McKay Correspondence from Quantum Arithmetics Replacing Sum and Product with Direct Sum and Tensor Product?

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Abstract

This article deals with two questions.

1. The ideas related to topological quantum computation suggests that it might make sense to replace quantum states with representations of the Galois group or even the coefficient space of state space with a quantum analog of a number field with tensor product and direct sum replacing the multiplication and sum.

   Could one generalize arithmetics by replacing sum and product with direct sum $\oplus$ and tensor product $\otimes$ and consider group representations as analogs of numbers? Or could one replace the roots labelling states with representations? Or could even the coefficient field for state space be replaced with the representations? Could one speak about quantum variants of state spaces?

   Could this give a kind of quantum arithmetics or even quantum number theory and possibly also a new kind of quantum analog of group theory. If the direct sums are mapped to ordinary sums in quantum-classical correspondence, this map could make sense under some natural conditions.

2. McKay graphs (quivers) have irreducible representations as nodes and characterize the tensor product rules for the irreps of finite groups. How general is the McKay correspondence relating these graphs to the Dynkin diagrams of ADE type affine algebras? Could it generalize from finite subgroups of $SL(k, C)$, $k = 2, 3, 4$ to those of $SL(n, C)$ at least. Is there a deep connection between finite subgroups of $SL(n, C)$, and affine algebras. Could number theory or its quantum counterpart provide insights to the problem?

In the TGD framework $M^8 - H$ duality relates number theoretic and differential geometric views about physics: could it provide some understanding of this mystery? The proposal is that for cognitive representations associated with extended Dynkin diagrams (EEDs), Galois group Gal acts as Weyl group on McKay diagrams defined by irreps of the isotropy group Gal_I of given root of a polynomial which is monic polynomial but with roots replaced with direct sums of irreps of Gal_I. This could work for p-adic number fields and finite fields. One also ends up with a more detailed view about the connection between the hierarchies of inclusion of Galois groups associated with functional composites of polynomials and hierarchies of inclusions of hyperfinite factors of type II_1 assignable to the representation of super-symplectic algebra.

1 Introduction

This article deals with two questions.

1. The ideas related to topological quantum computation [L13] suggests that it might make sense to replace quantum states with representations of the Galois group or even the coefficient space of Hilbert space with a quantum analog of a number field with tensor product and direct sum replacing the multiplication and sum. I have considered this kind of idea already earli [K7].

   Could one generalize arithmetics by replacing sum and product with direct sum $\oplus$ and tensor product $\otimes$ and consider group representations as analogs of numbers? Could one replace the roots labelling states with group representations? Or could even the coefficient field for the state space be replaced with a ring of representations? Could one speak about quantum variants of state spaces?

   Could this give a kind of quantum arithmetics or even quantum number theory and possibly also a new kind of quantum analog of group theory. If the direct sums are mapped to ordinary sums of algebraic numbers in quantum-classical correspondence interpreted as a kind of category theoretic morphism, this map could make sense under some natural conditions.

2. McKay graphs (quivers) have irreducible representations as nodes and characterize the tensor product rules for the irreps of finite groups. How general is the McKay correspondence relating these graphs to the Dynkin diagrams of ADE type affine algebras? Could it generalize from finite subgroups of $SL(k, C)$, $k = 2, 3, 4$ [A6, A5] to those of $SL(n, C)$. Is there a deep connection between finite subgroups of $SL(n, C)$, and affine algebras. Could number theory or its quantum counterpart provide insights to the problem?

1.1 Could one generalize arithmetics by replacing sum and product with direct sum and tensor product?

In the model for topological quantum computation (TQC) [B2, B1] quantum states in the representations of groups are replaced with entire representations (anyons). One can argue that this helps to guarantee stability: this generalization could be regarded as error correction code. In TGD, these representations would correspond to irreps of Galois groups or of discrete subgroups of the covering group for automorphisms of quaternions. Also discrete subgroups of $SL(2, C)$ assignable naturally to the tessellations of H^3 can be considered.

Tensor product $\otimes$ and direct sum $\oplus$ are commutative operations and very much like operations of ordinary arithmetics. One can also speak of positive integer multiples of representation. The algebras of irreps of various algebraic structures generated by $\oplus$ and $\otimes$ are applied quite generally in mathematics and especially so in gauge theories and conformal field theories and are known as fusion algebras (https://cutt.ly/TLU3hvJ) and quivers (https://cutt.ly/xLU3zrM).

Could the replacement of the roots of the EDD of the ADE group with representations of the finite subgroup of $SL(2, C)$ associated with the diagram make sense? The trivial representation would correspond to an additional node and lead to an extended Dynkin diagram (EDD).

Could one regard the irreps as quantum roots of an ordinary monic polynomial so that the ordinary algebraic numbers would have representation as state spaces? Could one obtain the full root diagram by a generalization of the Weyl group operation as reflection of root with respect to root? The first guess is that the isotropy group Gal_I of a root acts as a subgroup of Gal defines the polynomial, which gives the roots replaced by irreps and that Gal itself acts in the same role as the Weyl group.

McKay graph characterizes the rules for the tensor product compositions for the irreps of a finite group G, in particular Galois group. There is an excellent description of McKay graphs on the web (see https://cutt.ly/zLzoAwF). The article describes first the special McKay graphs for finite subgroups of $SL(2, C)$ and their geometric interpretation in terms of the geometry of Platonic solids and their denerate versions as regular polygons and shows that they turn out to correspond to EDDs for ADE type Lie algebras. Also general McKay graphs are considered.

1.2 McKay graphs and McKay correspondence

The McKay graphs are a special case of quiver diagrams (https://cutt.ly/xLU3zrM) and code for the tensor product decomposition rules for the irreps of finite groups [A9, A8].

For a general finite group, McKay graphs can be constructed in the following way. Consider any finite group G and its irreducible representations (irreps) ξ_i and assign to ξ_i vertices. Select one irrep V and assign also to it a vertex. For all tensor products $\xi_i \otimes V$ and decompose them to a direct sum of irreps ξ_j. If ξ_j is contained to $V \otimes \xi_i$ a_{ij} times, draw a_{ij} directed arrows connecting vertex i to vertex j. One obtains a weighted, directed graph with incidence matrix a_{ij}. Adjacency matrix plays a central role in graph theory.

McKay correspondence is only one of the mysteries related to MacKay graphs for finite subgroups of $SL(k, C)$, $k = 2, 3, 4$ and presumably also $k > 4$ [A6, A5]. The MacKay graphs correspond to EDDs for ADE type Lie groups having interpretations as Dynkin diagrams for ADE type affine algebras.

The classification of singularities of complex surfaces represents another example of McKay correspondence.

1. ADE Dynkin diagrams provide a classification of Kleinian singularities of complex surfaces having real dimension 4 and satisfying a polynomial equation $P(z_1, z_2, z_3) = 0$ with $P(0, 0, 0) = 0$ so that the singularity is at origin [A8] (https://cutt.ly/5LQPyhy). The finite subgroups of $SL(2, C)$ naturally appear as symmetries of the singularities at origin.

2. In the TGD framework, this kind of complex surfaces could correspond to surfaces with an Euclidean signature of induced metric as 4-surfaces in $E^2 \times CP_2 \subset M^4 \times CP_2$. What I call CP_2 type extremals have light-like M^4 projection as deformations of the canonically imbedded CP_2. These surfaces could correspond to deformations of CP_2 type extremals. One can ask whether one could assign ADE type affine algebras as affine algebras with these singularities.

2 Could the arithmetics based on direct sum and tensor product for the irreps of the Galois group make sense and have physical meaning?

The idea about the generalization of the mathematical structures based on integer arithmetics with arithmetics replacing $+$ and $\times$ with direct sum $\oplus$ and tensor product $\otimes$ raises a bundle of questions. This idea makes sense also for the finite subgroups of $SU(2)$ defining the covering group of quaternion automorphism having a role similar to that of the Galois group.

What motivates this proposal is that the extensions of rationals and their Galois groups are central in TGD. Polynomials P with integer coefficients are proposed to determine space-time surfaces by $M^8 - H$ duality in terms of holography based on the realization of dynamics in M^8 in terms of roots of P having interpretation as mass shells. Holography is realized in terms of the condition that the normal space of the space-time surface going through the mass shells has associative normal space [L7, L8].

2.1 Questions

The following questions and considerations are certainly very naive from the point of view of a professional mathematician and the main motivation for the mathematical self ridicule is that there are fascinating physical possibilities involved.

The basic question is whether $\otimes$ and $\oplus$ can give rise to quantum variants of rings of integers and even algebraic integers defined in terms of quantum roots of ordinary polynomial equations and could one even generalize the notion of number field: do quantum variants of extensions of rationals, finite fields, and p-adic number fields make sense?

Recall that also p-adic number fields and the adelic physics relying on the fusion of p-adic physics and real physics play a central role in TGD [L2, L3] [K6, K1, K2].

2.1.1 Quantum polynomials

To build extensions of rationals, one must have polynomials. The notion of polynomial playing central role in $M^8 - H$ duality [L7, L8], or rather the notion of a root of polynomial, generalizes.

1. Polynomials would look exactly like ordinary monic polynomials, with the real unit replaced with identity representation but their quantum roots would be expressible as direct sums of irreps associated with a given extension of rationals.

2. One would obtain roots as direct sums of the generators of the extension which could correspond to irreps of the isotropy group Gal_I of Galois group Gal. McKay graph would define the multiplication rules for the tensor products appearing in the polynomial whose coefficients would be quantum counterparts of ordinary (positive) integers.

3. Also a generalization of an imaginary unit could make sense for p-adic ring and finite fields as a root of a polynomial. Note that $\sqrt{-1}$ can exist for p-adic number fields. Also p-adic number fields and the adelic physics relying on the fusion of p-adic physics and real physics play a central role in TGD [L2, L3] [K6, K1, K2].

2.1.2 Does one obtain additive and multiplicative group structures, rings, and fields?

Could one give to the space spanned by irreps a structure of ring or even field?

1. Could one replace algebraic integers of the ordinary extension of rationals with direct sums of the n_C irreps of Galois group G, where n_C is the number of classes of G? Note that the dimensions n_i of irreps satisfy the formula $\sum n_i^2 = n_C$.

   If $\oplus$ corresponds to $+$ for ordinary integers, only non-negative integers can appear as coefficients so that one would have semigroups with respect to both $\oplus$ and $\otimes$.

2. The inverse with respect to $\oplus$ requires that negative multiples of quantum integers make sense. This is possible in p-adic topology: the number -1 would correspond to the quantum part of the integer $(p-1)\sum_{\oplus} p^{\oplus n}$. The summands in this expression would have p-adic norms p^{-n}. This allows to define also the negatives of other roots playing the role of generator of the quantum extension of rationals.

3. Is even the quantum analog of a number field possible? If one requires multiplicative inverse, only the finite field option remains under consideration since the quantum variant of $1/p^k$ does not make sense since one has $p \equiv= 0$. If one requires group structure for only $\oplus$, quantum p-adics remain under consideration.

2.1.3 Can one map the numbers of quantum extensions of rationals the numbers of ordinary extensions?

Concerning the physical interpretation, it would be important to map the quantum variants of algebraic integers to their real counterparts. Mathematicians might talk of some kind of category theoretical correspondence.

1. Since the same polynomial would have ordinary roots and quantum roots, the natural question is whether the quantum roots can be mapped to the ordinary roots.

2. If the quantum roots correspond to roots of the Dynkin diagram as quantum numbers in quantum extension of rationals, it should be able to map all quantum roots of the ADE type affine algebra to ordinary roots. This requires that sums with respect to $\oplus$ correspond to sums with respect to $+$: additivity of quantum numbers would hold true at both levels and one would have category theoretic correspondence as algebraic isomorphisms.

   Note that Galois confinement means that 4-momenta and other quantum numbers of states are integer valued, when one uses the momentum scale defined by causal diamond (CD). This means that they would correspond to $\oplus$ multiples of trivial representation of the Galois group acting as Weyl group.

3. What about the tensor products of roots appearing in the McKay graph? Can one require that the products with respect to $\otimes$ correspond to products with respect to $\times$. Only $\otimes$ does appear in the generation of the quantum roots of a given KM algebra representation.

   What about quantum variants of quantum states? If the quantum variants of p-adic integers or finite fields appear also as a coefficient field of quantum states, one can always express the coefficients as direct sums of quantum roots and map these sums to sums of ordinary polynomial roots, that is algebraic numbers. Extensions of rationals can appear as coefficient fields for Hilbert spaces.

   If one assumes that only quantum variants of p-adic numbers with a finite number of the pinary digits and their negatives are possible, they can be mapped to numbers in algebraic extension. One could overcome the problems related to the definition of inner product when finite field or p-adic numbers define the coefficient field for Hilbert state.

4. For generalized finite fields, the notions of vector space and matrix algebra, hermiticity and unitarity, and eigenvalue problem could be generalized. For instance, eigenvalues of a Hermitian operator could be just real numbers. Also a relatively straightforward looking generalization of group theory can be imagined, and would be obtained by replacing the elements of the matrix group with the elements of a generalized finite field.

2.2 Could the notion of quantum arithmetics be useful in the TGD framework?

These ideas might find an application in TGD.

1. The quantum generalization of the notion of rationals, p-adic number fields, and finite fields could be defended as something more than a mere algebraic game. In particular, in TGD the ramified primes of extension of rationals correspond to physically important p-adic primes,

especially the largest ramified prime of the extension. Algebraic prime is a generalization of the notion of ordinary prime. Also its generalization could make sense and give rise to the notion of quantum prime.

Unfortunately, the extension of finite field F_p induced by a given extension of rationals does not exist for the ramified primes appearing as divisors in the discriminant determined by the product of root differences.

Could the generalization of the notion of finite field save the situation? Topological quantum computations (TQC) relying on Galois representations as counterparts for anyons would mean an increase of the abstraction level replacing numbers of algebraic extension with representations of Galois group as their cognitive representations.

One can assign also to the possibly unique monic polynomial P_c defining the n_c-dimensional extension, a discriminant, call it D_c. For the primes dividing the discriminant D of P but not D_c, the quantum counterpart of the finite-field extension could make sense.

2. In TGD, the roots of polynomials define 3-D mass and energy shells in M^8 in turn defining holographic data defining 4-D surface in M^8 mapped to space-time surfaces in H by M^8-H duality. Could one consider quantum variants of the polynomial equations defining space-time surfaces by holography in the generalized extensions of rationals based on representations of Galois groups?

   Could monic polynomials define quantum variants of 4-surfaces or at least of discretizations of hyperbolic spaces H^3 as 3-D sections of 4-surface in M^8 defined as roots of polynomial P and containing holographic data as cognitive representation? Mass shells would be mapped by M^8-H duality to light-cone proper time hyperboloids in H.

   The interiors of 4-surfaces in M^8 would contain very few points of cognitive representation as momentum components in the extension of rationals defined by the polynomial P. Mass shells and their H images would be different and represent a kind of cognitive explosion. The presence of fermions (quarks) at the points of cognitive representation of given mass shells would make them active.

3. Could the transition from the classical to a quantum theory, which also describes cognition, replace discrete classical mass shells as roots of a polynomial in M^8 with roots with direct sums of irreps of the Galois group?

   This idea would conform with category theoretic thinking which leaves the internal structure of the basic object, such as point, open. That points of cognitive representations would be actually irreducible representations of the Galois groups would reveal a kind of cognitive hidden variables and quantum cognition.

These ideas are now completely new. I have earlier considered the possibility that points could have an infinite complex internal structure and that the "world of classical worlds" could be actually M^8 or H with points having this structure [K9]. I have also considered the possibility that Hilbert spaces could have arithmetic structure based on $\otimes$ and $\oplus$ with Hilbert spaces with prime dimension defining the primes [K7].

"Do not quantize" has been my motto for all these years but in this framework, it might be possible to talk about quantization of cognition as a deformation of number theory obtained by replacing $+$ and $\times$ with $\oplus$ and $\otimes$ and ordinary numbers with representations of Galois group. Perhaps this quantization could apply to cognition.

3 What could lurk behind McKay correspondence?

The appearance of EDDs in so many contexts having apparently no connection with affine algebras is an almost religious mystery and one cannot avoid the question of whether there is a deep connection between some finite groups G, in particular finite subgroups of $SL(n,C)$, and affine algebras. In the TGD framework M^8-H duality relates number theoretic and differential geometric views about physics and the natural question whether it could provide some understanding of this mystery.

$M^8 - H$ duality also suggests how to understand the Langlands correspondence: during years I have tried to understand Langlands correspondence [A4, A3] from the TGD perspective [K3, K4].

3.1 McKay correspondence

There is an excellent article of Khovanov [A9] describing the details of McKay correspondence for the discrete subgroups of $SL(2, C)$ (`https://cutt.ly/1LQDqce`). There is also an article "McKay correspondence" by Nakamura about various aspects of McKay correspondence [A8] (`https://cutt.ly/5LQPyhy`).

1. Consider finite subgroups G of $SL(2, C)$. The McKay graph for the tensor products of what is called canonical (faithful) 2-D representation V of G with irreps ξ_i of G corresponds to an extended Dynkin diagram with one node added to a Dynkin diagram. Note that V need not be always irreducible.

   The constraints on the graph come from the conditions for the dimension $d = 2d_j$ of the tensor product $V \otimes \xi_i$ satisfies $2d_i = \sum_j a_{ij} d_j$, where the sum is over all vertices directed away from the vertex i. If arrows in both directions are present, there is no arrow. This implies that the dimensions d_j associated with the vertex have G.C.D equal to 1.

2. Dynkin diagram in turn describes the minimal set of roots from which the roots of Lie algebra can be generated by repeated reflections with respect to roots. EDDs can be assigned to affine algebras and for them the eigenvalues of the adjacency matrix are not larger than 2. The maximum of the eigenvalues measures the complexity of the graph.

3. The Weyl group characterizes the symmetries of the root diagram and is generated by reflections of roots with respect to other roots. The Dynkin diagram contains a minimal number of roots needed to generate all roots by reflections as Weyl orbits of the roots of the Dynkin diagram. The action of the Weyl group leads away from the Dynkin diagram since otherwise this set of roots would not be minimal.

   The number of lines characterizes the angle between the roots i and j. For ADE groups $a_{ij} = 1$ codes for angle of 120 degrees $2\pi/3$, $a_{ij} = 2$ corresponds to 135 degrees, and $a_{ij} = 3$ to 150 degrees. $a_{ij} = 0$ means either angle π or $\pi/2$. In the general case, there are 2-valent and 3-valent nodes depending on the number of oriented lines emerging from the node.

   For instance, in the case of a triangle group with 6 elements with irreps $1, 1_1, 1_2$. The canonical representation to 2-D reducible representation decomposes to $1_1 + 1_2$ so that there are 3 vertices involved corresponding to 1_1 and 1_2 and 1. It is easy to see that the adjacency matrix is symmetric and gives rise to an EDD with 3 vertices. From the corresponding Dynkin diagram, representing 2 neighboring roots of the root diagram one obtains the entire root diagram by repeated reflections having 6 roots characterizing the octet representation of A_2 ($SU(3)$).

4. What kind of McKay graphs are associated with other than canonical 2-D representations in the case of rotation groups? Every representation of G belongs to some minimal tensor power $V^{\otimes k}$ and one can study the MacKay diagrams assignable to $V^{\otimes k}$. It is easy to see that the number of paths connecting vertices i and j in the McKay graph $M^k(V)$ for $V^{\otimes k}$ can be understood in terms of the McKay graph $M(V)$ for V. The paths leading from i to j are all k-edged paths along $M(V)$ leading from i to j.

   The symmetry of the adjacency matrix A implies that forth and back movement along $M(V)$ is possible. The adjacency matrix has the same number of nodes and equals the $k : th$ power A^k of A so that extended ADE type Dynkin diagrams are not in question.

3.2 Questions

McKay correspondence raises a series of questions which I have discussed several times from the TGD point of view several times [L1, L6, L5]. In the following these questions are discussed by introducing the possibility of quantum arithmetics and cognitive representations as new elements.

3.2.1 Why would $SL(2,C)$ be so special?

$SL(2,C)$ is in a very special role in McKay correspondence. Of course, also the finite subgroups of other groups could have a special role and it is actually known that $SL(n,C)$ $n<5$ are in the same role, which suggests that all groups $SL(n,C)$ have this role [A6, A5].

Why? In the TGD framework, a possible reason for the special role of $SL(2,C)$ acts as the double covering group of the isometries of the mass shell $H^3 \subset M^4 \subset M^8$ and its counterpart in $M^4 \times CP_2$ obtained by $M^8 - H$ correspondence. $SL(2,C)$ has also natural action on the spinors of H. The finite subgroups relate naturally to the tessellations of the mass shell H^3 leaving the basic unit of tessellation invariant.

The tessellations could naturally force the emergence of ADE type affine algebras as dynamical symmetries in the TGD framework. In fact, the icosa-tetrahedral tessellation plays a key role in the proposed model of the genetic code based on Hamiltonian cycles at icosahedron and tetrahedron [L9].

3.2.2 Why does the faithful representation have a special role?

The mathematical reason for the special role of the faithful canonical representation V is that its tensor powers contain all irreps of the finite group: the tensor product structure for other choices of V can be deduced from that for canonical representations. It is known that any irrep V, which is faithful irrep of G, generates the fusion algebra.

However, this kind of irrep might fail to exist. If G has a normal subgroup H and the irrep χ has H as kernel then the powers of χ contain only the irreps of G/H. In the article "McKay Connectivity Properties of McKay Quivers" by Hazel Brown [A7] (`https://arxiv.org/pdf/2003.09502.pdf`) it was shown that the number of connected components of the McKay quiver is the number of classes of the G, which are contained in H. For instance, the classes associated with the center of G are such (Z_n for $SL(n,C)$).

For simple groups this does not happen but in the case of Galois groups assignable to composite polynomials one has a hierarchy of normal subgroups and this kind of situation can occur since the number of classes of G contained in normal subgroups can be non-vanishing.

2-D representation is also in a special role physically in the TGD framework, the ground states of affine representation correspond to a 2-D spinor representation since quarks are the fundamental particles.

The irreps of the affine representation are obtained as tensor products of the irrep associated with the affine generators with it. Cognitive representations imply a unique discretization and this forces discrete subgroups of $SL(2,C)$ and implies that the irreps of $SL(2,C)$ decompose to irreps of a discrete subgroup. Therefore the quivers for their tensor products appear naturally.

Electroweak gauge group $U(2)$ corresponds to the holonomy group $U(2)$ for CP_2 and for $SU(2)_w$ the McKay correspondence holds true. Also the isometry group $SU(3)$ of CP_2 is assumed to appear as affine algebra. Discretization due to cognitive representations in M^8 induces discretization in H and CP_2. The replacement of $SU(3)$ with its discrete subgroups would decompose irreps for $SU(3)$ to irreps of $SU(3)$. $SL(3,C)$ allows analog of McKay correspondence [A6] so that also the finite subgroups of $SU(3)$ allow it.

3.2.3 What about McKay graphs for more general finite groups?

The obvious question concerns the generality of McKay correspondence. What finite groups and therefore corresponding Galois groups correspond to representations of affine type algebras.

In the general case, the McKay graphs look very different from Dynkin diagrams. The article "Spectral measures for G_2" of Evans and Pugh [A2] (`https://cutt.ly/hLQO7HE`) is of special interest from the TGD point of view since G_2 is the automorphism group of octonions. G_2 however naturally reduces to $SU(3)$ corresponding to color isometries in H. The article discusses in detail McKay graphs for the finite subgroups of G_2. These finite subgroups correspond to those for $SU(2) \times SU(2)$ and $SU(3)$ plus 7 other groups. The McKay graphs for the latter groups contain loops are very complex and contain loops.

What can one say about finite groups, which allow McKay correspondence.

1. ADE diagrams are known to classify the following three finite simple groups, the derived group F'_{24} of the Fischer F_{24}, the Baby monster B and the Monster M are related with E_6, E_7 and E_8 respectively [A8] (https://cutt.ly/5LQPyhy). In the TGD framework, this finding inspires the question whether these groups could appear as Galois groups of some polynomial and give rise to E_6, E_7 and E_8 as dynamical symmetries.

   In the TGD framework, one can ask whether also the above mentioned simple groups could appear as Galois groups. What is fascinating that monster would relate to icosahedron and dodecahedron: icosahedron and tetrahedron play key role in TGD inspired model of genetic code, in particular in the proposal that it relates to tetra-icosahedral tessellation of hyperbolic space H^3 [L9].

2. The article [A10](https://cutt.ly/jLQPgkQ) mentioned the conjecture that the tensor product structure for the finite subgroups of $SU(3)$ could relate to the integrable characters for some representations of affine algebra associated with $SU(3)$. This encourages the conjecture that this is true also for $SU(n)$.

   In TGD, this inspires the question whether finite Galois groups representable as subgroups of $SU(3)$ could give rise to corresponding affine algebras as dynamical symmetries of TGD.

3. Butin and Perets demonstrated McKay correspondence in the article "Branching law for finite subgroups of $SL(3,C)$ and McKay correspondence" [A6] (https://cutt.ly/CLQPvp2) for finite subgroups of $SL(3,C)$ in the sense that branching law defines a generalized Cartan matrix. In the article "Branching Law for the Finite Subgroups of SL(4,C) and the Related Generalized Poincare Polynomials" [A5] (https://cutt.ly/mLQPQnT) shows that the same result holds true for $SL(4,C)$, which suggests that it is true for all $SL(n,C)$.

   A generalization to finite subgroups of $SL(n,C)$ is a natural guess. Therefore Galois groups with this property could be assigned with affine algebras characterized by the generalized Cartan matrices and could correspond to physically very special kind of extensions of rationals,

3.3 TGD view about McKay correspondence

The key idea is that one replaces quantum numbers representable as sums of the roots of Lie algebra with representations of the isotropy group of Galois group which is same as a finite subgroup of say $SL(2,C)$ and that Galois groups acts as Weyl group. The Weyl group codes for the differential geometric notion of symmetry realized by Lie groups and Galois group codes for the number theoretic view of symmetry. This correspondence would represent a facet of the duality between number theory and differential geometry.

3.3.1 Quantum roots as direct sums of irreps

Consider first the correspondence between quantum roots (or more generally weights defined as dual space of roots) and ordinary roots (weights) as quantum numbers.

1. The representations of finite group G (say subgroup of $SL(2,C)$) represented by the isotropy group Gal_I of Galois group for a given root, would appear as labels of states rather than as counterparts of states. Galois group Gal itself would act as Weyl group on the roots.

2. Quantum numbers as labels of quantum states would be replaced with representations of Gal_I. The additivity of quantum numbers would correspond to the additivity of representations with respect to $\oplus$. Tensor product for the representations would be analogous to multiplication of quantum quantum numbers so that they would form an algebra. An abstraction or cognitive representation would be in question. Since the roots of the Dynkin diagram correspond to roots of a monic polynomial, one could map them to ordinary algebraic numbers. Same applies to the root of affine representations.

3.3.2 Could also the quantal version of the coeffient field of the state space make sense?

Could also the coefficient field of state space be replaced with a quantum variant of p-adic numbers or of finite field?

1. Here one encounters a technical problem that is encountered already at the level of ordinary p-adics and finite fields. Inner products are bilinear. If norm squared is defined as a sum for the squares of the coefficients of the state in the basis of n states, the non-well-ordered character of p-adics implies that one can have states for which this sum vanishes in p-adic and finite fields.

   In the p-adic case, allowance of only finite number of non-vanishing binary digits for the coefficients might help and would conform with the idea about finite measurement resolution as a pinary cutoff. One could even allow negatives of integers with finite number of pinary digits if the p-adic quantum integers are mapped to the real counterparts.

2. There is also a problem associated with the normalization factors of the states, which cannot be p-adic integers in general. Overall normalization does not however matter so that this problem might be circumvented.

   Physical predictions would require the map of the quantum integers to real ones. The fact that quantum integers are $\oplus$ sums of quantum roots of ordinary monic polynomials, makes this possible. The irreps appearing as coefficients of states would be mapped to ordinary algebraic numbers and the normalization of the states could be carried out at the level of the ordinary algebraic numbers.

3.3.3 What about negative multiples of quantum roots

If the quantum roots of a polynomial correspond to irreps of the Galois group, one encounters a technical problem with negative multiples of quantum roots.

1. The negatives of positive roots correspond to -1 multiples of irreps. This does not make sense in ordinary arithmetics. p-Adically -1 corresponds to $(p-1)(1+p+p^2+...)$ and would correspond to infinite $\oplus$-multiple of root but decompose to p^n multiples to which one can assign norm p^{-k} so that the sum converges: $-\xi_i = (p-1)(Id \oplus pId \oplus p^2Id \oplus ...)\xi_i$.

   One has finite measurement resolution so that the appearance of strictly infinite sums is highly questionable. Should one consider only finite sums of positive roots and their negatives but how should one deal with the negatives?

   Could the creation operators labelled by negative roots correspond to annihilation operators with positive roots as in the case of super-Virasoro and affine algebras. Note that if one restricts to ordinary integers at the level of algebra as one must to for supersymplectic and Yangian algebras, one must consider only half-algebras with generators, which have only non-negative conformal weights. This does not make sense for ordinary affine generators.

2. The most plausible solution of the problem relies on the proposed categorical correspondence between quantum roots and ordinary roots as roots of the same monic polynomial. One could map the quantum roots and their direct summands to sums of ordinary roots and this would make sense also for the negatives of positive roots with a finite number of summands. It would be essential that p-adic integers correspond to finite ordinary integers and to their negatives and are mapped to numbers in an extension of rationals. As found, this map would also allow us to circumvent the objections against the quantum variant of the state space.

3. Could zero energy ontology (ZEO) come to the rescue? In zero energy ontology creation and annihilation operators are assigned with the opposite boundaries of causal diamond (CD). Could one assign the negative conformal weights and roots with the members of state pairs located at the opposite boundary of CD?

   This works for the Virasoro and affine generators but this kind of restriction is unphysical in the case of eigenvalues of L_z with both signs? Why would opposite values of L_z be assigned to opposite boundaries of CD?

3.3.4 Wheels and quantum arithmetics

Gary Ehlenberg gave a link to a Wikipedia article telling of Wheel theory (`https://cutt.ly/RZnUB5y`). Wheel theory could be very relevant to the TGD inspired idea about quantum arithmetics.

I understood that Wheel structure is special in the sense that division by zero is well defined and multiplication by zero gives a non-vanishing result. The wheel of fractions, discussed in the Wikipedia article as an example of wheel structure, brings into mind a generalization of arithmetics and perhaps even of number theory to its quantum counterpart obtained by replacing + and - with direct sum $\oplus$ and tensor product $\otimes$ for irreps of finite groups with trivial representation as multiplicative unit: Galois group is the natural group in TGD framework.

Could wheel structure provide a more rigorous generalization of the notions of the additive and multiplicative inverse of the representation in order to build quantum counterparts of rationals, algebraic numbers and p-adics and their extensions?

1. One way to achieve this is to restrict consideration to the quantum analogs of finite fields $G(p,n)$: + and x would be replaced with $\oplus$ and $\otimes$ obtained as extensions by the irreps of the Galois group in TGD picture. There would be quantum-classical correspondence between roots of quantum polynomials and ordinary monic polynomials.

2. The notion of rational as a pair of integers (now representations) would provide at least a formal solution of the problem, and one could define non-negative rationals.

   p-Adically one can also define quite concretely the inverse for a representation of form $R = 1 \oplus O(p)$ where the representation $O(p)$ is proportional to p (p-fold direct sum) as a geometric series.

3. Negative integers and rationals pose a problem for ordinary integers and rationals: it is difficult to imagine what direct sum of -n irreps could mean.

   The definition of the negative of representation could work in the case of p-adic integers: $-1 = (p-1) \otimes (1 \oplus p * 1 \oplus p^2 * 1 \oplus ...)$ would be generalized by replacing 1 with trivial representation. Infinite direct sum would be obtained but it would converge rapidly in p-adic topology.

4. Could $1/p^n$ make sense in the Wheel structure so that one would obtain the quantum analog of a p-adic number field? The definition of rationals as pairs might allow this since only non-negative powers of p need to be considered. p would represent zero in the sense of Wheel structure but multiplication by p would give a non-vanishing result and also division with p would be well-defined operation.

3.3.5 Galois group as Weyl group?

The action of the Weyl group as reflections could make sense in the quantum arithmetics for quantum variants of extensions of p-adics and finite fields. The generalized Cartan matrix $C_{ij} = d\delta_{ij} - n_{ij}$, where n_{ij} is the number of lines connecting the nodes i and j and d is the dimension of V, is indeed well-defined for any finite group and has integer valued coefficients so that Weyl reflection makes sense also in quantum case.

Can one identify the Weyl group giving the entire root diagram number theoretically? The natural guess is $Gal = W$: Gal would define the Weyl group giving the entire root diagram from the Dynkin diagram by reflections of the roots of the EDD. One can assign to Gal an extension defined by a monic polynomial P with Galois group Gal.

3.3.6 How the group defining the McKay graph is represented?

How the group G defining the McKay graph is represented? The irreps of G should have natural realization and the quarks at mass shells would provide these representations.

One can consider two options. The first option is based on the isotropy group G_I of $Gal = W$ leaving a given root invariant. Second option is based on the finite subgroup of $SU(2)$ as a covering group of quaternion automorphisms.

1. The subgroup $Gal_I \subset Gal$ acting as an isotropy group of a given root of Gal would naturally define the EDD since the action of $Gal = W$ would not leave its nodes as irreps of Gal_I invariant.

   The root diagram should be the orbit of the EDD under $Gal = W$. The irreps of the EDD would correspond to the roots of a monic polynomial P_I associated with Gal_I and having $n_c + 1$ quantum roots. The quantum roots would be in the quantum extension defined by a monic polynomial P for Gal so that the action of Gal on EDD would be well-defined and non-trivial.

2. In the TGD framework, the mass mass squared values assignable to the monic polynomial representing the EDD correspond to different mass squared values. There is no deep reason for why the irreps of Gal_I could not correspond to different mass squared values and in the TGD framework the symmetry breaking $Gal \rightarrow Gal_I$ is the analog for the symmetry breaking in the Higgs mechanism.

   In the recent case this symmetry breaking would be associated with $Gal_I \rightarrow Gal_{I,I}$ and imply that quantum roots correspond to different mass squared values. At the level of affine algebra this could mean symmetry breaking since the different roots would not have different mass squared values.

   If Gal acts as a Weyl group, the McKay graph associated with Gal_I corresponds to the EDD. Gal_I is a subgroup of Gal so that the action of $Gal = Weyl$ on the quantum roots of the monic polynomial P_I would be non-trivial and natural. Could Gal_I be a normal subgroup in which case Gal/Gal_I would be a group and one would have a composite polynomial $P = Q \circ P_1$? This cannot be true generally: for instance for A_p, p prime and E_6 the W is simple. For E^7 and E^8 W is a semidirect product.

3. There is an additional restriction coming from the fact that Gal_I does not affect the rational parts of the 4-momenta. Is it possible to have construct irreps for a finite subgroup of $SL(2, C)$ or even $SL(n, C)$ using many quark states at a given mass shell? The non-rational part of 4-momentum corresponds to the "genuinely" virtual part of virtual momentum and for Galois confined states only the rational parts contribute to the total 4-momentum. Could one say that these representations are possible but only for the virtual states which do not appear as physical states: cognition remains physically hidden.

The very cautious, and perhaps over-optimistic conclusion, would be that only Galois groups, which act as Weyl groups, can give rise to affine algebras as dynamical symmetries. For this option, one would obtain cognitive representations for the isotropy groups of all Galois groups. For Galois groups acting as Weyl groups, EDDs could define cognitive representations of affine algebras. Also cognitive representations for finite subgroups of $SL(n, C)$ and groups like Monster would be obtained.

For the second option in which the subgroup G of quaternionic automorphisms affecting the real parts of 4-momenta is involved. This representation would be possible only for the subgroups of $SL(2, C)$. In this case one would have 3 different groups $Gal = W$, Gal_I and G rather than $Gal = W$ and Gal_I.

1. Quaternionic automorphisms are analogous to the Galois group and one can ask whether the finite subgroups G of quaternionic automorphisms could be directly involved with cognitive representations. This would give McKay correspondence for $SL(2, C)$ only. The quaternionic automorphism would affect the rational part of the 4-momentum in an extension of rationals unlike the Galois group which leaves it invariant. The irrep of G would be realized as many-quark states at a fixed mass shell. Different irreps would correspond to different masses having interpretation in terms of symmetry breaking.

2. Also now one would consider the extension defined by the roots of a monic polynomial P having Galois group $Gal = W$ associated with the corresponding EDD. P_I would give quantum roots defining the Dynkin diagram and define the mass squared values assignable to irreps of G.

3. The situation would differ from the previous one in that the action of G_I on irreps would be replaced by the action of G. Indeed, since G_I leaves the rational part of the 4-momentum invariant, G_I cannot represent G as a genuine subgroup of rotations.

4. The roots would correspond to irreps of a subgroup G of quaternionic automorphisms, which would affect the 4-momenta with a given mass shell and define an irrep of G. Different roots of P would define the mass shells and irreps of G associated with EDD as a McKay graph.

3.3.7 Information about Weyl groups of ADE groups

The Wikipedia article about Coxeter groups (`https://en.wikipedia.org/wiki/Coxeter_group#Properties`), which include Weyl groups, lists some properties of finite irreducible Coxter groups and contains information about Weyl groups. This information might be of interest in the proposed realization as a Galois group.

- $W(A_n) = S_{n+1}$, which is the maximal Galois group associated with a polynomial of degree $n+1$.
- $W(D_n) = Z_2^{n-1} \rtimes S_n$.
- $W(E_6)$ is a unique simple group of order 25920.
- $W(E^7)$ is a direct product of a unique simple group of order 2903040 with Z_2.
- $W(E_8)$ acts as an orthogonal group for F_2 linear automorphisms preserving a norm in Ω/Z_2, where Ω is E_8 lattice (`https://mathoverflow.net/questions/230120/the-weyl-group-of-e8-versus-o-230130#230130`)
- $W(B_n) = W(C_n) = Z_2^n \rtimes S_n$.
- $W(F_4)$ is a solvable group of order 1152 , and is isomorphic to the orthogonal group $O_4(F_3)$ leaving invariant a quadratic form of maximal index in a 4-dimensional vector space over the field F_3.
- $W(G_2) = D_6 = Z_2 \rtimes Z_6$.

3.3.8 Candidates for symmetry algebras of WCW, inclusions of hyperfinite factors, and Galois groups acting as Weyl groups

TGD allows several candidates for the symmetry algebras acting in WCW. The intuitive guess is that the isometries and possibly also symplectic transformations of the light-cone boundary $\delta M^4_+ \times CP_2$ define isometries of WCW whereas holonomies of H induce holonomies of WCW.

1. In TGD, supersymplectic algebra SSA could replace affine algebras of string models.

2. By the metric 2-dimensionality of the light-cone boundary δM^4_+, one can assign to it an infinite-dimensional conformal group of sphere S^2 in well-defined sense local with respect to the complex coordinate z of S^2. These transformations can be made local with respect to the light-like coordinate r of δM^4_+. Also a S^2-local radial scaling making these transformations isometries is possible. This is possible only for M^4 and makes it unique.

   Whether SSA or this algebra or both act as isometries of WCW is not clear: see the more detailed discussion in the Appendix of [L11].

3. One can assign this kind of hierarchy also to affine algebras assignable to the holomies of H and Virasoro algebras and their super counterparts. The geometric interpretation of these algebras would be as analogs of holonomy algebras, which serve at the level of H as the counterparts of broken gauge symmetries: isometries would correspond to non-broken gauge symmetries.

All these algebras, refer to them collectively by A, define inclusion hierarchies of sub-algebras A_n with the radial conformal weights given by n-ples of the weights of A.

1. I have proposed that the hierarchy of inclusions of hyperfinite factors of type II_1 to which one could perhaps assign ADE hierarchy could correspond to the hierarchies of subalgebras assignable to SSA and labelled by integer n: the radial conformal weights would be multiples of n. Only non-negative values of n would be allowed.

2. For a given hierarchy A_n, one has $n_1 \mid n_2 \mid .....$, where $\mid$ means "divides". At the n:th level of the hierarchy physical states are annihilated by A_n and $[A_n, A]$. For isometries, the corresponding Noether charges vanish both classically and quantally.

3. The algebra A_n effectively reduces to a finite-D algebra and A_n would be analogous to normal subgroup, which suggests that this hierarchy relates to a hierarchy of Galois groups associated with composite polynomials and having a decomposition to a product of normal subgroups.

4. These hierarchies could naturally relate to the hierarchies of inclusions of hyperfinite factors of type II_1 and also to hierarchies of Galois groups for extensions of rationals defined by composites $P_n \circ P_{n-1} \circ ...P_1$ of polynomials.

The Galois correspondence raises questions.

1. Could the Dynkin diagrams for A_n be assigned to the McKay graphs of Galois groups acting as Weyl groups?

2. The Galois groups acting as Weyl group could be assigned to finite subgroups of $SU(2)$ acting as the covering group of quaternion automorphisms and of $SL(2, C)$ as covering group of H^3 isometries acting on tessellations of H^3. Also the finite subgroups of $SL(n, C)$ can be considered.

The proposed interpretation for the hierarchies of inclusions of HFFs is that they correspond to hierarchies for the inclusions of Galois groups defined by hierarchies of composite polynomials $P_n \circ ... \circ P_1$ interpreted as number theoretical evolutionary hierarchies.

If the relative Galois groups act as Weyl groups, they would be associated with the inclusions of HFFs naturally and the corresponding affine algebra (perhaps its finite field or p-adic variant) would characterize the inclusion. The proposed interpretation of the inclusion is in terms of measurement resolution defined by the included algebra. This suggests that a finite field version of the affine algebra could be in question.

This picture would suggest that hierarchies of polynomials for which the relative Galois groups act as Weyl groups are very special and could be selected in the number theoretical fight for survival.

One could argue that since number theoretic degrees of freedom relate to cognition, the quantum arithmetics for the irreps of Galois groups could make possible cognitive representations of the ordinary quantum states: roots would be represented by irreps. Irreps as quantum roots would correspond to ordinary roots as roots of the same monic polynomial and the direct sums of irreps would correspond to ordinary algebraic numbers.

3.3.9 About the interpretation of EDDs

An innocent layman can wonder whether the tensor products for 2-D spinor ground states for the discrete subgroups of the covering group of quaternionic automorphisms or of $SL(2, C)$ as covering group of H^3 isometries could give rise to representations contained by ADE type affine algebras characterized by the same EDD. These representations would be only a small part of the representations and perhaps define representation from which all states can be generated.

1. The reflections for the roots represented as irreps of Gal_I by Weyl group represented as Gal should assign to the irreps of G new copies so that the nodes of the entire root diagram would correspond to a set of representations obtained from the ground state. Infinite number of states labelled by conformal weight n is obtained.

2. Adjacency matrix A should characterize the angles between the roots represented as irreps? If the irreps of Gal_I and their Weyl images correspond to roots of a monic polynomial, they can be mapped to roots of an ordinary algebraic extension of rationals and the angles could correspond to angles between the points of extension regarded as vectors.

How the EDD characterizing the tensor products of the irreps of finite subgroups G with 2-D canonical representation V could define an ADE type affine algebra?

1. Roots are replaced with representations of G, which are in the general case direct sums of irreps. The identity representation should correspond to the scaling generator L_0, whose eigenvalues define integer value conformal weights.

   The inner products between the roots appearing in the Cartan matrix would correspond to the symmetric matrices defined by the structure constant n_{2ij} characterizing the tensor product. One might say that the inner products are matrix elements of the operator $\langle \xi_j | V \otimes \xi_i \rangle$ defined by the tensor product action of V. The diagonal elements of the Cartan matrix have value $+2$ and non-diagonal elements are negative integers or vanish.

2. Weyl reflections of roots with respect to roots involve negatives of the non-diagonal elements of Cartan matrix, which are negative so that the coefficient of the added root is positive represented as a direct sum. The negatives of the positive roots would correspond to negative integers and make sense only p-adically or for finite fields.

   The expression for the generalized Cartan matrix for McKay graph is known (`https://cuttly/QLRqrGt`) for the tensor products of representation with dimension d and multiplicities n^d_{ij} and is given by

   $$C^d_{ij} = d\delta_{ij} - n^d_{ij} \quad .$$

   For Dynkin diagrams the Cartan matrix satisfies additional conditions.

   Weyl reflection (`https://cutt.ly/kLRuXBP`) of the root v with respect to root α in the space of roots is defined as

   $$s_\alpha v = v - 2\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha \quad .$$

   where $(.,.)$ is the inner product in V, which now corresponds to extension of rationals associated with Gal.

   The Weyl chamber is identified as the set of points of V for which the inner products (α, v) are positive. The Weyl group permutes the Weyl chambers.

3. The root system would be obtained from the roots of the quantum Dynkin diagram by Weyl reflections (Galois group as Weyl group) with respect to other roots. The number N of these roots is $n = d_C + 1$,where d_C is the dimension of Cartan algebra of the Dynkin diagram. The number N_I of irreps is the same: $N = N_I$. The Cartan matrix defines metric in the roots so that the reflections are well-defined also in the generalized picture.

4. It would seem that one must introduce an infinite number of copies of the Lie algebra realized in the usual manner (in terms of oscillator operators) with copies labelled by the conformal weight n. The commutators of these copies would be like for an ordinary affine algebra. Only the roots as labels of generators and possibly also the coefficient field would be replaced with their quantum variants.

5. What about the realization of the scaling generator L_0, whose Sugawara representation involves bilinears of the generators and their Hermitian conjugates with negative conformal weight? In the case of finite fields there are no obvious problems. Also the analog of Virasoro algebra can be realized in the case of finite fields. If one restricts consideration to finite quantum integers and their negatives as conformal weights, the map of the roots to algebraic numbers in extension of rationals is well defined.

3.4 Could the inclusion hierarchies of extensions of rationals correspond to inclusion hierarchies of hyperfinite factors?

I have enjoyed discussions with Baba Ilya Iyo Azza about von Neumann algebras. Hyperfinite factors of type II_1 (HFF) (https://cutt.ly/lXp6MDB) are the most interesting von Neumann algebras from the TGD point of view. One of the conjectures motivated by TGD based physics, is that the inclusion sequences of extensions of rationals defined by compositions of polynomials define inclusion sequences of hyperfinite factors. It seems that this conjecture might hold true!

Already von Neumann demonstrated that group algebras of groups G satisfying certain additional constraints give rise to von Neuman algebras. For finite groups they correspond to factors of type I in finite-D Hilbert spaces.

The group G must have an infinite number of elements and satisfy some additional conditions to give a HFF. First of all, all its conjugacy classes must have an infinite number of elements. Secondly, G must be amenable. This condition is not anymore algebraic. Braid groups define HFFs.

To see what is involved, let us start from the group algebra of a finite group G. It gives a finite-D Hilbert space, factor of type I.

1. Consider next the braid groups B_n, which are coverings of S_n. One can check from Wikipedia that the relations for the braid group B_n are obtained as a covering group of S_n by giving up the condition that the permutations σ_i of nearby elements e_i, e_{i+1} are idempotent. Could the corresponding braid group algebra define HFF?

   It is. The number of conjugacy classes $g_i \sigma_i g_i^{-1}$, $g_i == \sigma_{i+1}$ is infinite. If one poses the additional condition $\sigma_i^2 = U \times 1$, U a root of unity, the number is finite. Amenability is too technical a property for me but from Wikipedia one learns that all group algebras, also those of the braid group, are hyperfinite factors of type II_1 (HFFs).

2. Any finite group is a subgroup G of some S_n. Could one obtain the braid group of G and corresponding group algebra as a sub-algebra of group algebra of B_n, which is HFF. This looks plausible.

3. Could the inclusion for HFFs correspond to an inclusion for braid variants of corresponding finite group algebras? Or should some additional conditions be satisfied? What the conditions could be?

Here the number theoretic view of TGD comes to rescue.

1. In the TGD framework, I am primarily interested in Galois groups, which are finite groups. The vision/conjecture is that the inclusion hierarchies of extensions of rationals correspond to the inclusion hierarchies for hyperfinite factors. The hierarchies of extensions of rationals defined by the hierarchies of composite polynomials $P_n \circ ... \circ P_1$ have Galois groups which define a hierarchy of relative Galois groups such that the Galois group G_k is a normal subgroup of G_{k+1}. One can say that the Galois group G is a semidirect product of the relative Galois groups.

2. One can decompose any finite subgroup to a maximal number of normal subgroups, which are simple and therefore do not have a further decomposition. They are primes in the category of groups.

3. Could the prime HFFs correspond to the braid group algebras of simple finite groups acting as Galois groups? Therefore prime groups would map to prime HFFs and the inclusion hierarchies of Galois groups induced by composite polynomials would define inclusion hierarchies of HFFs just as speculated.

   One would have a deep connection between number theory and HFFs.

4 Appendix: Isometries and holonomies of WCW as counterparts of exact and broken gauge symmetries

The detailed interpretation of various candidates for the symmetries of WCW [L4] has remained somewhat obscure. At the level of H, isometries are exact symmetries and analogous to unbroken gauge symmetries assignable to color interactions. Holonomies do not give rise to Noether charges and are analogous to broken gauge symmetries assignable to electroweak interactions. This observation can serve as a principle in attempts to understand WCW symmetries.

The division to isometries and holonomies is expected to take place at the level of WCW and this decomposition would naturally correspond to exact and broken gauge symmetries.

4.1 Isometries of WCW

The identification of the isometries of WCW is still on shaky ground.

1. In the H picture, the conjecture has been that symplectic transformations of δM^4_+ act as isometries. The hierarchies of dynamically emerging symmetries could relate to the hierarchies of sub-algebras (SSA$_n$) of super symplectic algebra SSA [L4] acting as isometries of the "world of classical worlds" (WCW) [K8] [L10].

   Each level in the hierarchy of subalgebras SSA$_n$ of SSA corresponds to a transformation in which SSA_n acts as a gauge symmetry and its complement acts as genuine isometries of WCW: gauge symmetry breaking in the complement generates a genuine symmetry, which could correspond to Kac-Moody symmetry. By Noether's theorem, the isometries of WCW would give rise to local integrals of motion: also super-charges are involved. These charges are well-defined but they need not be conserved so that the interpretation as dynamically emerging symmetries must be considered.

   The symmetries would naturally correspond to a long range order. The hierarchies of SSA$_n$:s, of relative Galois groups and of inclusions of hyperfinite factors [?, ?] could relate to each other as M^8-H duality suggests [L12].

   What can one say about the algebras SSA_n and the corresponding affine analogs KM_n (for affine algebras the generalized Cartan matrix is a product of a diagonal matrix with integer entries with a symmetric matrix). If n is prime, one can regard these algebras as local algebras in a finite field $G(p)$. Also extensions $G(p,n)$ of $G(p)$ induced by extensions of rationals can be considered. KM algebras in finite fields define what are called the incomplete Kac-Moody groups. Some of their aspects are discussed in the article "Abstract simplicity of complete Kac-Moody groups over finite fields" [A1]. It is shown that for $p>3$, affine groups are abstractly simple, that is, have no proper non-trivial closed subgroups. Complete KM groups are obtained as completions of incomplete KM groups and are totally disconnected: this suggests that they define p-adic analogs of Kac-Moody groups. Complete KM groups are known to be simple.

2. There are also different kinds of isometries. Consider first the light-cone boundary $\delta M^4_+ \times CP_2$ as an example of a light-like 3-surface. The isometries of CP_2 are symmetries. ΔM^4_+ is metrically equivalent with sphere S^2. Conformal transformations of S^2, which are made local with light-like coordinate r of δM^4_+, induce a conformal scaling of the metric of S^2 depending on r. It is possible to compensate for this scaling by a local radial scaling of r depending on S^2 coordinates such that the transformation acts as an isometry of δM^4_+.

   These isometries of ΔM^4_+ form an infinite-D group. The transformations of this group differ from those of the symplectic group in that the symplectic group of δM^4_+ is replaced with the isometries of δM^4_+ consisting of r-local conformal transformations of S^2 involving S^2-local radial scaling. There are no localizat of CP_2 isometries. This yields an analog of KM algebra.

   This group induces local spinor rotations defining a realization of KM algebra. Also super-KM algebra defined in terms of conserved super-charges associated with the modified Dirac action is possible. These isometries would be Noether symmetries just like those defined by SSA.

3. What about light-like partonic orbits analogous to $\delta M^4_+ \times CP_2$. Can one assign with them Kac-Moody type algebras acting as isometries?

   The infinite-D group of isometries of the light-cone boundary could generalize. If they leave the partonic 2-surfaces at the ends of the orbit X^3_L, they could be seen as 3-D general coordinate transformations acting as internal isometries of the partonic 3-surface, which cannot be regarded as isometries of a fixed subspace of H. These isometries do not affect the partonic 3-surface as a whole and cannot induce isometries of WCW.

   However, if X^3_L is connected by string world sheets to other partonic orbits, these transformations affect the string world sheets and there is a real physical effect, and one has genuine isometries. Same is true if these transformations do not leave the partonic 2-surfaces at the ends of X^3_L invariant.

4.2 Holonomies of WCW

What about holonomies at the level of WCW? The holonomies of H acting on spinors induces a holonomy at the level of WCW: WCW spinors identified as Fock states created by oscillator operators of the second quantized H spinors. This would give a generalized KM-type algebra decomposing to sub-algebras corresponding to spin and electroweak quantum numbers. This algebra would have 3 tensor-factors. p-Adic mass calculations imply that the optimal number of tensor factors in conformal algebra is 5 [K5]. 2 tensor factors are needed.

1. SSA would give 2 tensor factors corresponding to δM^4_+ (effectively S^2) and CP_2. This gives 5 tensor factors which is the optimal number of tensor factors in p-adic mass calculations [K5]. SSA Noether charges are well-defined but not conserved. Could SSA only define a hierarchy of dynamical symmetries. Note however that for isometries of H conservation holds true.

2. Also the isometries of δM^4 and of light-like orbits of partonic 2-surfaces give the needed 2 tensor factors. Also this alternative would give inclusion hierarchies of KM sub-algebras with conformal weights coming as multiples of the full algebra. The corresponding Noether charges are well-defined but can one speak of conservation only in the partonic case? One can even argue that the isometries of $\delta M^4_+ \times CP_2$ define a more plausible candidate for inducing WCW isometries than the symplectic transformations. p-Adic mass calculations conform with this option.

To sum up, WCW symmetries would have a nice geometric interpretation as isometries and holonomies. The details of the interpretation are however still unclear and one must leave the status of SSA open.

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