Does $\mathbb{M}^8 - \mathbb{H}$ duality reduce to local G_2 symmetry?

Matti Pitkänen

orcid:0000-0002-8051-4364.
email: matpitka6@gmail.com,
url: http://tgdtheory.com/public_html/,
address: Valtatie 8, as. 2, 03600, Karkkila, Finland.

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Abstract

The idea of $\mathbb{M}^8 - \mathbb{H}$ duality has progressed through frustratingly many twists and turns and I have discussed several variants of $\mathbb{M}^8 - \mathbb{H}$ duality. There are 2 options, call them $\mathbb{T}$ and $\mathbb{N}$: either the local tangent space $\mathbb{T}$ or normal space $\mathbb{N}$ of $\mathbb{Y}^4 \subset \mathbb{M}^8$ is quaternionic and contains a complex subspace $\mathbb{C}$. This makes it possible to map $\mathbb{Y}^4 \subset \mathbb{M}^8$ to the space-time surface $\mathbb{X}^4 \subset \mathbb{H} = \mathbb{M}^4 \times \mathbb{CP}_2$. Which of them or possibly both? Any integrable distribution of quaternionic normal spaces $\mathbb{N}$ is allowed whereas for tangent spaces this is not the case. This led to a too hasty rejection of the $\mathbb{T}$ option.

The second problem relates to the lack of the concrete realization of the $\mathbb{M}^8 - \mathbb{H}$ duality. Is the $\mathbb{M}^8 - \mathbb{H}$ duality between 4-D surfaces in $\mathbb{M}^8$ and space-time surfaces in $\mathbb{H}$ or is it enough that only the 3-D holographic data in $\mathbb{H}$ are fixed by $\mathbb{M}^8 - \mathbb{H}$ duality.

A modification of the original form of the $\mathbb{M}^8 - \mathbb{H}$ duality formulated in terms of a real octonion analytic functions $f(o) : \mathbb{O} - \mathbb{O}$ leads to a possible solution of these problems. All the conditions $f(o) = 0$, $f(o) = 1$ and $Imf)(o) = 0$, and $Ref)(o) = 0$ are invariant under local G_2 and the local G_2 acts as a dynamical spectrum generating symmetry group since $f \circ g_2 = g_2 \circ f$ holds true. The task reduces to that of finding 4-surfaces with constant quaternionic normal space $\mathbb{N}$ or tangent space $\mathbb{T}$. $\mathbb{Y}^4 = \mathbb{E}^4 \subset \mathbb{M}^8$ and $\mathbb{Y}^4 = \mathbb{M}^4 \subset \mathbb{M}^8$ provide the simplest examples of them. Local G_2 transformations give more general surfaces $\mathbb{Y}^4$.

The roots of $\mathrm{Im}(f)(o) = 0$ *resp.* $\mathrm{Re}(f)(o) = 0$ are unions $\cup_{o_0} \mathbb{S}^6(o_0)$ of 6-spheres, where o_0 is octonionic real coordinate o_0. The 3-surface $\mathbb{Y}^3 = \mathbb{S}^6(o_0) \cap \mathbb{E}^4(o_0) = \mathbb{S}^3(o_0)$ defines holographic data for $\mathbb{Y}^4 \subset \mathbb{E}^4(o_0)$ as its boundary. The union $\mathbb{Y}^3 = \cup_{o_0} \mathbb{S}^6(o_0) \cap \mathbb{M}^4(o_0) = \cup_{o_0} \mathbb{S}^2(o_0)$ in turm defines holographic data for $\mathbb{Y}^4 \subset \mathbb{M}^4(o_0)$ as its boundary. Therefore both the $\mathbb{N}$ - and $\mathbb{T}$ option can be realized.

One can choose the function $f(o)$ to be an analytic function of a hypercomplex coordinate of $\mathbb{M}^4$ and 3 complex coordinates of $\mathbb{M}^8$. The natural conjecture is that the image $\mathbb{X}^4$ of $\mathbb{Y}^4$ has the same property and satisfies holography = holomorphy principle.

The simultaneous roots of $\mathrm{Im}(f)(o) = 0$ *resp.* $\mathrm{Re}(f)(o) = 0$ are 6-spheres with fixed value of o_0 and the radius r_7 of $\mathbb{S}^6(o_0)$. Two 4-surfaces $\mathbb{Y}^4_1$ and $\mathbb{Y}^4_2$, both of type $\mathbb{N}$ or $\mathbb{T}$, and satisfying $\mathrm{Im}(f)(o) = 0$ *resp.* $\mathrm{Re}(f)(o) = 0$ along $\mathbb{S}^3(o_0)$ or $\mathbb{S}^2(o_0)$. This makes it possible to build Feynman diagram-like structures with lines which have Minkowskian or perhaps even Euclidean number theoretic metric signatures. At the vertices smoothness is violated and this supports the view that they give rise to exotic smooth structures as defects of the standard smooth structure.

1 Introduction

The idea of $\mathbb{M}^8 - \mathbb{H}$ duality ($\mathbb{H} = \mathbb{M}^4 \times \mathbb{CP}_2$) has progressed through frustratingly many several twists and turns.

1.1 The evolution of the ideas

Consider first the development of the key ideas and the related problems.

1. The first key idea [L1, L2, L3] was that one can interpret octonions $\mathbb{O}$ as Minkowski space $\mathbb{M}^8$ [A4] by using the number theoretic inner product defined by the real part $\mathrm{Re}(o_1 o_2)$ of the octonion product. Later I gave up this assumption and considered complexified octonions, which do not form a number field, but finally found that the original option is the only sensible option.

2. The second key idea was that if either the tangent spaces $\mathbb{T}$ or normal spaces $\mathbb{N}$ of $\mathbb{Y}^4 \subset \mathbb{M}^8$ are quaternionic and therefore associative, and also contain a commutative subspace $\mathbb{C}$, they can be parameterized by points of $\mathbb{CP}_2$ and mapped to $\mathbb{H} = \mathbb{M}^4 \times \mathbb{CP}_2$. This would be the first half or $\mathbb{M}^8 - \mathbb{H}$ duality.

3. How to map the $\mathbb{M}^4 \subset \mathbb{M}^8$ projection to $\mathbb{M}^4 \times \mathbb{CP}_2$? This question did not have an obvious answer. The simplest map is direct identification whereas the inversion with respect to the cm or tip of causal diamond $cd \subset \mathbb{M}^4 \subset \mathbb{H}$ is strongly suggested by Uncertainty Principle and the interpretation of $\mathbb{M}^8$ coordinates as components of 8-momentum [L15]. Note that one can considerably generalize the simplest view by replacing the fixed commutative subspace of quaternion space $\mathbb{M}^4$ with an integrable distribution of them in $\mathbb{M}^8$.

4. I considered first the $\mathbb{T}$ option in which $\mathbb{T}$ was assumed to be associative. The cold shower was that there might be very few integrable distributions of associative tangent spaces [L6, L7]. As a matter of fact, $\mathbb{M}^4$ and $\mathbb{E}^4$ were the only examples of associative 4-surfaces that I knew of. On the other hand, any distribution of quaternionic normal spaces is integrable and defines an associative surface $\mathbb{Y}^4$. This led to a too hasty conclusion that only the $\mathbb{N}$ option might work.

5. If $\mathbb{M}^8$ is not complexified, the surfaces $\mathbb{Y}^4$ in $\mathbb{M}^8$ are necessarily Euclidean with respect to the number theoretic metric [L15]. This is in sharp conflict with the original intuitive idea that $\mathbb{Y}^4$ has a number theoretic Minkowski signature. Is it really the normal space $\mathbb{N}$, which must have a Minkowskian signature? Is also $\mathbb{T}$ possible.

6. The minimal option in which $\mathbb{M}^8 - \mathbb{H}$ duality determines only the 3-D holographic data as 3-surfaces $\mathbb{Y}^3 \subset \mathbb{M}^8$ mapped by $\mathbb{M}^8 - \mathbb{H}$ duality to $\mathbb{H}$. The images of $\mathbb{Y}^3$ could define holographic data consistent with the holography = holomorphy (H-H) vision [L14, L19, L15, L24, L18]. Both $\mathbb{M}^8$ and $\mathbb{H}$ sides of the duality would be necessary.

1.2 Interpretational problems

There are also interpretational problems.

1. The proposed physical interpretation for the 4-surface $\mathbb{Y}^4 \subset \mathbb{M}^8$ was as the analog of momentum space for a particle identified as a 3-D surface. In this interpretation the $\mathbb{Y}^4$ would be an analog of time evolution with time replaced with energy. A more concrete interpretation of the 3-D holographic data would be as a dispersion relation and $\mathbb{Y}^4$ could also represent off-mass shell states. Momentum space description indeed relies on dispersion relations and space-time description to the solutions of classical field equations.

2. For $\mathbb{N}$ option $\mathbb{Y}^4$ must be Euclidean in the number theoretic metric. Therefore the momenta defined in terms of the tangent space metric are space-like. What does this mean physically? Momenta are also co-quaternionic: does this exclude the Euclidean option?

   Could the problem be solved if the momentum assignable to a given point of $\mathbb{Y}^4$ is identified as a point of its quaternionic normal space as proposed in [L15].

   Or should one accept both $\mathbb{T}$ and $\mathbb{N}$ options and interpret the Euclidean $\mathbb{Y}^4$ as as a counterpart of a virtual particle with space-like momenta and of $\mathbb{CP}_2$ type extremals at the level of $\mathbb{H}$. At vertices belonging to $\mathbb{Y}^4_1 \cap \mathbb{Y}^4_2$ me, quaternionic $\mathbb{N}(\mathbb{Y}^4_1)$ would contain points of quaternionic $\mathbb{T}(\mathbb{Y}^4_2)$ so that the earlier proposal would not be completely wrong.

3. A further criticism against the $\mathbb{M}^8 - \mathbb{H}$ duality is that its explicit realization has been missing. For $\mathbb{N}$ option, the distributions of the quaternionic normal spaces $\mathbb{N}$ are always integrable but their explicit identification has been the problem. For $\mathbb{T}$ option even the existence of integrable distributions of $\mathbb{T}$ has remained open.

1.3 A possible solution of the problems of the earlier view

Consider now how the view to be described could solve the listed problems.

1. There are two options, which could be called $\mathbb{T}$ and $\mathbb{N}$: either the local tangent space $\mathbb{T}$ or normal space $\mathbb{N}$ is quaternionic. Which one is correct or are both correct?

   Any integrable distribution of quaternionic normal spaces is allowed whereas for tangent spaces this is not the case. This does not mean that non-trivial solutions would not exist. Perhaps the rejection of the $\mathbb{T}$ option was too hasty.

   Furthermore, for $\mathbb{X}^4 \subset H$ both Minkowskian and Euclidean signatures of the induced metric are possible: could $\mathbb{T}$ and $\mathbb{N}$ option be their $\mathbb{M}^8$ counterparts?

2. Holography= holomorphy vision (H-H) allows an explicit construction of the space-time surfaces $\mathbb{X}^4 \subset \mathbb{H}$. For $\mathbb{Y}^4 \subset mathbbM^8$ the situation has been different. The very nature of duality concept suggests that the explicit construction must be possible also at the level of $\mathbb{H}$.

3. Is $\mathbb{M}^8-\mathbb{H}$ duality between 4-D surfaces $\mathbb{Y}^4\subset\mathbb{M}^8$ and space-time surfaces $\mathbb{X}^4\subset\mathbb{H}$ or only between the 3-D holographic data $\mathbb{Y}^3\subset\mathbb{H}$ and $\mathbb{X}^3\subset mathbbM^8-\mathbb{H}$?

It turns out that a modification of the original form of the $\mathbb{M}^8-\mathbb{H}$ duality, formulated in terms of a real octonion analytic functions $f(o):\mathbb{O}\rightarrow\mathbb{O}$, leads to a possible solution of these problems.

1. All the conditions $f(o)=0$, $f(o)=1$ and $\mathrm{Im}f)(o)=0$, and $\mathrm{Re}f)(o)=0$ are invariant under local G_2 acting as as a dynamical spectrum generating symmetry group since $f\circ g_2=g_2\circ f$ holds true. The task reduces to that of finding the 4-surfaces with a constant quaternionic $\mathbb{T}$ or $\mathbb{N}$.

2. In particular, $\mathbb{M}^4\subset\mathbb{M}^8$ has been hitherto the only known $\mathbb{Y}^4$ of type $\mathbb{T}$ and the action of local G_2 generates a huge number of $\mathbb{Y}^4$ of type $\mathbb{T}$. Both $\mathbb{T}$ and $\mathbb{N}$ option are possible after all! G_2 symmetry applies also to the $\mathbb{N}$ option for which $\mathbb{E}^4$ is the simplest representative!

3. The roots of $\mathrm{Im}(f)(o)=0$ *resp.* $\mathrm{Re}(f)(o)=0$ are unions $\cup_{o_0}\mathbb{S}^6(o_0)$ of 6-spheres, where o_0 is octonionic real coordinate o_0. The 3-D union $\mathbb{Y}^4=\cup_{o_0}\mathbb{S}^6(o_0)\cap\mathbb{M}^4=\mathbb{S}^2(o_0)\subset M^4$ has quaternionic tangent space $\mathbb{T}=\mathrm{H}=\mathbb{M}^4$. The interpretation as holographic data and the $\mathbb{M}^8$ counterpart of a partonic orbit is suggestive.

4. The Euclidean 3-surface $\mathbb{Y}^3=\mathbb{S}^6(o_0)\cap\mathbb{E}^4(o_0)=\mathbb{S}^3(o_0)$ could serve as a holographic data for $\mathbb{Y}^4$ with quaternionic normal spaces and with an Euclidean number theoretic signature of the metric. Obviously, the option $\mathbb{Y}^4=\cup_{o_0}\mathbb{Y}^3(o_0)$ fails to satisfy this condition. The interpretation would be as the $\mathbb{M}^8$ counterpart of $\mathbb{CP}_2$ type extremal with a Euclidean signature of the induced metric. The identification as the $\mathbb{M}^8$ counterpart of a virtual particle with space-like momentum is suggestive.

   If $\mathbb{T}$ *resp.* $\mathbb{N}$ contains a commutative hyper-complex subspace, it corresponds to a point of $\mathbb{CP}_2$. Hence $\mathbb{Y}^4$ can be mapped to $\mathbb{X}^4\subset\mathbb{H}=\mathbb{M}^4\times CP_2$ as $\mathbb{M}^8-\mathbb{H}$ duality requires.

5. What could be the counterpart of H-H vision in $\mathbb{M}^8$? One can choose the function $f(o)$ to be an analytic function of a hypercomplex coordinate u or v of $\mathbb{M}^4$ and 3 complex coordinates of $\mathbb{M}^8$. The natural conjecture is that the image $\mathbb{X}^4$ of $\mathbb{Y}^4$ has the same property and satisfies H-H.

This view solves the interpretational problems.

1. The proposed physical interpretation for the 4-surface $\mathbb{Y}^4\subset\mathbb{M}^8$ was as the analog of momentum space for a particle identified as a 3-D surface. The interpretation the $\mathbb{Y}^4$ as the analog of time evolution with time replaced with energy looks range. A more concrete interpretation of the 3-D holographic would be as a dispersion relation emerges and $\mathbb{Y}^4$ could also represent off-mass shell states. Momentum space description indeed relies on dispersion relations and space-time description to the solutions of classical field equations.

   Number theoretic discretization as a selection of points as elements of the extensions of rationals defining the coefficient field for $f(o)$ and the replacement of fermions to the "active" points of discretization would realize many fermion states at the level of $\mathbb{H}$. Galois confinement [L16, L17, L8] stating that the total momenta are rational numbers would provide a universal mechanism for the formation of bound states.

2. For $\mathbb{N}$ option $\mathbb{Y}^4$ must be Euclidean in the number theoretic metric. Therefore the momenta defined in terms of the tangent space metric are space-like. What does this mean physically?

   Could the problem be solved if the momentum assignable to a given point of $\mathbb{Y}^4$ is identified as a point of its quaternionic normal space as proposed in [L15].

   Or should one accept both $\mathbb{T}$ and $\mathbb{N}$ options and interpret the Euclidean $\mathbb{Y}^4$ as as a counterpart of a virtual particle with space-like momenta and of $\mathbb{CP}_2$ type extremals at the level of $\mathbb{H}$.

   At 2-D vertices belonging to the intersection $\mathbb{Y}^4_1\cap\mathbb{Y}^4_2$, quaternionic $\mathbb{N}(\mathbb{Y}^4_1)$ would contain points of quaternionic $\mathbb{T}(\mathbb{Y}^4_2)$ so that the first proposal would not be completely wrong.

3. Could the TGD analogs of Feynman diagrams be built by gluing together $\mathbb{T}$ and $\mathbb{N}$ type surfaces $\mathbb{Y}^4$ along 3-surfaces $\mathbb{Y}^3$ defining analogs of vertices. In the role of consciousness theorist, I have called them "very special moments in the life of self" [L5] at which the non-determinism of the classical field equations in H-H vision is localized. At these 3-surfaces the smoothness of $\mathbb{Y}^4$ fails and could give a connection to the notion of exotic smooth manifold [A5, A6, A3], conjectured to make possible particle vertices and fermion pair creation in TGD despite the fact that fermions in $\mathbb{H}$ are free [L17, L9, L22].

In the following a formulation of $\mathbb{M}^8-\mathbb{H}$ duality possibly solving these problems in terms of local G_2 invariance is discussed in detail.

2 Understanding associativity in terms of local G_2 invariance

The motivation for reconsidering the $\mathbb{M}^8-\mathbb{H}$ duality came from the fact that the H-H hypothesis [L14, L19, L15, L24, L18] works extremely nicely for the space-time surfaces $\mathbb{X}^4\subset\mathbb{H}$. The roots of two generalized analytic functions f_1, f_2 of hypercomplex coordinate and 3 complex coordinates of $\mathbb{H}$ give as their roots space-time surfaces as minimal surfaces and the ansatz works for any action, which is general coordinate invariant and expressible in terms of the induced geometry. One would expect that H-H hypothesis appears also at the level of $\mathbb{M}^8$: How?

One can also argue that there might be problems with the 3-D holographic data. How to fix them in such a way that they are consistent with functions f_1 and f_2 as analytic functions of $\mathbb{H}$ coordinates involving hypercomplex coordinate and 3 complex coordinates?

2.1 About the basic notions of TGD

It is instructive to start with an explanation of the physical content of the basic notions related to $\mathbb{M}^8-\mathbb{H}$ duality.

2.1.1 What is the interpretation of 4-surfaces of $\mathbb{M}^8$ interpreted as a momentum space?

The basic objection against the momentum space interpretation of $\mathbb{M}^8$ can be formulated as a simple question. What does time - or rather, energy evolution mean? At $\mathbb{Y}^4$, the time evolution would mean change of momenta as points of $\mathbb{S}^3$ as the radius of $\mathbb{S}^3$ changes. Momentum has momentum. Could this be seen as an "acceleration" of an off-mass-shell 8-momentum with energy appearing in the role of time?

A more natural interpretation than evolution is in terms of dispersion relation between momentum and energy. Indeed, the conditions $\mathrm{Re}(f)=0$ or $\mathrm{Im}(f)=0$ give rise to 3-D surfaces defining dispersion relation. The 4-surface $\mathbb{Y}^4$ in turn could be interpreted as a kind of off-mass shell dispersion relation.

What could be the identification of free on-mass shell states? $\mathbb{M}^8$ masslessness means $\mathbb{M}^4$ massivation and off mass shell property! This gives a set of 3-surfaces with fixed $\mathbb{M}^8$ mass squared and also $\mathbb{M}^4$ mass squared. For $f=o^2-m^2$ the condition $\mathrm{Re}(f)=0$ reduces to a mass shell for the $\mathbb{T}$ option.

In this framework $\mathbb{M}^8-\mathbb{H}$ duality relates two views of physics: the description in terms of classical fields and geometry and the description of quantum physics in terms of energy, momentum and dispersion relations. Dispersion relations are algebraic equations so that this description rather naturally extends to a number theoretic vision of physics [L4].

2.1.2 Moduli space for the octonion structures

The identification of the 8-D momentum space $\mathbb{M}^8$ as octonions raises non-trivial conceptual challenges.

1. Causal diamond in $\mathbb{H}$ is $cd\times CP_2$. Consider first the moduli space for cd:s in $\mathbb{M}^4\subset H$ [L10]. The interpretation of cd as a particle-like entity in $\mathbb{H}$, as a kind of perceptive field is natural.

Poincare group and scalings generate new cd:s. "Mass shells" are mapped to "mass shells" also in scalings and conformal transformations. Whether conformal invariance can be allowed or needed is not obvious.

2. In $\mathbb{M}^8$, cd_8 is the natural candidate for the counterpart of $CD \subset \mathbb{H}$. The problem with $\mathbb{M}^8$ regarded as octonions is that it selects a preferred time direction. This does not conform with Lorentz invariance. If linearity and homogeneity are required, only $\mathbb{S}O(1,7)$ makes sense. Also scalings and even inversions with respect to origin could be allowed.

   At the $\mathbb{M}^8$ side the position of CD_8 and its translations do not make physical sense.

3. $\mathbb{M}^4$ inversion $m^k \rightarrow p^k = h_{eff} m^k / m^l m_l$ favoured by Uncertainty Principle in the $\mathbb{M}^8 - \mathbb{H}$ duality. The origin of $\mathbb{M}^4$ coordinates could correspond to the cm or tip of $cd \subset M^4$, which can be seen as a sub-CD of a larger CD. The size of CD_8 and cd would correlate. In this case, the map $\mathbb{H} \rightarrow \mathbb{M}^8$ would be a special conformal transformation as a scaling with respect to an arbitrary point of $\mathbb{M}^4$.

2.1.3 Is $\mathbb{M}^8 - \mathbb{H}$ duality bijective map or something more general?

The original intuitive view was that $\mathbb{M}^8 - \mathbb{H}$ duality is 1-valued map $\mathbb{M}^8 \rightarrow \mathbb{H}$. This works for space-time surfaces X^4 with Minkowskian signature, that is for option $\mathbb{T}$. It fails for CP_2 type extremals with Euclidean signature of the induced metric and naturally corresponding to option $\mathbb{N}$. The proposal was that these surfaces can be seen as singularities at which the quaternionic normal space N is not-unique.

A more elegant view is that for CP_2 extremals the map $\mathbb{M}^8 - \mathbb{H}$ duality defines a 1-valued map $\mathbb{H} \rightarrow \mathbb{M}^8$ for option $\mathbb{N}$ and a-valued map $\mathbb{M}^8 \rightarrow \mathbb{H}$ for option $\mathbb{T}$.

2.2 Could the roots of real analytic octonion function give rise to $\mathbb{M}^8 - \mathbb{H}$ duality after all?

The issues mentioned above led back to the original idea that the associative 4-surfaces $\mathbb{Y}^4 \subset \mathbb{M}^8$ might be definable in terms of real analytic functions $f(o)$ of octonions as an octonionic generalization of the notion of holomorphy. The conditions $f(o) = 0$, $f(o) = 1$, $\mathrm{Re}(f)(o) = 0$ and $\mathrm{Im}(f)(o) = 0$ are are invariant under local octonionic automorphism group G_2. The argument goes as follows.

1. Since G_2 acts as automorphisms, one has $f(g_2(o)) = g_2(f(o))$, where g_2 is any *local* G_2 automorphism. If $f(o) = x \; x \in \{0,1\}$ is true then also $f(g_2(o)) = x$ is true for any $g_2 \subset \mathrm{G}_2$. This is true also for the roots of $\mathrm{Re}(f(o)) = 0$ *resp.* $\mathrm{Im}(f(o)) = 0$, where "*Re*" *resp.* "*Im*" refers to the octonionic real *resp.* imaginary part. Since G_2 maps the decomposition of octonion to quaternion and to a part orthogonal to it, also the conditions $\mathrm{RE}(f(o) = 0$ and $\mathrm{IM}(f) = 0$, where $\mathrm{RE}(f)$ and $\mathrm{IM}(f)$ refer to the quaternionic co-quaternionic parts of the octonion, preserve their character under local G_2.

   One has a huge dynamical spectrum generating symmetry analogous to the holomorphic symmetries of H-H vision. It maps the quaternionic normal spaces to quaternionic normal spaces and complex subspaces to complex subpaces.

2. Consider first the condition $f(o) = x$, $x \in \{0,1\}$. The Taylor (or even Laurent -) expansion in powers of $\mathbb{O}$ gives only two terms. The first term is proportional to the octonionic real unit 1 of $\mathbb{O}$ and the second term to the octonionic imaginary part of $\mathrm{Im}(o) = o_7$ of $\mathbb{O}$.

   For o^2 one obtains $o^2 = o_0^2 - o_7 \cdot o_7 + 2 o_0 o_7$. The coefficients of these parts depend on the real part o_0 of $\mathbb{O}$ and the length r_7 of the imaginary $\mathrm{Im}(o)$. The higher powers of o involve products of two octonions of form $o_1 = \alpha_1 + \beta_1 o_7$ and $o_2 = \alpha_2 + \beta_2 0_7$ and the product is of form $o_1 o_2 = (\alpha_1\alpha_2 - \beta_1\beta_2) + (\alpha_1\beta_2 + \alpha_3\beta_1) o_7$. By induction, one finds that the coefficients for any power depend only on o_0 and the radius r_7 of 6-sphere only. In particular, the function $f(o)$ is expressible has the general form

$$f(o) = f_1(o_0, r_7) + f_2(o_0, r_7) \quad . \tag{2.1}$$

The detailed forms of these functions have been discussed in the earlier articles [L6, L7, L13] and will be described also below.

3. The condition $\mathrm{Im}(f(o)) = 0$ *resp.* $\mathrm{Re}(f(o)) = 0$ fixes the relationship between o_0 and r_7 and gives a "time evolution" of the radius r_7 of a 6-sphere as function of the real coordinate o_0 having identification as energy. The condition $\mathrm{RE}(f(o)) = 0$ requiring the vanishing of the quaternionic part implies the vanishing of both $\mathrm{RE}(f)$ and $\mathrm{IM}(f)$ and reduces to the condition $f(o) = 0$. The condition $\mathrm{IM}(f(o)) = 0$ implies the vanishing of $\mathrm{Im}(f(o))$.

2.3 Various G_2 invariant options

Consider now various G_2 invariant options.

2.3.1 The conditions $\mathrm{Re}(f) = 0$, $\mathrm{Im}(f) = 0$, and $f(o) = 0$

The conditions $\mathrm{Re}(f) = 0$ and $\mathrm{Im}(f) = 0$ has the roots $o_0 = h_1(r_7)$ and $o_0 = h_2(r_7)$. These roots define a union of 6-spheres $\mathbb{S}^6$ with radius $r_7 = r_7(o_0)$.

It deserves to be noticed that $\mathbb{S}^6$ can be represented as a coset space $G_2/SU(3)$. $\mathbb{S}^6$ has an almost complex structure induced by the octonionic cross product, which makes it nearly Kähler manifold.

Can one assign an associative 4-surface $\mathbb{Y}^4 \subset \mathbb{M}^8$ of type $\mathbb{N}$ or $\mathbb{T}$ to the set of roots of $\mathrm{Re}(f) = 0$ or $\mathrm{Im}(f) = 0$ or to a given $\mathbb{S}^6$?

Since $\mathbb{E}^4(o_0)$ and $\mathbb{M}^3(o_0)$ are the simplest examples of quaternionic 4-surfaces $\mathbb{Y}^4$ of type $\mathbb{N}$ *resp.* $\mathbb{T}$, it is natural to look what the intersections of $\mathbb{S}^6(o_0)$ with these spaces are.

1. Since both surfaces in the intersection $\mathbb{E}^4(o_0)\cap\mathbb{S}^6(o_0)$ are contained in the hyper-plane $\mathbb{E}^7(o_0)$, the dimension of $\mathbb{E}^4(o_0)\cap\mathbb{S}^6(o_0)$ is from the basic rule $6+6-7 = 3$. Clearly, the intersection is identifiable as 3-sphere $\mathbb{S}^3(o_0)$ and holography is needed to construct $\mathbb{Y}^4$ of type $\mathbb{N}$. $\mathbb{E}^4(o_0)$ solves the holography.

2. The set $\cup_{o_0}\mathbb{S}^6(o_0)$ is contained in the hyper-plane $\mathbb{E}^7(o_0)$ and the dimension of $\mathbb{M}^4(o_0) \cap \cup_{o_0}\mathbb{S}^6(o_0) = \cup_{o_0}\mathbb{S}^2(o_0)$ is from the basic rule $6 + 6 - 7 = 3$. The intersection, identifiable as a 3-D orbit of 2-sphere $\mathbb{S}^2$, defines the holographic data giving $\mathbb{Y}^4$ of type $\mathbb{T}$. $\mathbb{M}^4$ solves the holography.

3. In both cases, G_2 dynamical symmetry allows to construct more general solutions.

The conditions $f(o) = 0$ give the roots of f_1 and f_2 as $o_0 = h_1(r_7)$ and $o_0 = h_2(r_7)$. These roots define a discrete set of 6-spheres $\mathbb{S}^6$ with o_0 constant and $r_7 = constant$ as "very special moment in the life of self" [L5]. At these surfaces, the solutions of $\mathrm{Im}(f) = 0$ and $\mathrm{Re}(f) = 0$ can meet and the interpretation as the analog of a vertex of Feynman graph is suggestive. If both $\mathbb{N}$ and $\mathbb{T}$ type solutions are allowed, the maximal number of meeting 4-surfaces is 4, which bring in mind Yang-Mills theory.

It should be noted that the condition $\mathrm{IM}(f(o)) = 0$ is equivalent with the condition $\mathrm{Im}(f(o)) = 0$. The condition $\mathrm{RE}(f(o)) = 0$ gives a discrete set of 3-spheres as roots is equivalent with the condition $f(o) = 0$.

2.3.2 Explicit treatment of the conditions $\mathrm{Re}(f(o)) = 0$ and $\mathrm{Im}(f(o)) = 0$

It is straightforward to find an explicit general solution for the condition $f(o) = 0$ in the general case. The expression $o^2 = o_0^2 - r_7^2 + 2r_7oe$ where e satisfying $e^2 = -1$ is the octonionic imaginary unit defined by the imaginary part of $o = 0_0 + r_7e$ allows to write

$$\begin{aligned} o^{2n} &= (o_0^2 - r_7^2 + 2r_7o_0e)^n = a_n + b_ne \ , \\ o^{2n+1} &= (a_n + b_ne(o_0 + r_7e) = a_no_0 - b_nr_7 + (a_nr_7 + b_no_0)e \ . \end{aligned} \tag{2.2}$$

The coefficients $a_n = a_n(o_0^2, r_n^2)$ and $b_n = b_n(o_0^2, r_n^2)$ can be deduced from binomial coefficients. If the condition $o_0^2 - r_n^2$ giving $o_0 = \epsilon r_n$ is satisfied, this gives

$$\begin{aligned} o^{2n} &= (2r_7^2)^n \epsilon^n e^n \ , \\ o^{2n+1} &= (2r_7^2)^n r_7(\epsilon + e)e^n \ . \end{aligned} \tag{2.3}$$

One can decompose f as $f = f_{even} + f_{odd}o$, where one has $f_{even} = \sum_n f_{2n}o^{2n}$ and $f_{odd} = \sum_n f_{2n+1}o^{2n}$. One has

$$\begin{aligned} f &= f_1 o_0 + f_2 r_7 e \ , \\ f_1 &= f_{even} + f_{odd} o_0 = \sum_n f_{2n}(a_{2n} + b_{2n}e) + \sum_n f_{2n+1}(a_{2n} + b_{2n}e)o_0 \\ f_2 &= f_{odd} r_7 = \sum_n f_{2n+1}(a_{2n} + b_{2n}e)r_7 \ . \end{aligned} \tag{2.4}$$

This gives

$$\begin{aligned} f &= Re(f) + Im(f)e \ , \\ Re(f) &= \sum_n f_{2n}a_{2n} + f_{2n+1}a_{2n}o_0 - \sum_n f_{2n+1}b_{2n}r_7 \ , \\ Im(f) &= \sum_n f_{2n}b_{2n} + f_{2n+1}b_{2n}o_0 - \sum_n f_{2n+1}a_{2n}r_7 \ . \end{aligned} \tag{2.5}$$

The octonion analytic function reduces by its symmetries to a sum of real part and imaginary part such that the imaginary part is proportional to the imaginary part of o. Both real and imaginary parts depend only on 0_0 and r_7 which are analogous to energy and magnitude of 8-momentum.

2.3.3 Illustrative examples

The following illustrative examples help to understand the physical picture.

1. The case $f(o) = o^2$ with $\mathrm{Re}(f) = 0$ serves as an illustrative example. The condition $o_0^2 = r_7^2$ gives an expanding 6-sphere with radius $r_7 = \pm o_0$. The restriction of E^4 coordinates to $(e_4, e_5, e_6, e_7) = 0$ for $\mathbb{T}$ option gives $r_3 = \pm o_0$. One obtains union of sub-mass-shells $E = o_0 = p = r_3$ of a massless particle with a fixed length of 3-momentum p.

   For $\mathbb{N}$ option, this represents 3-D mass shells of a massless particle analogous to the mass shell of a massive particle. Now however momentum can have an additional component orthogonal $\mathbb{M}^4$ so that a virtual particle is in question in this case.

2. For $f(o) = o^2 - m^2$ $\mathrm{Re}f) = 0$ as a restriction guaranteeing tangent space quaternionicity gives $o_o^2 - r_7^2 = m^2$ giving 3-D positive and negative energy mass shells of a massive particle. Also now there is an analogy with partonic orbits for $\mathbb{T}$ option Note that the images of these mass shells in $\mathbb{H}$ under $\mathbb{M}^8 - \mathbb{H}$ duality are mass shells.

3. For more general functions $f(o)$, the dispersion relation given by $\mathrm{Re}(f) = 0$ the dispersion relation given by $r_2 = h(o_0)$ is rotationally invariant but is more general that that for a massless particle.

2.4 How to realize Lorentz invariance for on mass shell states?

If the tangent spaces of $\mathbb{Y}^4$ are quaternionic, the condition $\mathrm{Re}(f) = 0$ or $\mathrm{Im}(f) = 0$ has as a solution the union of $\cup_{o_0} S^6(o_0)$ of 6-spheres with radius $r_7(o_0)$. The intersection $Y^3 = E^3(o_0) \cap \cup_{o_0} S^6(o_0) \cup_{o_0} S^2(o_0)$ defines the holographic data. For the 2-spheres $S^2(o_0)$, the 3-momentum squared is constant but depends on the energy o_0 via a dispersion relation that is in general not Lorentz invariant. $\mathbb{M}^8 - \mathbb{H}$ duality suggests how to obtain Lorentz invariant mass shell conditions $E^2 - p^2 = m^2$.

1. The modes of the Dirac equation in $\mathbb{H}$ [L21, L20] are massless in the 8-D sense. This is a natural additional condition also in $\mathbb{M}^8$ and could define on mass shell states consistent with Lorentz invariance and distinguish them from the other points of $\mathbb{Y}^4$ having an interpretation as off-mass-shell momenta allowed by $\mathbb{Y}^4$ as a representation of a dispersion relation.

2. 8-D masslessness corresponds in $\mathbb{M}^8$ to the condition $o_0^2 - r_7^2 = 0$, where r_7^2 is the counterpart of the $\mathbb{CP}_2$ mass squared as the eigenvalue of the $\mathbb{CP}_2$ spinor Laplacian. The additional condition $o_0^2 - r_7^2 = 0$ picks up a discrete set of values $(o_0(r_7), r_7)$. The 4-D mass squared would be $m_4^2 = r_7^2$ and a discrete mass spectrum is predicted for a given $f(o)$ and a given selection a $\mathrm{Re}(f) = 0$ or $\mathrm{Im}(f) = 0$.

3. An interesting question is whether the eigenvalue spectrum of $\mathbb{CP}_2$ spinor Laplacian is realized at the level of $\mathbb{M}^8$ as on mass-shell states.

4. A natural guess would be that the eigenvalue spectrum of $\mathbb{CP}_2$ spinor Laplacian is realized at the level of $\mathbb{M}^8$ as on-mass-shell states.

   The TGD based proposal [L21, L20] for color confinement producing light states involves tachyonic states. These states would naturally correspond to 4-surfaces Y^4 with Euclidean signature and bound states would be formed by gluing together the tachyonic and non-tachyonics states to Feynman graph-like structures. Note that the on-mass-shell 2-spheres are in general different from those satisfying the conditions $(Re(f), Im(f)) = (0, 0)$ proposed to define vertices for the generalized Feynman graphs.

   Note that the on mass shell 2-spheres are in general different from those satisfying the conditions $(Re(f), Im(f)) = (0, 0)$ proposed to define vertices for the generalized Feynman graphs.

3 About the realization of $\mathbb{M}^8 - \mathbb{H}$ duality

The realization of $\mathbb{M}^8 - \mathbb{H}$ duality as map $\mathbb{Y}^4 \subset M^8 \rightarrow H = M^4 \times CP_2$ involves some non-trivial aspects.

3.1 $\mathbb{M}^8 - \mathbb{H}$ duality in $\mathbb{CP}_2$ coordinates

The $\mathbb{M}^8 - \mathbb{H}$ duality in $\mathbb{CP}_2$ coordinates would look like follows.

1. $\mathbb{M}^8 - \mathbb{H}$ duality requires that $\mathbb{M}^4$ contains $\mathbb{M}^2 \subset \mathbb{M}^4$ defining a commutative sub-space. Since $U(2) \subset SU(3)$ respects this choice, the normal spaces satisfying this condition are parameterized by $\mathbb{CP}_2 = SU(3)/U(2)$ and $\mathbb{M}^8 - \mathbb{H}$ duality allows to assign to a given point of $\mathbb{Y}^4$ a point of $\mathbb{CP}_2$.

2. An integrable distribution of these subspaces is possible. The local elements of G_2 map these distributions to each other. The subgroup leaving the distribution invariant corresponds to local SU(3), which at the $\mathbb{H}$ side has interpretation as color group whereas U(2) leaving the normal space invariant corresponds to the electroweak gauge group.

   The integrable distribution of these choices together with generalized complex coordinates for $\mathbb{M}^8$ defines the analog of Hamilton-Jacobi structure (H-J) [L18] in $\mathbb{M}^4 \subset \mathbb{M}^8$ mapped to its counterpart in $\mathbb{H}$ and playing a key role in H-H vision [L15].

3. Rather remarkably, the local $G_2/U(2)$ can therefore be identified as the moduli space of H-J structures [L18]. The division by U(2) is because the quaternionic normal space with complex subspace is invariant under $U(2) \subset G_2$. Note that $G_2/U(2)$ is 10-D.

4. The integrable division of the quaternionic normal space $\mathbb{M}^4$ to complex sub-space $\mathbb{M}^2$ and its complement $\mathbb{E}^2$ allows also to identify a number theoretic analog of Kähler structure in terms of the quaternionic cross product for $\mathbb{E}^2$ projections of the vectors of $\mathbb{M}^4$ in the simplest situation when $\mathbb{M}^4$ is constant. This Kähler structure is trivial in the longitudinal hypercomplex degrees of freedom assigned $\mathbb{M}^2$. This conforms with the physical intuition provided by gauge theories, string models and TGD: the longitudinal polarizations have zero Hilbert space norm. This decomposition induces a similar decomposition of $\mathbb{M}^4 \subset \mathbb{H}$ and of the tangent space of the space-time surface $\mathbb{X}^4 \in \mathbb{H}$ essential for the H-J structure.

3.2 $\mathbb{M}^8 - \mathbb{H}$ duality for $\mathbb{M}^4$ coordinates

What about $\mathbb{M}^8 - \mathbb{H}$ duality for $\mathbb{M}^4$ coordinates?

1. Could the $\mathbb{M}^4 \subset \mathbb{H}$ point correspond to the projection of the $\mathbb{Y}^4 = \mathbb{E}^4 \times \mathbb{S}^6$ point to $\mathbb{M}^4 \subset \mathbb{M}^8$ as such or is an inversion suggested by Uncertainty Principle and the interpretation of $\mathbb{M}^8$ as 8-D momentum space? This question remains open.

2. What can one say about the elements g_2 of the local G_2? The action of G_2 on octonions allows a matrix representation but the matrix elements are octonions [A1] so that the rules of multiplication are not standard and the product is non-associative. Associativity is obtained if one considers only elements of G_2 belonging to a local SU(3) subgroup having physical interpretation as a color group.

3. Holomorphy= holography vision [L14, L19, L23] inspires the question whether $g_2(o)$ can be regarded as a real part of an analytic function of the generalized complex coordinates of $\mathbb{M}^8$ (hypercomplex coordinate and 3 complex coordinates) for the Hamilton-Jacobi structure in question. Could this guarantee that the image of $\mathbb{Y}^4$ in $\mathbb{H}$ is consistent with the holomorphy in $\mathbb{H}$?

4. The real analytic functions $f(o)$ and $g(o)$ can be multiplied and summed so that the analog of a function field is in question. 4-surface in $\mathbb{M}^8$ become analogs of numbers as they do also in $\mathbb{H}$ [L23]. Also iterations of $f(o)$ are possible. The roots $\mathrm{Im}(g) = 0$ of $g = f \circ f ... \circ f$ contain the roots of f plus roots of higher iterates. A complexity hierarchy analogous to that appearing for function pairs (f_1, f_2) at $\mathbb{H}$ sides emerges and the interpretation in terms of cognitive hierarchies is suggestive. An interesting question is whether there is a simple relationship between functions $f(o)$ and function pairs (f_1, f_2) defined in $\mathbb{H}$.

3.3 Is the local G_2 invariance a symmetry of action and is the action exponential a number theoretic invariant?

The ramified primes of a polynomial $f(o) = P(o)$ having rational or algebraic coefficients are expected to play an important role in the number theoretic view of TGD.

1. If f assigned to $\mathbb{M}^8 - \mathbb{H}$ duality is a polynomial P with rational coefficients, the ramified primes would be assigned with the discriminant of P. The conjecture has been that the classical action defining the space-time surface is expressible as a power of discriminant of some polynomial P defined by the differences of ramified primes [L11]. This would be a central aspect of the 4-D version of Langlands duality [L14, L19]. The notion of discriminant makes sense also for real analytic functions f.

2. This would imply a huge degeneracy since all space-time surfaces related by local G_2 transformations as analogs of conformal transformations would have the same classical action defining the Kähler metric of WCW and give excellent hopes that also the functional integral over the 4-D "Bohr orbits" predicts by the holography = holomorphy principle reduces to a discrete sum (there is a slight failure of non-determinism as also for 2-D minimal surfaces) is calculable [L11]. Local G_2 would define zero modes for the WCW metric and symplectic degrees of freedom would correspond to non-zero modes as also conjectured [L12].

3. Besides space-time surfaces $\mathbb{X}^4$ representable as graphs of maps $\mathbb{M}^4 \to \mathbb{CP}_2$ also surfaces for which $\mathbb{M}^4$ projection has dimension smaller than 4, are possible. These could correspond to the singularities of the map G_2 such that the quaternionic normal space $\mathbb{M}^4$ labelled by a $\mathbb{CP}_2$ point depends on the direction in which one approaches a lower-dimensional surface $\mathbb{X}$ of $\mathbb{M}^8$. This would give rise to $\mathbb{CP}_2$ type extremals with 1-D $\mathbb{X}$ and cosmic strings with 2-D $\mathbb{X}$.

The above mentioned conjecture that the classical action equals some kind of discriminant and is thus a number theoretic invariant, can be sharpened in the recent picture.

1. The condition $\mathrm{Im}(f) = 0$ ($\mathrm{Re}(f) = 0$) has a discrete set of roots $\mathbb{Y}^4(n) \subset \mathbb{M}^8$ as time evolutions $r_7 = h_n(o_0)$ of $\mathbb{S}^6$, in turn giving rise to 4-surfaces $\mathbb{Y}^4(n)$ as time evolutions $\mathbb{S}^3(o_0) = \mathbb{E}^4(o_0) \cap \mathbb{S}^6(o_0)$ with respect to time coordinate o_0 mapped. Different roots $\mathbb{Y}^4(i)$ as 4-surfaces can be interpreted as free particles, mapped to space-time surfaces $\mathbb{X}^4(i)$ in $\mathbb{H}$ by $\mathbb{M}^8 - \mathbb{H}$ duality.

2. For each orbit $\mathbb{Y}^4(i)$ of $\mathbb{S}^3$, the condition $f(o) = 0$ defines a discrete set of "very special moments of time" $o_0(n, i)$ as its roots. The roots can be also complex but for real polynomials appear as complex conjugate pairs. One can define discriminant D as the product of

differences of squares of roots in the usual manner [L14, L19]. This is true also when f is analytic function rather than only polynomial.

One can assign a discriminant $D(i)$ to each $\mathbb{Y}^4(i)$. The product $\prod_{i\in U} D(i)$ is well defined for the system of all $\mathbb{Y}^4(i)$ or a subset U of them. These discriminants would define exponents of "free" actions for each $\mathbb{Y}^4(i)$.

Interactions are not taken into account yet.

1. At the level of $\mathbb{H}$ interactions reduce to a generalization of contact interactions for the Bohr orbits $\mathbb{X}^4$. In the generic case the intersection $\mathbb{X}^4 \cap \mathbb{Y}^4$ consists of a discrete set of points. If $\mathbb{X}^4$ and $\mathbb{Y}^4$ have the same H-J structure they have a common hypercomplex coordinate and the intersection consists of 2-D string world sheets so that string model type description for the interactions emerges.

2. How to assign "interaction action" to this system as a discriminant? The proposal is that the interactions between particles at the level of $\mathbb{H}$ are contact interactions made possible by the intersection of space-time surfaces. For identical H-J structures the intersection $\mathbb{X}^4_1 \cap \mathbb{X}^4_2$ consists of 2-D string world sheets rather than a discrete set of points. Identical H-J structures would mean that they correspond to the same element of local $G_2/U(2)$ since U(2) leaves the quaternionic normal space containing a preferred commutative plane invariant.

3. As found, the H-J structures of $\mathbb{M}^8$ and $\mathbb{H}$ naturally correspond to each other. If so, then also the intersection $\mathbb{Y}^4_1 \cap \mathbb{Y}^4_2$ consists of string world sheets. One should be able to assign to the intersection an "interaction action". The conditions $o_0(1) = o_0(2)$ and $r_7(1) = r_7(2)$ (radii of $\mathbb{S}^6$) must be satisfied. This gives a set $\{o_0(n)\}$ of roots. These 6-spheres now define "very special moments" for the interaction. The 3-spheres $\mathbb{S}^3(o_0, i) = \mathbb{E}^4(o_0, i) \cap \mathbb{S}^6(o_0)$, $i = 1, 2$ must intersect in $\mathbb{Y}^4_1 \cap \mathbb{Y}^4_2$. The intersection of two 3-spheres should consist of 2-D string world sheets for the same H-J structures. This looks sensical since the hypercomplex $\mathbb{M}^4$ coordinates appearing in the functions f_1 and f_2 are the same and one condition is eliminated. Also self-intersections for $\mathbb{Y}^4$ are possible and would contribute to the action terms having an interpretation in terms of self-interactions of $\mathbb{Y}^4$.

   One can assign discriminant D_{12} to the intersection $\mathbb{Y}^4_1 \cap \mathbb{Y}^4_2$ as a product of squares of root differences in $\{o_0(n)\}$. This would define an additional multiplicative contribution to the action exponential.

3.4 A possible connection with exotic smooth structures

A connection with the work of Michel Planat [A2] (see https://www.mdpi.com/2073-8994/10/12/773) is suggestive. The proposal of Michel Planat is that local exotic smooth structures could provide a representation for qubits reducing to bits and magic qubits for which the reduction does not occur. The Pauli group consisting of generalized spin matrices in the case of qdit generates both bits and magic qubits.

3.4.1 Planat's view of exotic smooth structures

Consider first a summary of the ideas of Planat's article.

1. The construction of 3-manifolds from $\mathbb{S}^3$ relies on the handle body decomposition based on knots K of $\mathbb{S}^3$. Consider removal of $\mathbb{B}^3 \times S^1$, where $\mathbb{S}^1$ gives rise to a knot, and replacement by $\mathbb{B}^2 \times S^2$. The gluing is along the boundary $\mathbb{S}^2 \times S^1$ of the knot. The homotopy group $G = \pi(S^3 \setminus K)$ of the knot complement having two generators and serves as a knot invariant. Michel Planat demonstrates that if it has subgroup $\mathbb{H}$ with index $d = |G|/|H|$, it is possible to construct magic qudits by using G.

   Rather remarkably, the 3-manifolds $\mathbb{S}^3 \setminus K$ can be mapped to double coset spaces $\mathbb{H}^3/G = SO3) \setminus SO(1,3)/G$, where the homotopy group G is an infinite subgroup of $\mathbb{S}L(2, C)$.

2. A surgery by replacing 2- handle $\mathbb{B}^2 \times S^2$ with $\mathbb{B}^3 \times S^1$ along a suitably defined knot yields so-called Brieskorn manifolds with exotic smooth structures. Consider a four-manifold with boundary $\mathbb{S}^3$. The construction of the exotic manifold homeomorphic with the original manifold involves gluing a pair of $\mathbb{S}^3 \setminus K$ with handles removed giving 3-manifolds. The smooth structure is standard in the interior but cannot be continued to the boundary $\mathbb{S}^3$ of $\mathbb{Y}^4$.

   Akbulut cork is a simple example of a fake R^4 and corresponds to the simplest Brieskorn manifold. Connection with hyperbolic 3-space $\mathbb{S}^3 \setminus K$ corresponds to a coset space $\mathbb{H}^3/G$, $G = \pi_1(S^3 \setminus K)$.

3.4.2 How could the Planat's view of exotic smooth structures relate to TGD?

Some background in TGD needed to understand the possible connection of Planat's description of exostic smooth structures with the TGD view of exotic smooth structures [A5, A6, A3], conjectured to make possible particle vertices and fermion pair creation in TGD despite the fact that fermions in $\mathbb{H}$ are free [L17, L9, L22].

1. As describedl, the interactions are contact interactions between space-time surfaces identified as slightly non-deterministic Bohr orbits. If H-J structure exists, self intersections are 2-D string world sheets. This holds true also for self interactions generating string world sheets as counterparts of 2-knots. This makes sense at both $\mathbb{M}^8$ - and $\mathbb{H}$ sides.

2. In TGD, quantization takes place only for the free spinor fields of H. The creation of fermion pairs would be made possible by the smooth defects of the standard spinor structure at which the fermion line can change its direction and thus violate standard smoothness. Fermion would turn backwards in time. Exotic smooth structures are indeed characterized as defects of the standard smooth structure and can be assigned to 3-spheres S^3.

Consider now the connection with $\mathbb{H}^8 - \mathbb{HS}$ duality.

1. In TGD, $\mathbb{S}^3$ represents a "very special moment in the life of self" as a root of $f = 0$ at which the Euclidean and Minkowskian 4-surfaces associated with $\mathrm{Im}(f) = 0$ and $\mathrm{Re}(f) = 0$ can meet. Meeting occurs along $\mathbb{S}^3$ if both surfaces are of type $\mathbb{N}$ and along $\mathbb{S}^2 \subset \mathbb{S}^3$ is either of them is of type T. The surfaces $\mathbb{S}^3$ would relate to the classical non-determinism of holography and would be 4-D analogs for the non-deterministic frames of ordinary soap films at which several branches can meet.

2. In TGD, the interpretation $\mathbb{S}^3$ would be as generalized vertices. Pairs of fermion lines associated with the different branches can emerge meaning the creation of virtual fermion pairs in the classical fields defined by the induced spinor connection and the trace of the second fundamental form as analog of Higgs field becoming singular at the vertex but vanishing elsewhere by the minimal surface property. Fermion lines would correspond to boundaries of 2-D string world sheets associated with the intersection of the 4-surfaces Y_i^4 involved.

3. In the TGD view of cognition based on zero energy ontology (ZEO), the localization for the superposition of the slightly non-deterministic Bohr orbits would result in a sequence of "small" state function reductions (SSFRs) involving a measurement in cognitive degrees of freedom due to the slight classical non-determinism of the space-time surface as a "Bohr orbit". This would lead to particle decay when the state of the system is measured in "big" SFRs (BSFRs) involving interaction with the space-time surface representing the observer.

4. From the point of view of consciousness, this state function reduction means the death of self represented by the particle as it decays at vertex and produces particles as decay products. In this BSFR the self in question would reincarnate with the opposite arrow of geometric time. It could naturally correspond to the $\mathbb{S}^3$ appearing as an on-mass shell state. If 8-D masslessness condition $o_0^2 = r_7^2$ is satisfied at $\mathbb{S}^3$ ($f = o^2$), it implies for $\mathbb{Y}^4$ of type $\mathbb{T}$, Minkowskian massless at $\mathbb{S}^2$ meaning on mass shell property in $\mathbb{M}^4$ sense. For $f = o^2 - m^2$ one has a massive mass shell in point-point correspondence with $\mathbb{S}^3 \subset \mathbb{H}^3$. The $\mathbb{T}$ surface $\mathbb{Y}^4$ defines region of hyperbolic 3-space $\mathbb{H}^3$ realized in H proper time = constant surface of $\mathbb{M}^4$.

This allows to interpret the vision of Michel Planat in the TGD framework.

1. For $\mathbb{Y}^4$ and $\mathbb{X}^4$ 2-knots as string world sheets would correspond to self-intersections of the space-time surfaces. At the fundamental level, the 2-knot would correspond to a local doubling of the space-time sheet. These 2-knots could relate to defects of the standard smooth structure as 2-knots. At the string world sheet $\mathbb{Y}^4$ or $\mathbb{X}^4$ branches and transversal derivatives are discontinuous. These self intersections would occur at generalized vertices identified as roots of $f = 0$. The ordinary 1-knot K corresponds to the intersection of the string world sheet with $\mathbb{S}^3$ or $\mathbb{S}^2 \subset S^3$.

2. This suggests that the connection between knots K in $\mathbb{S}^3$ and hyperbolic coset space space $\mathbb{H}^3$ could come from $\mathbb{M}^8 - \mathbb{H}$ duality. Both $\mathbb{S}^3(o_0)$ for $f = 0$ and the union $\cup_{0_0}\mathbb{S}^2(o_0)$ define subsets of $\mathbb{H}^3$. $\mathbb{H} - \mathbb{M}^8$ can be injective in either direction: from $\mathbb{H}$ to $\mathbb{M}^8$ or vice versa. The not $K \subset \mathbb{S}^3$ considered by Planat could correspond to knots K at $\mathbb{S}^3(o_0)$ or at at the orbit $\cup_{o_0}\mathbb{S}^2(o_0)$. In both cases, they correspond to a region of $\mathbb{H}^3$.

   What is interesting is that time-like and space-like knots and braids appear in the TGD based model of quantum computation [K2, K1] based on flux tubes and the motion of their ends. The model involves ance metaphor: the dance defines a time-like braid and if the dancers are connected by space-like strings to a wall, also a space-like braid is generated.

3. This would allow us to understand why the manifolds $\mathbb{S}^3 \setminus K$ correspond to double coset spaces $\mathrm{G} \setminus \mathbb{H}^3$, where G is the homotopy group $\pi_1(K)$. The ends of the open knot at $\mathbb{H}^3$ are identified in the definition coset space and it would be mapped to a closed knot K at $\mathbb{S}^3$ subset $\mathbb{M}^8$? At the level of $\mathbb{H}$, this would be a relative homotopy. The natural assumption is that the ends of K correspond to the same point of $\mathbb{CP}_2$ in the relative homotopy.

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