally a quantum superposition of categories? Should one replace time evolution as a functor with its quantum counterpart, which generates a quantum superposition of categories? If so, then state function reduction to opposite boundary of CD would mean localization in the set of categories! This is quite an abstraction from simple localization in 3-space in wave mechanics.

3. Categories form categories with functors between categories acting as morphisms. In principle one obtains an infinite hierarchy of categories identifiable as quantum states. This would fit nicely with various hierarchies associated with TGD, most of which are induced by the hierarchy of extensions of rationals.

4. The language of categories fits like glove also to TGD inspired theory of consciousness. The fermionic strings and associated magnetic flux tubes would serve as correlates of attention. The associated morphism would define the direction of attention and also define sensory maps as morphisms. Conscious intelligence relies crucially on analogies and functors realize mathematically the notion of analogy. Categorification means basically classification and this is what cognition does all the time.

2.3 Yangian as a generalization of symmetries to multilocal symmetries

Mere networks of arrows are not enough. One needs also symmetry algebra associated with them giving flesh around the bones.

1. Various quantum algebras, in particular Yangians are naturally related to physically interesting categories. The article of Jimbo [B6], one of the pioneers of quantum algebras, gives a nice summary of Yang-Baxter equation central in the construction of quantum algebras. R-matrix performs is an endomorphism permuting two tensor factors in quantal matter.

2. One of the nice features of Yangian is that it gives hopes for a proper description of bound states problematic in quantum field theories (one can argue that QCD cannot really describe hadrons and already QED has problems with Bethe-Salpeter equation for hydrogen atom). The idea would be simple. Yangian would provide many-particle generalization of single particle symmetry algebra and give formulas for conserved charges of many-particle states containing also interaction terms. Interactions would reduce to kinematics. This - as I think - is a new idea.

The iteration of the co-product Δ would map single particle symmetry operator by homomorphism to operator acting in N-parton state space and one would obtain a hierarchy of algebra generators labelled by N and Yangian inariance would dictate the interaction terms completely (as it indeed does in $\mathcal{N}=4$ SUSY in twistor Grassmannian approach [B3]).

3. There is however a delicacy involved. There is a mysterious looking doubling of the symmetry generators. One has besides ordinary local generators T_0^A generators T_1^A: in twistor Grassmann approach the latter correspond to dual conformal symmetries. For T_0^A the co-product is trivial: $\Delta(J_0^A) = J_0^A \otimes 1 + 1 \otimes J_0^A$, just like in non-interacting theory. This is true for all iterates of Δ.

For J_1^A one has $\Delta(J_1^A) = J_1^A \otimes 1 + 1 \otimes J_1^A + f^A_{BC} J_0^B \otimes J_0^C$. One has two representations and the duality suggests that the eigenvalues J_0^A and J_1^A are same (note that in Witten's approach [B2] $J_1^A = 0$ holds true so that it does not apply as such to TGD). The differences $T_0^A - T_1^A$ would give a precise meaning for "interaction charges" if the duality holds true, and more generally, to the perturbation theory formed by a pair of free and interacting theory. This picture raises hopes about first principle description of bound states: interactions described in wave mechanics in terms of phenomenological interaction Hamiltonians and interaction potentials would be reduced to kinematics.

For instance, for four-momentum $\Delta(P_1^k)$ would contain besides free particle term $P_0^k \otimes 1 + 1 \otimes P_0^k$ also the interaction term involving generators of - say - conformal group.

4. What about the physical interpretation of the doubling? The most natural interpretation would be in terms of SSA and the extended super-conformal algebra assignable to the light-like orbits of partonic 2-surfaces. An attractive interpretation is in terms of a generalization of Equivalence Principle (EP) stating that inertial and gravitational charges are identical for the physical states.

5. The tensor summands of Kac-Moody algebra would have different coupling constants k_i perhaps assignable to the 4 fundamental interactions and to the dynamical gauge group emerging from the SCA would give further coupling constant. This would give 5 tensor factors strongly suggested by p-adic mass calculations - p-adic masses depend only on the number of tensor factors [K12].

3 Some mathematical background about Yangians

In the following necessary mathematical background about Yangians are summarized.

3.1 Yang-Baxter equation (YBE)

Yang-Baxter equation (YBE) has been used for more than four decades in integrable models of statistical mechanics of condensed matter physics and of 2-D quantum field theories (QFTs) [A7]. It appears also in topological quantum field theories (TQFTs) used to classify braids and knots [B2] (see `http://tinyurl.com/mcvvcqp`) and in conformal field theories and models for anyons. Yangian symmetry appears also in twistor Grassmann approach to scattering amplitudes [B3, B4] and thus involves YBE. At the same time new invariants for links were discovered and new braid-type relation was found. YBEs emerged also in 2-D conformal field theories.

Yang-Baxter equation (YBE) has a long history described in the excellent introduction to YBE by Jimbo [B6] (see `http://tinyurl.com/l4z6zyr`, where one can also find a list of references). YBE was first discovered by McGuire (1964) and 3 years later by Yang in quantum mechanical many-body problem involving delta function potential $\sum_{i<j} \delta(x_i - x_j)$. Using Bethe's Ansatz for building wave functions they found that the scattering matrix factorized that it could be constructed using as building brick 2-particle scattering matrix - R-matrix. YBE emerged for R-matrix as a consistency condition for factorization. Baxter discovered 1972 solution of the eight vertex model in terms of YBE. Zamolodchikov pointed ot that the algebraic mechanism behind factorization of 2-D QFTs is same as in condensed matter models.

1978-1979 Faddeev, Sklyanin, and Takhtajan proposed quantum inverse scattering method as a unification of classical and quantum integrable models. Eventually the work with YBE led to the discovery of the notion of quantum group by Drinfeld. Quantum group can be regarded as a deformation $U_q(g)$ of the universal enveloping algebra $U(g)$ of Lie algebra. Drinfeld also introduced the universal R-matrix, which does not depend on the representation of algebra used.

R-matrix satisfying YBE is now the common aspect of all quantum algebras. I am not a specialist in YBE and can only list the basic points of Jimbo's article. Interested reader can look for details and references in the article of Jimbo.

In 2-D quantum field theories R-matrix $R(u)$ depends on one parameter u identifiable as hyperbolic angle characterizing the velocity of the particle. $R(u)$ characterizes the interaction experienced by two particles having delta function potential passing each other (see the figure of `http://tinyurl.com/kyw6xu6`). In 2-D quantum field theories and in models for basic gate in topological quantum computation (for early TGD vision see [K1] were also R-matrix is discussed in more detail) the R-matrix is unitary. One can interpret R-matrix as endomorphism mapping $V_1 \otimes V_2$ to $V_2 \otimes V_1$ representing permutation of the particles.

3.1.1 YBE

R-matrix satisfies Yang-Baxter equation (YBE)

$$R_{23}(u)R_{13}(u+v)R_{12}(v) = R_{12}(v)R_{13}(u+v)R_{23}(u) \tag{3.1}$$

having interpretation as associativity condition for quantum algebras.

At the limit $u, v \to \infty$ one obtains R-matrix characterizing braiding operation of braid strands. Replacement of permutation of the strands with braid operations replaces permutation group for n strands with its covering group. YBE states that the braided variants of identical permutations (23)(13)(12) and (12)(13)(23) are identical.

The equations represent n^6 equations for n^4 unknowns and are highly over-determined so that solving YBE is a difficult challenge. Equations have symmetries, which are obvious on basis of the topological interpretation. Scaling and automorphism induced by linear transformations of V act as symmetries, and the exchange of tensor factors in $V \otimes V$ and transposition are symmetries as also shift of all indices by a constant amount (using modulo N arithmetics).

One can pose to the R-matrix some boundary condition. For $V \otimes V$ the condition states that $R(0)$ is proportional to permutation matrix P for the factors.

3.1.2 General results about YBE

The following lists general results about YBE.

1. Belavin and Drinfeld proved that the solutions of YBE can be continued meromorphic functions to complex plane and define with poles forming an Abelian group. R-matrices can be classified to rational, trigonometric, and elliptic R-matrices existing only for $sl(n)$. Rational and trigonometric solutions have pole at origin and elliptic solutions have a lattice of poles. In [B6] (see `http://tinyurl.com/l4z6zyr`) simplest examples about R-matrices for $V_1 = V_2 = C^2$ are discussed, one of each type.

2. In [B6] it is described how the notions of R-matrix can be generalized to apply to a collection of vector spaces, which need not be identical. The interpretation is as commutation relations of abstract algebra with co-product Δ - say quantum algebra or Yangian algebra. YBE guarantees the associativity of the algebra.

3. One can define quasi-classical R-matrices as R-matrices depending on Planck constant like parameter $\hbar$ (which need have anything to do with Planck constant) such that small values of u one has $R = constant \times (I + \hbar r(u) + O(\hbar^2))$. $r(u)$ is called classical r-matrix and satisfies CYBE conditions

$$[r_{12}(u), r_{13}(u+v)] + [r_{12}(u), r_{23}(v)] + [r_{13}(u+v), r_{23}(v)] = 0$$

obtained by linearizing YBE. $r(u)$ defines a deformation of Lie-algebra respecting Jacobi-identities. There are also non-quasi-classical solutions. The universal solution for r-matrix is formulated in terms of Lie-algebra so that the representation spaces V_i can be any representation spaces of the Lie-algebra.

4. Drinfeld constructed quantum algebras $U_q(g)$ as quantized universal enveloping algebras $U_q(g)$ of Lie algebra g. One starts from a classical r-matrix r and Lie algebra g. The idea is to perform a "quantization" of the Lie-algebra as a deformation of the universal enveloping algebra $U_q(g)$ of $U(g)$ by r. Drinfeld introduces a universal R-matrix independent of the representation used. This construction will not be discussed here since it does not seem to be so interesting as Yangian: in this case co-product Δ does not seem to have a

natural interpretation as a description of interaction. The quantum groups are characterized by parameter $q \in C$.

For a generic value the representation theory of q-groups does not differ from the ordinary one. For roots of unity situation changes due to degeneracy caused by the fact $q^N = 1$ for some N.

5. The article of Jimbo discusses also fusion procedure initiated by Kulish, Restetikhin, and Sklyanin allowing to construct new R-matrices from existing one. Fusion generalizes the method used to construct group representation as powers of fundamental representation. Fusion procedure constructs R-matrix in $W \otimes V^2$, where one has $W = W_1 \otimes W_2 \subset V \otimes V^1$. Picking W is analogous to picking a subspace of tensor product representation $V \otimes V^1$.

3.2 Yangian

Yangian algebra $Y(g(u))$ is associative Hopf algebra (see `http://tinyurl.com/qfl8dwu`) that is bi-algebra consisting of associative algebra characterized by product μ: $A \otimes A \to A$ with unit element 1 satisfying $\mu(1, a) = a$ and co-associative co-algebra consisting of co-product $\Delta A \in A \otimes A$ and co-unit $\epsilon : A \to C$ satisfying $\epsilon \circ \Delta(a) = a$. Product and co-product are "time reversals" of each other. Besides this one has antipode S as algebra anti-homomorphism $S(ab) = S(b)S(a)$. YBE has interpretation as an associativity condition for co-algebra $(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$. Also ϵ satisfies associativity condition $(\epsilon \otimes 1) \circ \Delta = (1 \otimes \epsilon) \circ \Delta$.

There are many alternative formulations for Yangian and twisted Yangian listed in the slides of Vidas Regelskis at `http://tinyurl.com/ms9q8u4`. Drinfeld has given two formulations and there is FRT formulation of Faddeev, Restetikhin and Takhtajan.

Drinfeld's formulation [B6] (see `http://tinyurl.com/qfl8dwu`) involves the notions of Lie bi-algebra and Manin triple, which corresponds to the triplet formed by half-loop algebras with positive and negative conformal weights, and full loop algebra. There is isomorphism mapping the generating elements of positive weight and negative weight loop algebra to the elements of loop algebra with conformal weights 0 and 1. The integer label n for positive half loop algebra corresponds in the formulation based on Manin triple to conformal weight. The alternative interpretation for $n + 1$ would be as the number of factors in the tensor power of algebra and would in TGD framework correspond to the number of partonic 2-surfaces. In this interpretation the isomorphism becomes confusing.

In any case, one has two interpretations for $n + 1 \geq 1$: either as parton number or as occupation number for harmonic oscillator having interpretation as bosonic occupation number in quantum field theories. The relationship between Fock space description and classical description for n-particle states has remained somewhat mysterious and one can wonder whether these two interpretation improve the understanding of classical correspondence (QCC).

3.2.1 Witten's formulation of Yangian

The following summarizes my understanding about Witten's formulation of Yangian in $\mathcal{N} = 4$ SUSYs [B2], which does not mention explicitly the connection with half loop algebras and loop algebra and considers only the generators of Yangian and the relations between them. This formulation gives the explicit form of Δ and looks natural, when n corresponds to parton number. Also Witten's formulation for Super Yangian will be discussed.

It must be however emphasized that Witten's approach is not general enough for the purposes of TGD. Witten uses the identification $\Delta(J_1^A) = f^A_{BC} J_0^B \times J_0^C$ instead of the general expression $\Delta(J_1^A) = J_1^A \otimes 1 + 1 \times J_1^A + f^A_{BC} J_0^B \times J_0^C$ needed in TGD strongly suggested by the dual roles of the super-symplectic conformal algebra and super-conformal algebra associated with the light-like partonic orbits realizing generalized EP. There is also a nice analogy with the conformal symmetry and its dual twistor Grassmann approach.

The elements of Yangian algebra are labelled by non-negative integers so that there is a close analogy with the algebra spanned by the generators of Virasoro algebra with non-negative conformal weight. The Yangian symmetry algebra is defined by the following relations for the generators labeled by integers $n = 0$ and $n = 1$. The first half of these relations discussed in very clear manner in [B2] follows uniquely from the fact that adjoint representation of the Lie algebra is in question

$$\left[J^A, J^B\right] = f_C^{AB} J^C \quad , \quad \left[J^A, J^{(1)B}\right] = f_C^{AB} J^{(1)C} \quad . \tag{3.2}$$

Besides this Serre relations are satisfied. These have more complex form and read as

$$\begin{aligned}
&\left[J^{(1)A},\left[J^{(1)B},J^C\right]\right]+\left[J^{(1)B},\left[J^{(1)C},J^A\right]\right]+\left[J^{(1)C},\left[J^{(1)A},J^B\right]\right]\\
&\qquad = \frac{1}{24} f^{ADK} f^{BEL} f^{CFM} f_{KLM}\{J_D, J_E, J_F\} \ ,\\
&\left[\left[J^{(1)A},J^{(1)B}\right],\left[J^C,J^{(1)D}\right]\right]+\left[\left[J^{(1)C},J^{(1)D}\right],\left[J^A,J^{(1)B}\right]\right]\\
&\qquad = \frac{1}{24}(f^{AGL} f^{BEM} f_K^{CD}\\
&\qquad + f^{CGL} f^{DEM} f_K^{AB}) f^{KFN} f_{LMN}\{J_G, J_E, J_F\} \ .
\end{aligned} \tag{3.3}$$

The indices of the Lie algebra generators are raised by invariant, non-degenerate metric tensor g_{AB} or g^{AB}. $\{A, B, C\}$ denotes the symmetrized product of three generators.

The right hand sides have often as a coefficient $\hbar^2$ instead of $1/24$. $\hbar$ need not have anything to do with Planck constant. The Serre relations give constraints on the commutation relations of $J^{(1)A}$. For $J^{(1)A=J^A}$ the first Serre relation reduces to Jacobi identity and second to antisymmetry of Lie bracket. The right hand sided involved completely symmetrized trilinears $\{J_D, J_E, J_F\}$ making sense in the universal covering of the Lie algebra defined by J^A.

Repeated commutators allow to generate the entire algebra whose elements are labeled by non-negative integer n. The generators obtain in this manner are n-local operators arising in $(n-1)$-commutator of $J^{(1)}$: s. For $SU(2)$ the Serre relations are trivial. For other cases the first Serre relation implies the second one so the relations are redundant. Why Witten includes it is for the purposed of demonstrating the conditions for the existence of Yangians associated with discrete one-dimensional lattices (Yangians exists also for continuum one-dimensional index).

Discrete one-dimensional lattice provides under certain consistency conditions a representation for the Yangian algebra. One assumes that each lattice point allows a representation R of J^A so that one has $J^A = \sum_i J_i^A$ acting on the infinite tensor power of the representation considered. The expressions for the generators J^{1A} in Witten's approach are given as

$$J^{(1)A} \quad = \quad f_{BC}^A \sum_{i<j} J_i^B J_j^C \quad . \tag{3.4}$$

This formula gives the generators in the case of conformal algebra. This representation exists if the adjoint representation of G appears only one in the decomposition of $R \otimes R$. This is the case for SU(N) if R is the fundamental representation or is the representation of by k^{th} rank completely antisymmetric tensors.

This discussion does not apply as such to $\mathcal{N} = 4$ case the number of lattice points is finite and corresponds to the number of external particles so that cyclic boundary conditions are needed guarantee that the number of lattice points reduces effectively to a finite number. Note that the Yangian in color degrees of freedom does not exist for SU(N) SYM.

As noticed, Yangian algebra is a Hopf algebra and therefore allows co-product. The co-product Δ is given by

$$\begin{aligned}
\Delta(J^A) &= J^A \otimes 1 + 1 \otimes J^A \ ,\\
\Delta(J^{(1)A}) &= J^{(1)A} \otimes 1 + 1 \otimes J^{(1)A} + f_{BC}^A J^B \otimes J^C
\end{aligned} \tag{3.5}$$

Δ allows to imbed Lie algebra to the tensor product in non-trivial manner and the non-triviality comes from the addition of the dual generator to the trivial co-product. In the case that the single spin representation of $J^{(1)A}$ is trivial, the co-product gives just the expression of the dual generator using the ordinary generators as a non-local generator. This is assumed in the recent case and also for the generators of the conformal Yangian.

3.2.2 Super-Yangian

Also the Yangian extensions of Lie super-algebras make sense. From the point of physics especially interesting Lie super-algebras are $SU(m|m)$ and $U(m|m)$. The reason is that $PSU(2,2|4)$ (P refers to "projective") acting as super-conformal symmetries of $\mathcal{N}=4$ SYM and this super group is a real form of $PSU(4|4)$. The main point of interest is whether this algebra allows Yangian representation and Witten demonstrated that this is indeed the case [B2].

These algebras are Z_2 graded and decompose to bosonic and fermionic parts which in general correspond to n- and m-dimensional representations of $U(n)$. The representation associated with the fermionic part dictates the commutation relations between bosonic and fermionic generators. The anti-commutator of fermionic generators can contain besides identity also bosonic generators if the symmetrized tensor product in question contains adjoint representation. This is the case if fermions are in the fundamental representation and its conjugate. For $SU(3)$ the symmetrize tensor product of adjoint representations contains adjoint (the completely symmetric structure constants d_{abc}) and this might have some relevance for the super $SU(3)$ symmetry.

The elements of these algebras in the matrix representation (no Grassmann parameters involved) can be written in the form

$$x=\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \quad .$$

a and d representing the bosonic part of the algebra are $n\times n$ matrices and $m\times m$ matrices corresponding to the dimensions of bosonic and fermionic representations. b and c are fermionic matrices are $n\times m$ and $m\times n$ matrices, whose anti-commutator is the direct sum of $n\times n$ and $n\times n$ matrices. For $n=m$ bosonic generators transform like Lie algebra generators of $SU(n)\times SU(n)$ whereas fermionic generators transform like $n\otimes\overline{n}\oplus\overline{n}\otimes n$ under $SU(n)\times SU(n)$. Supertrace is defined as $Str(x)=Tr(a)-Tr(b)$. The vanishing of Str defines $SU(n|m)$. For $n\neq m$ the super trace condition removes identity matrix and $PU(n|m)$ and $SU(n|m)$ are same. That this does not happen for $n=m$ is an important delicacy since this case corresponds to $\mathcal{N}=4$ SYM. If any two matrices differing by an additive scalar are identified (projective scaling as now physical effect) one obtains $PSU(n|n)$ and this is what one is interested in.

Witten shows that the condition that adjoint is contained only once in the tensor product $R\otimes\overline{R}$ holds true for the physically interesting representations of $PSU(2,2|4)$ so that the generalization of the bilinear formula can be used to define the generators of $J^{(1)A}$ of super Yangian of $PU(2,2|4)$. The defining formula for the generators of the Super Yangian reads as

$$\begin{aligned} J_C^{(1)} &= g_{CC'}J^{(1)C'}=g_{CC'}f^{C'}_{AB}\sum_{i<j}J_i^AJ_j^B \\ &= g_{CC'}f^{C'}_{AB}g^{AA'}g^{BB'}\sum_{i<j}J^i_{A'}J^j_{B'} \quad . \end{aligned} \tag{3.6}$$

Here $g_{AB}=Str(J_AJ_B)$ is the metric defined by super trace and distinguishes between $PSU(4|4)$ and $PSU(2,2|4)$. In this formula both generators and super generators appear.

4 Yangianization in TGD framework

Yangianization of quantum TGD is quite challenging. Super-conformal algebras are much larger than in say $\mathcal{N}=4$ SUSY and even in superstring models and reconnection and 2-braiding are new topological elements.

4.1 Geometrization of super algebras in TGD framework

Super-conformal algebras allow a geometrization in TGD framework and this should be of considerable help in the Yangianization.

1. The basic generators of various Super-algebras follow from modified Dirac action as Noether charges and their super counterparts obtained by replacing fermion field Ψ (its conjugate $\overline{\Psi}$) by a mode u_m ($\overline{u}_n$) of the induced spinor field [K24, K17]. The anti-commutators of these Noetherian super charges labelled by n define WCW gamma matrices. The replacement of both Ψ and $\overline{\Psi}$ with modes u_m and $\overline{u}_n$ gives a collection of conserved c-number currents and charges labelled by (n, m). These c-number charges define the anti-commutation relations for the induced spinor fields so that quantization reduces to dynamics thanks to the notion of modified gamma matrices forced by super-conformal symmetry.

2. The natural generalization of Sugawara formula to the level of Yangian of SKMA starts from the Dirac operator for WCW defined like ordinary Dirac operator in terms of the contractions of WCW gamma matrices with the isometry generators (SCA) replacing the Super Virasoro generators G_r and WCW d'Alembert operator defined as its square replacing Virasoro generators L_n. Anti-commutators of WCW gamma matrices defined by super charges for super-symplectic generators define WCW Kähler metric [K24] for which action for preferred extremal would define Kähler function for WCW metric [K10].

3. Quarks and leptons give rise to a doubling of WCW metric if associated with same space-time sheet that is with the same sector of WCW. The duplication of the super algebra generators - in particular WCW gamma matrices - does not seem to make sense. Do quarks and leptons therefore correspond to different sectors of WCW and live at different space-time surfaces? But what could distinguish between 3-surfaces associated with quarks and leptons?

   Could quarks be associated with homologically non-trivial partonic 2-surfaces with CP_2 homology charges 2,-1,-1 proportional to color hypercharges $2/3, -1/3, -1/3$ and leptons with partonic 2-surfaces with vanishing homology charges coming as multiples of 3? Vanishing of color hypercharge for color-confined states would topologize to a vanishing of total homology charge. Could spin/isospin half property of fundamental fermions topologize to 2-sheeted structure of the space-time surface representing elementary particle consisting of elementary fermions?

SSA acting as isometries of WCW is not the only super-conformal algebra involved.

1. Partonic 2-surfaces are ends of light-like 3-surfaces- partonic orbits - and give rise to a generalization of SKMA of isometries of H so that they act as local isometries preserving the light-likeness property of the orbits. At the ends of the partonic 2-surface SKMA is associated with complex coordinate of partonic 2-surface. What is the role of this algebra, which is also extended SKMA (already christened PSCA) but with light-like coordinate parameterizing the SKMA generators?

   Is it an additional symmetry combining with string world sheet symmetries to a symmetry involving complex coordinate and complex or hypercomplex coordinate? Or is it dual to the string world sheet symmetry? How do these symmetries relate to SSA? Does SGCI implying SH leave only SKMAs associated with isometries, holonomies of CP_2 (electroweak interactions) and dynamical SKMA remaining as remnant of SCA.

2. I have earlier proposed that Equivalence Principle (EP) as identity of inertial and gravitational charges could reduce to the duality between these SSA assignable to strings and the partonic super-conformal algebra. This picture conforms with the expected form of the generators associated with these algebras. The dual generating elements T_0^A *resp.* T_1^A associated with generic Yangian could naturally correspond to isomorphic sub-algebras of super-conformal algebra associated with orbits of partonic 2-surfaces *resp.* super-symplectic algebra assignable to string world sheets.

4.2 Questions

There are many open questions to be answered.

Q1: What Yangianization could mean in TGD framework? The answer is not obvious and one can consider two options.

1. Assuming that SH leads to an effective reduction of super-symplectic algebra to finite-D Kac-Moody algebra, assign to partonic 2-surfaces direct sum of Kac-Moody type algebras $L(g) = g(z, z^{-1})$ assigned with complex coordinate z of partonic 2-surface. One could perform Yangianization for this algebra meaning that these symmetries become multi-local with locus identified as partonic 2-surface.

   In Drinfeld's approach this would mean Yangianization of $L(g)$ rather than g and would involve double loop algebra $L(L(g))$ and its positive and negative energy parts. In Minkowskian space-time regions the generators would be functions of complex coordinate z and hypercomplex coordinate u associated with string world sheet: in Euclidian space-time regions one would have 2 complex coordinates z and w. This would conform with holography. I do not know whether mathematicians have considered this generalization and whether it is possible. In the following this is assumed.

2. Physical states at partonic 2-surfaces consist of pointlike fermions and one can ask whether this actually means that one can consider just the Lie algebra g so that in Drinfeld's approach one would have just string world sheets and $Y(g)$. Already this option requires the algebraization of reconnection mechanism as a new element. Whether this simpler approach make sense for fermions and by QQC for quantum TGD, is not clear.

Q2: Can one really follow the practice of Grassmannian twistor approach and say that T_1^A and TA^0 are dual?

One has $[T_0^A, T_1^B] = f_C^{AB} T_1^C$. Witten's definition $T_1^A = f_{BC}^A T^B \otimes T^C \equiv T_1^A = f_{BC}^A T^B T^C$ with T_1^A identified as total charges for lattice, identifies T_1^A as 2-particle generators of Yangian. One the other hand, in TGD T_0^A would correspond to partonic super-conformal algebra and T_1^A to bi-local super-symplectic algebra and the general definition to be used regards also T_1^A as single particle generators in Yangian sense and defines the generators at 2-particle level as $\Delta(T_0^A) = T_0^A \otimes 1 + 1 \otimes T_0^A$ and $\Delta(T_1^A) = T_1^A \otimes 1 + 1 \otimes T_1^A + f_{BC}^A T_0^B \otimes T_0^C$.

For the Witten's definition one cannot demand that T_0^A and T_1^A have same eigenvalues for the physical states. For the more general definition of Δ to be followed in the sequel it seems to be possible require that T_0^A and T_1^A obey the same commutation relations for appropriate sub-algebras at least, and that it is possible to diagonalize Cartan algebras simultaneously and even require same total Cartan charges. This issue is not however well-understood.

Q3: What algebras are Yangianized in TGD framework?

The Yangians of SKMAs associated with isometries of $M^4 \times CP_2$ and with the holonomy group $SU(2) \times U(1)$ of CP_2 appear as symmetries. M^4 should give SKMA in transversal degrees of freedom for fermionic string. CP_2 isometries would give SKMA associated with SU(3). $SU(2) \times U(1)$ would be assignable to electroweak symmetries. This gives 4 tensor factors.

Five of them are required by p-adic mass calculations [K12], whose outcome depends only on the number of tensor factors in Virasoro algebra. The estimates for the number of tensor factors has been a chronic head ache: in particular, do M^4 SKMA correspond to single tensor factor or two tensor factors assignable to 2 transversal degrees of freedom.

Supersymplectic algebra (SSA) is assumed to define maximal possible isometry group of WCW guaranteeing the existence of Kähler metric with a well-defined Riemann connection. The Yangian of SSA could be the ultimate symmetry group, which could realize the dream about the reduction of all interactions to mere kinematics. If SSA effectively reduces to a finite-D SKMA for fermionic strings, one would have 5 tensor factors.

Q4: What does SSA mean?

1. SSA is associated with light-cone boundary $\delta M^4_\pm$ with one light-like direction. The generators (to be distinguished from generating elements) are products of Hamiltonians of symplectic transformations of CP_2 assignable to representations of color SU(3) and Hamiltonians for the symplectic transformations of light-cone boundary, which reduce to Hamiltonians for symplectic transformations of sphere S^2 depending parametrically on the light-like radial coordinate r. This algebra is generalized to analog of Kac-Moody algebra defined by finite-dimensional Lie algebra.

2. The radial dependence of Hamiltonians of form r^h. The naïve guess that conformal weights are integers for the bosonic generators of SSA is not correct. One must allow complex conformal weights of form $h = 1/2 + iy$: 1/2 comes from the scaling invariant inner product for functions at $\delta M^4_\pm$ defined by integration measure dr/r [K7, K17].

3. An attractive guess [L2] is that there is an infinite number of generating elements with radial conformal weights given by zeros of zeta. Conformal confinement must holds true meaning that the total conformal weights are real and thus half-odd integers. The operators creating physical states form a sub-algebra assignable by SH and QCC to fermionic string world sheets connecting partonic 2-surfaces.

4. SH inspires the assumption that preferred extremal property requires that sub-algebra H of SSA isomorphic to itself (conformal weights are integer multiples of SSA) and its commutator SH with SH annihilate physical states and classical Noether charges vanish. This could reduce the symmetry algebra to SKMA for a finite-dimensional Lie group. SSA could be replaced also with the sub-algebra creating physical states having half-odd integer valued radial conformal weights.

   Similar conditions could make sense for the generalization of super-conformal KM algebra associated with light-like partonic orbits.

Q5: What is the precise meaning of SH in the fermionic sector?

Are string world sheets with their ends behaving like pointlike particles enough or are also partonic 2-surface needed. For the latter option a generalization of conformal field theory (CFT) would be needed assigning complex coordinate with partonic 2-surfaces and hyper-complex or complex coordinates with string world sheets. Elementary particle vacuum functionals depend on conformal moduli of partonic 2-surface [K6], which supports the latter option.

There could be however duality between partonic 2-surfaces and string world sheets so that either of them could be enough [L8]. There is also uncertainty about the relationship between induced spinor fields at string world sheets and space-time interior. Are 4-D induced spinor fields obtained by process analogous to analytic continuation in 2-complex dimensional space-time or do 2-D induced spinor fields serve as sources for 4-D induced spinor fields?

Quantum algebras are characterized by parameters such as complex parameter q characterizing R-matrices for quantum groups. Adelic physics [L6] demands number theoretical universality and in particular demands that the parameters - say q - of quantum algebraic structures involved are products $q = e^{m/n}xU$, where U is root of unity (note that e^p exists as ordinary p-adic number for Q_p) and x is real number in the extension. This guarantees that the induced extensions of p-adic numbers are finite-dimensional (the hypothesis is that the correlates of cognition are finite-D extensions of p-adic number fields) [K17].

In the recent view about twistorial scattering amplitudes [L8] the fundamental fermionic vertices are $2 \rightarrow 2$ vertices. There is no fermionic contact interaction in the sense of QFT but the fermions coming to the topological vertex defined by partonic 2-surface at which 3 partonic orbits meet (analogy for the 3-vertex for Feynman diagram) are re-distributed between partonic two surfaces. Also in integrable 2-D QFTs in M^2 the vertices are $2 \rightarrow 2$ vertices characterized by R-matrix. The twistorial vertex is however not topological.

4.3 Yangianization of four-momentum

The QFT picture about bound states is unsatisfactory. The basic question to be answered is whether one should approach the problem in terms of Lorentz invariant mass squared natural in conformal field theories or in terms of Poincare algebra. It is quite possible that the fundamental formulation allowing to understand binding energies is in terms of SCA and PSCA.

Twistor lift of TGD [L8] however suggests that Poincare and even finite-D conformal transformations associated with M^2 could play important role. These longitudinal degrees of freedom are non-dynamical in string dynamics. Maybe there is kind of sharing of labor between these degrees of freedom. In the following we consider two purely pedagogical examples about Yangianization of four-momentum in M^4 and in 8-D context regarding four-momentum as quaternionic 8-momentum in M^8.

4.3.1 Yangianization of four-momentum in conformal algebra of M^4

Consider as an example what the Yangianization for four-momentum P^k could mean. This is a pedagogical example.

1. The first thing to notice is that the commutation relations between P_0^k and P_1^k are inherited from those between P_0^k and force P_1^k and P_0^k to commute. This holds true quite generally for Cartan algebra so that if the correspondence between T_0^A and T_1^A respects Cartan algebra property then Cartan algebras of T_0^A and T_1^A can be simultaneously diagonalized for the physical states. The Serre relations of Eq. 3.3 are identically satisfied for Cartan algebra and its image. This is consistent with the assumption that Cartan algebra is mapped to Cartan algebra but does not prove it.

2. The formula $f^A_{BC}T_0^A \otimes T_0^C$ for the interaction term appearing in the expresion of Δ should be non-trivial also when T^A corresponds to four-momentum. Already the Poincare algebra gives this kind of term built from Lorentz generators and translation generators.

   The extension of Poincare algebra extended to contain dilatation operator D can be considered as also M^4 conformal algebra with generators of special conformal transformations M^A included (see `http://tinyurl.com/nxlmfug`). One has doubling of all algebra generators. The interpretation as gravitational and inertial momenta is one possibility, and EP suggests that the two momenta have same values. In twistor Grassmannian approach the conformal algebras are regarded as dual and suggests the same. Hence one would have $P_0^k = P_1^k$ at the level of eigenvalues.

3. For conformal group the proposed co-product for P_i^k would read as

$$\begin{aligned}
\Delta(P_0^k) &= P_0^k \otimes 1 + 1 \otimes P_0^k \ , \\
\Delta(P_1^k) &= P_1^k \otimes 1 + 1 \otimes P_1^k + K f^k_{Al}(L_0^A \otimes P_0^l - P_0^k \otimes L_0^A) + K f^k_{Al}(M_0^A \otimes P_0^l - P_0^l \otimes M_0^A) \\
&+ K(D_0 \times P_0^k - P_0^k \times D_0) \ .
\end{aligned} \tag{4.1}$$

   This condition could be combined with the condition for mass squared operator. For $K = 0$ one would have additivity of mass squared requiring that P_1 and P_2 are parallel and light-like. For $K \neq 0$ it might be possible to have a simultaneous solution to the both conditions with massive total momentum.

The Δ-iterates of P_0^k contain no interaction terms. For P_1 one has interaction term. This holds true for all symmetry generators. Assume $P_0 = P_1$: does this mean that the interacting theory associated with P_1 is dual to free theory? The difference $\Delta P_0^k - \Delta(P_1^k)$ defines the analog interaction Hamilton, which would therefore be not due to a somewhat arbitrary decomposition of four-momentum to free and interaction parts. It should be possible to possible to measure this difference and its counterpart for other quantum numbers. One can only make questions about the interpretation for this duality applying to all quantum numbers.

1. In Drinfeld's construction the negative and positive energy parts of loop algebra would be related by the duality. In ZEO it might be possible to relate them to positive and negative energy parts of zero energy states at the opposite boundaries of CD.

2. If n is interpreted as number of partonic surfaces and the generators are interpreted as in Witten's construction then the duality could be seen as a geometric duality in plane mapping edges and vertices (partonic 2-surfaces ordered in sequence and string between them) to each other. In super-conformal algebra of twistor Grassmannian approach the generators T_0^A and T_1^A are associated with vertices and edges of the polygon defining the scattering diagram and this suggests that T_0^A corresponds to partonic 2-surfaces and T_1^A to the strings world sheets.

3. Could the duality be a generalization of for Equivalence Principle identifying inertial and gravitational quantum numbers? This interpretation is encouraged by the presence of SSA action on space-like 3-surfaces at the ends of CDs and extended super-conformal algebra associated with the light-like orbits of partons: SGCI would suggest that these algebras or at least their appropriate sub-algebra are dual. This interpretation conforms also with the above geometric interpretation and twistor Grassmannian interpretation.

Consider for simplicity the situation in which only scaling generator D is present in the extension.

1. Suppose that one has eigenstate of total momentum $\Delta(P_0^k)$ resp. $\Delta(P_1^k)$ with eigenvalue p_0^{tot} resp. p_1^{tot} and that

$$p_0^{tot} = p_1^{tot} \tag{4.2}$$

holds true.

2. Since D_0 and P_0^k do not commute, the action of D_0 must be realized as differential operator $D_0 = ip_0^k d/dp_0^k$ so that one has following eigenvalue equations

$$\begin{aligned} \Delta(P_0^k)\Psi &= (p_{0,1}^k + p_{0,2}^k)\Psi = p_0^{tot}\Psi \ , \\ \Delta(P_1^k)\Psi &= (p_{1,1}^k + p_{1,2}^k)\Psi + K(ip_{0,1}^k \otimes p_{0,2}^r \frac{d}{dp_{0,2}^r} - ip_{0,1}^r \frac{d}{dp_{0,1}^r} \otimes p_{0,2}^k)\Psi = p_1^{tot}\Psi \ . \end{aligned} \tag{4.3}$$

Ψ must be a superposition of states $|p_{0,1}, p_{0,2}\rangle$. One has non-trivial interaction. Analogous interaction terms mixing states with different momenta emerge from the terms involving Lorentz generators and special conformal generators.

4.3.2 Four-momenta as quaternionic 8-momenta in octonionic 8-space

In octonionic approach to twistorial scattering amplitudes particles can be regarded as massless in 8-D sense [L8]. The light-like octonionic momenta are actually quaternionic and one would obtain massive states in 4-D sense. Different 4-D masses would correspond to discrete set of quaternionic momenta for 8-D massless particle. Could the above conditions generalize to this case?

1. Suppose that the symmetries reduce to Poincare symmetry and to a number theoretic color symmetry acting as automorphisms of octonions. In this case the four-momentum for a given $M^4 \subset M^8$ decomposes to a sum of to a direct sum of M^2 invariant under SU(3) and E^2 invariant under $SU(2) \times U(1) \subset SU(3) \subset G_2$. ΔP_1 would be non-trivial for the transversal momentum and of form

$$\begin{aligned} \Delta(P_0^{L,k})\Psi &= (p_{0,1}^{L,k} + p_{0,2}^{L,k})\Psi = p_0^{tot}\Psi \ , \\ \Delta(P_0^{T,k})\Psi &= (P_0^{T,k} \otimes 1 + 1 \otimes P_0^{T,k})\Psi \ , \\ \Delta(P_1^{L,k})\Psi &= (p_{1,1}^{L,k} + p_{1,2}^{L,k})\Psi = P_1^{L,tot}\Psi \ , \\ \Delta(P_1^{T,k})\Psi &= (P_1^{T,k} \otimes 1 + 1 \otimes P_1^{T,k} + Kf_{Al}^k(ip_{0,1}^l \otimes t_{0,2}^A - i(ip_{0,2}^l \otimes t_{0,2}^A)\Psi \ . \end{aligned} \tag{4.4}$$

Here P_0^L resp. P_0^T represents longitudinal resp. transversal momentum and T_0^b denotes $SU(2) \subset SU(3)$ generator representable as differential operator acting on complexified momentum and $p_0^T = p_0^{T,x} + ip_0^{T,y}$ and its conjugate.

2. In transversal degrees of freedom the assumption about momentum eigenstates would be probably too strong. String model suggests Gaussian in transversal oscillator degrees of freedom. Hadronic physics suggests an eigenstate of transversal momentum squared. TGD based number theoretic considerations suggest that the transversal state is characterized by color quantum numbers.

   Hence the conditions

$$p_0^{L,tot} = p_1^{L,tot} \quad , \quad (p_0^{T,tot})^2 = (p_1^{T,tot})^2 \tag{4.5}$$

   are natural. It would be nice if the momenta p_{01} and p_{02} could be chosen to be on mass shell and satisfy stringy formula for mass squared where transverse momentum squared would correspond to stringy contribution.

One can also add to $\Delta(P)$ the terms coming from conformal group of M^4 or its subgroup. Since octonionic momentum is light-like M^2 momentum for a suitable choice of M^2, one must consider the possibility that the conformal group is that of $M^2 \subset M^4$. Twistorialization supports this view [L8]. The action of conformal generations would be on longitudinal momentum only.

One can wonder how gauge interactions and gravitational interaction do fit to this picture. Is the extension to super-conformal algebra and supersymplectic algebra the only manner to obtain gauge interactions and gravitation into the picture?

4.4 Yangianization for mass squared operator

It would be nice to have universal mass formulas as a generalization of mass squared formula for string models in terms of the conformal scaling generator $L_0 = zd/dz$. This operator should have besides single particle contributions also many particle contributions in bound states analogous to interaction Hamiltonian and interaction potential. Yangian as an algebra containing multi-local generators is a natural candidate in this respect.

One can consider Yangianization of Super Virasoro algebra (SVA). The Yangianization of various Super Kac-Moody algebras (SKMA) seems however more elegant if it induces the Yangianization of SVA. Consider first direct Yangianization of SVA. The commutation relations for SVA will be used in the sequel. They can be found in Wikipedia (see `http://tinyurl.com/klsgquz`) so that I do not bother to write them here. It must be emphasized that there might be delicate mathematical constraints on algebras which allow Yangianization as the article of Witten [B2] shows. The considerations here rely on physical intuition with unavoidable grain of wishful thinking.

What about the Yangian variant of mass squared operator m^2in terms of the conformal scaling generator $L_0 = zd/dz$? Consider first the definition of various Super algebras in TGD framework.

1. In standard approach the basic condition at single particle level $L_0\Psi = h_{vac}\Psi$ giving the eigenvalues of m^2. Massless in generalize sense requires $h_{vac} = 0$. One would have $m^2_{op} = L_0^{vib} + h_{vac}Id$, where "vib" refers to vibrational degrees of freedom of Kac-Moody algebra (KMA). Sugawara construction [A5] allows to express the left-hand side of this formula in terms of Kac-Moody generators - one has sum over squares $T^a_n T_a^{-n}$. One can say that mass squared is Casimir operator vibrational degrees of freedom for KMA

2. In absence of interactions - and always for $L_{0,0}$ - mass squared formula gives $m_1^2 + m_2^2 = L_0^{vib,1} + L_0^{vib,2}$ for vanishing vacuum weights. It is important to notice that this does *not* imply the additivity of mass squared since one does not have $(p_1 + p_2)^2 = m_1^2 + m_2^2$, which can hold true only for massless and parallel four-momenta. I have considered the possible additivity of mass mass squared for mesons [K15] but it of course fails for systems like hydrogen atom.

One can look what Yangianization of Super Virasoro algebra could mean.

1. One would have doubling of the generators of SKMA and SVA: one possible explanation is in terms of generalized EP. The difference $\Delta(T_0^A) - \Delta(T_1^A)$ would define the analog of interaction Hamiltonian of the duality holds true.

One has $L_0 = G_0^2/2$. Quite generally, one has $\{G_r, G_{-r}\} = 2L_0$ apart from the central extension term. Generalization Yangian to Super Algebra suggests that one has

$$\begin{aligned} \Delta(L_{0,0}) &= L_{0,0} \otimes 1 + 1 \otimes L_{0,0} \ , \\ \Delta(L_{1,0}) &= L_{1,0} \otimes 1 + 1 \otimes L_{1,0} + K \sum_n G_{0,r} \otimes G_{0,-r} \end{aligned} \tag{4.6}$$

Both operators give the value of h_{vac} expected to vanish when acting on physical states and the eigenvalues of the interaction mass squared $K \sum_n G_2 \otimes G_{-r}/2$ would represent the difference $m_{0,1}^2 + m_{0,2}^2 - m_{2,1}^2 - m_{2,2}^2$. By Lorentz invariance the interaction energy is expected to be proportional to the inner product $P_1 \cdot P_2$ and the interpretation in terms of gravitational interaction energy is attractive. The size scale of K would be determined by $l_P^2/R^2 \simeq 2^{-12}$, where l_P is Planck length and R is CP_2 radius gravitational constant [K2, L8].

2. The action of $k \sum_n G_{0,n} \otimes G_{0,-n}/2$ on state $|p_1, p_2\rangle$ is analogous to the action of a tensor product of Dirac operators on tensor product of spinors. Since Dirac operator changes chirality, this suggests that the states are superpositions of eigenstates of chirality of form

$$\Psi = G_{0,0}\Psi_1 \otimes \Psi_2 + \epsilon \times \Psi_1 \otimes G_{0,0}\Psi_2 \ , \ \ \epsilon = \pm 1 \ .$$

$L_{0,0}\Psi_i = 0$ and $\Delta(L_{0,0})\Psi = 0$ holds true. $\Delta(G_{0,0})$ and $\Delta(G_{1,0})$ are given by

$$\begin{aligned} &\Delta(G_{0,0}) = G_{0,0} \otimes 1 - \epsilon \times 1 \otimes G_{0,0} \ , \\ &\Delta(G_1, 0) = G_{1,0} \otimes 1 - \epsilon \times 1 \otimes G_{1,0} - \tfrac{3K}{2} \textstyle\sum_r r(L_{0,r} \otimes G_{0,-r} - (G_{0,-r} \otimes L_{0,r}) \ , \end{aligned} \tag{4.7}$$

and should annihilate Ψ. This is true if $L_{1,r}$ and $L_{0,r}$ annihilate the states.

3. Perhaps the correct approach reduces to the Yangianization of SKMAs (including the dynamically generated SKM two which SSA effectively reduces by gauge conditions) provided that it induces Yangianization of SVA. Momentum components would be associated with KM generators for M^4 excitations of strings such that only transversal excitations are dynamical.

   For fermionic and bosonic generators of SKMA one would have

$$\begin{aligned} \Delta(F_0^a) &= F_0^a \otimes 1 + 1 \times F_0^a \ , \\ (F_1^a) &= F_1^a \otimes 1 + 1 \times F_1^a + K f_a^{Ab}(T_0^A \otimes F_0^b - F_0^b \otimes T_0^A) \ , \\ \\ \Delta(T_0^A) &= T_0^A \otimes 1 + 1 \otimes T_0^A \ , \\ \Delta(T_1^A) &= T_1^A \otimes 1 + 1 \otimes T_1^A + f_{BC}^A(T_0^B \otimes T_0^C \ . \end{aligned} \tag{4.8}$$

Yangianization of SKMA would introduce interaction terms.

5 Category theory as a basic tool of TGD

I have already earlier developed ideas about the role of category theory in TGD [K5, K4, K3]. The hierarchy formed by categories, categories of categories, ... could allow to keep book about the complexity due to various hierarchies. WCW geometry with its huge symmetries combined with adelic physics; quantum states identified in ZEO as WCW spinor fields having topological interpretation as braided fusion categories with reconnection; the local symmetry algebras of quantum TGD extended to Yangians realizing elegantly the construction of interacting many-particle states in terms of iterated Δ operation assigning fundamental interactions to tensor summands of SKMAs: these could be the pillars of the basic vision.

5.1 Fusion categories

While refreshing my rather primitive physicist's understanding of categories, I found an excellent representation of fusion categories and braided categories [B1] introduced in topological condensed matter physics. The idea about product and co-product as fundamental vertices is not new in TGD [K3, K21, L8] but the physicist's view described in the article provided new insights.

Consider first fusion categories.

1. In TGD framework scattering diagrams generalize Feynman diagrams in the sense that in 3-vertices the 2-D ends for orbits of 3 partonic 2-surfaces are glued together like the ends of lines in 3-vertex of Feynman diagram. One can say that particles fuse or decay. 3-vertex would be fundamental vertex since higher vertices are unstable against splitting to 3-vertices. Braiding and reconnection would bring in additional topological vertices. Note that reconnection represents basic vertex in closed string theory and appears also in open string theory.

   Also fusions and splittings of 3-surfaces analogous to stringy trouser vertex appear as topological vertices but they do not represent particle decays but give rise to two paths along, which particles travel simultaneously: they appear in the TGD based description of double slit experiment. This is a profound departure from string models.

   The key idea is that scattering diagrams are analogous to algebraic computations: the simplest computation corresponds to tree diagram apart from possible braiding and reconnections to be discussed below giving rise to purely topological dynamics. One has a generalization of the duality of the hadronic string model: one does not sum over all diagrams but takes only one of them, most naturally the simplest one. This is highly reminiscent to what happens for twistor Grassmann amplitudes.

   One can eliminate all loops by moves and modify the tree diagram by moving lines along lines [?] Scattering diagrams would reduce to tree diagrams having in given vertex either product μ or its time reversal Δ plus propagator factors connecting them. The scattering amplitudes associated with tree diagrams related by these moves were earlier assumed to be identical. With better understanding of fusion categories I realized that the amplitudes corresponding to equivalent computations need not be numerically identical but only unitarily related and in this sense physically equivalent in ZEO.

2. Fusion categories indeed realize algebraically in very simple form the idea that all scattering diagrams reduce to tree diagrams with 3-vertices as basic vertices. Fusion categories [B1] (the illustrations `http://tinyurl.com/l2jsrzc` are very helpful) involve typically tensor product $a \otimes b$ of irreducible representations a and b of an algebraic structure decomposed to irreducible representations c. This product is counterpart for the 3-parton vertex generalizing Feynmanian 3-vertex.

   The article gives a graphical representation for various notions involved and these help enormously to concretize the notions. Fusion coefficients in $a \otimes b = N_{ab}^c c$ must satisfy consistency conditions coming from commutativity and associativity forcing the matrices $(N_a)_{bc} = N_{ab}^c$ to commute. One can diagonalize N_a simultaneously and their largest eigenvalues d_a are so called quantum dimensions. Fusion category contains also identity object and its presence leads to the identification of gauge invariants defining also topological invariants.

   The fusion product $a \otimes b$ has decomposition $V_{ab}^{c\alpha}|c, \alpha\rangle$ for each c. Co-product is an analog of the decay of particle to two particles and product and co-product are inverses of each other in a well-defined sense expressed as an algebraic identities. This gives rise to completeness relations from the condition stating that states associated with various c form a complete basis for states for $a \otimes b$ and orthogonality relations for the states of associated with various c coefficients. Square roots of quantum dimensions d_a appear as normalization factors in the equations.

   Diagrammatically the completeness relation means that scattering $ab \to c \to cd$ is trivial. This cannot be the case and the completeness relation must be more general. One would expect unitary S-matrix instead of identity matrix. The orthogonality relation says that loop diagram for $c \to ab \to c$ gives identity so that one can eliminate loops.

Further conditions come from the fact that the decay of particle to 3 particles can occur in two ways, which must give the same outcome apart from a unitary transformation denoted by matrix F (see Eq. (106) of `http://tinyurl.com/l2jsrzc`). Similar consistency conditions for decay to 4 particles give so called pentagon equation as a consistency condition (see Eq. (107) and Fig. 9 of `http://tinyurl.com/l2jsrzc`). These equations are all that is needed to get an internally consistent category.

In TGD framework the fusion algebra would be based on Super Yangian with super Variant of Lie-algebra commutator as product and Yangian co-product of form already discussed and determining the basic interaction vertices in amplitudes. Perhaps the scattering amplitude for a given space-time surface transforming two categories at boundaries of CD to each other could be seen as a diagrammatic representation of category defined by zero energy state.

5.2 Braided categories

Braided categories [B1] (see `http://tinyurl.com/l2jsrzc`) are fusion categories with braiding relevant in condensed matter physics and also in TGD.

1. Braiding operation means exchange of braid strands defining particle world-lines at 3-D light-like orbits of partonic 2-surfaces (wormhole throats) defining the boundaries between Minkowskian and Euclidian regions of space-time surface. Braid operation is naturally realized in TGD for fermion lines at orbits of partonic 2-surfaces since braiding occurs in codimension 2.

2. For quantum algebras braiding operation is algebraically realized as R-matrix satisfying YBE (see `http://tinyurl.com/l4z6zyr`). R-matrix is a representation for permutation of two objects represented quantally. Group theoretically the braid group for n-braid system is covering group of the ordinary permutation group.

   In 2-D QFTs braiding operation defines the fundamental $2 \rightarrow 2$ scattering defining R-matrix as a building brick of S-matrix. This scattering matrix is trivial in the sense that the scattering involves only a phase lag but no exchange of quantum numbers: particles just pass by each other in the 2-particle scattering. This kind of S-matrix characterizes also topological quantum field theories used to deduce knot invariants as its quantum trace [A3, A1, A4]. I have considered knots from TGD point of view in [K11] [L1].

3. For braided fusion categories one obtains additional conditions known as hexagon conditions since there are two ways to end up from $1 \rightarrow 3$ fusion diagram involving two 3-vertices and 2 braidings to an equivalent diagram using sliding of lines along lines and braiding operation (see Fig. 10 of `http://tinyurl.com/l2jsrzc`).

5.3 Categories with reconnections

Fusion and braiding are not enough to satisfy the needs of TGD.

1. In TGD one does not have just objects - point like particles, whose world lines define braid strands in time direction. One has also the morphisms represented by the strings between the particles. Partonic 2-surfaces are connected by strings and these strings have topological interaction: they can reconnect or just go through each other. Reconnection is in key role in TGD inspired theory of consciousness and quantum biology [K16].

   Reconnection is an additional topological reaction besides braiding and one must assign to it a generalization of R-matrix. Reconnection and going through each other are just the basic operations used to unknot ordinary knots in the construction of knot invariants in topological quantum field theories. Now topological time evolution would be a generalization of this process connecting the knotted and linked structures at boundaries of CD and allowing both knotting and un-knotting.

2. Although 2-knots and braids are difficult to construct and visualize, it seems rather obvious (to me at least) that the reconnections correspond in 4-D space-time surface to basic operations giving rise to 2-knots [A2] - a generalization of ordinary knot that is 1-knot. 2-knots

could be seen as a cobordism between 1-knots and this suggests a construction of 2-knot invariants as generalization of that for 1-knots [K11]. 2-knot would be the process transforming 1-knot by re-connections and "going through" the second 1-knot. The trace of the topological unitary S-matrix associated with it would give a knot invariant. If this view is correct, a generalization of TQFT for ordinary braids to include reconnection could give a TQFT for 2-braids with invariants as invariants of knot-cobordism. It must be however emphasized that the identification of 2-braids as knot-cobordisms is only an intuitive guess.

3. From the point of view of braid strands at the ends of strings, reconnection means exchange of braid strands. Composite particles consisting of strands would exchange their building bricks - the analogy with a chemical reaction is obvious and various reactions could be interpreted as knot cobordisms. Since exchange is involved also now, one expects that the generalization of R-matrix to algebraically describe this process should obey the analog of YBE stating that the two braided versions of permutation $abc \rightarrow cba$ are identical.

   If the strings are oriented, one could have YBEs separately for left and right ends such that braid operation would correspond to the exchange of braid between braid pairs. The topological interaction for strings AB and CD could correspond to a) trivial operation "going through" (AB + CD $\rightarrow$ AB+CD) visible in in the topological intersection matrix characterizing the union of string world sheets, exchanges of either left (AB+CD$\rightarrow$ CB+AD) or right ends (AB+CD$\rightarrow$ AD+CB), or exchange or right and left ends (AB+CD$\rightarrow$ CD+AB) representable as composition of braid operation for string ends and exchange of right or left ends and giving rise to braiding operation for pairs AB and CD.

   The following braiding operations would be involved.

   (a) Internal braiding operation $A \otimes B \rightarrow B \otimes A$ for string like object.

   (b) Braiding operation $(A \otimes B) \otimes (C \otimes D) \rightarrow (C \otimes D) \otimes (A \otimes B)$ for two string like objects.

   (c) Reconnection as braiding operation: $(A \otimes B) \otimes (C \otimes D) \rightarrow (A \otimes D) \otimes (C \otimes B)$ and $(A \otimes B) \otimes (C \otimes D) \rightarrow (C \otimes B) \otimes (A \otimes D)$.

   I have not found by web search whether this generalization of YBE exists in mathematics literature or whether it indeed reduces to ordinary braiding for the exchanged braids for different options emerging in reconnection. One can ask whether the fusion procedure for R-matrices as an analog for the formation of tensor products already briefly discussed could allow to construct the R-matrix for the reconnection of 2 strings with braids as boundaries.

4. The intersections of braid strands are stable against small perturbations unless one modifies the space-time surface itself (in TGD 2-braids are 2-surfaces inside 4-surfaces). Also the intersections of world lines in M^2 integrable theories are stable. Hence it would be natural to assign analog of R-matrix also to the intersections.

5. Light-like 3-D partonic orbits can contain several fermion lines identifiable as boundaries of string world sheets so that reconnections could induce also more complex reactions in which partonic 2-surfaces exchange fermions. Quite generally one would have braid of braids able to braid and also exchange their constituent braids. This would give rise to a hierarchy of braids within braids and presumably to a hierarchy of categories. This might provide a first principle topological description of both hadronic, nuclear, and (bio-)chemical reactions. For instance, the mysterious looking ability of bio-molecules to find each other in dense molecular soup could rely on magnetic flux tubes (and associated strings) connecting them [K16].

6. Reconnection requires a generalization of various quantum algebras, in particular Yangian, which seems to be especially relevant to TGD since it generalizes local symmetries to multi-local symmetries with locus identifiable as partonic 2-surface in TGD. Since braid strands are replaced with pairs of them, one might expect that the generalization of R-matrix involves two parameters instead of one.

6 Trying to imagine the great vision about categorification of TGD

The following tries to summarize the ideas described. This is mostly free play with the ideas in order to see what objects and arrows might be relevant physically and whether category theory might be of help in understanding poorly understood issues related to various hierarchies of TGD.

6.1 Different kind of categories

Category theory could be much more than mere book keeping device in TGD. Morphisms and functors could allow to see deep structural similarities between different levels of TGD remaining otherwise hidden.

6.1.1 Geometric and number theoretic categories

There are three geometric levels involved: space-time, CDs at embedding space level, sectors of WCW assignable with CDs their subsectors characterized by a point for moduli space of CDs with second boundary fixed.

There are also number theoretic categories.

1. Adelic physics would define a hierarchy of categories defined by extensions of rationals and identifiable as an evolutionary hierarchy in TGD inspired theory of consciousness. Inclusion of extensions parameterized by Galois group and ramified primes defining preferred p-adic primes would define a functor. The parameters of quantum algebras should be number theoretically universal and belong to the extension of rationals defining the adele in question. Powers or roots of e, roots of unity, and algebraic numbers would appear as building bricks. The larger the p-adic prime p the higher the dimension of extension containing e and possibly also some of its roots, the better the accuracy of the cognitive representation.

2. These inclusions should relate closely to the inclusions of hyperfinite factors of type II_1 assignable to finite measurement resolution [K23]. The measurement resolution at space-time level would characterize the cognitive representation defined in terms of points with embedding space coordinates in the extension of rationals defining the adele. The larger the extension, the larger the cognitive representation and the higher the accuracy of the representation.

   Should the points of cognitive representation be assigned

   (a) only with partonic 2-surfaces (each point of representation is accompanied by fermion)

   (b) or also with the interior of space-time surface (it is not natural to assign fermion to the point unless the point belongs to string world sheet, even in this case this is questionable)?

   Many-fermion states define naturally a tensor product of quantum Boolean algebras at the opposite boundaries of CD in ZEO and the interpretation of time evolution as morphism of quantum Boolean algebras is natural. If cognition is always Boolean then the first option is more plausible.

3. The hierarchy of Planck constants $h_{eff}/h = n$ with $n \leq ord(G)$ naturally the number of sheets and dividing the order $ord(G)$ of the Galois group G of the extension would relate closely to the hierarchy of extensions. n would be dimension of the covering of space-time surface defined by the action of Galois group to space-time sheet. Ramified primes for extensions are in special position for given extension. The conjecture is that p-adic primes near powers of two or more generally of small primes ramified primes for extensions, which are winners in number theoretic fight for survival [L6].

4. The hierarchy of infinite primes [K19] might characterize many-sheeted space-time and leads to a generalization of number concept with infinitely complex number theoretic anatomy provided by infinite rationals, which correspond to real and p-adic units. The inclusion of

lower level primes to the higher level primes would define morphism now. One can assign hierarchy of infinite primes with primes of any extension of rationals.

6.1.2 Consciousness and categories

Categories are especially natural from the point of view of cognition. Classification is the basic cognitive function and category is nothing but classification by defining objects as equivalence classes. Morphisms and functors serve as correlates for analogies and would provide the tool of understanding the power of analogies in conscious intelligence. Also attention could involve morphism and its direction would correlate with the direction of attention. Perhaps isomorphism corresponds to the state of consciousness in which the distinction between observer and observed is reported by meditators to cease. Cognitive representations would be provided by adelic physics at both space-time level, embedding space level, and WCW level (the preferred coordinates for WCW would be in extension of rationals defining the adele).

One would have a hierarchy of increasingly complex cognitive representations with inclusions as arrows and their sub-WCWs labelled by moduli of CDs and arrow of geometric time telling which boundary is affected in the sequence of state function reductions defining self as generalized Zeno effect [L9].

6.2 Geometric categories

Geometric categories appear at WCW level, embedding space level, and space-time level.

6.2.1 WCW level

The hierarchies formed by the categories defined by the hierarchies of adeles, space-time sheets and hierarchy of CDs would be mapped also to the level of WCW. The preferred coordinates of WCW points would be in extension of rationals defining the adele and one would form inclusion hierarchy. The extension at the level of WCW would induce that at the level of embedding space and space-time surface. Sub-CDs would correspond to sub-WCWs and the moduli space for given CD would correspond to moduli space for corresponding sub-WCWs. The different arrows of embedding space time would correspond to sub-WCW and its time reflection. By the breaking of CP,T, and P the space-time surfaces within time reversed sub-WCWs would not be mere CP, T and P mirror images of each other [L7, L5].

6.2.2 Embedding space level

ZEO emerges naturally at embedding space level and CDs are key notion at this level. Consider next the categories that might be natural in ZEO [K13].

1. Hierarchy of CDs could allow interpretation as hierarchy of categories. Overlapping CDs would define an analog of covering of manifold by open sets: one might speak of atlas with CDs defining conscious maps. Chart maps would be morphisms between different CDs assignable to common pieces of space-time surfaces. These morphisms would be also realized at the level of conscious experience. The sub-CD associated with CD would correspond to mental image defined by sub-self as image of the morphism.

2. Quantum state of single space-time sheet at boundary of CD would define a geometric and topological representation for categories. States at partonic 2-surfaces would be the objects connected by fermionic strings and the associated flux tubes would serve as space-time correlates of attention in TGD inspired theory of consciousness. The arrows represented by fermionic strings would correspond to some morphisms, at least thre Hilbert space isometries defined by entanglement with coefficients in an extension of rationals. Unitary entanglement gives rise to a density matrix proportional to unitary matrix and maximal entanglement in both real and p-adic sense. Much more general entanglement gives rise to maximal entanglement in p-adic sense for some primes.

3. Zero energy states the states at passive boundary would be naturally identifiable as categories. At active boundary quantum superpositions of categories could be in question. Maybe one

should talk about quantum categories defined by the superposition of space-time sheets with category assigned with an equivalence class of space-time sheets satisfying the conditions for preferred extremal.

4. One can imagine a hierarchy of zero energy states corresponding to the hierarchy of space-time sheets. One can build zero energy states also by adding zero energy states associated with smaller sub-CDs near the boundaries of CD to get an infinite hierarchy of zero energy states. The interpretation as a hierarchy of reflective levels of consciousness would be natural.

5. Zero energy states would correspond to generalized Feynman diagrams interpreted as unitary functors between initial and final state categories. Scattering diagram would be seen as algebraic computation in a fusion category defined by Yangian. All diagrams would be reducible to braided tree diagrams with braidings and reconnections. The time evolution between boundaries could be seen as a topological evolution a of tensor net [L3].

Category theory would provide cognitive representations as morphisms. Morphisms would become the key element of physics completely discarded in the existing billiard ball view about Universe: Universe would be like Universal computer mimicking itself at all hierarchy levels. This extends dramatically the standard view about cognition where brain is seen as an isolated seat of cognition.

6.2.3 Space-time level

Many-sheeted space-time is the most obvious application for categorification.

1. Smaller space-time sheets condensed at large space-time surface regarded as categories become objects at the level of larger space-time sheet. Functors between the categories defined by smaller space-time sheets define morphisms between them. Also now fermion lines and flux tubes connecting the condensed space-time sheets to each other via wormhole contacts with flux going along another space-time sheet could define functors. Closed loops involving larger space-time sheets and smaller space-time sheets are needed if monopole flux in question. The loop could visitat smaller space-time sheets.

2. Interactions would reduce to product and co-product. Interaction term in Δ for generalized Yangian would characterize fundamental interactions with dynamically generated SKMAs assignable to SSA as additional interactions. The coupling parameters with Δ assigned to a direct sum of SKMAs would define coupling constants of fundamental interactions. Iteration of the co-product Δ would give rise to a hierarchy of many-particle states. The fact that morphism is in question would map the structure of single particle states to that of many-particle states.

SH would involve a functor mapping the category of string world sheets (and partonic 2-surfaces) to that of space-time surfaces having same points with coordinates in extension of rationals. In p-adic sectors this morphism presumably exists for all p-adic primes thanks to p-adic pseudo-constants. In real sector this need not be the case: all imaginations are not realizable.

The morphisms would be mediated by either continuation of strings world sheets (and partonic 2-surfaces) to space-time interiors (morphism would be analogous to a continuation of holomorphic functions of two complex coordinates from 2-D data at surfaces, where the functions are real). Possible quaternion analyticity [K21] encourages to consider even continuation of 1-D data to 4-D surfaces and twistor lift gives some support for this idea.

In the fermionic sector one must continue induced spinor fields at string world sheets to those at space-time surfaces. The 2-D induced spinor fields could also serve as sources for 4-D spinor fields.

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