A fresh look at $M^8 - H$ duality and Poincare invariance

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Abstract

M^8-H duality is a proposal to integrate geometric and number theoretic visions of TGD. M^8-H duality has several questionable features. For various reasons it seems that M^8 must be replaced with its complexification M^8_c interpreted as complexified octonions O_c. This however leads to several problems. The modified variant of M^8-H duality identifying M^8 as a quaternionic sub-space of octonions O with a number theoretic norm defined by $Re(o^2)$, rather than $o\overline{o}$, solves these problems.

The proposal has been that octonionic polynomials $P(o)$ define the number theoretic holography. Their roots would define 3-D mass shells for which mass squared values are in general complex and the initial data for the holography would correspond to 3-surfaces at these mass shells. Also this assumption has problems. There is however no need for this assumption: the holography on the H side is induced by the M^8-H duality!

The hierarchy of polynomials defines a hierarchy of algebraic extensions defining an evolutionary hierarchy central for all applications of TGD and one must have it. Luckily, the recent realization that a generalized holomorphy realizes the holography at the H side as roots for pairs of holomorphic functions of complex (in generalized sense) coordinates of H comes to rescue. It can be strengthened by assuming that the functions form a hierarchy of pairs of polynomials.

Twistor lift strongly suggests that M^4 and space-time surfaces allow a Kähler structure and what I call Hamilton-Jacobi structure. These structures force a breaking of Poincare and even Lorentz invariance unless they are dynamically generated. It indeed turns out that M^8-H duality generates them dynamically.

1 Introduction

This article was motivated by the realization that the time is ripe for a critical discussion of both M^8-H duality and the notion of Poincare invariance.

1.1 From $O-H$ duality to M^8-H duality to $O-H$ duality: the history of the term "M^8-H-duality"

Before beginning, it is useful to discuss the meaning of the term M^8-H duality, whose interpretation and precise definition have been the longest lasting aberration in the development of TGD. I talked originally of $O-H$ duality, where O denotes the number field of octonions and one has $H=M^4\times CP_2$ is the embedding space containing space-times as 4-surfaces.

1. The original idea [K8, K9, K10] was that quaternions correspond to the tangent space of $Y^4\subset O$: the direction of the quaternion unit would naturally correspond to the time direction and number theoretic norm defined as $Re(o^2)$ gives a Minkowskian metric. This means that O behaves in a number theoretic sense like M^8 and one could speak of $O-H$ duality or M^8-H duality.

   The proposal was that the surface $Y^4\subset O$ has a quaternionic tangent space at each of its points. The cold shower [A1] was that the only solution to the associativity conditions for tangent spaces is trivial: just a quaternionic subspace of O [L4, L5]. The distribution of quaternionic normal spaces is however integrable.

2. Since the signature of octonions is Euclidean with respect to the standard Euclidean norm, I concluded that M^4 must be represented with a subspace of complexified octonions $O_c=M^8_c=E^8_c$ with Minkowskian signature.

   This motivated the erratic assumption that the quaternionic subspace corresponds to a normal space and tangent space of Y^4 to Minkowskian subspace in M^8_c in order to obtain non-trivial solutions to the associativity conditions. I continued to talk of M^8-H duality since M^8_c-H duality sounded too formal.

   Complexification forced the introduction of an imaginary unit i commuting with the octonionic imaginary units. Note that O_c fails to be a number field. I carefully replaced the "$O-H$ duality" with "M^8-H duality" in all chapters except one [K15]! As noted, O is M^8 with respect to the number theoretic norm.

3. A further motivation for the complexification of M^8 was rooted to the idea that the roots of real polynomials P continued to octonionic polynomials (o) defining mass shells could define the number theoretic holography. The roots of polynomials are in general complex and the idea that the imaginary unit could correspond to an octonionic imaginary unit looked unattractive: hence i.

   There was however the problem with the fact that the roots are complex: does the notion of complex mass shell make sense: here twistorialization suggested that this might make sense. Only physical states would have real masses. The complex counterpart of the hyperbolic space H^3 obtained by making the mass squared complex is however not an attractive notion.

4. As noticed, the real part of the square o^2 of the octonion defines the 8-D Minkowski norm. This interpretation is very natural in the number theoretic context, where one does not have the notion of Riemann metric. In the recent formulation of M_c^8-H duality [L4, L5, L20], this view of number theoretical norm is actually used. Together with the complexification it however produces a lot of problems, in particular the problem due to the fact that all possible metric signatures for the space-time surface are in principle possible.

5. I could blame myself for not realizing that the interpretation of M^8-H duality as $O-H$ duality might have worked after all. It will be found that the interpretation of M^8-H duality as $O-H$ duality solves a long list of problems related to M^8-H duality and also the explicit form of the duality simplifies.

   One must however give up the idea that the identification of mass shells as roots of octonionic polynomials $P(o)$ is an essential part of number theoretic holography. Polynomials are however needed since they define hierarchy of extensions of rationals providing a universal realization of evolution as increase of algebraic complexity, partially characterized by the dimension of an extension of rationals defining effective Planck constant $h_{eff}=nh_0$ [K1] [L1]. Polynomials should characterize space-time regions. How?

6. The rather recent discovery that holography=holomorphy vision [L23, L26, L25] makes it possible to construct exact solutions of field equations in terms of generalized dynamically generated complex structure for H, M^4 and space-time surfaces X^4 comes to rescue: the holomorphic functions of H coordinates defining the surfaces as their roots can be realized in terms of pairs of polynomials and the algebraic extensions emerge in this way naturally.

   The new view is also simpler because in this framework $O-H$ duality induces the number theoretic holography and there is no need to define it separately. The dynamically generated Hamilton-Jacobi (HJ) structure and Kähler structure of M^4 [L19] are also necessary for the definition of $O-H$ duality and the worries related to whether Poincare invariance is exact (it fails if Kähler structure of M^4 is not dynamically generated) disappear.

1.2 What could be the hard core of the M^8-H duality shared by $O-H$ duality?

1. M^8-H duality states that number-theoretic and geometric descriptions of physics provided by TGD are in some sense dual or complementary. Number theoretical vision involves besides the algebraic extensions of rational numbers also octonions and quaternions and p-adic number fields. The idea has been that partial differential equations in H correspond to algebraic equations in the space of possibly complexified octonions ($O_c=M_c^8=E_c^8$).

   The simplest algebraic equations are for polynomials $P(o)$ with coefficients with for number theoretic reasons would be integers possibly smaller than the degree of P. This assumption has many problems: for instance complex mass squared values and the fact that the choices of $M_c^4\subset M_c^8$ are parametrized by 6-sphere $G_2/SU(3)$.

   Can this picture be simplified? Could just octonions be enough: this indeed was the original proposal when I talked about $O-H$ duality?

2. The key observation is that algebraic norm for the octonions defined as the real part $Re(o^2)$ of o^2 is Minkowski norm and corresponds to a Riemannian metric in M^8. M_c^8 is not necessary and $O-H$ duality might make sense after all!

3. Associativity is attractive as a dynamic principle realizing the number-theoretic holography. The normal space of 4-surface $E^4 \subset O$ would be associative. The key mathematical result [L4, L5, L20] [A1] is that any distribution of associative normal spaces integrates to a 4-surface. So, in accordance with the original view, for O-H duality the normal space could be associative (quaternionic) and would have Minkowskian number theoretic norm squared defined as $Re(o^2)$!

4. Hierarchy for extensions of rational numbers is a central element of the number theoretic vision. The extension characterizes a given space-time region. Effective Planck constant $h_{eff} = nh_0$, where n is the dimension of the extension serving as a measure for the algebraic complexity. This inspires the number theoretic vision about evolution [K1] [L1].

   Number-theoretic discretization at the O or M^8_c side is natural. Discretization consists of momenta, whose components are algebraic numbers, maybe even algebraic integers. Galois confinement [L6, L8, L27, L10, L7] states that physical states have integer valued momentum components using a suitable unit.

5. $O - H$ duality requires that the quaternionic normal space $Q(x)$ has an integrable slicing by 2-D commutative subspaces $C(x)$ and their orthogonal complements $CC(x)$. This slicing $Q(x)$ defines the HJ structure in M^4 and X^4.

   The complement can is obtained by multiplying $C(x)$ by an imaginary unit $J(x)$ in the complement. The analogy to decomposition of the tangent space of Minkpowski space determined by a polarization ϵ and massless wave vector k is obvious.

   In addition, an analogous mirroring of the quaternion subspace $Q(x)$ to the co-quaternionic subspace $CQ(x)$ is needed. This is obtained by multiplying $Q(x)$ by the co-quaternionic unit $L(x)$. A possible interpretation is that this takes the CQ to a standard form in which the slicing of CQ does not depend on position. This operation can be composed of a quaternionic rotation doing the same in Q and of constant reflection L. A possible interpretation is as a gauge choice associated with the gauge group $SO(3)$ defined by local quaternionic automorphisms.

6. The generalization of holomorphy on the H side combined with the $O - H$ duality could induce number-theoretic holography as associativity. There would be no need for a separate realization of the number theoretic holography at O.

In the sequel I will discuss the $M^8 - H$ duality and its problems and discuss the $O - H$ duality. Also the possible problems related to the realization of Poincare invariance are discussed.

2 The two interpretations of $M^8 - H$ duality

I the following I will used the term $O - H$ duality, which could be interpreted as $M^8 - H$ duality if number theoretic norm is used for octonions. This unless the term $M^8 - H$ duality is needed to avoid confusions. First two possible interpretations of $O - H$ duality will be discussed.

1. The first interpretation is as a generalization of momentum-position duality forced by the generalization of a point-like particle to 3-surface.

2. The second interpretation is as TGD analog of the duality encountered in the twistor Grassmannian approach.

Are these interpretations mutually exclusive or equivalent?

Although $H = M^4 \times CP_2$ provides Poincare invariance, the realization of Poincare invariance has turned out to be far from trivial. Twistorialization strongly suggests that M^4 has Kähler structure and Hamilton-Jacobi structure [L19], which are closely related. One has two options.

1. If these structures are static rather than dynamical, Poincare invariance, the key motivation for TGD, is broken at the level of $M^4 \subset H$. This would be visible for the Dirac equation for H spinors. For instance, the right-handed covariantly constant neutrino becomes a tachyon.

This need not be a catastrophe since p-adic mass calculations assume that the ground state is tachyonic [K12, K5] [L14].

The breaking of Poincare invariance at the level of M^4 need not be a catastrophe. Zero energy ontology (ZEO) [K17] [L3, L11] suggests that Poincare invariance could be realized in the space of causal diamonds (CDs) [L21] rather inside CD. One can however argue that the non-uniqueness of the M^4 Kähler structure at the fundamental level is a fatal feature.

2. The general solution ansatz, realizing holography as holomorphy [L25, L26], strongly suggests that HJ structure [L19] and M^4 Kähler structure are one and the same thing and implied by M^8-H duality dynamically. They would be associated with the holomorphic dynamics of the space-time surface rather than the fundamental geometry of $M^4 \subset H$. Poincare invariance would remain exact.

   $M^8 - H$ duality requires an integrable slicing of the local normal space of the 4-surface $Y^4 \subset M^8_c$ by (quaternionically) commutative 2-D subspaces $C(x)$ and their orthogonal complements $CC(x)$. This allows to parameterize the normal space by a point of CP_2. This slicing could be reflected by a multiplication of the normal space by a preferred octonionic unit $J(x)$ to a slicing of M^4 defining the Hamilton-Jacobi and Kähler structure. This kinds of slicing characterizes the known extremals irrespective of action [K2] such as massless extremals.

2.1 The interpretation of $O-H$ duality as a generalization of momentum-position duality

In the sequel the motivations for $M^8 - H$ duality, or equivalently $O - H$ duality assuming that O is endowed with a number theoretic norm, are disucess. The emphasis is on problems and poorly understood aspects.

2.1.1 Why $O - H$ duality?

Consider first the motivations for $O - H$ duality.

1. In TGD point particles are replaced with 3-surfaces. The intuitive interpretation of M^8, or equivalently O with number theoretic norm, would be as a generalization of the momentum space. $O - H$ duality would be a generalization of the momentum-position duality forced by the generalization of the particle concept.

2. The semiclassical argument suggests the form of the $O-H$ duality in M^4 degrees of freedom. $O - H$ duality is local in M^4 degrees of freedom unlike in CP_2 degrees of freedom. M^4 point is mapped to its inversion $m^k \rightarrow p^k = \hbar_{eff} m^k/m_l m^l$ interpreted as a momentum [L21, L22, L17, L20, L18], which is a conformal transformation. $\hbar_{eff}$ is the effective Planck constant determined number theoretically. The transformation is defined in the interior of light-cone singular at the boundary of the light-cone, where one can consider inversion as $m^0 \rightarrow p^0 = \hbar_{eff}/m^0$, $r_M \rightarrow p = \hbar_{eff}/r_M$. This corresponds to the naive semiclassical interpretation of the Uncertainty Principle.

3. $O - H$ duality in this form is not Fourier transform. Fourier transform is a correspondence between function spaces. A completely localized wave function in position space is mapped to a superposition of plane waves characterized by momentum in momentum space. It has become clear that plane waves make sense for the positions of CDs [L21] defining the analogs of perceptive fields in H and that Poincare invariance is realized in the space of CDs rather than in H.

   The space of CDs is finite-dimensional and would form the backbone for the "world of classical worlds" (WCW) [K11, K6, K13] [L23, L13]: CD would contains 4-surfaces having ends at its boundaries. This interpretation is the only possible option if $M^4 \subset H$ has the analog of Kähler structure at the fundamental level since Kähler structure leads to a violation of the Poincare invariance becoming manifest for H spinor modes.

$O-H$ duality should not only assign to the 4-momentum a point of $CD \subset M^8$, but also the equivalent of a plane wave in the space of CDs associated with H. This plane wave would correspond to a plane wave in the space of the translates of CD containing the translates of the space-time surfaces analogous to Bohr orbits.

4. Since the CD corresponds to a many-particle state, a reasonable guess is that the momentum assignable to the plane wave in the space of CDs corresponds to a total momentum of a state assignable to CD, for instance the total momentum assignable to the 3-surface with time coordinate corresponding to the center of mass for CD in the case that the Poincare invariance is violated.

2.1.2 Critical questions concerning the assignment of a plane wave in the space of CDs to a given momentum

Does it make sense to assign a plane wave to a space-time surface and to a CD containing it? If so, how to make this assignment?

1. For CD the (4-)momentum of the plane wave should correspond to the total momentum of the physical state. Does the momentum correspond to the total momentum related to the fermions of H or to the classical total momenta as Noether charges on either boundary of the CD or perhaps at the $t = t_{cm}$ section corresponding to its cm of CD?

2. Inertial and gravitational masses, which are identical if Equivalence Principle is satisfied, correspond to 4-momenta. Gravitational and inertial momenta need not however be identical. In fact, the induction of the second quantized spinors from H to X^4 assigns to the modes of H spinors those of X^4, which satisfy a modified Dirac equation.

   By the generalized conformal invariance, the modified Dirac equation can be solved as in string models [L25] and the mass squared corresponds to a conformal weight, which could correspond to the mass squared at the level of H. Conserved conformal quantum numbers could correspond to the gravitational momentum.

   Which of these two 4-momenta corresponds to the momentum of the plane wave in the space of CDs? Could it correspond to the classical Noether momentum or to total fermionic conformal momentum as operator? Is it the conserved conformal momentum or inertial momentum? Are classical and fermionic momenta as eigenvalues identical by quantum-classical correspondence?

3. Does it really make sense to consider the CD, interpreted as a geometric correlate of a perceptive field, as a particle-like object moving in a plane wave? The identification of the momentum as a Noether charge assignable to either boundary of the CD or to the cm of the CD and the possible exact momentum conservation at the level of the space-time surfaces would suggest that CD has vanishing four-momentum at the infinite volume limit. This motivated the term zero energy ontology (ZEO).

4. One can also ask whether the 4-momentum associated with the CD could correspond to the difference of the 4-momenta associated with the boundaries of the CD, which could be non-zero for finite volume. The finite size of the CD brings with it the non-conservation of total fermionic momentum at the quantum level for the scattering amplitudes. Also the Kähler structure of M^4, strongly suggested by twistorization, suggests a violation of Poincare invariance, which certainly occurs at the level of the quantum numbers of the spinor modes of H.

2.2 The interpretation inspired by the twistor Grassmannian approach

In the twistor Grassmannian approach, the conformal symmetry of M^4 is realized at two levels corresponding to space-time and momentum space. These realizations lead to the Yangian symmetry, which extends the conformal symmetry involving poly-local symmetry generators.

2.2.1 Yangian symmetry of twistor Grassmannian approach as a guideline

1. In twistorialization there are two different conformal symmetries, which are dual and operate in position and momentum spaces. The generators of these symmetries generate Yangian symmetry [?] [B3, B1, B2] [?, K14] involving both symmetry algebra and coalgebra in which the commutator has a "time-reversed" analog.

   Could $O-H$ duality be the TGD counterpart for this duality rather than the counterpart of momentum-position duality? Or could it correspond to both in some sense. Could the inversion, which is a conformal transformation [L21, L22, L17, L20, L18], which cannot be generated continuously from an identity map, locally relate these 2 spaces and two conformal algebras to each other.

2. In TGD these two conformal symmetries would act in H and M^8. Yangian symmetry would be the central notion and involve n-local infinitesimal transformations. The gravitational Planck's constant [E1] and its variants for different interactions are two-particle parameters and one can ask whether they could relate to the structure of Yangian [K14, K16, K7, K4].

3. Could the correspondence between outgoing massless momenta and circulating virtual momenta in the twistor diagrams help? In the twistor Grassman approach, the outgoing momenta at a given vertex is the difference of the circulating momenta for the line leaving and for the line entering the vertex. Should the difference be replaced with its inversion in M^8-H duality. If this is the case M^8-H duality is bilocal.

4. The map $H \to M^8$ is naturally bilocal in $M^4 \subset H$ since the difference of two points of $M^4 \subset H$ defines classically the direction of the momentum. All point pairs with the same difference would correspond to the same momentum in M^8. The same point in $M^4 \subset M^8_c$ would correspond to an infinite number of pairs of points in $M^4 \subset H$. This is what happens in the Fourier transform.

5. To get M^8-H duality as a one-valued map $M^8 \to H$, one must attach the second member of the point point pair $M^4 \subset H$. It should be some preferred point of CD. Either tip of the CD or the center of mass CD (cm) are natural candidates. CD has a natural rest frame and time coordinate t. If the 4-momentum is not conserved (M^4 would have a Kähler form at the fundamental level), the classical momentum as Noether charges for the time value t_{cm} of cm could define the CD momentum for the space-time surface. The fixed point of the point pair to the cm.

6. $O-H$ duality should also assign to a point $Q \subset O$ (momentum) a plane wave in the space of CDs characterized by the momentum. This would realize the Fourier transform aspect and Uncertainty Principle. The natural idea is that the pairs of M^4 points with a varying first point correspond to different CDs so that $O-H$ duality would define a plane wave in the space of CDS.

7. Classical view of relativistic causality would be realized in the sense that the momentum direction would correspond to a direction defined by the difference of the M^4 coordinates.

   For the $O-H$ duality defined by the inversion, the plane wave in the space of CDs would have a value equal to one, when the direction determined by the pair of points is the same as the direction of the momentum.

2.2.2 The interpretation of the Yangian in the TGD framework

What could be the TGD counterparts of the Yangian algebra in TGD and their physical interpretation. Yangian has higher poly-local generators of the Yangian labelled by integers?

1. In TGD 3-surface replaces point particle. n-local generators could act on unions of n 3-surfaces, n-particle states in topological sense (each 3-surface can carry many-fermion states). They would describe the many-particle contributions to quantum numbers produced by the interactions.

2. The simplest manifestation of these poly-local contributions would be potential energy, which is a bilocal contribution to energy. The effective Planck constants related to gravitation [E1, E2] [K3, ?] [L2] and other interactions [L16] are proportional to charges of two particles and could be associated with multilocal contributions to the quantum numbers, while the usual Planck's constant would correspond to a single-particle contribution. The multilocal contributions should have a geometric representation and magnetic flux tubes would be a natural candidate in this regard.

3. Could one consider the following alternative physical interpretation? Free particles correspond to their own space-time surface and to their own CD each. When they interact, the 3-surfaces are placed in the same CD as initial value surfaces defining the holographic data and holography determines a connected 4-surface that describes the interactions. The difference of the symmetry charges generators for the interacting and free state can be identified as the polylocal contribution.

4. How could poly-locality for Yangian be realized at CD level? Could the vertices of the twistor diagram correspond to separate CDs with associated zero energy states? could the points of $M^4 \subset M^8$ correspond to the vertices of the CDs in $M^4 \subset H$.

3 Why should $M_c^8 - H$ duality be replaced with $O - H$ duality?

Consider next the weak points of $M^8 - H$ duality, or rather $M_c^8 - H$ duality.

3.1 The problems of the realization of holography using octonionic polynomials $P(o)$

Octonionic polynomials $P(o)$ would be the simplest way to obtain a hierarchy of rational extensions [L4, L5, L20]. There are arguments suggesting that the coefficients of the polynomials could be taken to be rationals, integers, or even integers smaller than the degree of P. However, the realization of the number theoretic holography using octonionic polynomials $P(o)$ with rational or even integer coefficients might be unnecessary.

1. The original wrong intuition was that in M_c^8, the 4-surface Y^4 could correspond to the vanishing of the real or imaginary part of an 8-variable polynomial P. The naive dimensional counting suggested that the dimension of this surface would be 4. This guess turns out to be wrong. One obtains 7-D surfaces.

   Furthermore, by setting the $P(o)$ to zero, one obtains instead of a discrete points, set of 3-D mass shells with mass squared values equal to the roots of P. The reason is Lorentz invariance. The argument of the real polynomial is effectively replaced by mass squared. The root becomes the entire mass shell H^3.

   There are also interpretational problems: the roots are in general complex: for the physical states the total mass squared should be real. Stringy mass spectrum would suggest integer values for mass squared. The spectrum defined by the roots does not correspond naturally to the mass spectrum of the Dirac operator of H.

2. One encounters also the problem of whether H_c^3 defined as complex mass shell with real 3-momenta has the extremely nice properties of H^3, which are in a key role in various applications of TGD to biology [L15] and to cosmology and astrophysics [L24]. But how does on obtain the algebraic extension of rationals without them?

3. If the number-theoretic holography works, the roots as mass shells determine part of the 3-D holographic data in the sense that 3-surfaces reside at them. It is however not clear whether the associative holography works. Do the data as 3-surfaces at mass shells really determine Y^4? Even more, $M^8 - H$ duality induces number theoretic holography from the holography at H side.

It seems that polynomials produce only troubles. If complex roots of polynomials are accepted, M_c^8 is required. If the roots of polynomials do not occur at all at the M^8 level, the problem disappears.

3.2 $O-H$ duality as a simpler version of M^8-H duality

The above considerations suggest a simplification of the M^8_c-H duality to $O-H$ duality.

1. The basic motivation are the problems related to different signatures produced by the complexification of O. For M^8_c-H duality the tangent space of Y^4 is Minkowskian and co-quaternionic and the possibility of both I_k and iI_k allows all signatures for the tangent space CQ as also for the normal space Q.

   The problems disappear if M^8_c is replaced by O and $Re(o^2)$ defines a number-theoretic Minkowski norm in Q. O is indeed M^8 in this sense! This is indeed in spirit with the algebraic thiking. In CQ, one has co-quaternionic structure and Euclidean number theoretic norm defined in the same way. The Q is characterized for the CP_2 point provided the normal space has the local slicing by complex space $C(x)$ and it complement $CC(x)$.

   Also the M^8_c option involves the number theoretic norm (the imaginary unit i defining the complexification was assumed to commute with octonionic imaginary units). One could get the picture proposed originally except that quaternionic normal space Q would replaced number theoretic M^4 on the octonionic side!

2. A given point of CQ would be mirrored by the co-quaternionic unit $L(x)$ into a point of the quaternionic normal space Q and from this by inversion to a point of $M^4\subset H$! The map in the opposite direction is defined in an obvious way. CQ cannot be identified as a momentum space and interesting questions relate to the physical interpretation of this point.

3. The slicing of $M^4\subset H$ required by the $O-H$ duality would automatically map to the slicing of $Q\subset O$ and further to a slicing of $CQ\subset O$. A possible interpretation of the slicing in $M^4\subset H$ is in terms of a local polarization ϵ and light-like wave vector k characterizing plane waves for massless fields. One can assign this kind of decomposition to massless extremals.

What is the interpretation of the 4-surface $Y^4\subset O$?

1. In the tangent space of Y^4, the number-theoretic norm is Euclidean. This is where $L(x)$ comes into the picture. CQ can have standard spatially constant slicing by a proper choice of $J(x)$?

2. A possible identification of H is as pairs $(N(m),m)$, where $N(m)$ is the associative normal space for the surface $Y^4\subset O$ parametrized by a point of CP_2, and $m\in M^4$ is its point: some kind of flag manifold would be in question.

   The action of $L(m)$ maps the point in Q to a point in the CQ. The energy would not be affected and the direction of 3-momentum would change by a local $SO(3)$ rotation (quaternionic automorphism) and would belong to an Euclidean mass shell in CQ, actually an energy shell. Q and CQ would change their roles. CP_2 point would be mapped to Q! There might be a connection to Wick rotation in which M^4 is effectively replaced by E^4.

All space-time surfaces in H do not have a 4-D M^4 projection as the basic form of $O-H$ duality would suggest. String-like objects with 2-D M^4 projection and CP_2 type extremals are the basic examples. These surfaces must emerge as singularities at which the normal plane of Y^4 is not unique and the normal planes define CP_2 or its sub-manifold [L16]. This happens when the surface Y^4 has a sharp tip or edge. The treatment of singularities of $O-H$ duality requires no changes to the existing picture.

3.2.1 $O-H$ duality is consistent with the expected mass squared spectrum

$O-H$ duality, would automatically induce holography from generalized holomorphy.

1. $O-H$ duality respects Lorentz invariance and describes mass shells (hyperbolic spaces H^3) for surfaces for which the light-cone proper time is constant.

2. For $O - H$ duality, masses and momentum components would be real algebraic numbers and ordinary integers for the physical states. Generalized holomorphy attaches two integer valued conformal weights and mass squared values as differences of these [?] this solves the problems related to p-adic mass calculations (tachyonic ground state was required).

   Stringy mass spectrum would fix the mass shells in O and their counterparts in $M^4 \subset H$ would be fized by inversion [?]

3. Galois confinement would remain as it is except that algebraic integers as momentum components must be real. Physical states would still have momenta with integer components. Now the roots of polynomials do not correspond to mass squared values so that there are no restrictions on polynomials from this condition.

   The proposal that the coefficients are integers smaller than the degree of the polynomials remains an attractive hypothesis implying that at the given level of the hierarchy of extensions the number of polynomials is finite [?] This forces the increase of the dimension of the extension and degree of the polynomials involved.

4. The assumption that holographic data are given at mass shells of M^8, or their H counterparts in $M^8 - H$ duality, is too restrictive in the new picture. In ZEO, the minimal option is that the data are given at the light-like passive boundary of CD but one can also consider that some mass shells, or rather, their images in H are possible.

   In the earlier picture, the initial data was assumed to correspond to 3-surfaces at mass shells H^3. One can argue that this assumption is quite too strong. A more general assumption would be that 3-surfaces defining the holographic data can be taken to be deformations of regions of mass shells H^3 in CP_2 direction. The initial data describing deformations of unions of fundamental regions of the tessellations of H^3 would obey periodicity conditions analogous to those satisfied by lattices in condensed matter physics.

5. What about the interpretation of the roots of the polynomials assignable to points at partonic 2-surfaces and string world sheets? Now the roots can be complex in the generalized complex coordinates of H. In the case of partonic 2-surfaces, an attractive idea is that they correspond to the loci of fermions. In the case of string world sheets they would correspond to hypercomplex points. Do hypercomplex points correspond to 1-D curves identifiable as fermion lines at the 3-D orbits of the wormhole contacts. For instance, for M^2 hypercomplex point $u = t - z = c$ would correspond to a light-like geodesic. In this framework, 3-D light-like curves could correspond to points obtained by putting hypercomplex coordinate u to constant.

3.2.2 Hierarchy of algebraic extensions realized in terms of pairs of holomorphic polynomials

If had been aware of the holography=holomorphy idea as I proposed the M^8_c formulation of the $M^8 - H$ duality, I would have asked whether the hierarchy of holomorphic embeddings could be formulated using the hierarchy of holomorphic polynomials already at the H side? This would have made it possible to get rid of the complexified octonionic polynomials altogether.

1. If holography=holomorphy vision works, one has on the side of H 2 holomorphic (in generalized sense) polynomials P_1 and P_2 of the 4 complex (in generalized sense) coordinates of H. These functions could also be analytic functions. The conditions $P_1 = 0$ and $P_2 = 0$ give 2+2 conditions determining a space-time surface X^4.

   Note that the vanishing of either P_1 or P_2 defines a 6-D surface. These surfaces could closely relate to the twistor spaces of M^4 or CP_2. This is proposed also in the case of $M^8_c - H$ duality.

2. Also now one could assign a hierarchy of extensions of rationals assignable to P_i defining a polynomial of a single variable when 3 complex coordinates (in generalized sense) are fixed. This restriction could be interpreted as an identification of a partonic 2-surface or a string world sheet.

The HJ structure defining a slicing of $M^4 \subset H$ and X^4 by 2-D complex surfaces defines these 2-D surfaces and the roots would correspond to the points at these surfaces. By $O-H$ duality they would define momenta in O. This brings in mind light-like orbits of fermions at the light-like orbits of partonic surfaces. These orbits are identified as boundaries of string world sheets. The momenta of fermions would belong to an algebraic extension of rationals if the polynomials P_i have rational or integer coefficients (possibly smaller than the degree of P_i.)

It should be noticed that the $M^8 - H$ duality mapping the momenta to M^4 points defines a geometric representation of momenta as an analog of diffraction pattern characterizing the momenta of the scattered X rays at the heavenly sphere. Mass shell becomes an analog of the Fermi surface containing active points populated by fermions and has a counterpart at the space-time level [L12].

3. $Y^4 \subset O$ is an associative surface. The reflection $L(x)$ of the quaternionic subspace Q characterized by the CP_2 point to the co-quaternionic subspace CQ determines the tangent plane of Y^4. Q decomposes to complex plane and C its complement mapped to each other by reflection $J(x)$ at each point of Y^4.

4. P_1 and P_2 have different degrees. What determines the unique algebraic extension? Could these polynomials define two extensions assignable to (twistor spaces of) M^4 and CP_2 respectively? Could the full extension be a composite of these extensions containing both?

   Note that one obtains 2+2 different 2-surfaces by putting 3 complex coordinates to zero simultaneously. There would be 2 2-surfaces in M^4 *resp.* CP_2 and the corresponding roots would define corresponding extensions of rationals. In M^4 these would naturally correspond to string world sheets and partonic 2-surfaces. Perhaps the projections of the 2 surfaces to M^4 and CP_2 correspond to these 4 2-surfaces.

3.2.3 What does the choice of $Y^4 \subset O$ mean physically?

$M^8_c - H$ duality involves a choice $M^4 \subset M^8_c$ as co-associative tangent space. The space of choices for M^4 can be regarded as a Grassmann space $Gr(8,4) = SO(1,7)/SO(1,3) \times SO(4)$ and has dimension 16. For M^4_c it would be $Gr_c(8,4) = SU(1,7)/SU(1,3) \times SU(4)$.

Also $O - H$ duality involves this kind of choice at each point of Y^4 but now this choice is fixed partially by $O - H$ duality. The choices for different points of Y^4 are related by $SU(2)$ automorphism leaving the quaternionic space invariant. For a given Y^4 there the choices at different points are related by $SU(3)$ automorphism. This leaves non-equivalent choices of M^4 related to by octonion automorphisms, and the space of choices is $G_2/SU(3) = S^6$. Since M^8 momentum is like-like, the M^4 momentum as its projection can be made light-like for a suitable choice of M^4, that is use of suitable G_2 automorphism? What does this mean physically?

1. What problems emerge if the choice of Q_0 differs for two subsystems? Are the conservation laws associated with the Poincare invariance lost? Does particle mass become a relative notion depending on the M^8_c (O-) frame just like energy becomes relative notion in Special Relativity? Should one generalize Poincare invariance to $SO(1,7)$ invariance or to number theoretic G_2 invariance? Is it possible to realize G_2 boosts physically?

2. What fixes the O frame in which particle masses, as we understand them, are defined? Is this frame the frame in which the mass vanishes in the lowest order approximation, that is for the state that gives the largest contribution to the state, and does p-adic thermodynamics [L14] describe the small corrections to mass squared due to the fact that one has a superposition of states with different values of mass squared.

3. There is an interesting analogy with the Higgs mechanism. G_2 transformation changes the mass and $SU(3) \subset G_2$ transformations leave it invariant. One could say that the parameters of $G_2/SU(3) = S^6$ are analogous to the components of a Higgs field and define the analog of Higgs expectation. An interesting unanswered question is whether S^6 allows Kähler structure (see this) . If this were the case, one could speak of a Higgs field with 3 complex components as an analog of the standard model Higgs field with 2 complex components.

3.2.4 The physical interpretation of the 4-surfaces $Y^4 \subset O$

What could be the physical interpretation of the 4-surfaces $Y^4 \subset M^8_c$ or $Y^4 \subset O$ in the case that $O-H$ duality is possible. One might say that one has evolution with respect to energy or mass squared rather than with respect to time.

1. In classical physics one can talk about trajectories for point-like particles in both space-time and corresponding momentum space. Could the surfaces Y^4 satisfying number theoretic holography be interpreted as trajectories of 3-D particles in 8-D momentum space, kind of Bohr orbits? In free field theory, the mass of a particle is fixed so that the idea about trajectory connecting different mass shells sounds strange. The problem with this interpretation is that one has energy instead of time now.

   I have considered the possibility that the mass shells define an analog of coupling constant evolution with respect to mass scale as the counterpart of the time evolution defined by $X^4 \subset H$.

2. Could holography and ZEO solve this interpretational problem? By holography, which is not completely deterministic at the level of H, 4-surfaces as analogs of Bohr orbits become the fundamental objects instead of 3-surfaces. The quantum states are superpositions of these Bohr orbits. This would hold true also at the level of both M^8_c and O. The surface Y^4 defined by the number theoretic holography would be the fundamental object.

3. The number theoretically allowed momenta for fermions having components in the extension defined by, say, polynomial P_1 could define preferred points of Y^4, which would be "active" if the physical state contains fermion with this momentum. The number of the points of Y^4 satisfying in a given extension of rationals is typically small: a good example is Fermat's theorem: for there are no non-trivial integer solutions $x^n + y^n - z^n = 0$ for $n > 2$.

   Mass shells in Q, satisfying $p^2 - m^2 = m^2$, where m^2 is a real algebraic number in an extension of rationals, allow allow an infinite number of solutions as momenta, whose components are algebraic integers in the extension defined by (say) polynomial P_1. These point of Q would be mapped to the points CQ by $L(x)$ and should also be algebraic points. Here the condition that $SU(2)$ automorphism $J(x)$, which maps the slicing of Q to the constant standard slicing in CQ, is essential. The number of momenta mapped to algebraic points of CQ is expected to be small and this would put strong constraints on surfaces Y^4 allowing given momentum as an active point.

   It is not clear whether one can pose the additional condition that the coordinates of CP_2 are in the extension of rationals defined by the second polynomial P_2. The possible problems are due to the fact that CP_2 coordinates are not linear and the choice of CP_2 coordinates is not unique. Projective coordinates (z^1, z^2, z^3) are linear but determined apart from an overall complex scaling. One can however require the ratios z^i/z^j are in the extension. This would allow $SU(3)$ rotations for which the elements of the rotation matrices belong to the extension.

4. Could different mass shells defining a slicing of the orbit correspond to a set of length scale resolutions corresponding to the sub-extensions of the Galois extension defining number theoretic evolution, which in turn defines coupling constant evolution associated with the increase of the size scale of CD?

5. The increase in the size of the CD and the size of $Y^4 \subset O$ in the sequences of "small" state function reductions (SSFRs) at the level of O would bring in additional momenta and increase the complexity. The sequences of SSFRs would correspond to a subjective time evolution defining conscious entities. At the level of H these evolutions would improve the length scale resolution in short scales. In the quantum field theory framework this would mean coupling constant evolution by the addition of radiative corrections to the amplitudes. New mass shells made possibly by a larger extension of rationals roots appear and the dimension of the realized part of the algebraic extension would increase.

3.3 $O-H$ duality for the fermion fields

Concerning the realization of $O-H$ duality for fermions, there are many questions to be discussed.

1. How the spinor structures of H and O related. What does one mean with octonionic spinor structure? How the modified gamma matrics of $X^4 \subset H$ relate to the octonionic gamma matrices for Y^4. Should the gamma matries for Y^4 be associative in some sense.

2. How is the second quantized spinor field in H mapped to that in O? The naive guess is that one performs inversion [L21, L22, L17, L20, L18] for the $M^4 \subset H$ coordinates of the H spinor field.

3. H spinor fields are complex. What happens to the imaginary unit i associated with them in $M^8 - H$ duality? Can one assume that i commutes with octonionic imaginary units so that O_c would be introduced at the level of spinor fields. Or should one assume that imaginary unit i corresponds to an imaginary octonionic unit?

4. How are the fermion propagators defined in M^8_c. If the $M^8 - H$ duality corresponds to a Fourier transform the propagators would be diagonal in the momentum space $M^4 \subset M^8_c$ and there would be no propagation. This does not make sense. Of course, inversion for the argument of the H spinor field does not imply this.

3.3.1 About the notion of octonionic spinor structure

The notion of H spinor structure is rather delicate and the delicacies are essential for the emergence of standard model couplings. This suggests that one should define the notion of octonionic spinor structure carefully. I have considered this problem earlier [K9]. This requires the definition of octonionic γ matrices, octospinors and the octonionic Dirac equation.

1. For the quaternionic spinor structure one can indeed speak of γ *matrices* since quaternion units allow matrix representation as sigma matrices. The ordinary gamma matrices have the representation as matrices $\sigma_3 \otimes \sigma_i$, $\sigma_i \otimes \sigma_0$, $i = 1, 2$. The matrices σ_i, $i = 0, 1, 2, 3$ can be replaced with quaternionic units $\{I_0, I_i\}$ in the quaternionic representation. The metric signature is obtained correctly by multiplying I_i with an imaginary unit i commuting with them.

2. Octonionic imaginary units I_i, $i = 1, .., 7$ anticommute just like gamma matrices and sigma matrices but one cannot find a matrix representation for I_i. One should also define the gamma matrix assignable to the real octonion unit. This representation should contain a quaternionic representation as a special case. From this one easily deduces that for instance the objects $I_K \otimes e_i$, $i = 1, 2, 3, 4$, $K > 4$ and $I_i \otimes I_0$ define the octonionic analogs of gamma matrices in $O \otimes O$.

How are the gamma matrices of H mapped to those of O in $O - H$ duality?

1. The gamma matrices of H are defined as linear combinations of the flat space gamma matrices γ_k of M^8 identified as the tangent space of H. The coefficients define 8-bein. Could one replace H gamma matrices γ_k with their octonionic counterparts representable in terms of I_i to obtain octonionic gamma matrices?

2. What about modified gamma matrices of X^4? Should this replacement map them to quaternionic gamma matrices? Can this be true without additional conditions? Could generalized holomorphy guarantee this? Could this condition fix the variational principle for the space-time surfaces and therefore the modified gamma matrices?

   Could one map the flat space gamma matrices γ_k of H to the gamma matrices of O and project them to Q? This would be the analog for the projection of the H gamma matrices to X^4 if the action is volume action. For a more general action, the modified gamma matrices are not mere projections since they are contractions$T^k_\alpha \gamma_k$ of the canonical momentum currents with γ_k.

What about octospinors and the octonionic Dirac equation?

1. Could they correspond to octonions, complexified octonions or to octonionic gamma matrices. Algebraic form of massless Dirac equation $D\psi = p^k\gamma_k\Psi = 0$ should give rise to the condition $p_k p^k = m^2$ and this is true one has $\Psi = D\Psi_0$ and associativity holds true in the sense that $(DD)\Psi_0 = D(D\Psi_0)$. This is the case always. By M^8-H duality the momentum p^k should correspond to a point of quaternionic subspace Q.

   There seems to be no obvious object against massive octospinors. Situation seems to be the same as in H. Massive spinors would be massless in 8-D sense and massivation would mix different M^4 and CP_2 chiralities but not H chiralities (H chiralities correspond to conserved lepton and quark numbers).

2. The action of the octonionic gamma matrices on Ψ_0 must be well defined so that it must be expressible in the tensor product $O\otimes O$. Associativity must be realized so that Ψ must belong to a quaternionic subspace of $O\otimes O$, most naturally $Q\otimes Q$. This would give $4\times 4 = 16$ spinor components.

3. Can the spinor space of H be mapped in 1-1 way to the space of octospinors? H spinors with a fixed chirality have 8 complex components so that one has 16 complex spinor components. For complex octospinors with i commuting with the I_k one would have 8+8 complex components. This suggests that the octospinors are complexified in the sense that imaginary unit i commuting with I_k is allowed.

3.3.2 How are the second quantized spinor fields of H mapped to spinor fields in O?

If one takes seriously the notion of octonionic spinor structure discussed above, the map of the second quantized spinor fields would be rather simple.

1. Consider first the plane waves and CP_2 harmonics. CP_2 point defines a normal space of Y^4 and the value CP_2 harmonic with the corresponding point of O. M^4 point appearing as the M^4 argument of the spinor field is mapped by inversion to O. Spinor components are mapped to octospinors.

   Whether this gives a *quaternionic* octospinor as the associativity suggests, which would naturally correspond to Q, is not clear. What constraints this poses on the action defining the space-time surfaces and defining the modified gamma matrices is an open question.

2. As noticed, massive modes in the M^4 sense are possible octospinors as in the case of H. One can however ask whether the massless Dirac equation for octo spinor mode could be massless. Could a general state be massless in O by a suitable choice of Q guaranteeing that light-like momentum is parallel to Q.

3. $O-H$ duality involves a selection of a basic Q_0 such that $Q(x)$ is an $SU(3)$ transform of Q_0. The choice defines the analog of $SU(3)$ chiral field preserving the mass squared value.

   The automorphism group G_2 of O gives different choices of Q_0 and the space of choices for Q_0 is 6-dimensional space $G_2/SU(3) = S^6$. By applying a suitable G_2 transformation, one can always find a special choice of Q_0 making a given spinor mode or a more general state massless. The notion of M^4 mass would be relative. In p-adic thermodynamics the contribution to the state with the largest conformal weight would be massless for this choice of Q_0.

4. The G_2 relativity looks paradoxical. Is there any empirical evidence for it? The points of $G_2/SU(3) = S^6$ define the analog of Higgs field, which is complex if S^6 allows a Kähler structure (see this).

   p-Adic thermodynamics [K12, K5] [L14] could be seen as a generation of a vacuum expectation of the deviation of the S^6 valued Higgs field from the value corresponding to a vanishing mass. This deviation is unavoidable since the excited states of superconformal representation with different values of mass squared cannot be transformed to massless states for the same choice of Q_0. Therefore the Higgs mechanism could be seen as direct evidence for G_2 relativity.

3.4 About the interpretation of Galois confinement and number-theoretic discretization

Galois confinement [L6, L8, L10, L7] is realized for the physical states at the level of O and is an essential aspect of $O-H$ duality.

> The components of fermion momenta are algebraic integers when momentum unit is determined by be the scale of CD. For the physical states defined as many-fermion states the sum of the momenta has integer components. One can chose the momenta to be real algebraic integers so that mass squared is real and integer valued. This number theoretic condition is analogous to a particle-in-box quantization and guarantees that the values of the plane wave factors are roots of unity in the number theoretic discretization. The integer quantization for the physical momenta can be interpreted in terms of a finite measurement resolution. It is not possible to measure the algebraic momenta of the building blocks of the Galois confined physical states. One could of course consider a hierarchy of measurement resolutions defined by algebraic extensions of rationals and that integer quantization is realized only at the lowest level. The fact that algebraic numbers do not have finite or periodic binary expansion however favors integer quantization. One can ask however whether the higher levels of cognitive hierarchy could represent numbers in the algebraic extensions geometrically. For instance, the diagonal of the unit square would represent $\sqrt{2}$. Geometric thinking would make it possible to overcome the restrictions of rational number based algebraic cognition.

One can pose several questions.

3. One should associate the algebraic integer points at the mass shell of Q and Galois confinement should apply to the physical states. Do the 4-momenta correspond to the

   (a) four momenta of spinor modes of at the level of $M^4 \subset H$?

   (b) momentum-like quantum numbers of induced spinors at the level of X^4 ?

   (c) classical Noether charges at the level of X^4?

   Does the quantum -classical correspondence require that the total spinorial quantum numbers at the level of H are identical with the total classical Noether charges at the level of X^4?

3.5 Criticizing the zero-energy ontology

Also zero energy ontology (ZEO) deserves critical comments. CD is an intersection of future and past directed light-cones. Is the notion of CD really needed? Could mere truncated light-cone (half-CD) be enough for a given arrow of time assignable to a sequence of SSFRs increasing the size of the half-CD. In each SSFR the size of half-CD increases and one can say that CD is virtually present.

What would happen in "big" state function reduction (BSFR) [K17, L3] [L9, L11]? In order not to lose information, it is required that the new half-CD and the old half-CD combine to form a full CD. At the level of H, the common base of the half-CDs would naturally serve as the geometric counterpart for the moment of the BSFR event. The new past-directed CD begins to increase in size by SSFRs.

What happens to the CD on the O side? Since momenta are in question, CD is replaced with truncated double light-cone. The half-CD of H would be mapped to either future or past directed half-CD in Q. In BSFR this half-CD would change its direction. Virtual CD would be thus replaced with the virtual truncated double light-cone.

4 Problems related to Poincare invariance

TGD was motivated by the energy problem of the General Relativity, but the Poincare invariance is still a source of troubles.

4.1 Does the Hamilton-Jacobi and Kähler structures require a loss of Poincare invariance at the fundamental level?

The most natural option is that the HJ structure assignable to both the space-time surface and $M^4 \subset H$ emerges dynamically and defines a generalized complex structure for both M^4 and the space-time surface. M^4 Kähler structure would be equivalent with the dynamical HJ structure of M^4 and would not therefore appear in the H Dirac equation and in the second quantization of H spinor fields. The dynamical HJ structure would make possible the realization of holography as generalized holomorphy. Poincare invariance would not be a problem.

On the other hand, twistor lift led to the proposal that $M^4 \subset H$ has Kähler structure. If it is not dynamical, it violates Poincare symmetry at the level of Dirac equation of H. This option is not attractive.

1. For the dynamical option, the same flat M^4 metric serves as the background and Kähler structures and HJ structures would correspond to different local $M^2(x) \times E^2(x)$ decompositions in different complex coordinates defining the analog of the moduli space for complex structures.

2. In the 2-D case Kähler form and metric are numerically related in complex coordinates. The same should be true for the 4-D generalization based on self-dual Kähler form. The M^4 metric would still be Poincare invariant and the dynamically generated M^4 Kähler form would not break the Poincare invariance.

3. Note that the self-duality of M^4 Kähler structure (not present for the dynamical option) implies that M^4 Kähler action vanishes for canonically imbedded M^4 but is non-vanishing in the general case (contravariant induced metric replaces M^4 metric in M^4 part of the Kähler action).

The dynamically generated HJ structure, equivalent with M^4 Kähler structure, should have some natural interpretation.

1. The decomposition of the tangent space to longitudinal and transversal parts emerges in the description of the modes of gauge fields and HJ structure would generalize this decomposition by allowing polarization and local momentum direction to depend on position.

2. Holomorphic coordinates are defined by the HJ structure. Holomorphies are symmetries and give rise to conserved Noether charges. Complex translations are analogous to translations. It might be also possible to define the analogs of Lorentz transformations, at least the counterpart of a subgroup corresponding to the slicing $\{ M^2(x) \times E^2(x) \}$. Interestingly, infinitesimal boosts in M^2 and infitesimal rotations in E^2 correspond to a local choice of observables in quantum measurement.

3. The Equivalence Principle (EP)requires the identification of gravitational and inertial masses. Is EP a mere tautology or can one define gravitational and inertial 4-momenta such that EP would have a real content? Could one identify gravitational momenta as holomorphic Noether charges and inertial charges as Noether charges so that they need not be identical? Note that Poincare Noether charges can be defined even if they are not conserved as they would be if M^4 has a Kähler structure at the fundamental level.

The key questions, considered already earlier, are whether the HJ structure [L19] is identifiable as the M^4 Kähler structure and whether and how it is dynamically generated. I have earlier considered the possibility that M^8-H duality defines the HJ structure as a fixed Kähler structure.

1. M^8-H duality requires an integrable slicing of the local normal space of the 4-surface $Y^4 \subset O$ by quaternionic subsepaes 2-D commutative (complex in quaternionic sense) sub-spaces and their normal spaces. This allows to parametrize the normal space by a point of CP_2 .

2. The multiplication of the quaternionic normal space by a preferred octonionic unit $J(x)$ inducesa reflection taking the normal space to the tangent space of $M^4 \subset M^8_c$. This reflection maps the slicing of the normal space to a slicing of M^4 defining the dynamically generated Hamilton-Jacobi and Kähler structures.

The choice of $J(x)$ is defined only modulo a local $SU(2)$ rotation defining a local quaternionic automorphism and is therefore not unique. Is this a problem or something physically desirable?

1. Could one interpret the non-uniqueness as a number theoretic analog of the $SU(2)$ gauge invariance at the level of M_c^8. This brings in mind electroweak gauge invariance but the $U(1)$ factor of the electroweak gauge group is missing. One can still ask, whether the selection of $J(x)$ at the level M_c^8 could define a number theoretic and geometric realization of the fixing of weak gauge choice both at the level of $M^4 \subset M_c^8$ and $M^4 \subset H$.

2. Physically this would mean non-uniqueness of the local decomposition of the tangent space $M^2 \times E^2$ to longitudinal degrees of freedom defined by a light-like wave vector k and polarization degrees of freedom orthogonal to k. Gauge transformations indeed affect the polarization directions of plane waves by adding to the polarization vectors ϵ_i a term proportional to the wave vector k. This interpretation is not gauge group specific.

The slicing of M^4 would be forced by $M^8 - H$ duality and would be essential for the realization of holography as a generalized holomorphy. However, if one accepts Kähler structure of M^4 a the fundamental level, one can consider also holography based on surfaces of form $X^2 \times Y^2 \subset M^4 \times CP_2$ [L26]. X^2 *resp.* Y^2 would be Lagrangian rather than complex submanifolds of M^4 *resp.* CP_2.

In this case the complex structure would not be induced by the generalized holomorphy of the embedding but by the 2-dimensionality of X^2 and Y^2 and would be analogous to that in string models. In this case, one ends up with partial differential equations for X^2 and Y^2. It is not clear whether one should allow this realization of holography as holomorphy.

If one assumes exact Poincare invariance, this option is excluded as a solution of field equations. One could however allow solutions of field equations of form $X^2 \times Y^2 \subset M^4 \times CP_2$ if one allows the holography based on 2-dimensionality. If cosmological constant vanishes, the action reduces to Kähler action for CP_2, and X^2 and Y^2 would be minimal surfaces and Y^2 could be also Lagrangian sub-manifold, say homologically trivial geodesic sphere.

It is important to realize that for the Poincare invariant holography in its basic form, the M^4 projection of the space-time surface is 4-dimensional. Cosmic strings and CP_2 type extremals do not satisfy this condition.

I have proposed how the space-time regions having an M^4 projection with dimension lower than 4 emerge as singularities analogous to the singularity of electric field of a point charge. The quaternionic normal space of the surface $Y^4 \subset M_c^8$ would not be unique at the singularity and the normal spaces would be parameterized by a sub-manifold of CP_2 just as the directions of electric field of a point charge are parametrized by sphere S^2. For CP_2 type extremals all points of CP_2 would be involved. For cosmic strings the points would correspond to a geodesic sphere of CP_2.

If the Lagrangian surfaces $X^2 \times Y^2 \subset M^4 \times CP_2$ (with respect to dynamically generated M^4 Kähler form and symplectic structure) the generalized holomorphy would due to the 2-dimensionality of the Cartesian factors rather than holomorphic imbedding [L26]. It seems that one cannot interpret them as singularities of Lagrangian surfaces with 4-D M^4 projection.

4.2 The realization of Poincare invariance in the space of CDs

$M^8 - H$ duality defines a plane wave in the space of CDs. The CD, regarded in ZEO as a correlate for a perceptive field of a conscious entity, would become a representation of a particle at the level of H. The details of this realization are however not fixed.

Assuming Poincare invariance at the fundamental level, one can imagine two different ways to realize Poincare invariance.

1. Conformal weights could correspond to gravitational masses assumed to be identical with inertial masses. Inertial and gravitational momenta need not be identical. Classical momenta could correspond to either inertial momenta realized as ordinary Noether charges for isometries of M^4 or to gravitational momenta realized as conserved conformal Noether charges.

2. The inertial momentum could be identified as the eigenvalue of the total fermionic momentum for the quantum state or as the classical Noether charge assignable to the space-time surface.

The same applies to the gravitational momentum as a conformal Noether charge. Quantum classical correspondence requires that these two momenta are identical.

3. The 4-momentum assignable to CD would correspond to either inertial or gravitational 4-momentum. Since CDs are associated with H rather than X^4, the natural identification of the 4-momentum of CD would be as a conserved Noether charge for Poincare symmetries.

4.3 Conclusions

In this article, I have done my best to identify the weak points of the recent view of TGD with a special emphasis on M^8-H duality and Poincare invariance. It would seem that the most plausible answers to the questions posed in the introduction boil down to the following conclusions.

1. M^8_c-H duality relies on octonionic polynomials. There are strong physical and mathematical objections against it. If M^8 is identified as O with respect to number theoretic norm defined as $Re(o^2)$ and M^4 corresponds to the quaternionic normal space of Y^4, $O-H$ duality avoids these objections and it is a matter of taste whether one speaks of M^8-H duality or $O-H$ duality. The crucial role of polynomials in the number theoretic vision is not lost if the functions defining space-time surfaces in the holomorphic realization of holography form a hierarchy of pairs of polynomials. Number theoretic holography is induced by the holography at H side.

2. $O-H$ duality has equivalent interpretations as a generalization of momentum-position duality and the duality emerging in the twistor Grassmannian approach.

3. M^8-H duality in the fermionic sector is induced by the inversion [L21, L22, L17, L20, L18] and can be extended to assign to a given point of the mass shell containing fermion a plane wave in the space of CDs characterized by the corresponding momentum. Quantum classical correspondence suggests that the classical Noether momentum is identical with the eigenvalue of the fermionic momentum operator.

4. Induced Dirac action can give rise to gravitational momentum as conformal charge for which mass squared equals to inertial mass squared. Mass squared is the difference of two conformal weights corresponding to $M^2(x)\times E^2(x)$ decomposition of the tangent space. This resolves the longstanding problem related to p-adic mass calculations requiring tachyonic conformal weights for the ground state. The notion of induction for spinor fields generalizes to the M^8 level.

5. Poincare invariance is exact if the Kähler structure of $M^4\subset H$ is dynamically generated by M^8-H duality, which requires a decomposition of the normal space Q of $Y^4\subset O$ to complex subspaces $C(x)$ and its complement $CC(x)$. Here the realization of the generalized holomorphy by a holomorphic embedding is essential. The slicing of Q and M^4 X^4 induced in this way is not unique and has interpretation in terms of $SU(2)$ gauge invariance.

6. M^8-H duality realizes position momentum duality only semi-classically and can be extended so that it assigns to a given momentum a plane waves in the space of CDs and one can speak of a generalized Fourier transform.

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