About Dirac equation in $H = M^4 \times CP_2$ assuming Kähler structure for M^4

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Contents

Abstract

The holography= holomorphy vision has led to a dramatic progress in the understanding of Dirac equation in the TGD framework, in particular its variant involving M^4 Kähler form. This article summarizes the latest discoveries.

The first topic of this article is Hamilton-Jacobi $(H-J)$ structure, in particular its hypercomplex aspects. The physical implications of M^4 Kähler structure are discussed and the classification of Hamilton-Jacobi structures at the level of $H = M^4 \times CP_2$ and of space-time surfaces are discussed.

The main topic of the article are the solutions of H Dirac equation with a non-trivial M^4 Kähler structure.

1. The mass squared spectrum of the M^4 Dirac operator turns out to be an integer valued harmonic oscillator spectrum, being therefore "stringy". M^4 mass squared has also negative tachyonic values.

   By 8-D masslessness, M^4 mass squared spectrum must be for on-mass- shell states identical with CP_2 mass squared spectrum given by eigenvalues of $SU(3)$ Casimir operator with a spin term added with spin term added. For the color partial waves, both the Casimir operator contribution and spin term are always integer valued for a natural unit which is for leptons 3 and for quarks 1. For CP_2, the ratio of Kähler couplings q_K of leptons and quarks is 3 and this ratio is natural also for M^4 Kähler couplings.

2. The condition that the M^4 and CP_2 mass squared values are identical has strong implications. For $q_K(L, M^4)/q_K(q, M^4) = 3$, the M^4 - and CP_2 mass squared values can be identical if M^4 - and CP_2 chiralities are identical. For leptons this is a natural assumption. For quarks for which M^4 - and CP_2 chiralities would be opposite, and the spin terms for M^4- and CP_2 are necessarily different. It is however possible to have 8-D masslessness.

3. There are several problems to be solved. The mass scale of colored fermions is given by CP_2 mass scale, about 10^{-4} Planck masses. How to obtain massless states, which by p-adic thermodynamics would give rise to light states? Is it possible to obtain color triplet states for quarks and color singlets for leptons and weak screening?

   Could the tachyonic mass spectrum of the M^4 Dirac operator help here? Could tachyonic states of right-handed covariantly constant neurino analogous to a generator of supersymmetry allow us to construct massless states by adding pairs of left and tachyonic right handed neutrinos to the physical states. If mass squared is additive for the physical states one obtains massless states and weak screening but the difference of quark and antiquark numbers is predicted to be 3. One obtains baryons and mesons but gluons and quarks as massless states are not possible.

   This mechanism could be a special case of the mechanism proposed for the 3-D generalization of super-conformal algebras and the mechanism suggested by M^8-H duality and is consistent with the fractal hierarchy of conformal half-algebras with symmetry breaking transforming finite subspace of superconformal generators from gauge - to dynamical symmetries.

4. In the earlier article, the induced/modified Dirac equation for the induced spinors at the space-time surface in the special case $X^4 = M^4$ was discussed. The discussion generalizes to non-trivial Kähler structure of M^4. The solutions are holomorphic or antiholomorphic in the generalized sense. Holomorphic modes are possible also for the H Dirac operator but are not globally defined. In X^4, they are required by the induction of the spinor structure. Holomorphic massless modes are obtained only for the second M^4 chirality: this conforms with the intuitive view about massless fermions. This means a new kind of parity violation. The gauge theory analog of this phase would be a phase with a vanishing value of the Higgs field.

   A possible interpretation is that massless quarks correspond to the holomorphic modes whereas hadrons correspond to the modes of the H Dirac equation and have M^4 masses of order CP_2 mass but can form massless color singlet bound states. A description of confined and deconfined phases of hadrons emerges and generalizes to all interactions. The intersection of space-time sheets with the same Hamilton-Jacobi-structure consists of 2-D string world sheets: this implies a stringy description of interactions in the deconfined phase. This generalizes to a detailed QCD type picture for all interactions.

The third topic are elliptic curves, which correspond to 2-D lattices consisting of parallelograms. These kinds of lattices are very interesting since the 2 periods could correspond to wavelengths. The periodicity of the function pair (f_1, f_2) with respect to the hypercomplex coordinate u provides the third period. There are however 4 momentum components: could the hypercomplex sector also allow a second period? Space-time surfaces are located inside causal diamond (CD) in zero energy ontology (ZEO): could the size scale of CD provide the other period? Also various realizations of the lattice structure are discussed.

1 Introduction

The discovery of holography= holomorphy (H-H) vision meant a revolutionary leap in the development of TGD. H-H vision allows an exact solution of field equations by reducing the extremely complex non-linear field equations to algebraic equations [L8, L11]. Space-time surfaces correspond to the roots for pairs $f = (f_1, f_2) : H \rightarrow C^2$ of generalized analytic functions $H \rightarrow C^2$ of one hypercomplex and 3 complex coordinates of $H = M^4 \times CP_2$. This happens universally assuming that the classical action depends on the induced geometry and is general coordinate invariant. The space-time surfaces are minimal surfaces with lower-dimensional singularities at which the details of a particular action principle become visible.

An essential element is the notion of a generalized complex structure of the embedding space H inducing a similar structure to the space-time surface X^4. The generalized complex structure involves in the case of M^4 complex coordinate w and hypercomplex coordinate u and its dual v as its conjugate which does not however appear in the equations defining the space-time surface. The generalized complex structure involves the Hamilton-Jacobi $(H - J)$ structure, which can be defined for X^4 and M^4 [L5].

This allows also to realize the physical, 4-D analogs of geometric Langlands correspondence and in this framework space-time surfaces become in a well-define sense representations of numbers, both in a generalized sense as representations for elements of function field or algebra and in the usual sense [L8]. H-H vision has led to several developments. H-H vision in relation to twistor lift of TGD is discussed in [L15]. The twistorialization suggests that also M^4 has a Kähler structure but this aspect was not discussed. Elliptic surfaces are discussed as a kind of pedagogical example in [L14].

The dynamical symmetries as maps $g = (g_1, g_2) : C^2 \rightarrow C^2$ composed with $f = (f_1, f_2)$ make it possible to construct from a given space-time surface satisfying $(f_1, f_2) = (0, 0)$ an infinite hierarchy of increasingly complex space-time surfaces. This leads to a vision about geometric correlates of cognition. The relationship to Gödel's theorem is discussed [L13]. Also a functional generalization of the notion of p-adic number field emerges and allows finally to understand the origin of p-adic primes [L11] and of the p-adic length scale hypothesis [L4].

In TGD only the fermion fields are primary fields and all particles are composites of fundamental fermions. Therefore the Dirac propagators for free spinor fields of H and possibly also for the induced spinor fields are fundamental. In [L12], the Dirac propagator at the level of H and space-time surface was discussed. The outcome was the observation that H-spinor fields allow also holomorphic modes which are not globally defined and that they might be highly relevant as far induced spinor fields in X^4 are considered.

1.1 Topics of the article

The topics of the article are Hamilton-Jacobi (H-J) structure, the solutions of the H Dirac equation assuming M^4 Kähler structure and construction of 4-dimensional analogs of elliptic curves and associated lattices.

1.1.1 Hamilton-Jacobi structure as 4-dimensional analog of complex structure

The first topic of this article is Hamilton-Jacobi $(H - J)$ structure [L5], in particular the its hyper-complex aspects. The physical implications of M^4 Kähler structure are discussed and the classification of Hamilton-Jacobi structures at the level of $H = M^4 \times CP_2$ and of space-time surfaces are discussed.

1.1.2 Solutions of H Dirac equation for non-trivial M^4 Kähler structure

The main topic of the article are the solutions with non-trivial M^4 Kähler structure. These solutions for the non-trivial M^4 Kähler structure were studied in [L12].

1. The mass squared spectrum of the M^4 Dirac operator turns out to be an integer valued harmonic oscillator spectrum, being therefore "stringy". M^4 mass squared has also negative tachyonic values.

   By 8-D masslessness, M^4 mass squared spectrum must be for on-mass- shell states identical with CP_2 mass squared spectrum given by eigenvalues of $SU(3)$ Casimir operator with a spin term added with spin term added. For the color partial waves, both the Casimir operator contribution and spin term are always integer valued for a natural unit which is for leptons 3 and for quarks 1. For CP_2, the ratio of Kähler couplings q_K of leptons and quarks is 3 and this ratio is natural also for M^4 Kähler couplings.

2. The condition that the M^4 and CP_2 mass squared values are identical has strong implications. For $q_K(L, M^4)/q_K(q, M^4) = 3$, the M^4 - and CP_2 mass squared values can be identical if M^4 - and CP_2 chiralities are identical. For leptons this is a natural assumption. For quarks for which M^4 - and CP_2 chiralities would be opposite, and the spin terms for M^4- and CP_2 are necessarily different. It is however possible to have 8-D masslessness.

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   This mechanism could be a special case of the mechanism proposed for the 3-D generalization of super-conformal algebras and the mechanism suggested by M^8-H duality and is consistent with the fractal hierarchy of conformal half-algebras with symmetry breaking transforming finite subspace of superconformal generators from gauge - to dynamical symmetries.

4. Also the modified Dirac equation for the induced spinors at the space-time surface in the special case $X^4 = M^4$ is discussed for non-trivial Kähler structure. The solutions are holomorphic or antiholomorphic in the generalized sense. Holomorphic modes are possible also for the H Dirac operator and required by the induction of the spinor structure. Holomorphic massless modes are obtained only for the second M^4 chirality: this conforms with the intuitive view about massless fermions. This means a new kind of parity violation. The gauge theory analog of this phase would be a phase with a vanishing value of the Higgs field.

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1.1.3 The 4-dimensional analogs of elliptic curves

The third topic are elliptic curves, which correspond to 2-D lattices consisting of parallelograms. These kinds of lattices are very interesting since the 2 periods could correspond to wavelengths. The periodicity of the function pair (f_1, f_2) with respect to the hypercomplex coordinate u provides the third period. There are however 4 momentum components: could the hypercomplex sector also allow a second period? Space-time surfaces are located inside causal diamond (CD) in zero energy ontology (ZEO): could the size scale of CD provide the other period? Also various realizations of the lattice structure are discussed.

2 Hamilton-Jacobi structure and holography= holomorphy vision

In the following I will consider a more precise formulation of the notion of Hamilton-Jacobi structure as a generalization of 2-D complex structure to a 4-D situation with Minkowski signature on the basis of holography= holomorphy vision. Also the TGD counterparts of elliptic curves and corresponding lattices are discussed.

2.1 Basic questions

The basic questions relate to the generalization of conformal structure to the 4-D situation.

1. Do the notions of 2-D conformal structure, Teichmüller space, mapping class group, etc. generalize to the 4-D and hyperbolic case? Is there a hyperbolic counterpart part for the moduli space of complex structures of torus characterizing a lattice cell in a complex plane?

2. Hamilton-Jacobi (H-J) structure is a 4-D generalization of the complex structure [L5]. Locally it is a product of ordinary 2-D complex structure and hypercomplex structure and involves identification of local tangent spaces M^2 and E^2, which are orthogonal, at least in the simplest situation. The choices of these spaces must be integrable so that one obtains a slicing of M^4 by 2-dimensional string world sheets and partonic 2-surfaces transversal or even orthogonal to them. The physical interpretation is in terms of a local light-like momentum vector defining the local M^2 and an orthogonal local polarization vector defining the local E^2.

   The generalized complex coordinates are light-like hyperbolic coordinates u and v dual to each other and related by hypercomplex conjugation, and complex coordinate w for M^4. For general space-time surfaces also the CP_2 coordinate ξ_1 or ξ_2 can take the role of w. For the situation studied in [L14] either w or ξ_1 can be used as a complex coordinate.

One can raise questions and objections.

1. If the H-J structure for M^4 is fundamental, it can violate Lorentz invariance since selfdual or anti-self-dual analog of electric field is present implies preferred direction. Note however that metric of M^4 is Poincare invariant. One can also argue that there is a large number of H-J structures just as there is a large number of complex structures for Riemann surfaces. This is in conflict with the basic vision of TGD.

   Could the H-J structure of X^4 be dynamical rather than fundamental? Could the H-J structure for the space-time surface be an induced H-J structure defined as the sum of the M^4 and CP_2 parts of the induced Kähler form of CP_2 make this possible? Here the massless extremals (MEs) as counterparts of radiation fields characterized by local light-like momentum vector and local polarization suggest a guideline.

   Twistor lift of TGD [L15] motivated the introduction of the H-J structure holography= holomorphy vision seems to allows a representation of the twistor structure at the level of space-time surfaces without the introduction of 12-D twistor space as the Cartesian product of twistor spaces of M^4 and CP_2. This would require that the same is true for H-J structure.

2. Could one classify the H-J structures? If MEs define H-J structures and if this generalizes to all space-time surfaces, then the topological classification of the flow patterns of light-like currents associated with MEs based on braid invariants provides a first guess for the classification. A stronger classification would rely on the generalized conformal invariance.

2.2 Hypercomplex arithmetics

It is appropriate to start by summarizing the elements of hypercomplex arithmetics.

1. Hypercomplex coordinate and its dual correspond to two dual light-like directions (assignable to the lines $t+z=0$ and $t-z=0$. The analog of Kähler form, given by $(J_{uu}, J_{uv}; (J_{vu}, J_{vv} = \pm(0,1;1,0)$ in the simplest situation, represents the hypercomplex imagnitary unit and is the square root of metric tensor satisfying $J^2 = g$ rather than $J^2 = -g$ as in the complex case. It transforms dual light-like vectors to each other. g_{uv} is Kähler metric in the hypercomplex sense and one can assign to its negative a square root.

2. One can express hypercomplex vectors in the basis defined by the light-like vector $(1,0)$ and its dual $(0,1)$. Hypercomplex product is given by $(x,y) \times (u,v) = (xu + yv, xv + yu)$. The vanishing of the square of a light-like vector $(x,0)$ or $(0,x)$ reflects the failure of the number field property.

   Hypercomplex conjugate of the vector $V = (x,y)$ is given by $V = (x,y) \rightarrow V^* = (y,x)$. If the vector is light-like the conjugate is given by $(x,0) \rightarrow (0,x)$. The inner product of vectors $V_1 = (x,y)$ and $V_2 = (u,v)$ is given by

   $$\langle V_1|V_2\rangle = V_1 \times V_2 = (x,y) \times (u,v) = xu + yv$$

   and is the inner predicted defined by the metric tensor $g_{uu}, g_{uv}; g_{vu}, g_{vv} = (0,1;1,0)$ for M^2. The norm squared for a general vector $V = (x,y)$ is $V \times V^* = (x,y) * (y,x) = 2xy$ and is equal to the Minkowskian norm squared for M^2.

   The hyperbolic cosine of the hyperbolic angle η between $V_1 = (x,y)$ and $V_2 = (u,v)$ is given by

   $$cosh(\eta) = \frac{V_1 \times V_2}{\sqrt{V_1 \times V_1}\sqrt{V_2 \times V_2}} = \frac{xu + yv}{2\sqrt{xyuv}} \ .$$

3. Hypercomplex counterparts of analytic mapes $(u,v) \rightarrow (f(u), f(v))$ leave the hyperbolic angle invariant. For the simplest hypercomplex structure with $(t = u+v, z = u-v)$ this corresponds to $(t,z) \rightarrow (f(t+z) + f(t-z), f(t+z) - f(t-z))$.

2.3 About the definition of H-J structure

Twistorialization in terms of the twistor space of H more or less forces M^4 to have the analog of the Kähler structure. Self duality/anti-self-duality of M^4 Kähler form implies that one has magnetic and electric fields with the same/opposite direction and magnitude. In QCD, instantons imply CP breaking and also M^4 analog of Kähler field is expected to imply this. The smallness of CP breaking effects requires that the effects caused by the M^4 Kähler structure must be small.

Self-duality/anti-self duality indeedimplies the vanishing of the action density for M^4 Kähler gauge potential, which is linear in the generalized complex coordinates linear in variables $u, v, w, \overline{w}$ for the standard H-J structure. Situation changes for the induced Kähler structure at the level of space-time. For cosmic strings M^4 projection of the object is 2-D string world and M^4 part of Kähler action is large. This is expected to be the case also for the monopole flux tubes. For fermion lines identifiable as intersections of string world sheets with 3-D light-like partonic orbits the effects are also small and solely due to the Kähler potential of M^4.

2.3.1 Twistor lift predicts M^4 Kähler form

The twistor lift of TGD suggests also a modification of the neutral weak forces.

1. The twistor lift of TGD requires that there is a covariantly constant self-dual Kähler form also in M^4. Kähler gauge potential of M^4 would contribute to the $U(1)$ part of the electromagnetic and Z^0 fields coupling to hyper charge an additional term.

2. M^4 Kähler form contributes to the Kähler action an additional term. The M^4 contribution is fixed by the condition that the M^4 metric is the square of the Kähler form. Also H-spinors couple to M^4 Kähler gauge potential defining a self-dual Abelian field: essentially constant electric and magnetic fields, which are orthogonal and have the same strength, is in question.

The scale of the M^4 metric defines the normalization of $J(M^4)$. Here one however encounters a problem since M^4 does not have any inherent scale in its geometry. One can consider two basic options.

1. The size scale L causal diamond ($CD = cd \times CP_2$), where cd is the intersection of light-cones with opposite direction, serves as a first candidate for the scale allowing to identify dimensionless coordinates for M^4 in such a way that the range of variation for the dimensionless coordinates does not depend on the size of CD.

   In these coordinates the self-dual Kähler form would scale as $E = B = k/L^2$, k a constant near unity. At the limit of long length scales $E = B$ would approach zero. Could L be identified as a length scale determined by the cosmological constant? The breaking of Lorentz symmetry to that of M^4 for the Dirac operator $D(H)$ would be small in long length scales. In very short length scales associated with quarks, the breaking would be large.

2. The alternative option is $E = B = k/R^2$, where R is CP_2 scale for which the breaking of Lorentz invariance would be large in all scales. This interpretation turns out to be more plausible.

2.3.2 Some implications of M^4 Kähler structure

The presence of M^4 Kähler structure has highly non-trivial implications also at the level of particle physics and solves several long standing interpretational problems.

1. M^4 Kähler form is self-dual or anti-self-dual and analogous to $U(1)$ instanton and just as in the case of QCD it violates CP.

2. The prediction is that all particles have an additional M^4 contribution in their Z^0 and em force and also right-handed neutrinos couple to M^4 Kähler gauge potential.

   Remark: The Kähler gauge potential A does not correspond to a genuine gauge invariance and each choice defines a different physics. The proposal is that the so-called Hamilton-Jacobi structures could correspond to different choices of A.

   In particular, M^4 Kähler gauge potential $A(M^4)$ couples also to right-handed neutrinos unlike $A(CP_2)$, where the net coupling vanishes. The effects should be small in the TGD view about space-time sheets at particle level but could provide some understanding to the phenomenon of neutrino mixing which is still poorly understood.

3. The dramatic implication is that, by 8-D masslessness, the mass squared spectra of M^4 - and CP_2 Dirac operators must be identical for the allowed modes. The mass squared spectra of the M^4 operator is "stringy" and can contain tachyons although 8-D masslessness excludes them. This solves a long standing interpretational problem. The physical fermions should be massless apart from small p-adic thermodynamic mass squared. The screening by pairs of right-and left handed neutrinos allows the construction of massless ground states.

2.4 Hypercomplex counterpart of conformal invariance, 4-D conformal structure and H-J structure

Consider first the basic questions about generalized conformal structures.

1. Does the moduli space of conformal structures for Riemann surfaces generalize to the 4-D situation? For the Riemann surfaces, the topology is in a central role and genus and the number of holes determine the number of conformal equivalence classes [K2].

   The topology of the hypercomplex structure associated with the string world sheet is trivial unless it contains handles. In the case of the space-time surface, their presence would violate

Minkowski signature locally. There could be however an effective cylinder topology induced by the periodicity of with respect to the hyperbolic coordinate u or v. The action of a discrete subgroup of Lorentz boosts in the direction of v would be represented as a cyclic group and this group could serve as the counterpart of the $SL(2,Z)$ for the modular invariance in the case of torus. One could also interpret the second half of the CD as a part of light-cone.

2. Are the generalized complex structures of X^4 and H correlated? Can one regard H-J structure as being induced from H so there would be fundamental only single H-J structure at the level of H? What could the induction process mean?

H-J structure can be define both at the level of H and at the level of X^4 as induced H-J structure. The key question is whether the H-J structure of H is unique.

1. The Kähler form J_{uv} can be regarded as a representation of the hypercomplex unit represented as a matrix which is $\pm$ square root of the metric. CP_2 Kähler structure is unique. It would be very nice if also the Kähler structure of M^4 were unique.

   The H-J structure of M^4 should involve an integrable local $M^2 \times E^2$ decomposition of the tangent space of M^4, which is integrable and gives rise to string world sheets and partonic 2-surfaces orthogonal to them. H-J structure would characterize a particular family of solutions to field equations and a particular sector of WCW rather than M^4. The metric of $M^4 \times CP_2$ is indeed unaffected. If CD characterizes the H-J structure and also the sector of WCW, the local $M^2 \times E^2$ decomposition is fixed. This looks an attractive option at the level of $M^4 \subset H$.

2. There are very many choices of $M^2 \times E^2$ decomposition for M^4 and conformal equivalent choices must correspond to the same local choice of M^2. Is there some fundamental local choice of $M^2(x) \times E^2(x)$? Physical intuition suggests that all Lorentz boosts of this choice are possible. If there is such a choice, there would be a preferred direction. The choice of causal diamond (CD) limits the choices but still leaves the Lorentz group with respect to the tip of the CD.

   For the second half of the CD, natural choice of coordinates are radial coordinates $(t+r, t-r, w)$, where r is the radius of the sphere and w is the complex coordinate for the sphere and the metric is Robertson-Walker metric for the light-cone associated with the tip of the CD. In short scales this choice reduces at a given point to the first choice and E^2 would correspond to the tangent space of the sphere. This choice is rotational invariant but selects a preferred time direction which could define the rest system.

3. In H-H vision it is clear that the generalized conformal transformations affect the distribution of decomposition $M^2 \times E^2$. In particular, the coordinate curves have light-like tangents but are not anymore light-like geometries. The expectation is that the generalized conformal transformations act as gauge symmetries. Therefore by a suitable gauge choice the light-like curves can be transformed to light-like geodesics of H. TGD suggests that the Virasoro algebra involves only non-negative conformal weights and there is a hierarchy of symmetry breakings to sub-Virasoro algebra isomorphic with the full algebra [L6]. Finite sub-algebra would become dynamical spectrum generating algebra in these symmetry breakings.

   Although the conformal gauge invariance would partially solve the problem caused by the non-uniqueness of the H-J structure, there would be simply very many conformally equivalent choices of the H-J structure unless the H-J structure of the CD fixes the gauge.

4. One must distinguish between the H-J structure of M^4, or more generally of H, and that of X^4, which would be induced. The natural assumption is that the induced Kähler structure is defined by the sum of the induced Kähler forms of M^4 and CP_2. This makes possible a situation in which the coordinates u and v are associated with light-like geodesics, or even more general light-like curves, of H. For these the M^4 projection is time-like and this could be the case for massive particles as massless particles in 8-D sense [L12].

   $M^8 - H$ duality [L7] suggests a number theoretic interpretation for the H-J structure. The space-time surfaces in M^8 have quaternionic normal space with number theoretical Minkowski metric and have commutative subspace as 2-D subspace in M^4 identifiable as the string world sheet. At the level of H, the commutative subspace would correspond to the string world sheet and hypercomplex tangent space.

The classification of H-J structures of X^4 and H reduces to the classification of 4-D conformal structures equivalent under generalized conformal transformations involving hypercomplex and ordinary conformal transformations. How large is the moduli space of H-J structures? Can one understand it physically? Is there a generalization of the Virasoro conditions associated with the 2-D conformal invariance?

1. Hypercomplex transformations are of the form $(u,v) \rightarrow (f(u), f(v))$ and leave the distribution of M^2:s invariant. The generalized conformal transformations of complex coordinates of H depend parametrically on the hypercomplex coordinate u.

2. The light-likeness of the coordinate curve for u expressed as $m_{kl}dm^k/dudm^l/du = 0$, which defines an analog of the Virasoro conditions. By taking a derivative with respect to u, one indeed obtains the Virasoro conditions so that the light-likeness condition is stronger than Virasoro conditions.

   This gives two representations for Virasoro algebra. The first representation is in terms of differential operators $L_n = u^{n+1}nd/du$ acting on $m_{kl}dm^k/dudm^l/du = 0$. The second representation is in terms of Fourier transform obtained by expressing m^k in terms of plane waves $exp(inu)$. The representation in terms of oscillator operators could make sense as conditions for physical states. If only non-negative conformal weights n matter, one obtains a fractal hierarchy of Virasoros.

3. The light-like curves start from some 3-D surface, say at the boundary of CD. The initial point has $u = u_0$ and points have varying coordinates w (or ξ_1 or ξ_2) and v which do not appear in the dynamics. These light-like curves orthogonal to the associated polarization vectors give rise to an integrable flow with flow lines as various coordinate curves.

4. The first physical analogy is hydrodynamic flow with local light velocity involving transversal sound. The second physical analogy is in terms of massless extremals (MEs) representing induced gauge fields for which Kähler current is light-like. The interpretation of MEs is in terms of radiation fields. There would be conserved a light-like current and its dual. In the complex direction there would be an integrable distribution of planes determined by the local polarization direction orthogonal to the local light-like direction.

This inspires several questions and ideas.

1. The non-trivial H-J structures defined by the Kähler form of X^4 as the sum of induced M^4- and CP_2 Kähler forms in X^4 correspond to non-trivial dynamics for the space-time surface. Holography= holomorphy vision suggests that the topological and number theoretic invariants of the space-time surfaces defined as roots of the function pairs (f_1, f_2) serve as classifiers of $H-J$ structures.

   Also the generalization of the Galois group for the space-time surface transforming to each other the space-time regions corresponding to the roots of (f_1, f_2), would serve as an invariant [L8]. For space-time surfaces obtained by composing $g : C^2 \rightarrow C^2$ with prime pair $f = (f_1, f_2) : H \rightarrow C^2$, which is prime in the sense that it does not allow this kind of decomposition, the roots $g \circ f = (0,0)$ consist of a set of disjoint space-time surfaces corresponding to the roots of g and this gives additional invariants [L11].

2. Could the topology of the flowlines of the light-like total Kähler current serve as a partial classifier for H-J structures? The braiding of light-like flow lines is possible and would correspond to flow lines of non-dissipating em field satisfying $j^{mu}J_{\mu\mu} = 0$. In 3-D Euclidean case the field lines of magnetic fields for which Lorentz force vanishes correspond to this kind of situation and one speaks of Beltrami fields [B1, B3]. For instance, construction of a Beltrami field allowing all possible link and knot types is possible [B2]! The 4-dimensional generalization of Beltrami fields in the TGD framework is discussed [K1].

Braid theory for light-like braidings in M^4 defined by the flow lines of total Kähler current is a suggestive approach. The hypercomplex coordinate v is not visible in the flow. In this sense the flow is effectively 3-dimensional and the braiding is stable against generalized holomorphic transformations. The topological classification is rougher than that based on conformal invariance but one expects that topological equivalence class as braidings decompose to conformal equivalence classes.

3. Light-likeness condition implying the Virasoro conditions can fail locally. At the vertices for fermion pair creation identified as turning of a fermion backwards in time, the light-likeness fails at the vertex and $m_{kl}dm^k/dudm^l/du$ has a delta function singularity at the vertex and does not vanish. There Virasoro conditions have singularity, which is a derivative of delta function. The generalized conformal invariance is violated for the ordinary smooth structure. The notion of exotic smooth structure [A2, A3, A1] possible only in 4-D case and reducing to the standard smooth structure apart from defects identifiable as this kind of singularities allows these kinds of edges [L9, L3, L10].

3 About the solutions of Dirac equation in H assuming M^4 Kähler structure

In [L12], I discussed the construction of the Dirac propagator assuming that M^4 does not have Kähler structure. In this section I consister the solutions of the Dirac equation when M^4 Kähler structure is assumed.

3.1 Dirac equation in H in the presence of M^4 Kähler potential

I have already earlier [L2] [K3] considered Dirac equation in H assuming the presence of M^4 Kähler potential but failed to realize how far reaching the implications are.

Intuitively, in the case of the simplest H-J structure for $M^4 \subset H$, the presence of parallel E_K and B_K with constant values suggests for the mass squared cyclotron spectrum with the same scale. It turns out that the longitudinal and transversal harmonic oscillator contributions to the mass squared have the same scale B_K^2 and of opposite sign as one might. If the integers characterizing longitudinal and transversal oscillator modes are the same, their sum is zero and the mode is massless. In the general case the contribution can be negative or positive and this might have direct relevance to the tachyonic problem encountered in the original interpretation of p-adic mass calculations.

1. At the level of H the square $D^2(H)$ of the modified Dirac operator would allow spinors to be eigen states of energy and single momentum component. Self duality and covariant constancy imply that $D^2(H)$ contains a term proportional to the charge matrix $J^{kl}(M^4)\Sigma_{kl} \propto (\sigma_{03}+\Sigma_{12})$, which vanishes for the second M^4 chirality. One can decompose the square of the $D^2(H)$ to a sum of the analog of scalar d'Alembertian D_S^2 and of spin term S: $D^2 = D_S^2 + S$.

2. 2 components of the 3-momentum would correspond to harmonic oscillator states so that the states would be confined to a finite transversal volume to a harmonic oscillator state characterized by transversal momenta of order magnetic length $\sqrt{B_K}$.

   Suppose that for the transversal degrees of freedom in E^2 with signature (-1,-1), Kähler gauge potential can be chosen to be $A_x = B_K y$. For an eigenstate of p_x, one obtains for the square of the E^2 part of D_S^2

$$D_S^2(E^2) = -(\partial_x - iB_K y)^2 - \partial_y^2 = p_x^2 - \partial_y^2 + B_K^2 y^2 + 2p_x B_K y = -\partial_y^2 + B_K^2(y + p_x/B_K)^2 \quad . \tag{3.1}$$

   The sign of the harmonic oscillator term is correct and the complex shift does not produce problems if the notion of hermiticity is generalized so that PT replaces complex conjugation.

Eigenvalues of $p_y^2 + ..$ are essentially the eigenvalues of energy in harmonic oscillator potential and proportional to $2nB_K$ with $n = 1$ assignable to the ground state. Since $D^2(E^2)$ corresponds to $p_x^2 + p_y^2$, $D^2(E^2)$ gives a negative harmonic oscillator contribution to the mass squared eigenvalue proportional to B_K^2.

3. In the longitudinal degrees of freedom M^2, the signature of the metric is (1,-1). If A is given by $A_t = B_K z$, the M^2 part of the square of the Dirac operator for an energy eigenstate reduces to

$$D_S^2(M^2) = (iE - iB_K z)^2 - \partial_z^2 = -E^2 - \partial_z^2 - B_K^2 z^2 - 2iEB_K z = -\partial_z^2 - B_K^2(z + iE/B_K)^2 \ . \tag{3.2}$$

Energy disappears completely from the equation!

One obtains a harmonic oscillator potential with a wrong sign and has suffered a complex shift by $z \to z + iE/B_K$. Suppose that this complex shift can be understood in terms of generalized hermiticity [L1].

Since $D^2(M^2)$ corresponds to $-E^2 + p_z^2$, the contribution of the harmonic oscillator term to the mass squared is analogous to a harmonic oscillator energy and is positive.

Harmonic oscillator Gaussian would be replaced with an imaginary exponential - this is of course familiar from free quantum field theories based on path integral defined by Gaussian. The size scale of CD would bring to the theory an arbitrarily long p-adic length scale as a fundamental level scale but expressible in terms of CP_2 radius.

4. Besides this there is also a contribution $q_K(M^4)J^{kl}\Sigma_{kl}$ from the spin term, where q_K is the coupling to the Kähler gauge potential. A similar term appears in the $D(CP_2)^2$.

This picture has rather dramatic implications.

1. By self-duality M^2 and E^2 contributions to the mass squared of the right-handed neutrino are proportional to B_K^2 and of opposite sign. They cancel each other if the oscillator quantum numbers $n_{||}$ and $n_\perp$ have the same value. If they are different, the contribution to the mass squared is proportional to $n_{||} - n_\perp$ and can be positive or negative so that tachyonic states are possible. Rather remarkably, a stringy mass squared spectrum is obtained with string tension determined by B_K^2.

2. 8-dimensional masslessness has powerful implications for the mass spectrum of the Dirac operator of H. The stringy spectrum of M^4 mass squared must be equal to the mass spectrum of the CP_2 Dirac operator. This equivalence solves a long standing interpretational problem.

   The equivalence gives very strong constraints on the mass squared values of the CP_2 color partial waves and also fixes the value of B_K. Note that the mass squared values for color partial waves are integers as eigenvalues of the Casimir operator of color group.

3. For massless states, such as right-handed neutrino, 8-D masslessness implies that one must have $n_{||} - n_\perp = 0$. In one possible interpretation of the p-adic mass calculations based on 4-D generalization of conformal invariance [L5], the two integers would correspond to conformal weights assignable to the longitudinal and transversal degrees of freedom and for the right-handed neutrino these conformal weights would cancel.

   Although the situation looks very "stringy", no strings are involved at the level of the H Dirac operator. They can emerge at the space-time level for the modified Dirac operator.

3.2 A more detailed view

3.2.1 The contributions to the mass squared of the ground state fermion mode

Consider now in more detail the contribution of the ground state fermion mode.

1. The square of the CP_2 Dirac operator $D(CP_2)$ for a fundamental fermion is the sum of a Casimir contribution C and spin term J. or a representation (p,q), $C(p,q)$ is given by

$$C(p,q) = \frac{1}{3}(p^2+q^2+3(p+q)+pq) \ . \tag{3.3}$$

The value of $C \ mod \ 3$ equals to triality of the representation. The modes are (p,p) and $(p,p+3)$ for leptons and $(p,p+1)$ and $(p,p+4)$ for quarks. For leptons one has $p \ mod \ 3 = q$ giving $3(C(p,q) \ mod \ 3 = p^2+q^2+3(p+q)+pq \ mod \ 3 = 0$. For quarks one has $3C(p,q) \ mod \ 3 = 1$. Note that for quark triplet one has $(p,q) = (1,0)$ and $C(p,q) = 4/3$ is not integer valued. Rhe scales of Casimir operarators for leptons and quarks are in the ratio 3. The contribution of CP_2 Casimir operator to the mass squared is non-vanishing except for the color singlet states. This strongly suggests that only color singlet states are massless and colored states have CP_2 mass scale. Note that the dimension of the representation (p,q) is $d(p,q) = (p+1)(q+1)p+q+2)/2$.

2. $D^2(CP_2)$ has also spin contribution $J = J^{kl}\Sigma_{kl}$ which satisfies $J \in \{1,0,0,-1\}$ for quarks and $J \in 3\times\{1,0,0,-1\}$ for leptons. The sum of the Casimir and spin term can satisfy $m^2(q) \in Z_\times$ for quarks and $m^2(L) \in Z_\times$ for leptons. The contribution of CP_2 Casimir operator to the mass squared is non-vanishing except for the color singlet states. This suggests that only color singlet states are massless and colored states have CP_2 mass scale.

3. The fact that the natural mass squared units for quarks and leptons are in the ratio 1/3 suggests that the addition of neutrino pairs can give states with vanishing M^4 mass squared only if the number quarks defined as difference for the numbers of quarks and antiquarks is a multiple of 3. This would give baryons and mesons: note that also lepton pairs would give rise to meson-like states [K4].

   In the case of mesons and electroweak gauge bosons, the neutrino pairs can give rise to color singlet states for quarks and antiquarks if the tensor product of their color multiplets contains a color singlet. Gluons are not possible as massless states. Unlike in QCD, light physical states can be described in terms of hadrons rather than quarks and gluons and color would emerge only in CP_2 scale.

Consider next the spin terms in more detail.

1. The spin term $S = q_K J^{kl}\Sigma_{kl}$, where q_K is the coupling to the Kähler gauge potential, does not depend on color quantum numbers. One has leptons $q_K(CP_2) = 3$ for leptons and $q_K(CP_2) = 1$ for quarks.

2. The separate conservation of baryon and lepton number requires that H spinors have a well-defined H-chirality so that M^4 - and CP_2 chiralities ϵ_1 and ϵ_2 are either same or opposite so that $\epsilon_1\epsilon_2 = \epsilon = \pm 1$. $\epsilon = 1$ could correspond to leptons and $\epsilon =_1$ to quarks.

3. For $\epsilon = 1$, one has

$$\frac{S(M^4)}{S(CP_2)} = \frac{q_K(M^4)}{q_K(CP_2)} \quad \text{or} \quad S(M^4) = S(CP_2) = 0 \ . \tag{3.4}$$

   Therefore the values of $S(CP_2)$ and $S(M^4)$ are in the ratio $q_K(M^4)/q_K(CP_2)$ or both of them vanish. $q_K(M^4)/q_K(CP_2) = 1$, $S(CP_2)$ and $S(M^4)$ can cancel each other.

4. For $\epsilon = -1$ one has

$$S(M^4) = q_K(M^4)J \ \text{and} \ S(CP_2) = 0 \quad \text{or} \quad S(M^4) = q_K(M^4)J \ \text{and} \ S(CP_2) = q_K(CP_2)J \ . \tag{3.5}$$

   where one has $J = J^{kl}\Sigma_{kl} = 1$.

5. What can one say about the ratio $q_K(L, M^4)/q_K(q, CP_2)$? For CP_2 one has $q_K(L, CP_2) = 3q_K(q, CP_2)$. Could this be the correct option? For leptons, this option guarantees that the M^4 and CP_2 spin terms are identical so that mass squared values can be the same. For quarks with negative H-chirality, either M^4 or CP_2 contribution vanishes so that the spin terms cannot be identical.

   The Casimir and spin contributions are however integer valued when the unit is proportional to Kähler coupling (1 for quarks and 3 for leptons). Therefore 8-D masslessnes allows integer valued M^4 mass squared using this unit. In the case of a single quark, he integer-valued tachyonic M^4 contribution coming from a fermion pair cannot cancel the mass squared so that colored fundamental fermions have masses with CP_2 mass scale. Massless physical fermions are necessarily color singlets.

3.3 How to obtain massless states in M^4 sense and what these states are?

How to obtain massless states in M^4 sense and what theses massless states are? For spinor modes they are massless in 8-D sense but massless states in M^4 sense are required. I have considered several proposals.

1. The first proposal is that the generalized superconformal analogs of Kac-Moody algebras and super symplectic algebras are used to construct states from the ground states assignable to the modes of the Dirac equation for which states are massless in 8-D sense and have CP_2 mass scale if the mode is colored. A possible exception is covariantly constant right-handed neutrino analogous to a generator of supersymmetry: it might be tachyonic.

   Assume that at the fundamental level, the mass squared is identifiable as a conformal weight and is additive for states of consisting of fundamental fermions. The action of super conformal generators on the fermionic ground states generates an integer valued mass squared spectrum and mass squared corresponds to a conformal weight with CP_2 mass scale. For the 4-D generalization of superconformal algebra hypercomplex and complex degrees of freedom give rise to corresponding conformal weights as longitudinal and transversal conformal weights and the total conformal weight is the difference of these two. This difference could be also negative. This would make it possible to generate massless states from very massive states.

2. How to generate states with vanishing M^4 mass? There are two options.

   (a) Massless Dirac equation requires 8-D masslessness for the fundamental fermions defining the ground states of generalized super-conformal representations. If it is possible to have negative integer values of M^4 mass squared for covariantly constant right-handed neutrino, these off mass shell states could be allowed. The addition of a tachyonic right-handed covariantly constant neutrinos would be analogous to a supersymmetry affecting mass squared value. Could these tachyonic contributions to the positive mass squared from the superconformal degrees of freedom make possible M^4 masslessness?

   (b) The second option is that the contributions of longitudinal and transversal generators of the 4-D generalization of superconformal algebra sum up to a negative mass squared. Also this allow to build states with vanishing M^4 mass squared.

      If the addition of neutrino-pairs corresponds to the action of a tachyonic super-conformal generators reducing the mass squared, these intuitive views are equivalent. The objection is that the action of superconformal generators is gauge action and should not affect the states.

      However, TGD predicts that conformal algebras have non-negative conformal weights and define a hierarchy of isomorphic sub-algebras with conformal weights coming as integer multiples of the weights of the full algebra. The gauge action of the sub-space of A spanned by generators with conformal weights $h \leq n$ would transform to dynamical symmetries and only the generators A_{nk} and their commutators with A would annihilate the physical states.

3. $M^8 - H$ duality provides a second way to see the problem.

(a) $M^8 - H$ duality [L7, L12] predicts that $M^4 \subset H$ corresponds to some quaternionic normal space M^4 of a four-surface $Y^4 \subset M^8$ with Euclidian signature of number theoretically induced metric. Fermions correspond to discrete points of Y^4 and the deformation of Y^4 rotate the quaternionic normal space M^4 locally so that 8-D massless states are massless in M^4 sense. In other words, the 8-momentum is in the direction of the normal space. 8-D masslessness corresponds to M^4 masslessness. The coordinate system of M^8 could be regarded as analogous to a system in which the transverse part of light-like momentum is vanishing.

(b) What does this kind of deformation of Y^4 mean? Could it correspond to a local G_2 automorphism respecting quaternionicity. This automorphism modifies color representations. At the level this would suggest that the color representations of leptons and quarks combinbe to irreducible representations of G_2 with different values of M^4 mass squared and that one can obtain also vanishing mass squared value. One can however argue that local G_2 transformation can have no physical effect.

How the fractal hierarchy for the local G_2 deformations could allow to transform subgroup of gauge symmetries to dynamical symmetries realized as addition of pairs of tachyonic right handed neutrino and left handed neutrino.

The following picture is suggestive.

(a) For the massless physical states the difference between quark and antiquark numbers is a multiplet of 3. This allows baryons, mesons and electroweak gauge bosons and gravitons as massless states. Electroweak gauge bosons would consist of superpositions of lepton pairs and color singlet mesons. Gluons and quarks cannot appear as free particles.

(b) There is an infinite number of color partial waves for both leptons and quarks. Could there be an infinite hierarchy of lepton- and baryon like states, which would correspond to their own p-adic length scales? Also a hierarchy of meson-like and gauge boson-like states can be considered. Could this hierarchy correspond to the fractal hierarcy of isomorphic generalized conformal algebras for which finite-dimension sub-algebra would have transformed from gauge symmetry to a dynamical symmetry.

(c) p-Adic thermodynamics [L4] would give thermal mass squared assignable to higher excitations.

3.4 Possible physical implications

The proposed scenario differs rather dramatically from the stanard view.

4. There is interesting relation to the notion of quark liberation predicted by QCD to occur at LHC energies for the collisions of heavy nuclei and of protons. In TGD framework, quark liberation was not assumed to take place in this phase transition [K3, ?]. Instead of this, dark hadrons of M_{89} hadron physics with a value $h_{eff}/h = 512$, which is the mass scale ratio of M_{89} and M_{107} hadron physics would form. As a consequence, the Compton length scale for dark M_{89} would be the same as the ordinary Compton length for the ordinary nucleons. The is some evidence for the mesons of the M_{89} hadron physics, predicted to have masses of ordinary mesons scaled up by a factor 512.

   In the TGD framework, mesons would replace gluons. Weak bosons in turn should be treated as superpositions of lepton pairs (actually, leptonic mesons [K4]!) and mesons. The nice feature of this would be that empirical data is about baryons and mesons rather than quarks and gluons.

5. TGD view of hadrons, might have a direct relevance to the understanding of muon's anomalous g-2 anomaly. The scaled up hadron physics and possible scaled variants of weak bosons could give rise to this anomaly but TGD does not have a machinery to estimate it precisely enough. There are two approaches to the anomaly: the first one is based on experimental input about hadrons and the second one is based on QCD lattice calculations. QCD approach conforms with the most recent experimental result (2025) (see this). TGD supports the

approach based on the calculation using empirical data from hadronic vacuum polarization, which predicts that the anomalous magnetic moment of muon is larger than the standard model prediction by about factor 10^{-9}. This could be seen as a support for new physics. On the other hand, the existence of holomorphic massless modes for the induced/modified Dirac equation suggests that also the phase with 3 massless quarks is possible. Clearly, the situation is extremely interesting from the TGD point of view.

Could M^4 Kähler force have observable effects? M^4 Kähler potential should be felt also by covariantly constant right-handed neutrino so that right-handed neutrino would not completely decouple from gauge interactions. TGD predicts that both quarks and leptons, in particular right- and left-handed neutrinos have an infinite number of color partial waves with CP_2 mass scale. As found, there is a mechanism neutralizing color partial waves of leptons and giving rise to massless neutrinos, which become massive by p-adic thermodynamics. Right-handed covariantly constant neutrinos would already be color singlets and massless so that this mechanism would not be needed. There would however be coupling to the induced M^4 gauge potential. Could this coupling relate to the poorly understood massivation of neutrinos involving the mixing of right-handed and left-handed neutrinos?

The following simple model makes it possible to estimate the size of the effect of M^4 Kähler force for elementary fermions at space-time level. Induced Dirac equation is assumed.

1. Both nucleons and leptons create a classical induced M^4 Kähler potential, which contributes to the U(1) part of the induced electroweak gauge potentials in the space-time surace assignable to, say, nucleus. The gauge forces are felt at the light-like fermion lines at 3-D light-like partonic orbits. A string world sheet connecting say electron and nucleon could mediate the interaction.

2. Consider 1-D light-like fermion line at the partonic orbit of a fermion. Idealize the fermion line as a light-like geodesic line in $M^4 \times S^1$, where $S^1 \subset CP_2$ is a geodesic circle. 8-D masslessness implies $p^2 - R^2\omega^2 = 0$ [L12], where ω is expected to be the order of the particle mass and characterizes the rotation velocity associated with S^1. A physically motivated guess is that by 8-D light-likeness allowing massivation in 4-D sense, ω is a geometric correlate for the Compton time of the fermion so that fermion can be said to have an internal clock.

3. Consider M^4 and CP_2 contributions to the Kähler potential. Denote by u the CP_2 coordinate serving as a coordinate for the fermion line at the partonic orbit as the interface between Euclidean CP_2 type region identifiable as a wormhole contact connecting two Minkowskian space-time sheets and Minkowskian region. The CP_2 part of the induced Kähler potential is of order $A_u^{CP_2} \sim 1/R$, where R is CP_2 radius. The M^4 part of the induced Kähler potential is $A_k^{M^4}\partial m^k/\partial_u \sim \omega \sim m$. For electrons the ratio of the two contributions is $\omega R \sim m_e/m(CP_2) \sim 10^{-17}$ and therefore extremely small. This guarantees that the induced M^4 Kähler form has negligible effects.

3.5 Holomorphic solutions of Dirac equation for the induced spinors in X^4

TGD allows 3 Dirac type equations corresponding to the Dirac equation in H, Dirac equation for the induced spinors in X^4 and Dirac equation in the WCW. At the space-time level the general solution of the equation relies on generalized holomorphy, which also allows a family of holomorphic solutions at the level of H [L12]. It is not quite clear whether this massless solution family is orthogonal to the family of ordinary solutions, which are massive apart from right-handed neutrinos. The solutions are not globally finite in H but makes sense when restricted to the space-time surface by induction procedure.

The holomorphic solutions of Dirac equation are analogs of massless solutions and satisfy the conditions

$$\begin{aligned} &D_v\Psi = 0 \ , \quad D_{\overline{z}} = 0 \ , \\ &\Gamma_u\Psi = 0 \ , \quad D_z\Psi = 0 \ . \end{aligned} \tag{3.6}$$

$D_u\Psi$ and $D_z\Psi$ cannot vanish. The squared of Dirac equation gives d'Alembertian for self dual Kähler form with spin term which is constant.

It is instructive to study the situation for the for $M^4 \subset H$ realized as 4-surface with constant CP_2 coordinates. Somewhat surprisingly, the d'Alembertian without spin term gives a constant mass squared term for holomorphic solutions. It could vanish or is non-vanishing constant depending on whether the M^4 Kähler form is self dual or anti-self dual.

1. Apart from the scale factor, one can write the Kähler gauge potential in terms of Kähler function $K = uv + \epsilon\overline{w}$

$$\begin{aligned} A_u &= \partial_u K = \text{v} & A_v &= -\partial_v K = -u \\ A_w &= \epsilon\partial_u K = \epsilon\overline{w} & A_{\overline{w}} &= \epsilon\partial_{\overline{w}} K = -\epsilon w \ . \end{aligned} \tag{3.7}$$

   This gives $J_{uv} = 1, J_{w\overline{w}} = \epsilon$.

2. The action of d'Alembertian without the spin terms reads as

$$\Box\Psi = g^{uv}(D_u D_v + D_v D_u) + g^{z\overline{z}}(D_z D_{\overline{z}} + D_{\overline{z}} D_z))\Psi = g^{uv}(\partial_v - iu))(\partial_u + iv))\partial_u K = \tag{3.8}$$

   By the covariant constancy with respect to v and $\overline{z}$ the equation simplifies and one can transform $D_v D_u$ and $D_{\overline{z}} D_z$ to constant terms by commuting the covariant derivatives so that D_v and $D_{\overline{z}}$ act on Ψ and annihilate it. The commutator is given by

$$g^{uv} J_{uv} + g^{z\overline{z}} J_{z\overline{z}} = 1 + \epsilon$$

   for the normalization used. For $\epsilon = -1$ this term vanishes and equals to 2 for $\epsilon = 1$.

3. The square of the Dirac operator contains also the spin term, which has opposite sign for left and right handed chiralities. The spin term is the contraction of Kähler form and of signa matrices and given by $\Sigma_{03} \pm Sigma_{12}$ depending on whether the Kähler form is self-dual or anti-self-dual. Without additional constraints it has values proportional to $1, 0, 0, -1$. For the second chirality the spin term vanishes.

   The condition that the square of the Dirac equation is satisfied, requires that this term has value opposite to that for the spin term. Only the values 0,0 are allowed for $\epsilon = -1$. Only the second M^4 chirality would be possible in accordance with the conditions $\gamma_z\Psi = 0$ and $\gamma u\Psi = 0$ or their antiholomorphic counterparts. The restriction on chirality is natural for massless fermions.

4. This is possible if the spin term is $J^{kl(M^4)}\sigma_{kl}$ is the same for quarks and leptons. This condition makes sense also for the H Dirac equation, which does not allow single quark solutions with any choice of the spin term.

Again one can raise questions.

1. The induction of spinor structure requires that the holomorphic modes of the modified Dirac equation are expressible in terms of the H spinor modes. However, CP_2 spinor modes in H are not holomorphic. This suggests that holomorphy for X^4 modes fails for all fermions at the space-time surface except when the fermion is a right-handed or space-time surfaces is M^4.

   This raises the question whether also the holomorphic solutions of the Dirac equation in H could be physical. They can be realized but are not finite as global solutions. The modes restricted to a finite volume, say the volume of CD are possible and could be used in the induction [?] For CP_2, the non-compactness causes problems but typically the CP_2 type extremals have holes at their ends. This would require quantization of the holomorphic modes. An intuitive guess is that since they are massless modes they could be seen as local massless modes orthogonal to the global massive modes.

2. The induced Dirac equation makes sense also for the lower-dimensional submanifolds of the space-time surface such as string world sheets and partonic 2-surfaces and their light-like orbits. A possible model for the interactions of particles as Bohr orbit like entities defined by space-time surfaces is in terms of their intersections. During the interaction the space-time surfaces would share H-J structure and the intersections would be 2-D string world sheets. Also self-intersections could be described in terms of intersections for a space-time surface and its infinitesimal deformation.

   The string world sheets reduce effectively to the fermion lines at their boundaries since the dynamics is restricted to a single hypercomplex coordinate, say u. 1-D induced Dirac does not see the induced gauge fields but only the induced gauge potentials. One cannot eliminate it completely by a gauge transformation. Covariant constancy would hold true along fermion lines for the modes of the induced spinor field.

3. Do the above results for $X^4 = M^4$ hold true generally? This is not completely clear. The induced (and modified) gamma matrices need not be covariantly constant for a general X^4. The divergence of the induced gamma matrices however vanishes almost everywhere irrespective of action since space-time surfaces are minimal surfaces except at the point-like singularities, where the trace of the second fundamental form would have a delta function singularity. The divergence of the modified gamma matrices containing an additional term reflecting the deviation of the action from volume term, could vanish also at these points. These points would be singularities for the standard smooth structure and correspond to the vertices for fermion pair creation []holoholonumber, intsectform, whatgravitons.

4. The successes of QCD suggests that also the description in terms of massless quarks should make sense and should correspond to a phase different from the hadronic phase. The generalized holomorphic solutions are massless in the sense that the square of the modified Dirac operator annihilates them. The conjugates of the holomorphic gamma matrices annihilate these modes and implies that the spin term involving the induced Kähler form vanishes and does not give rise to mass squared term.

   (a) One can argue that the induction of spinor structure by restricting the H modes to WCW requires a generalized holomorphic solution basis for H, which makes sense only in finite regions of M^4 and CP_2 inside which the holomorphic modes remain finite. It is not clear whether this basis is locally orthogonal to the solution basis of ordinary Dirac equation in H. These modes must remain finite in X^4. Since space-time surfaces are enclosed inside CDs with a finite size scale and CP_2 type Euclidean regions connecting two Minkowskian space-time sheets [**?**]ave holes as 3-D light-like partonic orbits, it these modes can remain finite.

   (b) One can also argue that one can express the second quantized solution of induced/modified Dirac equation in X^4 as superposition of the H spinor modes and solve from this expression the oscillator oscillator operators in X^4 in terms of those in H. The overlaps of the H and X^4 modes appear in the relationship and later it will be found that the overlap matrix should in some sense be invertible.

Covariantly constant right-handed neutrino does not feel electro-weak and color forces and is in a very special physical and mathematical role both in standard model and its extensions and in TGD. Right-handed neutrino could be also important as far as CP breaking considered. I have considered neutrinos earlier [L2] [K3].

1. The physical neutrino is massive and in the TGD based model for elementary fermions it must correspond to a colored neutrino, whose weak charges and color could neutralized by a pair of antineutrino and right-handed neutrino and corresponds to a size scale of a large neuron.

2. The most recent upper bound for the neutrino mass is .2 eV and corresponds roughly to scale 5.6 μm, which is roughly twice the p-adic length scale $L(167) = 2.75$ μm, which is one of the miraculous p-adic length scales assignable to the Gaussian Mersenne primes $M_{G,k}$, $k = 151, 157, 163, 167$ and corresponds to the size of cell nucleus. The energy associated with

$L(167)$ corresponds also to the nominal value .5 eV of the metabolic energy currency. The ratio of the mass scale .2 eV to proton mass is $.2 \times 10^{-9}$ and could characterize the size of CP breaking since the number of CMB photons per one proton is roughly given by this number.

Could the right handed neutrino play a key role in CP breaking? Notice however that all fermions couple to the CP violating M^4 Kähler form.

3. Note that in the context of particle physics and specifically concerning the behavior of particles in a gas, 30 microns is a significant size. This corresponds to p-adic length scale $L(181)$ ($k = 181$ is prime) by a factor $2^7 = 128$ longer than $L(167)$. At this scale, particles are large enough to be affected by gravity and other macroscopic forces, but still small enough to be influenced by the motion of individual gas molecules. This means their movement is a mix of ballistic motion. Water blob with a size of about 10 μ m has Planck mass. Gravitational Planck constant h_{gr} becomes larger than h for Planck mass. Photon energy for 30 micron wavelength is $.04eV$.

4 Possible generalizations of elliptic curves and corresponding lattices to 4-D situation

There are also questions related to the generalization of elliptic curves and corresponding lattices. This kind of generalization was already discussed in [L14].

1. Elliptic curves are complex curves in C^2 defined by the roots of third order polynomials of two complex coordinates. They have the topology of torus parameterized by a single conformally invariant complex parameter as ratio of the complex periods and therefore can be realized also as lattices in C. The identification of the lattice cells by the periodic boundary conditions gives the elliptic surface. Lattice cell is a quadrangle with sides a and b. By conformal invariance only the ratio $\tau = a/b$ is conformal invariant and defines complex structure and a point of Teichmüller space of torus. The moduli related by the action of the modular group $SL(2, Z)$ on (a, b) identifies equivalent moduli so that also this moduli space is a torus.

2. The Teichmüller spaces of general Riemann surfaces and modular invariance are discussed in the TGD based topological explanation of the family replication of fermions (and also of bosons) in terms of the genus of the partonic 2-surface [K2, K3] associated with fermion at the fundamental level. By the way, muon and its neutrino and charmed and strange quark correspond to torus topology when topological mixing as CKM mixing is neglected.

3. Space-time representations of elliptic surfaces and their generalizations as 4-surfaces using a single space-time surface are discussed in [L14]. In TGD, the coefficients of functions defining an elliptic surface are functions of either the hypercomplex coordinate u or of its conjugate v so that the conformal moduli are time dependent.

4. Is it possible to define hypercomplex analog for the lattice periods assignable to the elliptic curve? Now one does not have a torus but light-cone or perhaps CD. Since the hypercomplex coordinate u is real, only one period is possible which suggests periodic boundary conditions in u for the function pair (f_1, f_2) defining the space-time surface as its root: M^2 would be effectively compactified to a cylinder.

4.1 Hypercomplex and 4-D analogs of lattices associated with elliptic surfaces

Elliptic surface is topologically a torus. Torus allows also Minkowskian metric so that also hypercomplex torus is possible. Periodic boundary conditions for "fields" in two directions could define torus topology as an effective topology. By hypercomplex analyticity the imbedding space coordinates are only functions of (say) u. Therefore only a single coordinate can be compactified and the effective topology is that of a cylinder.

However, the intuitive idea is that the periods correspond to four wave-lengths assignable to the counterparts of p-plane waves so that 4 periods are needed. Could some physical condition allow to identify the ends of the hypercomplex cylinder?

One can imagine several ways of how lattice-like structures could emerge in a 4-D situation.

1. The functions $(f_1, f_2) : H \rightarrow C^2$ with generalized complex coordinates u, w can be more general analytic functions than polynomials of w and could have the periodicities defined by a lattice in w plane assignable having an elliptic curve as its lattice cell. The periodicities assignable to an elliptic curve (which are not conformally invariant) could correspond to lattice in the complex plane having the complex coordinate w as a coordinate. The periodicity of functions (f_1, f_2) with respect to u would provide a third period. One period is missing.

2. Could one physically define 2-D lattices of M^2 or of its light-cone? Could the periodicity in the coordinate v emerge somehow even when (f_1, f_2) does not depend on v. Space-time surfaces are located inside causal diamond (CD) which has finite size. If v corresponds to a light-like coordinate along the boundary of the CD the cylinder has its ends at the boundaries of the CD. The second periodicity, defined by the range of v, would correspond to the size of the CD defined by the distance from the tip of the CD to the maximal radius of the expanding sphere.

4.2 4-D lattices formed by space-time surfaces

Holography= holomorphy vision [L8, L11] leads to the view that space-time surfaces as numbers. This suggests that also the lattice points could correspond to space-time surfaces!

1. One can define the lattice structure also in the space C^2. Consider function pairs $(g_1, g_2 = Id) : C^2 \rightarrow C^2$ such that $g_1 : C \rightarrow C$ has the periodicities of a 2-D complex structure acting on f_1. The condition $g_1(f_1) = 0$ defines a lattice of roots of g_1 and the condition $f_1 = r$ defines a lattice of space-time surfaces. These surfaces are disjoint. For H-J structure with natural coordinates u, w, this would give a lattice-like structure in w plane. For H-J structure with the natural coordinates are (u, ξ_1) , where ξ is a complex coordinate for a Riemann surface of CP_2, say geodesic sphere or torus, the infinite lattice would correspond to a set points at this surface. Note however that the the there are physical motivations to for the form $f_2 = w - \xi^n$ or $f_2 = \xi_1 - w^n$ of f_2 [L8, L11] so that this set would corresponds to a lattice in w plane.

2. One can worry whether the condition that they have the same H-J structure is stable against external perturbations. A physical intuition suggests that stability is achieved if these space-time surfaces should be connected by monopole flux tubes in the case that the H-J structure corresponds to the coordinates (u, w).

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