New findings related to the number theoretical view of TGD

M. Pitkänen,

February 2, 2024
Email: matpitka6@gmail.com.
http://tgdtheory.com/public_html/.
Postal address: Rinnekatu 2-4 A 8, 03620, Karkkila, Finland. ORCID: 0000-0002-8051-4364.

Contents

Abstract

The geometric vision of TGD is rather well-understood but there is still a lot of fog in the number theoretic vision.

1. There are uncertainties related to the interpretation of the 4-surfaces in M^8 what the analogy with space-time surface in $H = M^4 \times CP_2$ time evolution of 3-surface in H could mean physically?
2. The detailed realization of $M^8 - H$ duality involves uncertainties: in particular, how the complexification of M^8 to M^8_c can be consistent with the reality of $M^4 \subset H$.
3. The formulation of the number theoretic holography with dynamics based on associativity involves open questions. The polynomial P determining the 4-surface in M^8 doesn't fix the 3-surfaces at mass shells corresponding to its roots. Quantum classical correspondence suggests the coding of fermionic momenta to the geometric properties of 3-D surfaces: how could this be achieved?
4. How unique is the choice of 3-D surfaces at the mass shells $H^3_m \subset M^4 \subset M^8$ and whether a strong form of holography as almost $2 \rightarrow 4$ holography could be realized and make this choice highly unique.

These and many other questions motivated this article and led to the observation that the model geometries used in the classification of 3-manifolds seem to be rather closely related to the known space-time surfaces extremizing practically any general coordinate invariant action constructible in terms of the induced geometry.

The 4-surfaces in M^8 would define coupling constant evolutions for quantum states as analogs of and mappable to time evolutions at the level of H and obeying conservation laws associated with the dual conformal invariance analogous to that in twistor approach.

The momenta of fundamental fermions in the quantum state would be coded by the cusp singularities of 3-surfaces at the mass shells of M^8 and also its image in H provided by $M^8 - H$ duality. One can consider the possibility of $2 \rightarrow 3$ holography in which the boundaries of fundamental region of H^3/Γ is 2-D hyperbolic space H^2/Γ so that TGD could to high degree reduce to algebraic geometry.

1 Introduction

TGD could be seen as a holy trinity of three visions about quantum physics based on physics as geometry, physics as number theory, and physics as topology.

Quite recently I gave a talk on TGD and TGD inspired theory of consciousness and was asked about the motivations for the number theoretic vision. My response went roughly as follows.

1. The attempt to find a mathematical description for the physical correlates of cognition could have led to the vision of quantum TGD as a number theory. What are the possibly geometric/number theoretic/topological correlates of thought bubbles?

   A bold guess could have been p-adic numbers, $p = 2, 3, 5, 7, ...$ provide natural mathematical correlates for cognition. Rationals, algebraic extensions of rationals, and the extensions of p-adic number fields induced by them are natural candidates as also complex numbers, quaternions, and octonions. Also finite number fields emerged quite recently as natural ingredients of the number theoretic vision [K11, K12, K10] [L16].

   As a matter of fact, I ended up to a proposal that p-adic physics provides the correlates of cognition via a different route, by p-adic mass calculations based on p-adic thermodynamics, which turned out to have surprisingly high predictive power due to the number theoretic existence conditions [K5].

2. Sensory experience corresponds to real number based physics. There is a strong correlation between cognition and sensory experience, but it is not perfect. Sensations arouse thoughts, but cognition is also able to dream and imagine.

3. Cognition includes mathematical thought. The concretization of mathematical thinking as computation requires discretization. This suggests that discretization should correspond to what one might call a cognitive representation transforming thoughts to sensory percepts and it should have a number theoretic representation.

4. Mathematical thinking is able to imagine spaces with an arbitrary dimension, while the dimension of the perceptual world is fixed and is the dimension of three-space. How does cognition achieve this?

5. Cognition has evolved. Why and how can this be the case?

6. If the correlates of cognition are part of reality, then cognition must be optimally efficient. How?

This leads to the following questions and answers.

1. Could p-adic spacetime surfaces represent thought bubbles, equivalent to real 4-surfaces? They are a number-theoretic concept, they also involve a different topology than the sense-world, and p-adic space-time surfaces would be examples of algebraic geometry.

2. How is cognition able to imagine? p-Adic differential equations are non-deterministic: integration constants, which by definition have vanishing derivatives are only piecewise constants. Could this make imagination possible [K6]??

3. How the strong correlation between cognition and sensory experience could be realized? All p-adic number fields and their extensions must be allowed. Consider first the simplet book involving only reals and p-adic number fields. p-Adic number fields Q_p, $p = 2, 3, 5, ...$ can be combined into a book, an adele [L1, L2]. Different number fields as extensions of rationals represent the pages of this book. Real numbers correspond to sensory experience and various p-adic number fields to cognition. The back of the book corresponds to rational numbers that are common to all chapters.

   Every algebraic extension of rationals defines extensions of p-adic number fields. The p-adic pages of the algebraically extended book are algebraic extensions of various p-adic number fields. One obtains an infinite library with books labeled by algebraic extensions of rationals.

   Now the back of the book consists of algebraic numbers for the extension generated by the roots of a polynomial with integer coefficients. The back of the book gives a cognitive representation, a number theoretic discretization of the 4-surface that is unique for a given extension. The bigger the extension, the more accurate the discretization. Cognitive evolution would correspond to a refinement of cognitive representation induced by the increase of the order of the polynomial defining the extension.

4. How is cognition able to imagine higher-dimensional mathematical objects that do not exist at the level of sensory experience? algebraic extensions for p-adic numbers can have an arbitrarily high dimension if the corresponding polynomial has high enough degree. One can have p-adic 4-surfaces for which the associated algebraic dimension is arbitrarily high! p-Adic cognition is liberated from the chains of matter!

5. Why is evolution related to cognition? One gets an infinite number of books labeled algebraic extensions, a whole library. Does the evolution of cognition present a hierarchy? The bigger the algebraic extension, the better the approximation to real numbers and thus to sensory experience.

6. Can p-adic cognition be maximally effective? Here p-adic thermodynamics suggests the answer. p-Adic mass calculations assign to each elementary particle a p-adic prime. For instance, for electrons it is Mersenne prime $p = M_{127} = 2^{127} - 1 \simeq 10^{38}$. p-Adic mass squared value is expansion powers of p and its real counterpart is power series in negative powers of p. This series converges extremely rapidly for large primes such as $p \simeq 10^{38}$ and two lowest orders give a practically exact answer so all errors would be due to the assumptions of the model rather than due to computations.

How to realize number theoretic physics?

1. Number theoretic discretization does not resonate with the idea of general coordinate invariance. For $H = M^4 \times CP_2$ allows linear Minkowski coordinates but CP_2 coordinates are not

linear although also now complex coordinates consistent with the isometries of $SU(3)$ are natural.

What about M^8 or possibly its complexification suggested by twistorial considerations and also by the fact that classical TGD predicts that Euclidian space-time regions give an imaginary contribution to the conserved four momenta. M^8 allows highly unique linear Minkowski coordinates and the idea that M_c^8 corresponds to complexified octonions is very natural. The automorphism group G_2 of octonions poses additional conditions.

2. This leads to the idea that number theoretic physics is realized at the level of M_c^8 and that it is dual to the geometric physics realized at the level of H and that these physics are related by $M^8 - H$ duality mapping 4-D surfaces in M^8 to H. TGD can be regarded as a wave mechanics for point-like particles replaced with 3-D surfaces in H, which, by the failure of complete determinism for holography, must be replaced by analogs of Bohr orbits. Wave mechanics is characterized by momentum-position duality, which naturally generalizes to $M^8 - H$ duality [L4, L5, L15, L16].

3. The physics in M_c^8 should be purely algebraic as is also the ordinary physics at the level of momentum space for free fields. This physics should make sense also in all p-adic number fields. This suggests that polynomials with integer coefficients, in particular their roots, together with number theoretic holography based on associativity, partially characterize the 4-surfaces in M^8, which would make sense also as their p-adic variants.

   It is not clear whether the p-adicization is needed at the level of H: it might be enough to have it only at the level of M^8 so that only the p-adic variants of M^8 would be needed.

The geometric vision of TGD is rather well-understood (see for instance [L8]), but one need not think long to realize that there is still a lot of fog in the number theoretic vision (see for instance [K11, K12, K10] and [L4, L5, L6, L15, L16]).

1. There are uncertainties related to the interpretation of the 4-surfaces in M^8 what the analogy with space-time surface in $H = M^4 \times CP_2$ time evolution of 3-surface in H could mean physically?

2. The detailed realization of $M^8 - H$ duality [L4, L5] involves uncertainties: in particular, how the complexification of M^8 to M_c^8 can be consisted with the reality of $M^4 \subset H$.

3. The formulation of the number theoretic holography with dynamics based on associativity involves open questions. The polynomial P determining the 4-surface in M^8 doesn't fix the 3-surfaces at mass shells corresponding to its roots. Quantum classical correspondence suggests the coding of fermionic momenta to the geometric properties of 3-D surfaces: how could this be achieved?

4. How unique is the choice of 3-D surfaces at the mass shells $H_m^3 \subset M^4 \subset M^8$ and whether a strong form of holography as almost $2 \rightarrow 4$ holography could be realized and make this choice highly unique.

5. The understanding of 3-geometries is essential for the understanding of the holography in both M^8 and H. The mathematical understanding of 3-geometries is at a surprisingly high level: the prime 3-manifolds can be constructed using 8 building bricks. Do these building bricks, model geometries, have counterparts as prefered extremals of action in the TGD framework.

   The known extremals $X^4 \subset H$ satisfying holography should be analogues of Bohr orbits [L9]. They are proposed to satisfy a 4-D generalization of 2-D holomorphy and apart from lower-D singularities would be the same for any general coordinate invariant action based on induced geometry and spinor structure. They would be minimal surfaces both in H and M^8 except at singularities at which the details of the action principle would matter [L24]. This suggests that the preferred extremals could have maximal isometries and provide topological invariants as also do the model geometries in the classification of 3-geometries.

2 What does one mean with M^8 physics?

In TGD, the point-like particle is replaced by a 3-surface $X^3 \subset H = M^4 \times CP_2$, and the holography required by the general coordinate invariance requires the replacment of the 3-surfaces with the analogues of Bohr trajectories passing through them. The Bohr trajectories are not completely deterministic as already the case of hydrogen atoms demonstrates. The "World of Classical Worlds" (WCW) is thus the space of generalized Bohr orbits as the counterpart of the superspace of Wheeler which originally inspired the notion of WCW [L23, L24, L20, L21]).

In wave mechanics, the duality between the descriptions using momentum and position space applies in wave mechanics but does not generalize to field theory. The $M^8 - H$ duality [L4, L5] can be seen as a generalization of this duality. M^8 is the momentum space counterpart and $H = M^4 \times CP_2$ is the position space counterpart in this duality.

2.1 Physical interpretation of the 4-surfaces of the space M^8 and their singularities

The physical interpretation of 4-surfaces in the complexifixation of the momentum space M^8 is far from straightforward. There are many reasons for the complexification.

1. Complexified octonionicity requires that M^8, or equivalently E^8, is complexified: one has $M^8_c = E^8_c$ giving as its subspaces various real subspaces with various signatures of the number theoretical. The metric obtained from the Minkowski norm $\delta_{kl} z^k z^l$, where z^k are 8 complex coordinates. M^4 with signature $(1, -1, -1, -1)$ is in a special physical role and one can of course ask, whether also other signatures might be important.

2. If complex roots are allowed for polynomials P determining together with associativity the holography, complexification must be allowed. Virtual momenta could therefore be complex, but Galois confinement says that the total momenta of physical states are real and have integer components in the momentum scale determined by the size of the causal diamond (CD). Physical intuition suggests that the imaginary parts of the momenta code for the decay width of the particle. This is natural if the imaginary part is associated with the energy in the rest system.

3. The conserved momenta given by Noether's theorem at the level of H have real parts assignable to Minkowskian space-time regions. The fact that $\sqrt{g_4}$ appears in the integral defining a conserved quantity differs in Minkowski and Euclidean regions by an imaginary unit suggests that the contributions to momenta from the Euclidian regions are imaginary. The momenta from the Minkowskian space-time regions can be transferred to the light-like boundaries between Minkowskian and Euclidian regions identified as light-like partonic orbits. Quantum-classical correspondence requires that the classical total momenta, like all conserved quantities, correspond to the total momenta of the fermion state.

   Euclidian regions most naturally correspond to CP_2 type extremals as preferred extremals. They can be regarded as singularities resulting in the blow up of tip-like cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`) in M^8 . This would suggest that the real parts of momenta are associated with the Minkowskian regions of space-time surfaces and imaginary parts to the Euclidian regions. This applies also to other conserved quantities.

2.2 Number theoretic holography

Number theoretic holography has two forms.

1. The weak $3 \rightarrow 4$ form corresponds to the ordinary holography $Y^3 \subset M^8 \rightarrow Y^4 \subset M^8$, which is by $M^8 - H$ duality equivalent of the holography for $X^3 \subset H \rightarrow X^4 \subset H$ for space-time surfaces. The proposed interpretation of Y^3 is as a fundamental region of H^3/Γ.

2. For the strong $2 \rightarrow 4$ form of the holography Y^3 is determined by a 2-D data defined by the boundary of the fundamental region of H^3/Γ. The proposal to be considered is that the boundary of the fundamental region of H^3/Γ can be identified as 2-D hyperbolic space H^2/Γ.

Consider next the weak form of the holography.

1. The 4-surface $Y^4 \subset M_c^8$ is determined from number-theoretic dynamics and is an associative surface, i.e. its normal space is associative and therefore quaternionic.

2. There are also commutative 2-D surfaces that most naturally correspond to string world sheets, and for them commutativity of tangent space (as analog of associativity) as subspace of normal space of Y^2 defines holography. Holographic data corresponds now to strings connecting wormhole contacts assignable to Euclidian singularities inside $Y^3 \subset H_m^3$. One can also consider the possibility that partonic 2-surfaces correspond to co-commutative 2-surfaces. The situation is not completely clear here.

3. One must also identify the 3-surfaces $Y^3 \subset H_m^3$ defining the holography. Holography is subject to very strong conditions and I have proposed that these surfaces are hyperbolic 3-manifolds X^3 obtained as coset spaces H^3/Γ, where G is suitably chosen discrete but infinite subgroup of $SL(2,C)$ acting as Lorentz transformations in H^3. The spaces H^3/Γ are fundamental domains of H^3 tessellations.

   $Y^3 = H^3/\Gamma$ is counterpart for the unit cells of a lattice in E^3, which effectively has this topology and geometry due to boundary conditions stating that G leaves various "field configurations" invariant. The situation is the same as in the case of ordinary condensed matter, where periodic boundary conditions for a cube as a unit cell make it effectively a 3-torus.

   Also the crystal-like structures consisting of a finite number of copies of the fundamental domain of H^3/Γ glued together are possible choices for Y^3. They would be analogous to the unit cells of the lattices of Euclidian space E^3 or finite crystals formed from them. Therefore the analog of solid state physics would be realized at the fundamental level.

   One can also consider closed 3-manifolds $Y^3 = H^3/\Gamma$ obtained by gluing two copies of the fundamental region with different S^3 coordinates connected together along their 2-D boundaries. The gluing could be performed by a cylinder of $S^3 \subset E^4 \subset M^4 \times E^4$ connecting the boundaries.

4. The quantum state at H_m^3 consists of several Galois singlets assignable to 3-surfaces Y_i^3. The total momenta for X_i^3 would be real and have integer valued components for the momentum unit defined by the size scale of CD involved. This condition is analogous to the periodic boundary conditions.

5. Quantum classical correspondence requires that the many-fermion state on H_m^3, characterized partially by momenta, which are in the algebraic extension of rationals associated with the polynomial P, determines $Y_i^3 \subset H_m^3$. For a given Y_i^3, the accompanying fermions correspond to the points of H_m^3. The classical momentum of the state given by Noether theorem in H would the sum of the fermionic momenta.

   An attractive idea is that at least a subset of the fermionic momenta corresponds to cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`), which can be visualized geometrically as vertices of an algebraic surface at which the direction of normal space is ill-defined.

   The cusps correspond to parabolic subgroups $P \subset G \subset SL(2,C)$ (`https://rb.gy/b5t55`), where the $\Gamma \subset SL(2,C)$ defines the fundamental domain H^3/Γ of the tessellation. Parabolic subgroups P are automorphic to the subgroup of translations of upper half-plane H generated by $SL(2,C)$ matrix $(1a;0,1)$, a a real algebraic number. This particular P acts as Möbius transformations in H representing hyperbolic space H^2. The cusp singularities would encode at least a subset of fermionic momenta of the state into the hyperbolic geometry of Y_i^3. Each fermion would correspond to its own parabolic generator in the subgroup G.

   In the TGD view of hadron physics [L18], the fermions associated with the cusps could be identified as analogs of valence quarks. They would be associated with singularities identifiable as light-like 3-D partonic orbits serving as boundaries of 4-D CP_2 type extremals with Euclidian signature of the induced metric.

   Also fermionic momenta, which have algebraic integers as components but do not correspond to cusps, can be considered. These would be naturally associated with strings predicted to connect cusps at the throats of different wormhole contacts. The blow-up would be now 2-sphere relating to cusp singularity like line charge to point charge. It is not clear whether the sea partons could correspond to these string-like singularities. In any case, hyperbolic 3-manifolds have string-like singularities connecting cusps.

6. If Y_i^3 corresponds to a Galois singlet, then its total 4-momentum is real and integer-valued and should be mapped to a discrete plane wave in the finite lattice defined by the crystal like structure formed by the copies $X^3(Y^3)$ in H_a^3 given by inversion. Each Galois singlet Y_i^3 would define such a plane wave and one can imagine a hierarchy of such structures just as in the case of condensed matter with crystals of different sizes. The analogy with condensed matter physics suggests that Γ is a lattice. This follows also from the condition that H^3/Γ has a finite volume.

7. This picture would suggest that also $X^3(Y_i^3)$ is hyperbolic manifold of its fundamental region and perhaps isometric with Y^3. This would mean a geometric realization of Lorentz invariance analogous to the dual of conformal invariance encountered in the twistorialization.

2.3 Quantum classical correspondence for momenta

Quantum classical correspondence for momenta and also other conserved charges poses very strong conditions.

1. Noether charges for the classical action define momenta and other conserved charges. The classical contribution is a c-number. In addition, quantum contributions from fermions are included. They correspond to the momenta related to the second quantized spinor modes of H and from the orbital degrees of freedom associated with the "world of classical worlds" (WCW). The fermionic contributions are related to the ground states of the super symplectic representation characterized in terms of spinor modes for H spinor fields [K4, K8] [L16].

2. Are the classical contributions separate and do they add up to the total momentum? The fact that classical contributions are separately conserved, does not support this view.

3. Quantum classical correspondence would mean that the classical total momentum is the sum of the fermionic momenta determined by the multi-fermion state. This would hold quite generally for Cartan algebra of observables. For example, in the case of hadrons, the dominant classical contribution could correspond to the gluon sea, that is to multi-gluon state with gluons expressible in terms of quark-antiquark pairs. This picture is consistent with QCD and is therefore perhaps a more realistic guess.

4. Wormhole contact has Euclidean induce metric and the related classical conserved momentum is naturally imaginary. Could the sum of the imaginary parts of complex fermionic momenta of fermions for a wormhole throat correspond to the classical imaginary momentum assignable with the wormhole contact? Could the imaginary part of the fermionic momentum be assigned with the end of the euclidean string inside CP_2 type extremals, while the real momentum would be assigned with an end of a Minkowskian string?

5. Quantum-classical correspondence would be realized if the fermionic conserved four-momenta on the H side corresponded to M^8 points at hyperbolic 3-surfaces H_m^3. Their inversions in the M^8-H duality would be points of M_c^4 of the spacetime surface $H_m^3 \subset M_c^4 \times CP_2$. It would seem that one must map only the real parts of the momenta at H_m^3 to H_a^3.

   It would also seem that H_m^3 must be associated with the M^4 projection of M_c^4. Whether the variant of H_m^3 for complex valued m^2 makes even sense is not obvious.

2.4 The analog of time evolution in M^8 as a coupling constant evolution conserving dual quantum numbers

The proposal that 4-D surfaces appear at the level of M^8 suggests that it makes sense to talk about dynamics also in M^8 and the 4-D analogies of space-time surfaces make sense. This does not fit the usual classical intuition.

The twistor picture for conformal invariant field theories predicts that conformal invariance has a dual counterpart. This would mean that 4-momenta and other Poincare charges in H have dual counterparts in M^8. In TGD, the dual counterparts would be obtained by inversion from the defining the M^8-H duality and mapped to points of the space-time surface at the mass shells H^3_a in H. They would be analogs of the representation of lattice momenta as points of the heavenly sphere in crystal diffraction.

1. In zero energy ontology (ZEO), the time evolution at the level of H by "small" state function reductions (SSFRs), which are analogous to the so called weak measurements introduced by quantum opticians, would correspond to time evolution in terms of scalings rather than time translations. They would scale the size of the causal diamond (CD) and leave the passive boundary of CD invariant. These analogs of time evolutions would be generated by the scaling generator L_0. This would naturally also apply to M^8. This time evolution would be induced by scalings of the mass-scale, which need not be identical.

2. Could the "energy evolution", by identifying the square of the mass as the counterpart of time, correspond to the development related to the renormalization group? M^8-H duality would map the renormalization group evolution from M^8 to time evolution in H. This is quite a strong prediction.

3. Mass squared values for the fundamental fermions would not define particle masses but mass scales. 4-momenta for physical particles would correspond to total momenta for many fermion states, which obey Galois confinement, which requires that the momentum components are integers, when the mass unit is defined by the size scale of CD.

4. What would be the interpretation of the mass shells M^4_c determined by the roots of the polynomial P in the coupling constant evolution? Could the related hyperbolic 3-manifolds correspond to fixed points for the coupling constant evolution? With these mass values, something special would happen. Could H^3_a correspond to critical moments of light-cone proper time a when the SSFRs occur and a new unitary time evolution begins and ends with the next SSFR, as I have suggested?

5. What about the M^8 side? Could one talk about conserved quantities with respect to the evolution determined by scalings? Could these dual charges, dual momenta, and. also the charges related to E^4 isometries, be invariants of the renormalization group evolution?

   I have proposed that the $SO(4)$ symmetry of of hadrons in old-fashioned hadron physics involving notions like conserved vector current (CVC) and partially conserved axial current (PCAC) could correspond to the color symmetry of higher energy hadron physics by M^8-H duality in which the natural conserved charges on M^8 side are associated with the product of isometry groups of M^4 and E^4 and perhaps even with $SO(1,7)\times T^8$ or G_2 as automorphism group of octonions. At H side one would have a product of Poincare group and color group. Also holonomy groups are involved. At least $SO(4)$ symmetry could define invariants of the coupling constant evolution in M^8.

Consider now a more detailed interpretation of 4-surfaces $Y^4\subset M^8$ in terms of a generalized coupling constant evolution.

1. The changes $m_i^2\rightarrow m_{i+1}^2$ for the roots of P would define a discrete evolution in both M^8 and H. Discrete coupling constant evolution affects the mass resolution and brings in or deletes details and therefore would induce changes for the representations of the states. The 4-surfaces in M^8 would represent renormalization group flows. The failure of a complete determinism is expected and could be interpreted in terms of phase transitions occurring at critical masses.

2. A given mass shell m_i^2 determined by a root of P would define a discrete mass scale for the evolution having perhaps an interpretation as a fixed point or a critical point of the coupling constant evolution. It would be natural to assume that the evolution induced by the change of resolution conserves other total quantum numbers than 4-momentum.

3. What about the conservation of 4-momentum? At $m^2 = m_{i+1}^2$ the value of mass squared for fundamental fermions defining the mass scale changes. The structure of the state must change in $m_i^2 \to m_{i+1}^2$ if 4-momentum conservation is assumed.

   The normalization group evolution for the mass m^2 of the physical state, is typically logarithmic in QFTs, and must be distinguished from the discrete evolution for the mass scale m_i^2. Hence the change of m^2 in $m_i^2 \to m_{i+1}^2$ is expected to be small. This could be realized if n corresponds to a (possibly normal) subgroup of the Galois group of P perhaps spanned by the roots $m_k^2 \leq m_i^2$.

   Could the phase transition $m_i^2 \to m_{i+1}^2$ change the rest energy of the state? Does the change require an energy feed between the CD and its environment as ordinary phase transitions require? This is not the case if CD is interpreted as a perceptive field rather than a physical system.

4. Does it make sense to talk about the conservation of dual momentum $X^k = \sum_i X_i^k$, $X_i^k = Re(\hbar_{eff} p^k / p_{k,i} p_i^k) = Re(\hbar_{eff} p^k / m_i^2)$? The conservation of momentum p^k does not imply the conservation of dual momentum since it is proportional to $1/m_i^2$: X^k would scale as $1/m_i^2$. The size of the CD is assumed to increase in statistical sense during the sequence of small state function reductions (SSFRs). The increase of the size of the CD would gradually make the mass shells inside it visible.

   $M^4 \subset H$ center of mass position X^k therefore changes $m_i^2 \to mi + 1^2$ unless h_{eff} is not scaled to compensate the change $m_i^2 \to mi + 1^2$ in the formula for X_i^k. The integer n in $h_{eff} = nh_0$ is assumed to correspond to the order of the Galois group of P. It could also correspond to the order of a subgroup of the Galois group of P defined by the roots m_k^2, $k \leq i$. If so, h_{eff} would increase in evolution and one can even imagine a situation in which $Re(h_{eff}/m_i^2)$ remains constant.

3 $M^8 - H$ duality

The proposed $M^8 - H$ duality maps 4-surfaces $Y^4 \subset M_+^4 \subset M^8 = M^4 \times E^4$ to space-time surfaces $X^4 \subset M_+^4 \subset M^4 \subset M^4 \times CP_2$.

3.1 $M^8 - H$ duality as inversion

$M^8 - H$ duality relates also the hyperbolic spaces $H_m^3 \subset M_+^4 \subset M^8 = M^4 \times E^4$ and $H_a^3 \subset M_+^4 \subset M^4 \subset M^4 \times CP_2$. The hyperbolic space $H_m^3 \subset M_+^4 \subset M^8$ corresponds to the mass shell for which mass squared value m^2 is a root of a polynomial P. The hyperbolic space $H_a^3 \subset M^4 \subset= M^4 \subset M^4 \times CP_2$ corresponds to light-cone proper time constant surface $t^2 - r^2 = a^2$.

1. The root of P is in general a complex algebraic number. The first guess is that $M^8 - H$ duality is defined by inversion $p^k \to m^k = \hbar_{eff} p^k / p^l p_l$. Or briefly, $p \to \hbar_{eff} p / m^2$. The light-cone proper time $a = h_{eff}/m$ characterized the hyperboloid $H_a^3 \subset M^4$. $H_m^3 \to H_a^3$ is consistent with the Uncertainty Principle. In this case the image would be complex. This creates interpretational problems. There is no need for the complexification of CP_2, which also suggests that the image un H_a^3 must be real.

2. One can consider the possibility that only the real projection of the complex variant of H_m^3 to $H_{Re(m)}^3$ is involved. The image of the real part $Re(p^k)$ in H_a^3 obtained by inversion would be real but would not code information about the imaginary part $Im(p^k)$.

3. One could however take the real part of a complex inversion to get a point in H_a^3. $Re(\hbar_{eff} p^k / p^2$ would code information about the imaginary value of m^2.

Inversion fails at the light-cone boundary with $m = 0$. In this case, the inversion must be defined as the inversion of the energy of the massless state: $p^k \to \hbar_{eff} Re(p^k / E^2)$.

3.2 The technical problems posed $M^8 - H$ duality the complexification of M^8

Complexification of M^8 is highly desirable in the number-theoretic vision. But how to deal with the fact that fermion momenta, for which with components are algebraic integers in the algebraic extension determined by a polynomial P, are in general complex?

1. Without additional conditions, the mass shells in $M_c^4 \subset M_c^8$ for complex m^2 as a root of P are 6-D. There are 2 conditions coming from the conditions fixing the value of $Re(m^2) = Re(p^2) - Im(p^2)$ and $Im(m^2) = 2Re(p) \cdot Im(p)$. If one only energy is complex, the dimension of the mass shell is 3. This looks natural. The preferred time axis would be determined by the rest system for massive states. A possible interpretation for the imaginary part is as decay width in the rest system.

2. The complexified mass shells of complexified $M^4 \subset H$ must be considered. Does this make sense? Since the CP_2 point labelling tangent space of Y^4 does not depend on complexification there is no need to consider complexification of CP_2. Therefore the natural conclusion is that also the $M^4 \subset H$ images should be real.

The inversion $p^k \rightarrow p^k/m_r p_r p_s$ is the simplest realization for $M^8 - H$ duality and would naturally fit into a generalization of 2-D conformal invariance to 4-D context. $h_{eff} = nh_0$ hierarchy comes along in a natural way. The polynomial P determines the algebraic extension and the value of h_{eff}. The size of the CD would scale like h_{eff} on the H side. There would be no scaling on the M^8 side.

1. The first thing to notice is that one could understand classically complex momenta. On the H side, Euclidean regions could give an imaginary contribution to the classical momentum.

2. The complex inversion $p^k \rightarrow p^k/m_r p_r p_s$ maps complex H_m^3 to complex H_a^3. What would be the interpretation of the complex M_c^4 coordinates? The same problem is also encountered in twistorization. One can ask whether a complex time coordinate corresponds to, for example, the inverse temperature?

   However, since no complexification is needed for CP_2, it seems that the only natural option is that the $M^4 \subset H$ image is real.

3. One can consider 3 options guaranteeing the reality.

   (a) Only the real parts of the complex $M^4 \subset M^8$ momenta are mapped to H. The information about the imaginary parts would be lost.

   (b) The complex algebraic integer valued momenta are allowed and the real part $Re(p^k \rightarrow p^k/m_r p_r p_s)$ of the complex inversion defines the image points in H. The $M^4 \subset H$ complexification would not be needed for this option but the information about the imaginary part of the momenta would not be lost.

   (c) By Galois confinement, the physical multiparticle states consist of momenta with integer value components using the momentum unit assignable to CD at the M^8 level of space with mass shells. These would define the 3-D data for the holography, which determines the 4-surface $Y^4 \subset M^8$ through the associativity of the normals space of Y^4. Only the real, integer-valued momenta of Galois confined states would be mapped from the mass shells of $M8$ to their images in H.

   The information about the fermion composition of the many-particle states would be lost completely. Therefore this option does not look realistic.

   The realization of this view might be possible however. 4-momenta determine the 3-surfaces Y^3 with real mass shells H^3 as data for associative holography. Momenta could correspond to point-like singularities on $Y^3 \subset H^3$ and these should be assigned as CP_2 type extremals at H side as blow-ups of the singularities.

To conclude, the option $p^k \rightarrow Re(p^k \rightarrow p^k/m_r p_r p_s)$ seems to be the physically realistic option.

3.3 Singularities and $M^8 - H$ duality

Consider next the description of the cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`) in $M^8 - H$ duality. The condition that information is not lost, requires that the map is given by $p^k \to Re(p^k/m_{rs}p^r p^s)$.

1. The cusps in $M^8 - H$ duality would be mapped to a 3-D surface of CP_2. It would correspond to the 3-D section CP_2 of the extremum, which corresponds to a wormhole contact associated with fermions at its throats.

   At H^3_m there is a temptation to assign to the cusp singularity, identified as a blow-up, the 3-D sphere $S^3 \subset E^4$ of the normal space E^4 defined by the mass shell condition. The simplest option is that this sphere is mapped to $U(2)$ invariant sphere $S^3 \subset CP_2$ for which the radius would be fixed by the mass squared value.

   The metric of H^3/Γ is singular at the cusp. The elimination of the singularity requires that one must allow a hole Z^3 around the cusp. The boundary of X^3 can have any genus. The size scale of the hole should be determined by the mass squared value.

   This view conforms with the physical picture of the CP_2 type extremal as an orbit of an Euclidian wormhole contact connecting two Minkowskian space-time sheets. S^3 would be replaced with $S^3 \setminus Z^3$ mapped to CP_2, where it corresponds to a wormhole throat having arbitrary genus.

   This picture would suggest that a given cusp singularity can correspond only to a single wormhole throat. This is not in conflict with the recent view of what elementary particles having wormhole contacts as composites should be. Composite, involving 2 wormhole contacts (required by the conservation of the monopole flux forming a loop involving two space-time sheets) and therefore 2 wormhole throats, can have spin varying from 0 to 2 which conforms with the popular wisdom that elementary particle spins vary in this range.

2. In the case of string-like objects $Y^2 \times R \subset H^3_m$, that is $S^2 \times R$ and their higher genus counterparts $H^2/\Gamma \times R$, the counterpart of the blow-up would be $Y^2 \subset S^3 \subset E^4$. Y^2 would be mapped to $X^2 \subset CP_2$ such that the radius assignable to S^2 or the size scale assigned to H^2/Γ would correspond to the mass squared.

3. Fermion trajectories at the partonic orbits would be light-like curves defining boundaries of string world sheets. CP_2 extremal would be associated with a fermionic cusp by holography and $M^8 - H$ duality.

   Fundamental fermion as an analog of valence quark [L18] could correspond to a cusp at the boundary of the string world sheet. Cusps would be related to the boundary of X^3_a composed of partonic 2-surfaces.

4. In principle, fermion momenta in the interior of $Y^3 \subset H^3_m$ are also possible. The picture given by hadron physics would indicate that the interior contribution corresponds to the sea partons. They can also be associated with string world sheets and correspond to virtual bosons appearing as fermion-antifermion pairs. These singularities would be string-like objects of the form $X^2 \times R$ and $X^2 \subset CP_2$ would replace the sphere of CP_2. One could say that fermions are delocalized at string.

3.4 Realization of the Uncertainty Principle

Inversion alone is not enough to realize Uncertainty Principle (UP), which requires that $M^8 - H$ duality is analogous to the Fourier transform. However, with the help of H^3 tessellations, it is possible to understand how the UP is realized in a finite measurement resolution.

The invariance of points of H^3/Γ under the subgroup $G \subset SL(2,C)$ is analogous to the periodic boundary conditions that replace the cubic unit cell of a crystal lattice with a torus. Now the tessellation of H^3, which quantizes the momenta, would be replaced by H^3/Γ. The momentum lattice having the fundamental region for H^3_m/G as unit cell would be mapped by inversion to a position lattice having the fundamental region H^3_a/G as a unit cell. A point in H^3_m would correspond to an analog of plane wave as a superposition of all positions of

$X^3(Y^3)$ in a part of the tessellation in H_a^3: a wave function in finite crystal. One would have a superposition of 3 surfaces X_a^3 corresponding to different lattice points multiplied by the phase factor. For a multi-fermion state the cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`) assigned to the momenta of fermions would characterize H^3/G so that the information about ("valence") fermion state would be code geometrically. Similar coding would be realized also for the string-like entities $H^2/\Gamma \times R$ at H^3/Γ. What is new and surprising, and also challenges the interpretation, is that the genus of H^2/Γ would code for the momenta of many-fermion states. Does the number of fermion-antifermion pairs correlate with the genus which in turn is proposed to label fermion families? There would be one fermion-antifermion pair per single handle. This would conform with the quantum classical correspondence. The proposed explanation [K3] for the number of observed fermion families would be in terms of hyper-ellipticity meaning that Z_2 acts as a conformal symmetry for all genera smaller than 3. Genus two would correspond to a formation of a bound state of two handles. Could this mean a formation of a graviton-like bound state of 2 fermion pairs and that higher spin states are not possible as bound states of handle. If fermions correspond to cusp singularities surrounded by holes, this picture might make sense: fermion antifermion pair would correspond to two holes connected by a handle.

The $M^8 - H$ duality maps the surface $Y^4 \subset M^8$ to the space-time surface $X^4 \subset H$. The point of $M^4 \subset H$ is obtained as the real part of the inversion of the point of the M^4 projection of the surface Y^4.

There would be a direct analogy to the physics of condensed matter.

A hyperbolic 3-manifold would correspond to a fundamental domain of a tessellation. It would be the equivalent of a unit cell both in position space and momentum space. These unit cells would correspond to each other at the H^3 level by $M^8 - H$ duality. Both would involve discretization. By finite momentum and position resolution UP would be reduced to the interior of the finite tessellation analogous to finite crystal. Quantum-classical correspondence and inversion are consistent with the realization of the UP related to Bohr's orbitology. Momenta in H_m^3 would be mapped to equivalents of plane waves, i.e. superpositions of positions of the fundamental region in the tessellation. This picture generalizes to the multi-fermion states. Each fermion momentum defines a cusp and fermionic statistics makes it possible to avoid several cusps at the same points. Fermions for which other quantum numbers, such as spin differ, can however have the same momentum. They should correspond to the same cusp. How can this make sense? Could S^3 be involved somehow. Could they correspond to different holes in S^3 whose sizes and locations correlate with the other quantum numbers somehow? I have considered this problem earlier in the twistor picture where spin corresponds to a geometric degree of freedom in twistor space, which has identification at the level of M^8. The space of causal diamonds (CDs) as a kind of spine of WCW is discussed in [L23]. Lorentz transformations also occur at the level of CDs. The moduli of CD correspond to cm degrees of freedom in H. The finite volume of CD allows states for which Poincare quantum numbers are not exactly opposite for the boundaries of CD. Therefore the values of the total Poincare quantum numbers can be assigned to the CD. Only at the limit of infinitely large CDs does the zero energy property become exact. Therefore the CD wave function carries genuine information. At the p-adic level, translations and Lorentz transformations have the same effect as transformations of a compact group. Translations or Lorentz transformations of order $O(p^n)$ do not increase the p-adic norm of a point.

4 Holography

4-D general coordinate invariance forces holography at the level of $H = M^4 \times CP_2$ and one can regard space-time surfaces as analogues of Bohr orbits determined almost uniquely by 3-D surfaces. Quantum TGD is therefore very much like wave mechanics with point-like particles replaced with 3-surfaces in turn replaced with 4-D Bohr orbits. In fact, a wave-mechanical toy model for TGD would replace electron wave functions in atoms with wave functions in the space of its Bohr orbits.

M^8 is analogous to the momentum space in wave mechanics and the 4-surfaces in M^8 obey number theoretical holography based on associativity.

4.1 What does one mean with holography?

Consider now a more precise definition of holography.

1. The standard form of holography as $3 \rightarrow 4$ assigning to a 3-surface at the boundary of causal diamond (CD) an almost unique 4-D surface is the weakest form of holography. The non-uniqueness of the holography forces zero energy ontology (ZEO) in which analogues of Bohr orbits are basic geometrical objects.

2. $2 \rightarrow 4$ holography is the strongest form of holography. I have called it strong holography (SH). The 2-D partonic surfaces and possibly also the string world sheets would encode the data about the 4-surface and also the data about quantum holography. The strong form of holography could be realized as super symplectic and super-Kac Moody invariance and super-conformal invariance being minimally broken. Only the scaling generator L_0 would not annihilate the states. This condition is however too strong.

3. For the weaker form of SH super-symplectic and conformal symmetries are broken such that the algebras A_n (there are several of them), whose conformal weights are n-multiples of the conformal weights of the entire algebra A, and $[A_n, A]$ annihilate the physical states [L16, L3]. This requires half-algebra with non-negative conformal weights.

   The breaking hierarchy labelled by the values of n makes sense also for the ordinary conformal invariance but to my best albeit non-professional knowledge is not considered as a physical option. Hierarchies corresponding to the inclusion hierarchies of rational extensions and HFFs are obtained.

Both holographies set very strong conditions for the 3-surfaces appearing as holographic data.

4.1.1 Role of polynomials

At the level of M^8 physics is algebraic as it is also for the momentum space in the case of free field theory and reduces to algebraic conditions like mass shell condition and orthogonality of polarization vector and momentum. Polynomials P having integer coefficients determined the physics.

1. P as such does not fix the 4-surface nor even the 3-surface defining the data for number theoretic holography.

2. The polynomial P must have integer coefficients to guarantee number theoretical universality in the sense that they make sense also in p-adic number fields. If the coefficients are smaller than the degree of P, also finite fields become natural mathematical structures in TGD so that all number fields are involved. The roots of P give rise to the mass shells in M_c^8 with mass squared values defined by the roots of P. The roots define an extension of rationals.

3. Polynomials are also characterized by ramified primes as the divisors of the discriminant of the polynomial determined by the product of the differences of its roots [L24]. They are not a property of the algebraic extensions. They depend on P and the exponent of the classical action is proposed to correspond to the discriminant D. Ramified primes are identified as p-adic primes playing a central role in p-adic mass calculations [K5].

4.1.2 The role of fermions

Quantum classical correspondence requires that the 3-surfaces Y^3 at the mass shells are determined by the quantum numbers of fermions associated with the quantum states. What assumptions could provide this additional data and how could this data be coded to the geometry of Y^3?

The data in question correspond to fermion momenta, spins and electroweak quantum numbers. Color does not appear as spin-like quantum numbers but corresponds to color partial waves in CP_2. Consider next how momenta are coded to the properties of 3-surfaces.

1. I have proposed that the 3-surfaces Y^3 in H^3_m could correspond to the fundamental domains of tessellations of H^3. The unit cell of ordinary crystal in E^3 serves as an analog for the fundamental domain of a tessellation in H^3. The disjoint components of Y^3 would naturally correspond to surfaces Y^3_i at H^3_m and would correspond to fundamental domains of analogs of finite crystals formed by gluing them together.

   The points of $E^4 \subset M^4 \times E^4$ correspond to 3-sphere with radius determined by mass m and for given Y^3_i the values of E^4 coordinates would be constant. A stronger condition would be that the values are the same for all Y^3_i. At the cusp points the point would be replaced by S^3, which could touch two disjoint sheets with different values of S^3 coordinates. Since the metric becomes singular at cusp, a natural proposal is that a small hole is drilled around the point and to S^3 and they are glued along their boundaries. The scale of the hole would be determined by the mass.

2. TGD predicts as basic objects also string-like objects $X^2 \times R \subset H$ and their deformations to magnetic flux tubes. By $M^8 - H$ duality they are expected to be present also in M^8 in particular at the hyperboloids H^3_m.

   There are two kinds of string-like objects depending on whether their CP_2 projection is homologically trivial or not. In the latter case the string carries monopole flux.

Quantum classical correspondence suggests that the momenta of fermions as points of $H^3_m \subset M^8$ are coded into the geometry of Y^3 as singularities. $M^8 - H$ duality based on inversion in turn maps the momenta to singular points of H^3_a.

1. Singularities would be naturally cusps as analogs of tips of algebraic surfaces allowing all normal spaces of Y^4 at the singularity: M^8 duality would assign a 3-D subset of CP_2 to the tip.

2. Is it possible to have singularities, where the throats of the opposite wormhole throats touch? Or could the wormhole throats of the incoming partons fuse to single throat? This could occur in the topological counterpart of 3-vertex describing pair annihilation to a single particle. The singularities emerging in this way could relate to the description of the creation of fermion-antifermion pairs and would also define defects essential for exotic differential structures occurring only in dimension $D = 4$ [L14].

Can one code also spin to geometry or should it be regarded as a fermionic quantum number?

1. At the level of H one would have a product of twistor spaces $T(M^4)$ and $T(CP_2)$: these twistor spaces are unique in the sense that they have Kähler structure. This makes H a unique choice for the embedding space.

   Twistorialization replaces space-time surface with 6-D surface $X^6 \subset T(M^4) \times T(CP_2)$ having S^2-bundle structure as possessed also by $T(M^4)$ and $T(CP_2)$. Spinor description of spin and electroweak isospin is replaced by a wave function in twistor spheres S^2.

   The embedding corresponds to dimensional reduction producing X^6 as S^2 bundle. The twistor spheres associated with M^4 and CP_2 must be identified by the embedding of $X^6 \subset T(M^4) \times T(CP_2)$.

   The identification of the twistor spheres forces spin doublets to correlate almost completely with electroweak spin doublets apart from the directions of the two spins. This picture allows only spin- and electroweak spin doublest. Does this force a complete correlation between the values of spin and electroweak to be identical or do the details of the identification for the the embedding of $X^6 \subset T(M^4) \times T(CP_2)$ allow to regard spin and electroweak spin as independent?

   The identification of the two twistor spheres is not unique. The spin rotations and possibly also electroweak spin rotations (, which are not isometries) changes the identification of the two twistor spheres. This would make spin and electroweak spin independent quantum numbers.

   One can argue that only the relative rotation of the two spheres matters. Could this mean that electroweak spin axes can be thought of being completely fixed. Electroweak quantization axes are indeed completely fixed physically.

2. Something similar could happen at the level of M^8. Now one must consider twistor spaces of M^4 and E^4 and similar embedding of a 6-D surface $X^6 \subset M^8$ as twistor space with S^2 bundle structure.

   In M^8 one would have an algebraic description of spin and electroweak spin instead of a wave function at S^2. A direction of S^2 would define a quantization axis and the diametrically opposite points of S^2 associated with it would provide a geometric correlate for the spin and electroweak spin values of fermion. The relative rotations of the twistor spheres of M^4 and E^4 associated with their identification are also now possible so that the two quantum numbers can be regarded as independent but with the electroweak quantization axes fixed.

In the twistorial picture one would have $5 \rightarrow 6$ weak holography or even $4 \rightarrow 6$ strong almost unique holography.

4.2 What kind of 3-geometries are expected in the TGD framework?

To get a wider perspective, it is good to have an overall view of the Geometrization conjecture of Thurston `https://rb.gy/9x3pm` proven by Perelman by studying Ricci flows. Geometrization theorem implies Poincare conjecture and so called spherical space conjecture.

The inspiration comes from the classification of 2-D manifolds expressed by uniformization theorem (`https://rb.gy/ts8va`). There are only 3 closed simply connected Riemann manifolds: sphere, disk, and hyperbolic plane. These are constant curvature spaces with corresponding Lie groups of isometries. One can obtain connected closed 2-manifolds with a nontrivial fundamental group by identifying the points related by a discrete subgroup of isometries.

In the case of the hyperbolic plane the isometry group is infinite and gives rise to a non-trivial fundamental group. For the hyperbolic plane, one obtains 2-manifolds with nonvanishing genus allowing a negative constant curvature. Constant curvature can be normalized to be -1, 1 or 0 in various cases. For non-vanishing curvature, the area serves as a topological invariant. For torus this is not the case.

The following provides the summary of my non-professional understanding of the 3-D case. The TGD inspired comments rely on what I know from the universal preferred extremals of practically any variational principle which is general coordinate invariant and can be constructed from the induced geometric quantities. They are always minimal surfaces outside 3- or lower-dimensional singularities at which the field equations depend on the action. The known extremals are discussed in [K1, K2, L9].

1. Thurston's conjecture `https://rb.gy/9x3pm` states that every oriented and closed irreducible (prime) 3-manifold can be cut along tori, so that the interior of each of the resulting manifolds has a geometric structure with a finite volume which becomes a topological invariant in geometric topology. For instance, knots give rise to a 3-manifold in this way.

   An important difference is that the closed 3-manifold decomposes to a union of different types of 3-manifolds rather than only of single type as in the 2-D case.

2. The notion of model geometry is essential. There exists a diffeomorphism to X/Γ for some model geometry such that Γ is a discrete subgroup of a Lie group of isometries acting in G. There are 8 types of model geometries.

3. Irreducible 3-manifolds appear as building bricks of 3-manifolds using connected sum. There are 8 types of model geometries for closed prime 3-manifolds, which by definition do not allow a connected sum decomposition. These geometries are E^3, S^3, H^3, $S^2 \times R$, $H^2 \times R$, $SL(2, "R")$, Nil, and $Solve$.

4. The model geometries allow a constant curvature metric. The finite volume of the manifold becomes a topological characteristic if one has constant curvature equal to ± 1.

5. All types except one, $S^2 \times R$, which corresponds to a string-like objects in TGD, allow a 3-D Lie group as subgroup of isometries (Bianchi group).

6. All model geometries except hyperbolic manifold (`https://rb.gy/snpft`) and Solv manifold are Seifert fiber spaces (`https://rb.gy/uxszk`), which are fibered by S^1 fiber. Hyperbolic manifolds are atoroidal but have an infinite fundamental group since Γ must be infinite from the finite volume property. Atoroidality means that there is no embedding of torus which would not be parallel to the boundary of the hyperbolic manifold.

   The finite volume property of H^3/Γ also requires that Γ is a lattice: this implies a deep analogy with condensed matter physics. The group elements in the TGD frame-work be $SL(2,C)$ matrices with elements which are algebraic integers in an extension of rationals defined by the polynomial P defining 4-surface in M^8. Note that also momentum components are predicted to be algebraic integers using a unit defined by the scale of the causal diamond (CD).

   TGD leads to the proposal [K9] that the H^3 lattices could appear in cosmological scales and explain "God's fingers" [E1] discovered by Halton Arp. They are astrophysical objects appearong along a line and having quantized redshifts.

7. One can form the spaces of the orbits for a discrete subgroup $\Gamma \subset G$ to obtain 3-manifolds with non-trivial fundamental group or orbifolds as in the case of S^3 and $S^2 \times R$. For hyperbolic 3-manifolds, the fundamental group is infinite and generated by elements of parabolic subgroups of G (`https://rb.gy/b5t55`). Cusp point and cusp neighborhood (`https://rb.gy/fd4dz`) are related to the infinite part of the fundamental group. Since parabolic subgroups $P \subset \Gamma$ are infinite groups, the fundamental group of the hyperbolic manifold is infinite.

8. One can decompose an irreducible 3-manifold to pieces, which are either Seifert manifolds or atoroidal. All 8 model geometries except hyperbolic geometries and so called Solv manifolds are Seifert manifolds.

   Solv manifolds are fiber spaces over a circle with 2-D plane with Minkowski signature as a fiber. In TGD solv manifolds could correspond to the so-called massless extremals [K1, K2, K7] serving representing classical radiation fields having only Fourier components with wave vectors in a single direction: laser beam is a good analog for them. They are not embeddable to H^3.

   In Ricci flows the hyperbolic pieces expand whereas other pieces contract so that asymptotically the manifold becomes hyperbolic. In fact, the collapse occurs in some caes in a finite time as found already by Richard S. Hamilton. The flow "kills" the positive curvature geometries S^3 and $S^2 \times R$ in the connected sum. What is left at large times is "thick-thin" decomposition. The "thick" piece is a hyperbolic geometry whereas the "thin" piece is a so-called graph manifold.

4.2.1 Hyperbolic manifolds and Seifert fiber spaces

Hyperbolic space and Seifert fiber space (`https://rb.gy/uxszk`) are in a central role in the TGD framework and therefore deserve short discussion. The following just gives the basic definitions and brief TGD inspired comments.

1. *Hyperbolic manifolds*

A hyperbolic n-manifold (`https://rb.gy/2esup`) is a complete Riemannian n-manifold of constant sectional curvature. Every complete, connected, simply-connected manifold of constant negative curvature -1 is isometric to the real hyperbolic space H^n. As a result, the universal cover of any closed manifold M of constant negative curvature -1.

Every hyperbolic manifold (`https://rb.gy/snpft`) can be written as H^n/Γ, where Γ is a torsion-free discrete group of isometries on H^n. That is, Γ is a discrete subgroup of $SO^+_{1,n}$. The manifold has a finite volume if and only if Γ is a lattice.

Its "thick–thin" decomposition has a "thin" part consisting of tubular neighborhoods of closed geodesics and ends which are the product of a Euclidean $(n-1)$-manifold and the closed half-ray. The manifold is of finite volume if and only if its "thick" part is compact.

In the TGD framework, the lattice structure is natural and would mean that the elements of the matrices of Γ are algebraic extensions in the extension of rational defined by the polynomial P determining Y^4. The tubular neighborhoods of the "thin" part would correspond to string-like objects (tubular neighborhoods) as geodesics whereas the ends would correspond to cusp singularities inducing blow-up as 3-surface $S^3 \subset E^3$.
At the level of H the tubular neighborhoods correspond to a string-like object and their ends to CP_2 type extremals serving as building bricks of elementary particles. Hadronic strings would represent examples of string-like objects and all elementary particles would involve them as monopole flux tubes connecting wormhole contacts.

For $n > 2$ the hyperbolic structure on a finite volume hyperbolic n-manifold is unique by Mostow rigidity theorem and so geometric invariants are in fact topological invariants. One of these geometric invariants used as a topological invariant is the hyperbolic volume of a knot or link complement, which can allow us to distinguish two knots from each other by studying the geometry of their respective manifolds.

The identification of geometric invariants as topological invariants conforms with the TGD vision about "holy trinity" geometry-number theory-topology. Number theory would leak in through the identification of Γ as a lattice determined by the polynomial P.

2. Seifert fiber spaces

A Seifert manifold `https://rb.gy/uxszk` is a closed 3-manifold together with a decomposition into a disjoint union of circles (called fibers) such that each fiber has a tubular neighborhood that forms a standard fibered torus.

A standard fibered torus corresponding to a pair of coprime integers (a, b) with $a > 0$ is the surface bundle of the automorphism of a disk given by rotation by an angle of $2\pi b/a$ (with the natural fibering by circles). If $a = 1$ the middle fiber is called ordinary, while if $a > 1$ the middle fiber is called exceptional. A compact Seifert fiber space has only a finite number of exceptional fibers.

The set of fibers forms a 2-dimensional orbifold, denoted by B and called the base — also called the orbit surface — of the fibration. It has an underlying 2-dimensional surface B_0, but may have some special orbifold points corresponding to the exceptional fibers.

The definition of Seifert fibration can be generalized in several ways. The Seifert manifold is often allowed to have a boundary (also fibered by circles, so it is a union of tori). When studying non-orientable manifolds, it is sometimes useful to allow fibers to have neighborhoods that look like the surface bundle of a reflection (rather than a rotation) of a disk, so that some fibers have neighborhoods looking like fibered Klein bottles, in which case there may be one-parameter families of exceptional curves. In both of these cases, the base B of the fibration usually has a non-empty boundary. 6 of the 8 basic geometries of Thurston are Seifert fiber spaces.

In the TGD framework, the Seifert fiber spaces would correspond to string-like objects, which appear as several variants.

4.2.2 The eight simply connected 3-geometries appearing in the Thurston's conjecture from the TGD point of view

This section contains as almost verbatim the description of the 8 Thurston geometries provided by the Wikipedia article `https://rb.gy/9x3pm`. There is a good reason for this: I am not a professional and do not understand the technical details. There is a good reason for not giving a mere Wikipedia link: I have added comments relating to the TGD based identification of these model geometries as 3-surfaces.

It turns out that the geometries could correspond to fundamental regions of H^3, to energy $E = constant$ (M^4 time $t = constant$ in H) surfaces $D^3 \subset M^4_+ \subset M^8$, to string-like objects $X^2 \times R$ allowing Seifert fiber space structure, or to masless extremals with structure $M^2 \times E^2$ with M^2 and E^2 corresponding to the orthogonal planes defined by light-like momentum and polarization vector.

First some definitions:

1. A model geometry is a simply connected smooth manifold X together with a transitive action of a Lie group G on X having compact stabilizers (the isotropy group of a point is compact).

2. A model geometry is called maximal if G is maximal among groups acting smoothly and transitively on X with compact stabilizers.This condition can be aso included in the definition of a model geometry.

3. A geometric structure on a manifold M is a diffeomorphism from M to X/Γ for some model geometry X, where Γ is a discrete subgroup of G acting freely on X; this is a special case of a complete (G, X)-structure. If a given manifold admits a geometric structure, then it admits a structure, whose model is maximal.

   One can say that the spaces X provide the raw material from which one obtains various 3-geometries by identifications using a discrete subgroup of G.

A 3-dimensional model geometry X is relevant for the geometrization conjecture if it is maximal and if there is at least one compact manifold with a geometric structure modelled on X. Thurston classified the 8 model geometries satisfying these conditions; they are listed below and are sometimes called Thurston geometries. (There are also uncountably many model geometries without compact quotients.)

There is a connection with the Bianchi groups, which are the 3-dimensional Lie groups. Most Thurston geometries can be realized as a left invariant metric on a Bianchi group. However, $S^2 \times R$ does not allow Bianchi geometry; Euclidean space corresponds to two different Bianchi groups; and there are an uncountable number of solvable non-unimodular Bianchi groups, most of which give model geometries having no compact representatives.

1. Spherical geometry S^3

The point stabilizer is $O(3, R)$, and the group G is the 6-dimensional Lie group $O(4, R)$, with 2 components. The corresponding manifolds are exactly the closed 3-manifolds with a finite fundamental group. Examples include the 3-sphere, the Poincaré homology sphere, and lens spaces. This geometry can be modeled as a left invariant metric on the Bianchi group of type IX. Manifolds with this geometry are all compact, orientable, and have the structure of a Seifert fiber space (often in several ways). The complete list of such manifolds is given in the article on spherical 3-manifolds. Under Ricci flow, manifolds with this geometry collapse to a point in finite time.

In the TGD framework S^3 geometry could be associated with cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`) of hyperbolic 3-manifold and represent the blow-up of a the cusp to S^3 which can be regarded as sphere in $E^4 \subset M^8 = M^4 \times E^4$. This is mapped to a 3-sphere of CP_2 in $M^8 - H$-duality.

2. Euclidean geometry E^3

The point stabilizer is $O(3, R)$, and the group G is the 6-dimensional Lie group $R^3 \times O(3, R)$, with 2 components. Examples are the 3-torus, and more generally the mapping torus of a finite-order automorphism of the 2-torus; see torus bundle. There are exactly 10 finite closed 3-manifolds with this geometry, 6 orientable and 4 non-orientable. This geometry can be modeled as a left invariant metric on the Bianchi groups of type I or VII_0.

Finite volume manifolds with this geometry are all compact, and have the structure of a Seifert fiber space (sometimes in two ways). The complete list of such manifolds is given in the article on Seifert fiber spaces. Under Ricci flow, manifolds with Euclidean geometry remain invariant.

In M^8 one has two kinds of roots of polynomials. For the first option they correspond mass square values defining mass shells H^3 . For the second option applying to the light-cone boundary as mass shell, energy E replaces mass and roots correspond to discrete energies. $E = constant$ surface corresponds to E^3 as 3-balls inside the light-cone.

3. Hyperbolic geometry H^3

The point stabilizer is $O(3, R)$, and the group G is the 6-dimensional Lie group $O^+(1, 3, R)$, with 2 components. There are enormous numbers of examples of these, and their classification is not completely understood. The example with the smallest volume is the Weeks manifold. Other examples are given by the Seifert–Weber space, or "sufficiently complicated" Dehn surgeries on links, or most Haken manifolds.

The geometrization conjecture implies that a closed 3-manifold is hyperbolic if and only if it is irreducible, atoroidal, and has an infinite fundamental group. This geometry can be modeled as a left invariant metric on the Bianchi group of type V or $\mathrm{VII}_{h\neq 0}$. Under Ricci flow, manifolds with hyperbolic geometry expand.

In TGD H^3 has an interpretation as a mass shell in $M^4 \subset M^8$ determined by the roots of the polynomial P or as a light-cone proper time constant hyperboloid in M^4.

This geometry does not allow Seifert fiber space structure unlike most other geometries.

4. The geometry of $S^2 \times R$

The point stabilizer is $O(2,R)\times Z/2Z$, and the group G is $O(3,R)\times R \times Z/2Z$, with 4 components. The four finite volume manifolds with this geometry are: $S^2\times S^1$, the mapping torus of the antipode map of S^2, the connected sum of two copies of 3-dimensional projective space, and the product of S^1 with two-dimensional projective space.

The first two are mapping tori of the identity map and antipode map of the 2-sphere, and are the only examples of 3-manifolds that are prime but not irreducible. The third is the only example of a non-trivial connected sum with a geometric structure. This is the only model geometry that cannot be realized as a left invariant metric on a 3-dimensional Lie group.

Finite volume manifolds with this geometry are all compact and have the structure of a Seifert fiber space (often in several ways). Under normalized Ricci flow manifolds with this geometry converge to a 1-dimensional manifold.

In the TGD framework, these surfaces could correspond to the simplest string-like objects for which S^2 corresponds to a geodesic sphere (homologically trivial or non-trivial) with a finite length connecting fundamental regions of H^3 or finite tessellations formed by them. S^2, which would correspond to a 2-D surface in CP_2 would be the base and string the fiber. One might argue that S^2 is more natural as fiber.

5. The geometry of $H^2\times R$

The point stabilizer is $O(2,R)\times Z/2Z$, and the group G is $O^+(1,2,R)\times R \times Z/2Z$, with 4 components. Examples include the product of a hyperbolic surface with a circle, or more generally the mapping torus of an isometry of a hyperbolic surface.

Finite volume manifolds with this geometry have the structure of a Seifert fiber space if they are orientable. (If they are not orientable the natural fibration by circles is not necessarily a Seifert fibration: the problem is that some fibers may "reverse orientation"; in other words their neighborhoods look like fibered solid Klein bottles rather than solid tori.) The classification of such (oriented) manifolds is given in the article on Seifert fiber spaces. This geometry can be modeled as a left invariant metric on the Bianchi group of type III. Under normalized Ricci flow manifolds with this geometry converge to a 2-dimensional manifold.

In the TGD context, these geometries would correspond to closed string-like objects for which the CP_2 projection is a 2-surface with genus $g>0$. Seifert fiber space property corresponds to closed strings.

6. The geometry of the universal cover of $SL(2,"R")$

The universal cover of $SL(2,R)$ is denoted $\widetilde{\mathrm{SL}}(2,\mathbf{R})$. It fibers over H^2, and the space is sometimes called "Twisted $H^2\times R$". The group G has 2 components. Its identity component has the structure $(\mathbf{R}\times \widetilde{\mathrm{SL}}_2(\mathbf{R}))/\mathbf{Z}$. The point stabilizer is $O(2,R)$.

Examples of these manifolds include: the manifold of unit vectors of the tangent bundle of a hyperbolic surface, and more generally the Brieskorn homology spheres (excepting the 3-sphere and the Poincare dodecahedral space). This geometry can be modeled as a left invariant metric on the Bianchi group of type VIII or III. Finite volume manifolds with this geometry are orientable and have the structure of a Seifert fiber space. The classification of such manifolds is given in the article on Seifert fiber spaces. Under normalized Ricci flow manifolds with this geometry converge to a 2-dimensional manifold.

Also now the interpretation as a closed string-like entity is possible in TGD.

7. Nil geometry

This fibers over E^2, and so is sometimes known as "Twisted $E2 \times R$". It is the geometry of the Heisenberg group. The point stabilizer is $O(2, R)$. The group G has 2 components, and is a semidirect product of the 3-dimensional Heisenberg group by the group $O(2, R)$ of isometries of a circle. Compact manifolds with this geometry include the mapping torus of a Dehn twist of a 2-torus, or the quotient of the Heisenberg group by the "integral Heisenberg group". This geometry can be modeled as a left invariant metric on the Bianchi group of type II.

Finite volume manifolds with this geometry are compact and orientable and have the structure of a Seifert fiber space. The classification of such manifolds is given in the article on Seifert fiber spaces. Under normalized Ricci flow, compact manifolds with this geometry converge to R^2 with the flat metric. *In TGD also this geometry might be assigned with a closed string-like object as all Seifert fiber spaces.*

8. Sol geometry

This geometry (also called Solv geometry) fibers over the line with fiber the plane, and is the geometry of the identity component of the group G. The point stabilizer is the dihedral group of order 8. The group G has 8 components, and is the group of maps from 2-dimensional Minkowski space to itself that are either isometries or multiply the metric by - 1. The identity component has a normal subgroup R^2 with quotient R, where R acts on R^2 with 2 (real) eigenspaces, with distinct real eigenvalues of product 1.

This is the Bianchi group of type VI_0 and the geometry can be modeled as a left invariant metric on this group. All finite volume manifolds with solv geometry are compact. The compact manifolds with solv geometry are either the mapping torus of an Anosov map of the 2-torus (such a map is an automorphism of the 2-torus given by an invertible 2 by 2 matrix whose eigenvalues are real and distinct, such as

$$\left(\begin{array}{cc} 2 & 1 \\ 1 & 1 \end{array} \right)) \ ,$$

or quotients of these by groups of order at most 8. The eigenvalues of the automorphism of the torus generate an order of a real quadratic field, and the solv manifolds can be classified in terms of the units and ideal classes of this order. Under normalized Ricci flow compact manifolds with this geometry converge (rather slowly) to R^1.

Unlike in the case of Seifert fiber spaces, a plane or disk appears as a fiber. Could one consider the possibility whether boundary conditions guaranteeing conservation laws could allow string-like objects for which the cross section is disk rather than a closed 2-surface? The appearance of isometries of 2-D Minkowski space suggests that the disk X^2 must have Minkowski signature so that the embedding to H^3 would not be possible.
Could one assign this structure to massless extremals [K1, K7], which in the TGD framework define the representations for classical radiation fields, which involve the decomposition $M^4 = M^2 \times D^2$. The circle $S^1 \subset D^2$ defining its boundary would define the base space. Boundaries would be light-like and might allow to solve the boundary conditions. It is not clear how to obtain the counterparts of massless extremals at the M^8 level.

4.3 $3 \to 4$ form of holography

One can consider two forms of holography. The first, weak, form corresponds to the ordinary $3 \to$ holography in which 3-D boundaries provide the data defining the 4-surface. The second, strong form, corresponds to $2 \to 4$ holography in which conformal boundaries provide the data defining the 4-surface. In this section the $3 \to 4$ form of the holography is considered.

4.3.1 Fundamental domains of hyperbolic tessellations as data for $3 \to 4$ holography

Good candidates for the surfaces Y_i^3 are fundamental domains assignable to hyperbolic 3-manifolds H^3/Γ_i represented as surfaces in $H^3 \subset M^4 \subset M^8$ (or its complexification). In the case of string-like objects, the fundamental domains would correspond to the analogs of fundamental domains for $S^2/\Gamma \times R$ and $H^2/\Gamma \times R$. The treatment of this case is a rather straightforward modification of the first case so that the discussion is restricted to H^3/Γ_i.

The surfaces Y_i^3 would correspond topologically to many-particle states of free particles. Holography would induce topological interactions in the interior of Y^4 and $X^4(Y^4)$. The momenta (positions) of the fermions analogous to valence quarks correspond to the cusp singularities.

For fundamental fermions momenta would have components, which are algebraic integers. Galois confinement states that the momenta for many-fermion states are ordinary integers. This poses a condition for H^3/Γ and it would be interesting to understand what the condition means.

The degrees of freedom orthogonal to H^3 correspond to a complexified sphere S^3 of E^4, whose radius squared corresponds to the square of the complex mass squared.

1. Hyperbolicity is a generic property of 3-manifolds and probably preserved in small enough deformations. In other words, deformations of hyperbolic 3-manifold X_i^3 probably allow a hyperbolic metric although the induced metric for the deformation is not in general hyperbolic.

   Deformation of the hyperbolic manifold (`https://rb.gy/snpft`) could take place in its evolution defining Y_i^4 and $X^4(Y_i^4)$ and could lead to, for example, to singularities such as the touching of different surfaces and interaction vertices at which partonic 2-surfaces meet.

2. There is an interesting connection to the geometrization conjecture of Thurston (`https://rb.gy/9x3pm`), especially with the work of the Russian mathematician Grigori Perelman, who studied 3-D Ricci flows (`https://rb.gy/n6qlv`) for metrics and proved that, apart from scaling, they lead to hyperbolic geometries.

   Interestingly, hyperbolic manifolds decompose into "thin" and "thick" pieces and the "thin" piece corresponds to cusp neighborhoods (`https://rb.gy/fd4dz`). This decomposition brings in mind the notions of valence partons and sea partons with sea partons, in particular gluons assignable to the interior of Y_i^3 and giving the dominant contribution to the hadron mass.

Consider now what one can assume about Y_i^3.

1. The simplest assumption is that the S^3 coordinates are constant for Y_i^3 identified as the fundamental domain of a tessellation defined by H^3/Γ. It would represent a piece of H^3.

   Could one consider the allowance of S^3 deformations H_d^3 of H^3 in the direction of S^3, which are invariant under G so that the space H_d^3/G would exist. They would define what mathematicians would call a model of hyperbolic geometry.

2. Can one allow for a given Y_i^3 a multiple covering of H^3 by copies of Y_i^3 with different constant values of S^3 coordinates? Could this state correspond topologically to a many-sheeted covering naturally associated with the polynomial P?

   An interesting possibility is that Galois symmetry implies the existence of several copies of Y_i^3 with different S^3 coordinates as the orbit of the Galois group or its sub-group. Z_2 would be the simplest Galois group and give two sheets.

4.3.2 3-D data for $3 \to 4$ holography with 3-surfaces as hyperbolic 3-manifolds

It is good to start with questions.

1. Could the 3-surfaces X^3 associated with the mass shells $H_m^3 \subset M^8$ appearing as holographic data be fundamental domains (analogs of unit cell for crystals) of the tessellation H^3/Γ? Could a fermionic many-particle state for an algebraic extension determined by a given polynomial P assign a singularity to the fundamental domain and fix it?

   The TGD view of hadron physics provides some clues. Gluon sea consists of gluons identifiable as fermion-antifermion pairs and fermions and antifermions. Here is the data for the given hyperbolic 3-manifold of singularities. The valence fermions could reside at throats and virtual sea gluons could be associated with strings $Y^2 \times R$ inside flux tubes and would give to the classical string tension?

Hyperbolic 3-manifolds also have string-like singularities connecting the cusp singularities. In the physical picture of TGD, these would correspond to strings connecting wormhole throats of different wormhole contacts which in turn would correspond to blow-ups of cusps.

2. Is the situation the same in M^8 and H? Could 4-surfaces assignable to the X_i^3 be minimal surfaces in both H and M^8 having a generalization of holomorphic structure to dimension 4? It would be possible to map X_i^3 to each other by inversion. Note that M^8-H correspondence would map the $M^4 \subset M^8$ projections of the points of Y^4 by inversion to H also in the interior of 4-surface.

   Could this realize the dual conformal invariance proposed by the twistors, which would therefore be behind the analogy of Langlands duality and $M^8 - H$ duality?

4.4 Strong form of the hyperbolic holography

Holography roughly means an assignment of, not necessarily a unique 4-surface, to a set of 3-surfaces at the mass shells defined by roots of the polynomial P. The 4-surface is analogous to Bohr orbit.

A stronger form of the holography would be approximate $2 \rightarrow 4$ holography suggesting that the 3-surfaces allow $2 \rightarrow 3$ holography, which need not be completely deterministic. To understand what is involved one must have an idea about what kind of 3-surfaces could be involved.

1. Irreducible closed 3-surfaces Y_i^3 at H_m^3 consist of regions of 8 different types. Could these regions correspond to model geometries or at least have the symmetries of model geometries?

   This conjecture is natural if the 3-surfaces $Y_i^3 \subset H_m^3 \subset M^4 \subset M^8$ belong to (possibly complex) mass shells of M^4. In this case, the composites of fundamental regions of hyperbolic manifolds (`https://rb.gy/snpft`) as analogs of finite crystals would be natural.

   The interiors of these regions would correspond to the "thick" part of the 3-manifold whereas the cusp singularities and string singularities as boundaries of string world sheets would correspond to the "thin" part. The blow-ups of cusp singularities would give rise to 3-D regions of CP_2.

2. Also monopole flux tubes connecting hyperbolic regions to form a network should be involved. Here the natural model geometries would be of type $S^2 \times R$ or $H^2 \times R$ with the ends of R at the two hyperbolic regions. By replacing H^2 with H^2/Γ, one would obtain higher flux tubes with a cross section having a higher genus.

The natural idea is that hyperbolic holography gives rise to $2 \rightarrow 3$ holography. In the case of H^3/Γ, the holography would assign H^3/Γ its fundamental region Y^3 to H^2/Γ.

In the case of H^2/Γ, applying for string-like objects, holography would assign $Y^2 \times R$ to a union of circles H^1/Γ defining its boundary. The rule would be simple: Y^n/Γ is a union of fundamental regions Y_i^n having H^{n-1}/Γ as boundary.

4.4.1 Hyperbolic holography from H^2/G to the fundamental domain of H^3/Γ

The representation of M^4 momenta in terms of bispinors is possible only for massless particles. This raises the question whether one must assume a strong form of holography in which 2-D surfaces at the boundaries of H_m^3 dictate the 4-D surface almost completely. The hyperbolic 2-manifold H^2/G should define the boundary for Y_i^3 identifiable as a fundamental domain Y^3 of a hyperbolic 3-manifold H^3/Γ.

1. This would conform with the proposed realization of super-symplectic invariance and Kac-Moody type symmetries for light-like partonic orbits meaning that the interior degrees of freedom associated with the 3-surfaces X_i^3 and light-like orbits of partonic 2-surfaces are eliminated with a suitable gauge choice formulated in terms of a generalization the Virasoro and Kac-Moody conditions [L15, L16].

2. Physically this would mean that the fermion momenta at cusp point are light-like. This would conform with the view that fermions move along light-like curves inside the light-like partonic orbit.

3. If hyperbolic holography makes sense, the above formulation for H^2 would generalize to the case of H^3. Cusp neighborhood U/P as a projection $U \to H^2/G$ has a counterpart for H^3/Γ and the fundamental domain for H^2/G extends to a fundamental domain for H^3/Γ. For H^2/G as boundary it would correspond to the condition $p_3 > 0$ for the momentum component in the chosen direction.

4. The cusp singularity is analogous to a cusp of an algebraic surface. This suggests that near the cusp point of H^3/Γ the metric behaves like the induced metric of 3-D cusp in 4-D space. Near the cusp one has $t = k\sqrt{\rho}$ where t and ρ are vertical coordinate and transversal coordinates of the cusp in 4-D space. The radial component of the induced metric orthogonal to tip direction should behave like $g_{\rho\rho} = 1 + k^2/4\rho$ and the radial distance from the tip would diverge logarithmically. One cold say that this point is missing so that the hyperbolic manifold is compact but not closed since it has boundaries. The singularity of the metric is a good motivation for cutting off a small ball around the singularity in M^4 and a small ball from S^3 and for gluing the two together along boundaries. At the level of H this would correspond to wormhole throat.

4.5 An explicit formula for $M^8 - H$ duality

$M^8 - H$ duality is a generalization of momentum-position duality relating the number theoretic and geometric views of physics in TGD and, despite that it still involves poorly understood aspects, it has become a fundamental building block of TGD. One has 4-D surfaces $Y^4 \subset M^8_c$, where M^8_c is complexified M^8 having interpretation as an analog of complex momentum space and 4-D spacetime surfaces $X^4 \subset H = M^4 \times CP_2$. M^8_c, equivalently E^8_c, can be regarded as complexified octonions. M^8_c has a subspace M^4_c containing M^4.

Comment: One should be very cautious with the meaning of "complex". Complexified octonions involve a complex imaginary unit i commuting with the octonionic imaginary units I_k. i is assumed to also appear as an imaginary unit also in complex algebraic numbers defined by the roots of polynomials P defining holographic data in M^8_c.

In the following $M^8 - H$ duality and its twistor lift are discussed and an explicit formula for the dualities are deduced. Also possible variants of the duality are discussed.

4.5.1 Holography in H

$X^4 \subset H$ satisfies holography and is analogous to the Bohr orbit of a particle identified as a 3-surface. The proposal is that holography reduces to a 4-D generalization of holomorphy so that X^4 is a simultaneous zero of two functions of complex CP_2 coordinates and of what I have called Hamilton-Jacobi coordinates of M^4 with a generalized Kähler structure.

The simplest choice of the Hamilton-Jacobi coordinates is defined by the decomposition $M^4 = M^2 \times E^2$, where M^2 is endowed with hypercomplex structure defined by light-like coordinates (u, v), which are analogous to z and $\overline{z}$. Any analytic map $u \to f(u)$ defines a new set of light-like coordinates and corresponds to a solution of the massless d'Alembert equation in M^2. E^2 has some complex coordinates with imaginary unit defined by i.

The conjecture is that also more general Hamilton-Jacobi structures for which the tangent space decomposition is local are possible. Therefore one would have $M^4 = M^2(x) \times E^2(x)$. These would correspond to non-equivalent complex and Kähler structures of M^4 analogous to those possessed by 2-D Riemann surfaces and parametrized by moduli space.

4.5.2 Number theoretic holography in M^8_c

$Y^4 \subset M^8_c$ satisfies number theoretic holography defining dynamics, which should reduce to associativity in some sense. The Euclidian complexified normal space $N^4(y)$ at a given point y of Y^4 is required to be associative, i.e. quaternionic. Besides this, $N^4(i)$ contains a preferred complex

Euclidian 2-D subspace $Y^2(y)$. Also the spaces $Y^2(x)$ define an integrable distribution. I have assumed that $Y^2(x)$ can depend on the point y of Y^4.

These assumptions imply that the normal space $N(y)$ of Y^4 can be parameterized by a point of $CP_2 = SU(3)/U(2)$. This distribution is always integrable unlike quaternionic tangent space distributions. $M^8 - H$ duality assigns to the normal space $N(y)$ a point of CP_2. M^4_c point y is mapped to a point $x \in M^4 \subset M^4 \times CP_2$ defined by the real part of its inversion (conformal transformation): this formula involves effective Planck constant for dimensional reasons.

The 3-D holographic data, which partially fixes 4-surfaces Y^4 is partially determined by a polynomial P with real integer coefficients smaller than the degree of P. The roots define mass squared values which are in general complex algebraic numbers and define complex analogs of mass shells in $M^4_c \subset M^8_c$, which are analogs of hyperbolic spaces H^3. The 3-surfaces at these mass shells define 3-D holographic data continued to a surface Y^4 by requiring that the normal space of Y^4 is associative, i.e. quaternionic. These 3-surfaces are not completely fixed but an interesting conjecture is that they correspond to fundamental domains of tessellations of H^3.

What does the complexity of the mass shells mean? The simplest interpretation is that the space-like M^4 coordinates (3-momentum components) are real whereas the time-like coordinate (energy) is complex and determined by the mass shell condition. Is this deformation of H^3 in imaginary time direction equivalent with a region of H^3?

One can look at the formula in more detail. Mass shell condition gives $Re^2(E) - Im(E)^2 - p^2 = Re(m^2)$ and $2Re(E)Im(E) = Im(m^2)$. The condition for the real parts gives H^3, when $\sqrt{Re^2(E) - Im(E)^2}$ is taken as an effective energy. The second condition allows to solve $Im(E)$ in terms of $Re(E)$ so that the first condition reduces to a dispersion relation for $Re(E)^2$.

$$Re(E)^2 = \frac{1}{2}(Re(m^2) - Im(m^2) + p^2)(1 \pm \sqrt{1 + \frac{2Im(m^2)^2}{(Re(m^2) - Im(m^2) + p^2)^2}} \ . \tag{4.1}$$

Only the positive root gives a non-tachyonic result for $Re(m^2) - Im(m^2) > 0$. For real roots with $Im(m^2) = 0$ and at the high momentum limit the formula coincides with the standard formula. For $Re(m^2) = Im(m^2)$ one obtains $Re(E)^2 \to Im(m^2)/\sqrt{2}$ at the low momentum limit $p^2 \to 0$. Energy does not depend on momentum at all: the situation resembles that for plasma waves.

4.5.3 Can one find an explicit formula for $M^8 - H$ duality?

The dream is an explicit formula for the $M^8 - H$ duality mapping $Y^4 \subset M^8_c$ to $X^4 \subset H$. This formula should be consistent with the assumption that the generalized holomorphy holds true for X^4.

The following proposal is a more detailed variant of the earlier proposal for which Y^4 is determined by a map g of $M^4_c \to SU(3)_c \subset G_{2,c}$, where $G_{2,c}$ is the complexified automorphism group of octonions and $SU(3)_c$ is interpreted as a complexified color group.

1. This map defines a trivial $SU(3)_c$ gauge field. The real part of g however defines a non-trivial real color gauge field by the non-linearity of the non-abelian gauge field with respect to the gauge potential. The quadratic terms involving the imaginary part of the gauge potential give an additional condition to the real part in the complex situation and cancel it. If only the real part of g contributes, this contribution would be absent and the gauge field is non-vanishing.

2. A physically motivated proposal is that the real parts of $SU(3)_c$ gauge potential and color gauge field can be lifted to H and the lifts are equal to the classical gauge potentials and color gauge field proposed in H. Color gauge potentials in H are proportional to the isometry generators of the color gauge field and the components of the color gauge field are proportional to the products of color Hamiltonians with the induced Kähler form.

3. The color gauge field $Re(G)$ obeys the formula $Re(G) = dRe(A) + [Re(A), Re(A)] = [Re(A), Re(A)]$ and does not vanish since the contribution of $[Im(A), Im(A)]$ cancelling the real part is absent. The lift of $A_R = g^{-1}dg$ to H is determined by g using M^4 coordinates for Y^4 . The

M^4 coordinates $p^k(M^8)$ having interpretation as momenta are mapped to the coordinates m^k of H by the inversion

$$I : m^k = \hbar_{eff} Re(\frac{p^k}{p^2}) \ , \ p^2 \equiv p^k p_k \ ,$$

where p^k is complex momentum. $Re(A)_H$ is obtained by the action of the Jacobian

$$dI^k_l = \frac{\partial p^k}{\partial m^l} \ ,$$

as

$$A_H = dI \cdot Re(A_{M^8}) \ .$$

dI^k_l can be calculated as the inverse of the Jacobian $\partial m^k/\partial Re(p)^l$. Note that $Im(p^k)$ is expressible in terms of $Re(p^k)$.

For $Im(p^k) = 0$ the Jacobian for I reduces to that for $m^k = \hbar_{eff}\frac{p^k}{p^2}$ and one has

$$\frac{\partial m^k}{\partial p^l} = \frac{\hbar_{eff}}{p^2}(\delta^k_l - \frac{p^k p_l}{p^2}) \ .$$

This becomes singular for $m^2 = 0$. The nonvanishing of $Im(p^k)$ however saves from the singularity.

4. The $M^8 - H$ duality obeys a different formula at the light-cone boundaries associated with the causal diamond: now one has $p^0 = \hbar_{eff}/m^0$. This formula should be applied for $m^2 = 0$ if this case is encountered. Note that number theoretic evolution for masses and classical color gauge fields is directly coded by the mass squared values and holography.

How could the automorphism $g(x) \subset SU(3) \subset G_2$ give rise to $M^8 - H$ duality?

1. The interpretation is that $g(y)$ at given point y of Y^4 relates the normal space at y to a fixed quaternionic/associative normal space at point y_0, which corresponds is fixed by some subgroup $U(2)_0 \subset SU(3)$. The automorphism property of g guarantees that the normal space is quaternionic/associative at y. This simplifies the construction dramatically.

2. The quaternionic normal sub-space (which has Euclidian signature) contains a complex sub-space which corresponds to a point of sphere $S^2 = SO(3)/O(2)$, where $SO(3)$ is the quaternionic automorphism group. The interpretation could be in terms of a selection of spin quantization axes. The local choice of the preferred complex plane would not be unique and is analogous to the possibility of having non-trivial Hamilton Jacobi structures in M^4 characterized by the choice of $M^2(x)$ and equivalently its normal subspace $E^2(x)$.

   These two structures are independent apart from dependencies forced by the number theoretic dynamics. Hamilton-Jacobi structure means a selection of the quantization axis of spin and energy by fixing a distribution of light-like tangent vectors of M^4 and the choice of the quaternionic normal sub-space fixes a choice of preferred quaternionic imaginary unit defining a quantization axis of the weak isospin.

3. The real part $Re(g(y))$ defines a point of $SU(3)$ and the bundle projection $SU(3) \rightarrow CP_2$ in turn defines a point of $CP_2 = SU(3)/U(2)$. Hence one can assign to g a point of CP_2 as $M^8 - H$ duality requires and deduce an explicit formula for the point. This means a realization of the dream.

4. The construction requires a fixing of a quaternionic normal space N_0 at y_0 containing a preferred complex subspace at a single point of Y^4 plus a selection of the function g. If M^4 coordinates are possible for Y^4, the first guess is that g as a function of complexified M^4 coordinates obeys generalized holomorphy with respect to complexified M^4 coordinates in the same sense and in the case of X^4. This might guarantee that the $M^8 - H$ image of Y^4 satisfies the generalized holomorphy.

5. Also space-time surfaces X^4 with M^4 projection having a dimension smaller than 4 are allowed. I have proposed that they might correspond to singular cases for the above formula: a kind of blow-up would be involved. One can also consider a more general definition of Y^4 allowing it to have a M^4 projection with dimension smaller than 4 (say cosmic strings). Could one have implicit equations for the surface Y^4 in terms of the complex coordinates of $SU(3)_c$ and M^4? Could this give for instance cosmic strings with a 2-D M^4 projection and CP_2 type extremals with 4-D CP_2 projection and 1-D light-like M^4 projection?

4.5.4 What could the number theoretic holography mean physically?

What could be physical meaning of the number theoretic holography? The condition that has been assumed is that the CP_2 coordinates at the mass shells of $M^4_c \subset M^8_c$ mapped to mass shells H^3 of $M^4 \subset M^4 \times CP_2$ are constant at the H^3. This is true if the $g(y)$ defines the same CP_2 point for a given component X^3_i of the 3-surface at a given mass shell. g is therefore fixed apart from a local $U(2)$ transformation leaving the CP_2 point invariant. A stronger condition would be that the CP_2 point is the same for each component of X^3_i and even at each mass shell but this condition seems to be unnecessarily strong.

Comment: One can o criticize this condition as too strong and one can consider giving up this condition. The motivation for this condition is that the number of algebraic points at the 3-surfaces associated with H^3 explodes since the coordinates associated with normal directions vanish. Kind of cognitive explosion would be in question.

$SU(3)$ corresponds to a subgroup of G_2 and one can wonder what the fixing of this subgroup could mean physically. G_2 is 14-D and the coset space $G_2/SU(3)$ is 6-D and a good guess is that it is just the 6-D twistor space $SU(3)/U(1) \times U(1)$ of CP_2: at least the isometries are the same. The fixing of the $SU(3)$ subgroup means fixing of a CP_2 twistor. Physically this means the fixing of the quantization axis of color isospin and hypercharge.

4.5.5 Twistor lift of the holography

What is interesting is that by replacing $SU(3)$ with G_2, one obtains an explicit formula form the generalization of $M^8 - H$ duality to that for the twistorial lift of TGD!

One can also consider a twistorial generalization of the above proposal for the number theoretic holography by allowing local G_2 automorphisms interpreted as local choices of the color quantization axis. G_2 elements would be fixed apart from a local $SU(3)$ transformation at the components of 3-surfaces at mass shells. The choice of the color quantization axes for a connected 3-surface at a given mass shell would be the same everywhere. This choice is indeed very natural physically since 3-surface corresponds to a particle.

Is this proposal consistent with the boundary condition of the number theoretical holography mean in the case of 4-surfaces in M^8_c and $M^4 \times CP_2$?

1. The selection of $SU(3) \subset G_2$ for ordinary $M^8 - H$ duality means that the $G_{2,c}$ gauge field vanishes everywhere and the choice of color quantization axis is the same at all points of the 4-surface. The fixing of the CP_2 point to be constant at H^3 implies that the color gauge field at $H^3 \subset M^8_c$ and its image $H^3 \subset H$ vanish. One would have color confinement at the mass shells H^3_i, where the observations are made. Is this condition too strong?

2. The constancy of the G_2 element at mass shells makes sense physically and means a fixed color quantization axis. The selection of a fixed $SU(3) \subset G_2$ for entire space-time surface is in conflict with the non-constancy of G_2 element unless G_2 element differs at different points of 4-surface only by a multiplication of a local $SU(3)_0$ element, that is local $SU(3)$ transformation. This kind of variation of the G_2 element would mean a fixed color group but varying choice of color quantization axis.

3. Could one consider the possibility that the local $G_{2,c}$ element is free and defines the twistor lift of $M^8 - H$ duality as something more fundamental than the ordinary $M^8 - H$ duality based on $SU(3)_c$. This duality would make sense only at the mass shells so that only the spaces $H^3 \times CP_2$ assignable to mass shells would make sense physically. In the interior CP_2 would be replaced with the twistor space $SU(3)/U(1) \times U(1)$. Color gauge fields would be

non-vanishing at the mass shells but outside the mass shells one would have nonvanishing G_2 gauge fields.

There is also a physical objection against the G_2 option. The 14-D Lie algebra representation of G_2 acts on the imaginary octonions which decompose with respect to the color group to $1 \oplus 3 \oplus \overline{3}$. The automorphism property requires that 1 can be transformed to 3 or $\overline{3}$: this requires that the decomposition contains $3 \oplus \overline{3}$. Furthermore, it must be possible to transform 3 and $\overline{3}$ to themselves, which requires the presence of 8. This leaves only the decomposition $8 \oplus 3 \oplus \overline{3}$. G_2 gluons would both color octet and triplets. In the TDG framework the only conceivable interpretation would be in terms of ordinary gluons and leptoquark-like gluons. This does not fit with the basic vision of TGD.

The choice of twistor as a selection of quantization axes should make sense also in the M^4 degrees of freedom. M^4 twistor corresponds to a choice of light-like direction at a given point of M^4. The spatial component of the light-like vector fixes the spin quantization axis. Its choice together with the light-likeness fixes the time direction and therefore the rest system and energy quantization axis. Light-like vector fixes also the choice of M^2 and of E^2 as its orthogonal complement. Therefore the fixing of M^4 twistor as a point of $SU(4)/SU(3) \times U(1)$ corresponds to a choice of the spin quantization axis and the time-like axis defining the rest system in which the energy is measured. This choice would naturally correspond to the Hamilton-Jacobi structure fixing the decompositions $M^2(x) \times E^2(x)$. At a given mass shell the choice of the quantization axis would be constant for a given X_i^3.

5 Singularities, quantum classical correspondence, and hyperbolic holography

The point-like fermions and their 1-D trajectories appear as singularities of the minimal surfaces [L9]. Strings that connect fermions located at their ends, and string world sheets in the interior of X^4 appear also as singularities. Also partonic 2-surfaces separating Minkowskian and Euclidian regions should correspond to singularities of X_i^3 and their light-like radii.

There would therefore be singularities in dimensions $D = 0, 1, 2, 3$. These singularities should relate to the fundamental domains $Y_i^3 \subset H_m^3 \subset M^8$ and holography would suggest that they correspond to the singularities of 3-D hyperbolic manifolds (`https://rb.gy/snpft`).

5.1 Cusp singularities and fermionic point singularities

The singularities should be associated with hyperbolic manifolds Y_i^3 identified as fundamental domains of coset spaces H^3/Γ , that is, as effective geometries H^3/Γ defined by the boundary conditions for various "fields". In the same way as, for example, a torus geometry appears in condensed matter physics for a unit cell of lattice.

Cusp singularity (`https://rb.gy/fd4dz`) is a natural candidate for a point-like singularity and geometrically corresponds to a cusp. For abstract Riemann geometry, the cusp property would correspond to a singularity of the metric for a cusp (tip) and mean that the radial component of the metric diverges at the tip.

Consider first the basic concepts and ideas in the case of 2-D hyperbolic space H^2 and corresponding hyperbolic manifold H^2/G case.

1. Riemann surface can be regarded as a coset space H^2/G, which is represented as a fundamental region for a tessellation of H^2.

2. Cusp singularities of H^2/G correspond to parabolic subgroups P (`https://rb.gy/b5t55`) generated by a parabolic element for $G \subset SL(2, C)$. Parabolic subgroup P is isomorphic to a discrete group of translations along, say, the real axis as a boundary of the upper half-plane and is noncompact. It is represented as Möbius transformations induced by the matrices $(1, n : 0, n)$. P can be regarded as a subgroup generated by a Lorentz boost in a fixed direction.

The cusp singularity results from the identification of points related by the elements of P, which form a non-compact group. Let U denote the set with $Im(z) > 1$ which corresponds to the set $p^3 > 0$ in momentum space. U and $P(U)$ are disjoint. The cusp neighborhood (`https://rb.gy/fd4dz`) can be identified as the set U/P which is the projection of U to H/G.

3. In the simplest situation, one has $G \subset SL(2,Z) \subset SL(2,R) \subset SL(2,C))$, where $S(2,R)$ leaves the real axis invariant. Z could be replaced by an algebraic extension for rationals of algebraic integers in this extension.

$SL(2,C)$ and therefore also $SL(2,R)$ acts in M^4 as Lorentz transformations.

1. A given M^4 momentum has the representation $p^k = \overline{\Psi}\sigma^k\Psi$, $\Psi = (z_1, z_2)$. The representation is unique apart from a complex scaling of z_i so that $z = z_1/z_2$ can be taken as a complex coordinate for the plane and $SL(2,C)$ acts as Möbius transformation. $SL(2,R)$ leaves the real axis invariant.

   The automorphism of sigma matrices induced by $SL(2,C)$ transformation in turn induces Lorentz transformation in momentum space.

2. Under what conditions can bi-spinors correspond to M^4 coordinates? Bi-spinor can be assumed to be of the form $(z_1, z_2) = (z, 1)$. From the formula $p^k = \overline{\Psi}\sigma^k\Psi$, $\Psi = (z_1, z_2) = (z, 1)$ one can deduce an expression of the condition $Im(z_1/z_2) = Im(z_1) > 1$ in terms of p^k. The condition implies that the z-component of momentum satisfies $p^z = z\overline{z} - 1 > 0$.

   The description of M^4 momenta in terms of bi-spinors and H^2 identified as upper half-plane, denoted by H, is possible only for massless particles.

5.1.1 What happens at cusp singularity

What happens at the cusp singularity?

1. The normal space of the singularity is completely ill-defined as the direction of the electric field of a point-like charge. If so, CP_2 would always be a companion to the cusp. CP_2 would be a blow-up of the cusp points of X^3_i as a hyperbolic manifold (`https://rb.gy/snpft`). One would have $X^3_i \subset H^3$ and the cusp points would correspond to a 3-D sub-manifold of CP_2 defined by the normal spaces at the cusp singularity.

2. In the interior of the space-time surface the 3-D submanifold of CP_2 would extend to CP_2 type extremal with a light-like M^4 projection or its deformation. Several cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`) could be associated with a single CP_2 type extremal representing wormhole contact. This corresponds to the view that wormhole throats can carry more than one fermion although the recent model assumes only a single fermion.

3. CP_2 type extremal defines a wormhole contact connecting two Minkowskian space-time sheets in $H = M^4 \times CP_2$. This would mean that the 3-D submanifold of CP_2 as a blow up is deformed to CP_2 type extremal with 2 throats at opposite sheets: at them the Euclidian induced metric transforms to Minkowskian signature.

   The conservation of monopole flux indeed forces the presence of two Minkowskian space-time sheets in the picture based on H. If the throat as a boundary of the 3-D region of $X^4 \subset H$ involves an incoming radial monopole flux, there must be another throat CP_2, where this flux runs to the other space-time sheet.

4. How could the throats connecting the two Minkowskian space-time sheets emerge in the M^8 picture? Should one allow several copies of Y^3_i with the same H^3 projection but different constant S^3 coordinates and with a common cusp point. The blow-up would give several copies of 3-D regions of CP_2, and in holography they would define wormhole contact with 2 or even more throats.

   The simplest view is that quarks are the only fundamental fermions and leptons correspond to wormhole contacts carrying three antiquarks. They could have three throats associated with the same CP_2 type extremal but this is not the only possibility.

5.1.2 The singularities associated with string-like objects

For string-like objects, the fundamental domains would correspond to the analogs of fundamental domains for $S^2/\Gamma \times R$ and $H^2/\Gamma \times R$. For $S^2 \times R$ the spaces S^2/Γ, Γ a finite non-trivial subgroup of $SO(3)$ are orbifolds: the faces of Platonic solids are basic examples. For P^2/Γ one obtains $g > 0$ 2-manifolds with constant curvature metric with negative curvature.

The physical interpretation would be that $S^2 \times R$ and $H^2/\Gamma \times R$ are glued along their ends S^2 or P^2/Γ to partonic 2-surfaces associated with wormhole contacts.

For string-like objects, the fundamental domains would correspond to the analogs of fundamental domains for $S^2/\Gamma \times R$ and $H^2/\Gamma \times R$. For $S^2 \times R$ the spaces S^2/Γ are orbifolds if Γ is finite non-trivial subgroup of $SO(3)$: the triangular, quadrilateral, and pentagonal faces of Platonic solids are key examples. From these one can build finite lattices at S^2. For P^2/Γ one obtains $g > 0$ 2-manifolds with constant curvature metric with a negative curvature.

The physical interpretation would be that $S^2 \times R$ and $H^2/\Gamma \times R$ are glued along their ends S^2 or P^2/Γ to partonic 2-surfaces associated with wormhole contacts.

What could be the quantal counterpart for the geometric holography? This has been a long standing open question. Suppose that the strong form of holography is realized.

1. In [L18], I considered quantal holography as a counterpart of geometric holography discussed in this article. This led to a suggestion that valence quarks at the wormhole throats could pair with pairs of dark quark and antiquark at strings associated with magnetic flux tubes in the interior of the hadronic 3-surfaces. Could these strings correspond to string-like singularities assignable to geodesic lines inside fundamental regions of $H^3/Gamma$?

2. The flux tubes were proposed to have an effective Planck constant $h_{eff} > h$. The correspondence between valence quarks and dark quarks was proposed to be holographic. The spin and electroweak quantum numbers of dark antiquark would be opposite to those of valence quark and dark quark would have quantum numbers valence quark. There would be entanglement in color degrees of freedom for valence quark and dark antiquark to form color single: this would screen the color of valence quark and transfer it to the magnetic body. The holography in this way would allow a convergent perturbation theory. Nature would be theoretician friendly: a phase transition increasing h_{eff}, transferring color to dark quarks, and reducing color coupling strength to $\alpha_s = 2^2/4\pi\hbar_{eff}$ would occur.

   Whether the dark quark-antiquark pairs as analog for gluon pairs as explanation for hadron mass could explain most of hadron mass remained open: if the classical conserved quantities are identical with the quantum contribution from fermions for Cartan algebra, this could be the case. Whether they could correspond to sea gluons remains also an open question.

3. Quantal holography allowing to obtain a convergent perturbation theory might be realized quite generally, also for leptons which correspond to color partial waves in CP_2 neutralized by super symplectic generator [K5, K13] [L3].

It should be noticed that leptonic dark holography would be very natural if leptons consist of 3 antiquarks [L7]. This option would explain matter-antimatter asymmetry in a new way. Antimatter would be identifiable leptons. For the simplest option, the 3 antiquarks would be associated with a single single wormhole throat. The generalized Kähler structure assignable to M^4 in twistor lift [L10, L11] allows a CP violation, which could favor the condensation of quarks to baryons and antiquarks to leptons.

There are however objections against this idea. The considerations of this article inspire the question whether a single wormhole throat can carry only a single quark assignable to the cusp singularity, as suggested already earlier. Two wormhole contacts would be required. This is required also by the fact that stable wormhole contact must carry a monopole flux and monopole flux flux loops must be closed. Uncertainty Principle would suggest that the flux tube must have length of order lepton Compton length. Can this be consistent with the point-like nature of leptons? These arguments favor the option in which leptons and quarks as opposite H chiralities of H-spinors are the fundamental fermions.

5.1.3 Other kinds of point-like singularities and analogy with Fermi surface

Point-like singularities as cusps would naturally correspond to fundamental fermions at the light-like orbits of partonic 2-surfaces.

1. The 2-D boundaries of the fundamental region Y_i^3 associated with H^3/Γ would be analogues for 2-D pieces of the Fermi surface corresponding to atomic energy levels as energy bands.

   In condensed matter physics, the energy shells can deform and the components of the Fermi surface can touch. These singularities are central to topological physics.

2. At M^8 level the 2-D boundaries of Y_i^3 are analogues of energy bands. The evolution defined by the number theoretic holography, identifiable as a coupling constant evolution at the level of M^8, induces deformations of Y_i^3. One expects that this kind of touching singularities take place.

   At the level of H this would correspond to simple touching of the outer boundaries of the physical objects. In particular, these touchings could take place at the partonic 2-surfaces identified as vertices at which severa partonic orbits meet as the partonic surfaces as their ends are glued to single surface just like the ends of lines of a vertex of Feynman diagram are glued together along their ends.

   Could the meeting of fermion and antifermion cusp singularity in this way relate to an annihilation to a boson regarded as a fermion antifermion pair?

3. One can of course challenge the assumption that all fermions correspond to cusps, which correspond to parabolic subgroups of $G \subset SL(2,C)$ (`https://rb.gy/b5t55`). The proposal that all momenta, whose components are algebraic integers for the extension defined by P, are possible. What could be the interpretation of fermions which do not correspond to cusp.

   What the addition of a fermion to a particular allowed momentum could mean? Could it mean that its momentum defines a parabolic subgroup of G? Or is it true only for the "thin" part of Y_i^3 perhaps representing analogs of valence quarks.

   Or could the non-singular momenta correspond to the momenta for the analogues of sea partons, in particular analogs of sea gluons as fermion-antifermion pairs so that their total momentum would dominate in the total momentum of hadron. These would correspond to the "thick" part of Y_i^3. Could these interior momenta correspond to states delocalized at the string world sheets in the interior of monopole flux tubes and also states delocalized in the interiors of the flux tubes. Are these fermions present too?

   The presence of these states should be coded by the geometry of the hyperbolic manifold H^3/Γ (`https://rb.gy/snpft`) and Y_i^3 as its fundamental domain. Somehow the group $G \subset SL(2,C)$ should be responsible for this coding.

5.2 About the superconformal symmetries for the light-like orbits of partonic 2-surfaces

Are the cusp singularities (see `https://rb.gy/0p30o` and `https://rb.gy/fd4dz`) giving rise to CP_2 type extremals and the fermion momenta inside string world sheets and flux tubes associated with Y_i^3 sufficient to fix the 3-surfaces Y_i^3 in turn fixing number-theoretic holography?

1. The total energy for the classical action associated with these two kinds of fermions should correspond to the "sea" (thick part) and "valence fermions" assigned to the cusps (thin part).

2. Supersymplectic invariance and generalized conformal and Kac-Moody invariance assignable to light-like partonic orbits allows a large number of alternatives for the light-like surfaces [L8, L16, L15]. If supersymplectic and Kac-Moody symmetries act as gauge symmetries, the surfaces related by these symmetries are physically equivalent.

   The proposal is that these symmetries are partially broken and there is a hierarchy of breakings labelled by subalgebras $A_n \subset A$ of these algebras. The vanishing conditions for classical and quantal charges for A_n and $[A_n, A]$ serve as gauge conditions and also select the partonic 3-surfaces.

Interpretation of the partially broken gauge symmetries giving rise to dynamical symmetries is in terms of number theoretical measurement resolution and inclusion of hyperfinite factors of type II_1. These hierarchies relate to the hierarchies of extensions of rationals defined by the polynomials P defining the space-time surfaces apart from the effect of fermions.

If the preferred extremal property means generalization of holomorphy from 2-D case to 4-D case, one can conclude that the preferred extremals differ only at the singularities of space-time surfaces such as partonic orbits where the entire action comes into play. The regions outside the singularities would be universal: the minimal surface property would realize the 4-D generalization of the holomorphy.

3. Could different choices of the classical action, which determine the expressions of the classical and fermionic (quantal) Noether charges in terms of the modified Dirac action, correspond to different gauge choices selecting singular surfaces, in particular the CP_2 type extremals differently?

   The standard view would suggest that the change of the parameters of the action at the level of H corresponds to coupling constant evolution, which in the TGD framework is discrete and in terms of p-adic length scales. On the other hand, the existence of dual M^4 conformal invariance suggests that the coupling constant evolution at the level of M^8 is realized as "energy" evolution by using associativity as a dynamical principle. Can these two views be consistent?

   Note that the discriminant of the polynomial P is proposed to correspond to the exponent of action [L10, L12, L16, L15, L13]. The discriminant should change if the action changes. Does this mean that the change of the (effective) action in the discrete coupling constant evolution changes the polynomial?

6 Birational maps as morphisms of cognitive structures

https://en.wikipedia.org/wiki/Birational$_geometry$ and their inverses are defined in terms of rational functions. They are very special in the sense that they map algebraic numbers in a given extension E of rationals to E itself.

1. In the TGD framework, the algebraic extensions E are defined by rational polynomials P at the level of M^8_c identifiable as complexified octonions. E defines a unique discretization for the number theoretically preferred coordinates of M^8_c by the condition that the M^8 coordinates have values in E: I call these discretizations cognitive representations. They make sense also in the extensions of p-adic number fields induced by E serving as correlates of cognition in TGD inspired theory of conscious experience. Birational maps respect the extension E associated with the cognitive representations and map cognitive representations to cognitive representation of same kind. They are clearly analogous to morphisms in category theory.

2. M^8-H duality [L4, L5, L22, L26] is a number theoretic analogue of momentum-position duality. M^8_c serves as the analog of momentum space and $H = M^4 \times CP_2$ as the analog of position space. M^8-H duality maps the 4-surface defined in M^8_c by number theoretic holography based on 3-D data to a 4-D space-time surface in H.

3. Should M^8-H duality respect the algebraic extension? If so, it would map the cognitive representation defined by points belonging to 4-D surface $Y^4 \subset M^8$ with the values of preferred coordinates in E to points of $M^4 \subset H$ with coordinate values in E. One could say that M^8-H duality respects the number theoretical character of cognitive representations. The precise meaning of this intuition is however far from clear.

There are also questions related to the choice of preferred coordinates in which the cognitive representation is defined.

1. Number theoretic constraints fix the preferred coordinates at M^8 side rather uniquely and this induces a preferred choice also on $M^4 \subset H$. For hyperbolic spaces (mass shells) a cognitive explosion happens and a natural question whether cognitive explosion happens also

for the light-like curves assignable to the partonic orbits. If the light-like curve is geodesic, the explosion indeed occurs. For more general light-like curves this is not the case always: could these more general light-like curves be related by a birational map to light-like geodesics?

2. At the H side one can also imagine besides standard Minkowski coordinates also other physically preferred choices of coordinates: are they also theoretically preferred? The notion of Hamilton-Jacobi structure [L19] suggests that in the case of M^4 Hamilton-Jacobi coordinates are very natural for the holomorphic realization of holography. If these are allowed, a natural condition would be that the Hamilton-Jacobi coordinates are related to each other by birational maps mapping the point of E to points of E so that cognitive representations are mapped to cognitive representations.

6.1 $M^8 - H$ duality, holography as holomorphy, Hamilton-Jacobi structures, and birational maps as cognitive morphisms

In the sequel the questions raised in the introduction are considered. The basic notions are $M^8 - H$ duality [L4, L5, L22, L26], holography as a generalized holomorphy [L17, L25], Hamilton-Jacobi structures [L19], and birational maps as cognitive morphisms.

6.1.1 About more precise definitions of the basic concepts

Consider first more precise definitions of various notions involved.

1. What are the preferred coordinates of M^8_c in which the cognitive representation is constructed? M^8_c has a number theoretic interpretation in terms of complexified octonions and physical interpretation as 8-D momentum space. Linear Minkowski coordinates are number-theoretically preferred since octonionic multiplication and other arithmetic operations have a very simple form in these coordinates. Also the number theoretic automorphisms respect the arithmetic operations. The allowed automorphisms correspond to the group G_2 which is a subgroup of $SO(1,7)$. Physically Minkowski space coordinates are preferred coordinates in the momentum space and also in $M^4 \subset H$.

2. How the algebraic extension of rationals, call it E, is determined? The proposal is that rational polynomials characterize partially the 3-D data for number theoretic holography [L22]. The roots of a rational polynomial P define an algebraic extension of rationals, call it E. A stronger, physically motivated, condition on P is that its coefficients are integers smaller than the degree of P.

   The roots of P define mass shells $H^3_c \subset M^4_c \subset M^8_c$, which in turn assign to the mass shells a 4-D surface Y^4 of M^8_c going through the mass shells by associative holography requiring that the normal space of Y^4 is associative, that is quaternionic. It has been be assumed that the roots are complex although also the condition that the roots are real can be considered. The imaginary unit i associated with the roots commutes with the octonionic imaginary units.

3. How the cognitive representation is defined? The points of $Y^4 \subset M^8_c$ with M^4 coordinates in E define a unique discretization of Y^4, called a cognitive representation, making sense also in the extensions of p-adic number fields induced by E. In general, the number of algebraic points in the interior of Y^4 is discrete and even finite but at the mass shells H^3 a cognitive explosion takes place. All points of H^3 with coordinates in E are algebraic.

   The algebraic points with coordinates, which are algebraic *integers* are physically and cognitively in very special role in number theoretic physics and make sense also as points of various p-adic number fields making possible number theoretical universality. The points of H^3 have interpretation as momenta and for physical states the total momentum as sum of momenta at mass shells defined by the roots of P has components which are integers, called Galois confinement [L10, L11], would define fundamental mechanism for the formation of bound states.

4. $M^8 - H$ duality maps the points of $H^3_c \subset M^4_c \subset M^8_c$ to points of $H^3 \subset M^4 \subset M^4 \times CP_2 = H$ by a map, which is essentially an inversion: this form is motivated by Uncertainty Principle:

for the most recent formulation of the duality see [L26]. This map is a birational map and takes points of E points of E. Also the points of cognitive representation belonging to the interior of $Y^4 \subset M^8_c$ are mapped to the interior of $X^4 \subset M^8_c$. One can ask whether the discrete set of points of cognitive representations in the interiors are of special physical interest, say having interpretation as interaction vertices.

6.1.2 Questions to be pondered

There are many questions to be considered.

1. Also partonic orbits in $X^4 \subset H$ define 3-D holographic data in H. What are these partonic orbits? The simplest partonic orbits have light-like M^4 projection but also more general light-like H projection can be considered (note the analogy with a 2-D rigid body rotating along a light-like geodesic of H). A general light-like geodesic of H is a combination of time-like geodesic of M^4 and space-like geodesic of CP_2.

   A point of the light-like partonic orbit correspond at the level of M^8 to the 3-D blowup of a point of M^8 at which the quaternionic normal space parametrized by CP_2 point is not unique so that the normal spaces for a 3-D section of CP_2, whose union along (probably light-like) geodesic is CP_2 with two holes corresponding to the ends of the partonic orbit. This singularity is highly analogous to the singularity of the electric field of a point charge. Partonic orbits define part of the 3-D holographic data.

2. Could one associate cognitive representations also to the partonic orbits? Could also partonic orbits allow a cognitive explosion? The simplest way to guarantee light-likeness for the H projection is as a light-like geodesic and this indeed allows an infinite number of algebraic points in Minkowski coordinates. Same applies to more general light-like orbits. One would have at least 1-D explosion of the cognitive representation.

3. What can one say about the CP_2 and M^4 projections of the partonic 2-surface? Could also these projections to X^2 and Y^2 allow an infinite number of points with coordinates in E or do these kinds of points have some special physical meaning, say as vertices for particle reactions? Concerning cognitive representation, the blow-up would mean that the point has an infinite but discrete set of quaternionic normal spaces at the level of M^8. Since the partonic surface can have an arbitrary complex sub-manifold as CP_2, there is indeed information to be cognitively represented.

6.1.3 The most general cognitively preferred coordinate choices for space-time surfaces and H?

In the case of M^8_c, number theoretical considerations fix the preferred coordinates highly uniquely. In the case of H the situation is not so obvious and one cannot exclude alternative coordinate choices related by a birational map.

A possible motivation comes from the following argument.

1. String world sheets are candidates for singularities analogous to partonic orbits. At a given point of the string world sheet a blow up to a 2-D complex sub-manifold of CP_2 would occur. This would mean that the normal spaces of the point in M^8_c form this sub-manifold. Cosmic strings are the simplest objcts of this kind. Monopole flux tubes are deformations of the cosmic strings and allow also an interpretation in terms of maps from M^4 to CP_2.

   If string world sheets define part of the data needed to define holography, one could argue that it makes sense to assign cognitive explosion to the string world sheet.

2. Cognitive explosion takes place if the string world sheets are 2-D geodesic submanifolds of H. Planes $M^2 \subset M^4$ represent the simplest example. A more complex example is obtained by taking a space-like geodesic in H and rotating it along a time-like geodesic of H. One can also take a light-like geodesic in H and rotate it along a light-like geodesic in dual light-like

direction (ruler surface would be in question). In which case the gluing of the string world sheet along the boundary to the partonic orbit could be possible.

One might perhaps think of building string world sheets by gluing these kinds of ultrasimple regions along their boundaries so that one would have edges. An interpretation as a discretization would be appropriate. One might even go further and argue that the cognitive explosion explains why we are able to think of these kinds of regions in terms of simple formulas. One might argue that number theoretic physics realizes exactly what is usually regarded as approximation. One can however wonder whether life is so simple.

This argument encourages to consider a more complex option allowing more general string world sheets.

1. In the case of M^4 projection, the notion of the Hamilton-Jacobi structure [L19], generalizing the notion of ordinary complex structure, is highly interesting in this respect. It involves a generalization of complex coordinates involving local decompositions $M^4(x) = M^2(x) \times E^2(x)$ of the 4-D tangent space of M^4. The integrable distribution of $E^2(x)$ corresponds to complex coordinates $(w, \overline{w}$ integrating to a partonic 2-surface whereas the integrable distribution of $M^2(x)$ to light-like coordinate pairs (u, v) integrating to a string world sheet in M^4.

   Cognitive representation mean that the discretized values of the Hamilton-Jacobi coordinates $(u, v, w, \overline{w})$ are in E. Hamilton-Jacobi structure generalizes also to the level of $X^4 \subset H$ and now Y^2 can also correspond to CP_2 projection as in the case of cosmic strings and magnetic flux tubes. Note that in TGD one can use a subset of H coordinates as coordinates of X^4.

2. The simplest assumption is that the 1-D parton orbit corresponds to a light-like geodesic but could one map light-like geodesics to more general light-like curves by a birational map? Hamilton-Jacobi structure gives rise to a pair of curved (u, v) of light-like coordinates: could it relate to the standard flat light-like coordinates of M^2 by a birational map? Could a birational map relate standard complex coordinates of E^2 to the pair $(w, \overline{w})$? Could one also consider more general birational maps of $M^4 \to M^4$? If so, the Hamilton-Jacobi structures would be related by maps respecting algebraic extensions and cognitive representations. This would give a very powerful constraint on the Hamilton-Jacobi structures.

In the case of CP_2, projective coordinates are group-theoretically highly unique and determined apart from color rotations. Could one require that the CP_2 projection Y^2 associated with the partonic 2-surface and cosmic string or magnetic flux tube involves cognitive explosion. Are the allowed M^4 and CP_2 projections related by birational maps? Note that color rotations are birational maps.

These considerations suggest the following speculative view.

1. $M^8 - H$ duality, when restricted to 3-D holographic data at both sides, is analogous to a birational map expressible in terms of rational functions and respects the number theoretical character of cognitive representations.

2. Cognitive explosion occurs for the holographic data (this is very natural from the information theoretic perspective): this includes also string world sheets. Hamilton-Jacobi structures in the same cognitive class, partially characterized by the extension E of rationals, are related by a birational map.

3. $M^8 - H$ duality maps the quaternionic normal spaces to points of CP_2 and is an example of a birational map in M^4 degrees of freedom. It is not however easy to guess how the number theoretic holography is realized explicitly and how the 4-surfaces in M^8 are mapped to holomorphic 4-surfaces in H.

4. An interesting additional aspect relates to the non-determinism of partonic orbits due to the non-determinism of the light-likeness condition deriving from the fact that the action is Chern-Simons-Kähler action. The deformation of the partonic orbit induces the deformation of time derivatives of H coordinates at the boundary of $\delta M^4_+ \times CP_2$ to guarantee that

boundary conditions at the orbit are realized. This suggests a strong form of holography [L25]. Already the 3-surfaces at $\delta M^4_+ \times CP_2$ or partonic orbits would be enough as holographic data. This in turn suggests that the analog of birational cognitive correspondence between the holographic data at $\delta M^4_+ \times CP_2$ and at partonic orbits.

6.2 Appendix: Some facts about birational geometry

Birational geometry has as its morphisms birational maps: both the map and its inverse are expressible in terms of rational functions. The coefficients of polynomials appearing in rational functions are in the TGD framework rational. They map rationals to rationals and also numbers of given extension E of rationals to themselves (one can assign to each space-time region an extension defined by a polynomial).

Therefore birational maps map cognitive representations, defined as discretizations of the space-time surface such that the points have physically/number theoretically preferred coordinates in E, to cognitive representations. They therefore respect cognitive representations and are morphisms of cognition. They are also number-theoretically universal, making sense for all p-adic number fields and their extensions induced by E. This makes birational maps extremely interesting from the TGD point of view.

The following lists basic facts about birational geometry as I have understood them on the basis of Wikipedia articles about birational geometry and Enriques-Kodaira classification. I have added physics inspired associations with TGD.

Birational geometries are one central approach to algebraic geometry.

1. They provide classification of complex varieties to equivalence classes related by birational maps. The classification complex curves (real dimension 2) reduces to the classification of projective curves of projective space CP_n determined as zeros of a homogeneous polynomial. Complex surfaces (real dimension 4) are of obvious interest in TGD: now however the notion of complex structure is generalized and one has Hamilton-Jacobi structure.

2. In TGD, a generalization of complex surfaces of complex dimension 2 in the embedding space $H = M^4 \times CP_2$ of complex dimension 4 is considered. What is new is the presence of the Minkowski signature requiring a combination of hypercomplex and complex structures to the Hamilton-Jacobi structure. Note however the space-time surfaces also have counterparts in the Euclidean signature $E^4 \times CP_2$: whether this has a physical interpretation, remains an open question. Second representation is provided as 4-surfaces in the space M^8_c of complexified octonions and an attractive idea is that $M^8 - H$ duality corresponds to a birational mapping of cognitive representations to cognitive representations.

3. Every algebraic variety is birationally equivalent with a sub-variety of CP_n so that their classification reduces to the classification of projective varieties of CP_n defined in terms of homogeneous polynomials. $n = 2$ (4 real dimensions) is of special relevance from the TGD point of view. A variety is said to be rational if it is birationally equivalent to some projective variety: for instance CP_2 is rational.

4. A concrete example of birational equivalence is provided by stereographic projections of quadric hypersurfaces in n+1-D linear space. Let p be a point of quadric. The stereographic projection sends a point q of the quadric to the line going through p and q, that is a point of CP_n in the complex case. One can select one point on the line as its representative. Another exammple is provided by Möbius transformations representing Lorentz group as transformations of complex plane.

The notion of a minimal model is important.

1. The basic observation is that it is possible to eliminate or add singularities by using birational maps of the space in which the surface is defined to some other spaces, which can have a higher dimension. The zeros of a birational map can be used to eliminate singularities of the algebraic surface of dimension n by blowups replacing the singularity with CP_n. Poles in turn create singularities. Peaks and self-intersections are examples of singularities.

The idea is to apply birational maps to find a birationally equivalent surface representation, which has no singularities. There is a very counter-intuitive formal description for this. For instance, complex curves of CP_2 have intersections since their sum of their real dimensions is 4. The same applies to 4-surfaces in H. My understanding is as follows: the blowup for CP_2 makes it possible to get rid of an intersection with intersection number 1. One can formally say that the blow up by gluing a CP_1 defines a curve which has negative intersection number -1.

2. In the TGD framework, wormhole contacts have the same metric and Kähler structure as CP_2 and light-like M^4 projection (or even H projection). They appear as blowups of singularities of 4-surfaces along a light-like curve of M^8. The union of the quaternionic/associative normal spaces along the curve is not a line of CP_2 but CP_2 itself with two holes corresponding to the ends of the light-like curve. The 3-D normal spaces at the points of the light-like curve are not unique and form a local slicing of CP_2 by 3-D surfaces. This is a Minkowskian analog of a blow-up for a point and also an analog of cut of analytic function.

The Italian school of algebraic geometry has developed a rather detailed classification of these surfaces. The main result is that every surface X is birational either to a product $\mathbb{P}^1 \times$ $C for some curve$C$or to a minimal surface$Y$. Preferred extremals are indeed minimal surfaces so that space-time surfaces might define$ minimal $models. The absence of singularities (typically peaks or self-intersections) cha$

There are several birational invariants listed in the Wikipedia article. Many of them are rather technical in nature. The canonical bundle K_X for a variety of complex dimension n corresponds to n:th exterior power of complex cotangent bundle that is holomorphic n-forms. For space-time surfaces one would have $n = 2$ and holomorphic 2-forms.

1. Plurigenera corresponds to the dimensions for the vector space of global sections $H_0(X, K_X^d)$ for smooth projective varieties and are birational invariants. The global sections define global coordinates, which define birational maps to a projective space of this dimension.

2. Kodaira dimension measures the complexity of the variety and characterizes how fast the plurigenera increase. It has values $-\infty, 0, 1, ..n$ and has 4 values for space-time surfaces. The value $-\infty$ corresponds to the simplest situation and for $n = 2$ characterizes CP_2 which is rational and has vanishing plurigenera.

3. The dimensions for the spaces of global sections of the tensor powers of complex cotangent bundle (holomorphic 1-forms) define birational invariants. In particular, holomorphic forms of type $(p, 0)$ are birational invariants unlike the more general forms having type (p, q). Betti numbers are not in general birational invariants.

4. Fundamental group is birational invariant as is obvious from the blowup construction. Other homotopy groups are not birational invariants.

5. Gromow-Witten invariants are birational invariants. They are defined for pseudo-holomorphic curves (real dimension 2) in a symplectic manifold X. These invariants give the number of curves with a fixed genus and 2-homology class going through n marked points. Gromow-Witten invariants have also an interpretation as symplectic invariants characterizing the symplectic manifold X.

   In TGD, the application would be to partonic 2-surfaces of given genus g and homology charge (Kähler magnetic charge) representatable as holomorphic surfaces in $X = CP_2$ containing n marked points of CP_2 identifiable as the loci of fermions at the partonic 2-surface. This number would be of genuine interest in the calculation of scattering amplitudes.

What birational classification could mean in the TGD framework?

1. Holomorphic ansatz gives the space-time surfaces as Bohr orbits. Birational maps give new solutions from a given solution. It would be natural to organize the Bohr orbits to birational equivalence classes, which might be called cognitive equivalence classes. This should induce similar organization at the level of M_c^8.

2. An interesting possibility is that for certain space-time surfaces CP_2 coordinates can be expressed in terms of preferred M^4 coordinates using birational functions and vice versa. Cognitive representation in M^4 coordinates would be mapped to a cognitive representation in CP_2 coordinates.

3. The interpretation of $M^8 - H$ duality as a generalization of momentum position duality suggests information theoretic interpretation and the possibility that it could be seen as a cognitive/birational correspondence. This is indeed the case M^4 when one considers linear M^4 coordinates at both sides.

4. An intriguing question is whether the pair of hypercomplex and complex coordinates associated with the Hamilton-Jacobi structure could be regarded as cognitively acceptable coordinates. If Hamilton-Jacobi coordinates are cognitively acceptable, they should relate to linear M^4 coordinates by a birational correspondence so that M^8-H duality in its basic form could be replaced with its composition with a coordinate transformation from the linear M^4 coordinates to particular Hamilton-Jacobi coordinates. The color rotations in CP_2 in turn define birational correspondences between different choices of Eguchi-Hanson coordinates.

   If this picture makes sense, one could say that the entire holomorphic space-time surfaces, rather than only their intersections with mass shells H^3 and partonic orbits, correspond to cognitive explosions. This interpretation might make sense since holomorphy has a huge potential for generating information: it would make TGD exactly solvable.

REFERENCES

Cosmology and Astro-Physics

[E1] Sato H Fang LZ. Is the Periodicity in the Distribution of Quasar Red Shifts an Evidence of Multiple Connectedness of the Universe? *Gen Rel & Grav*, (11), 1985.

Books related to TGD

[K1] Pitkänen M. About Preferred Extremals of Kähler Action. In *Physics in Many-Sheeted Space-Time: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdclass1.html*. Available at: https://tgdtheory.fi/pdfpool/prext.pdf, 2023.

[K2] Pitkänen M. Basic Extremals of Kähler Action. In *Physics in Many-Sheeted Space-Time: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdclass1.html*. Available at: https://tgdtheory.fi/pdfpool/class.pdf, 2023.

[K3] Pitkänen M. Construction of elementary particle vacuum functionals. In *p-Adic Physics.* *https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/elvafu.pdf, 2023.

[K4] Pitkänen M. Construction of WCW Kähler Geometry from Symmetry Principles. In *Quantum Physics as Infinite-Dimensional Geometry.* *https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/compl1.pdf, 2023.

[K5] Pitkänen M. Massless states and particle massivation. In *p-Adic Physics.* *https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/mless.pdf, 2023.

[K6] Pitkänen M. p-Adic Physics as Physics of Cognition and Intention. In *TGD Inspired Theory of Consciousness: Part II.* *https://tgdtheory.fi/tgdhtml/Btgdconsc2.html*. Available at: https://tgdtheory.fi/pdfpool/cognic.pdf, 2023.

[K7] Pitkänen M. Quantum Antenna Hypothesis. In *Bio-Systems as Self-Organizing Quantum Systems.* *https://tgdtheory.fi/tgdhtml/BbioSO.html*. Available at: https://tgdtheory.fi/pdfpool/tubuc.pdf, 2023.

[K8] Pitkänen M. Recent View about Kähler Geometry and Spin Structure of WCW . In *Quantum Physics as Infinite-Dimensional Geometry.* *https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/wcwnew.pdf, 2023.

[K9] Pitkänen M. TGD and Cosmology. In *Physics in Many-Sheeted Space-Time: Part II.* *https://tgdtheory.fi/tgdhtml/Btgdclass2.html*. Available at: https://tgdtheory.fi/pdfpool/cosmo.pdf, 2023.

[K10] Pitkänen M. TGD as a Generalized Number Theory: Infinite Primes. In *TGD as a Generalized Number Theory: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/visionc.pdf, 2023.

[K11] Pitkänen M. TGD as a Generalized Number Theory: p-Adicization Program. In *Quantum Physics as Number Theory: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/visiona.pdf, 2023.

[K12] Pitkänen M. TGD as a Generalized Number Theory: Quaternions, Octonions, and their Hyper Counterparts. In *TGD as a Generalized Number Theory: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/visionb.pdf, 2023.

[K13] Pitkänen M. The Recent View about Twistorialization in TGD Framework. In *Quantum TGD: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdquantum3.html*. Available at: https://tgdtheory.fi/pdfpool/smatrix.pdf, 2023.

Articles about TGD

[L1] Pitkänen M. Philosophy of Adelic Physics. In *Trends and Mathematical Methods in Interdisciplinary Mathematical Sciences*, pages 241–319. Springer.Available at: https://link.springer.com/chapter/10.1007/978-3-319-55612-3_11, 2017.

[L2] Pitkänen M. Philosophy of Adelic Physics. Available at: https://tgdtheory.fi/public_html/articles/adelephysics.pdf., 2017.

[L3] Pitkänen M. Scattering amplitudes and orbits of cognitive representations under subgroup of symplectic group respecting the extension of rationals . Available at: https://tgdtheory.fi/public_html/articles/symplorbsm.pdf., 2019.

[L4] Pitkänen M. A critical re-examination of M^8-H duality hypothesis: part I. Available at: https://tgdtheory.fi/public_html/articles/M8H1.pdf., 2020.

[L5] Pitkänen M. A critical re-examination of M^8-H duality hypothesis: part II. Available at: https://tgdtheory.fi/public_html/articles/M8H2.pdf., 2020.

[L6] Pitkänen M. About the role of Galois groups in TGD framework. https://tgdtheory.fi/public_html/articles/GaloisTGD.pdf., 2021.

[L7] Pitkänen M. Can one regard leptons as effectively local 3-quark composites? https://tgdtheory.fi/public_html/articles/leptoDelta.pdf., 2021.

[L8] Pitkänen M. TGD as it is towards the end of 2021. https://tgdtheory.fi/public_html/articles/TGD2021.pdf., 2021.

[L9] Pitkänen M. What could 2-D minimal surfaces teach about TGD? https://tgdtheory.fi/public_html/articles/minimal.pdf., 2021.

[L10] Pitkänen M. About TGD counterparts of twistor amplitudes: part I. https://tgdtheory.fi/public_html/articles/twisttgd1.pdf., 2022.

[L11] Pitkänen M. About TGD counterparts of twistor amplitudes: part II. https://tgdtheory.fi/public_html/articles/twisttgd2.pdf., 2022.

[L12] Pitkänen M. About the number theoretic aspects of zero energy ontology. https://tgdtheory.fi/public_html/articles/ZEOnumber.pdf., 2022.

[L13] Pitkänen M. Finite Fields and TGD. https://tgdtheory.fi/public_html/articles/finitefieldsTGD.pdf., 2022.

[L14] Pitkänen M. Intersection form for 4-manifolds, knots and 2-knots, smooth exotics, and TGD. https://tgdtheory.fi/public_html/articles/finitefieldsTGD.pdf., 2022.

[L15] Pitkänen M. Some New Ideas Related to Langlands Program *viz.* TGD. https://tgdtheory.fi/public_html/articles/Langlands2022.pdf., 2022.

[L16] Pitkänen M. Trying to fuse the basic mathematical ideas of quantum TGD to a single coherent whole. https://tgdtheory.fi/public_html/articles/fusionTGD.pdf., 2022.

[L17] Pitkänen M. About the Relationships Between Weak and Strong Interactions and Quantum Gravity in the TGD Universe . https://tgdtheory.fi/public_html/articles/SW.pdf., 2023.

[L18] Pitkänen M. About the TGD based views of family replication phenomenon and color confinement. https://tgdtheory.fi/public_html/articles/emuanomaly.pdf., 2023.

[L19] Pitkänen M. Holography and Hamilton-Jacobi Structure as 4-D generalization of 2-D complex structure. https://tgdtheory.fi/public_html/articles/HJ.pdf., 2023.

[L20] Pitkänen M. Magnetic Bubbles in TGD Universe: Part I. https://tgdtheory.fi/public_html/articles/magnbubble1.pdf., 2023.

[L21] Pitkänen M. Magnetic Bubbles in TGD Universe: Part II. https://tgdtheory.fi/public_html/articles/magnbubble2.pdf., 2023.

[L22] Pitkänen M. New findings related to the number theoretical view of TGD. https://tgdtheory.fi/public_html/articles/M8Hagain.pdf., 2023.

[L23] Pitkänen M. New result about causal diamonds from the TGD view point of view. https://tgdtheory.fi/public_html/articles/CDconformal.pdf., 2023.

[L24] Pitkänen M. Reduction of standard model structure to CP_2 geometry and other key ideas of TGD. https://tgdtheory.fi/public_html/articles/cp2etc.pdf., 2023.

[L25] Pitkänen M. Symmetries and Geometry of the "World of Classical Worlds" . https://tgdtheory.fi/public_html/articles/wcwsymm.pdf., 2023.

[L26] Pitkänen M. The TGD view of the recently discovered gravitational hum as gravitational diffraction. https://tgdtheory.fi/public_html/articles/gravhum.pdf., 2023.