About Langlands correspondence in the TGD framework

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Abstract

The considerations of this article were inspired by an interview of Edward Frenkel relating to the Langlands correspondence and led to a considerably more detailed understanding of how number theoretic and geometric Langlands correspondences emerge in the TGD framework from number theoretic universality, holography=holomorphy vision leading to a general solution of field equations based on the generalization of holomorphy, and M^8-H duality relating geometric and number theoretic visions of TGD.

The space-time surfaces are realized as roots for a pair (P_1,P_2) of holomorphic polynomials of four generalized complex coordinates of $H=M^4\times CP_2$. In this view space-time surfaces are representations of the function field of generalized polynomial pairs in H and can be regarded as numbers with product induced from the product $(P_1,Q)*(P_2,Q)=(P_1P_2,Q)$. Same applies to other arithmetic operations.

A proposal for how to count the number of roots of the $(P_1,P_2)=(0,0)$, when the arguments are restricted to a finite field in terms of modular forms defined at the hyperboloid $H^3\times CP_2\subset M^4\times CP_2$. The geometric variant of the Galois group as a group mapping different roots for a polynomial pair (P_1,P_2) identifiable as regions of the space-time surface (minimal surface) would be in terms of homomorphisms of H.

The interpretation of space-time surfaces as numbers leads to a general construction recipe for quantum states in terms of a geometric analog $(P_1,Q)*(P_2,Q)=(P_1P_2,Q)$ of a tensor product with the property that the product corresponds to a union of the surfaces $(P_1,Q)=(0,0)$ and $(P_2,Q)=0$. The asymmetry between P_i and Q has a physical interpretation. The interactions would be associated with stable intersection points of these surfaces and a deformation making P_1P_2 irreducible replacing them with wormhole contacts is suggestive. One could say that TGD is exactly solvable at both classical space-time level and quantum level. The geometric Langlands correspondence extends to a trinity between number theory, geometry and physics.

1 Introduction

I listened to an extremely interesting interview with Edward Frenkel. Thanks to Marko Manninen for providing the link (see this). Frenkel explained more than two hours aspects of the number theoretic Langlands correspondence [A10, K5, A3, A2] in order to provide the background making it possible to get to the geometric Langland correspondence in the next interview! The fact of course is that one must learn the basic notions first.

If one should summarize the Langlands correspondence (LC) very briefly, one might say that LC can be seen as a kind of grand unification of mathematics. LC relates typically two totally different fields of mathematics. Modular forms is one aspect of the LC. Galois groups permuting the roots of polynomials are a second aspect, also mentioned in the interview.

1.1 Polynomials defining elliptic surfaces in finite fields and modular forms in hyperbolic plane

The example related to the number theoretic LC discussed in the interview relates number theory and so-called modular forms defined in the hyperbolic plane H^2 (see). The graphics of Escher gives a good idea of what a hyperbolic plane looks like.

1. The modular forms known as elliptic functions are doubly periodic functions in complex plane and serve analogs of plane waves appearing as solutions of field equations of free field theories.

2. Lorentz group $SL(2,C)$ acts as global conformal symmetries of the complex plane compactified to the Riemann sphere CP_1. $SL(2,C)$ consists of 2×2 matrices with unit determinant and acts linearly on 2-spinors and as Möbius transformations on the points of the complex plane CP_1, whose points correspond to the ratios z^1/z^2 of the complex spinor components. 2-D hyperbolic space H^2 can be represented as upper half-plane of the complex plane or as an interior of a unit disk and has the real Lorentz group $SL(2,R)$ as conformal symmetries mapping the real axis or the boundary of the disk to itself.

   For H^2, the modular group can be identified as some subgroup $SL(2,Z)$, having integer valued matrix elements satisfying some additional conditions. Modular forms have the modular group as their symmetries. The modular subgroup can correspond also to a group $SL(2, Z^2)$ where Z^2 consist of Gaussian integers $n_1 + in_2$. Modular transformations act as translations in the complex plane. There is an infinite number of analogs of modular groups since the elements could also be algebraic integers Z_E in some extension E of rationals.

The example discussed in the interview was very inspiring also from the TGD point of view.

1. The example was about the number of solutions to certain kinds of Diophantine equations defining roots of polynomials involving 2 variables x, y with integer coefficients in finite field defined by integers modulo prime p. Modulo p corresponds to a generalization of the clock arithmetics in which 13=1, 14=2, etc.: now there would be p hours per day. The polynomials considered are cubic polynomials, which third order polynomials of x and second order polynomials y satisfying some additional conditions, which guarantee that they allow a modular symmetry analogous to a 2-D translation symmetry.

   The independent variables x, y can be numbers in finite fields, integers, reals, complex numbers. In other words, the equations are number-theoretically universal. This is what makes algebraic geometry so beautiful.

2. If the variables x, y are complex numbers, the solutions are 2-D surfaces in 2-D complex space C^2 having complex coordinates x and y. With some additional assumptions about the integer coefficients they are elliptic curves, which have the topology of the torus. Elliptic curve, or a general cubic curve, can be written in rather general case in the canonical form (see this):

$$y^3 = x^2 + ax + b \ . \tag{1.1}$$

3. Elliptic curve intersects line (in complex case complex line, that is plane) in 3 points except in the situations, when the equation allows degenerate roots. This reflects the fact that the equation has 3 roots. A similar equation appears in the cusp catastrophe. The coefficients are however real numbers rather than integers in this case. The 4-D generalization of cusp catastrophe is very interesting also in the TGD framework,

4. The elliptic curve can be parameterized in terms of elliptic functions, which can be regarded as doubly periodic functions in the complex plane. They are non-linear analogs of plane waves. The rational points of an elliptic curve form a group which is finitely generated. Elliptic curves are modular, which means that it can be constructed as a quotient of a complex plane under modular grou, which is a subgroup of the group $SL(2, Z)$ analogous to a discrete subgroup of translational group but acting on H^2 rather than Euclidean space E^2.

5. With this background one can consider a general equation defining elliptic curve and restrict the solutions to integers and consider solutions x,y as integers modulo p.

   What is the number of solutions for a given prime p. This is the problem. Langland's discovery implies that one can find a modular form in the hyperbolic plane H^2, whose Taylor expansion contains terms with coefficients, which code for the numbers of the solutions for any prime p: the problem of finding the number of solutions is solved for almost all primes at the same time! There exists a finite number of exceptions but still this is quite an impressive achievement.

   Anyone can write a computer code listing the numbers of solutions for primes $p = 2, 3, 5, 7, ...,$ This is a gigantic leap in understanding and LC generalizes this. There is also a geometric variant of this correspondence.

1.2 Galois group, dual group and number theoretic LC

Galois groups define number theoretic symmetries. Consider a polynomial $P(x)$ of single variable x with coefficients, which are algebraic integers in some extension E of rationals. The standard definition for the Galois group is as a group acting in the extension of E as permutations of the roots acting trivially in E.

Let G be a reductive Lie group G represented in an extension K of field k, which has Galois group $Gal(K/k)$. LC relates the "good" irreducible representations G to the representations of Langlands dual ${}^LG$, which has the same root datum as G and is therefore infinitesimally equivalent with G. ${}^LG$ can be regarded as the semidirect product ${}^LG^o \ltimes Gal(K/k)$, where ${}^LG^o$ is the connected component of ${}^LG$. The number of connected components of ${}^LG$ corresponds to the order of the Galois group, which implies that ${}^LG$ depends on K via its Galois group.

LC states roughly that there is one-one correspondence between "good" representations of G and homomorphisms of the Galois group $Gal(K,k)$ to the Langlands dual ${}^LG$. One could say that the extension by the Galois group gives rise to a kind of number theoretic degree of freedom and that the irreducible representations of the Galois group become additional degrees of freedom.

1.3 LC and TGD

It must be emphasized that LC is extremely general and the forms of LC discussed in this article from the TGD perspective are not expected to be the only ones realized in TGD. Both number theoretic and geometric LC have counterparts in TGD.

1.3.1 Number theoretic LC and TGD

Consider first the number theoretic LC.

1. In the electric-magnetic duality (Montonen-Olive duality) of gauge theories [B1], electric and magnetic couplings are combined to a complex number $z = z_1/z_2$ representing a point of CP_1. Electric-magnetic duality corresponds to S-duality as the Möbius transformation $z \to -1/z$.

   More generally, $SL(2,Z)$ transformations act as Möbius transformations induced by the linear action on (z_1, z_2) in C^2 and could correspond to physical symmetries giving rise to different actions. $SL(2,Z)$ has 3 matrix generators $S = [0,1;-1,0]$,$T = [1,1;0,1]$ and $T^{-1} = [1,-1;0,1]$ acting linearly on (z^1, z^2). The product $U = ST$ is called $U = ST$ duality. S corresponds to inversion $z \to z$ and to electric magnetic duality. In M-theory there are 3 analogous dualities S, T, and U. One might guess that SL(2,Z) appears because it generates global conformal transformations of the conformal algebras.

2. In the TGD framework, the self-duality of CP_2 Kähler form implies that the electric and magnetic fluxes over a given 2-surface of CP_2 are indeed identical. This inspires the proposal that TGD is self-dual with respect to S [K1] a stronger condition would be that the entire $SL(2,Z)$ defines equivalent actions. In fact, in the TGD framework space-time surfaces are minimal surfaces apart from singularities irrespective of the action principle and the action makes itself manifest only through the singularities at which the minimal surface property fails [L30, L31, L32, L19].

   A weak form of this duality is proposed to be realized at the light-like interactions between Euclidean CP_2 type extremals and Minkowskian regions of the space-time surface. The corresponding parts in the action correspond to Kähler action and the associated topological instanton term.

3. In TGD, the geometry of the "world of classical worlds" (WCW) allows two kinds of symmetries.

   (a) The generalized super-conformal symmetries defined as a 4-D generalization of Kac-Moody symmetries appear since holography = holomorphy vision implies that space-time surfaces are holomorphic minimal surfaces of $H = M^4 \times CP_2$ irrespective of the action as long it is general coordinate invariant and constructed in terms of the induced geometry. One can assign the charges of these symmetries to the light-like boundaries of the causal diamond $CD = cd \times CP_2$ and also to the light-cone proper time constant surfaces of the light-cone.

   (b) Super-symplectic symmetries in turn have a natural action at the boundaries at the light-like orbits of the partonic 2-surfaces between the Euclidean interior of the CP_2 type extremals and the Minkowskian exterior. Could a strong form of holography, which would state that it is enough to define the theory using only the charges of either kind, make sense. Also this kind of duality might be seen as a kind of LC .

1.3.2 Geometric LC and TGD

The geometric Langlands duality emerges naturally from the holography = holomorphy vision.

1. The space-time surfaces are realized as roots for a pair (P_1, P_2) of holomorphic polynomials (or more general analytic functions) of four generalized complex coordinates of $H = M^4 \times CP_2$. In this view space-time surfaces are representations of the function field of generalized polynomial pairs in H and can be regarded as numbers with arithmetic operations induced from those for the polynomial pairs. Product is always well defined by inverse is ill-defined if either function vanishes.

2. A proposal for how to count the number of roots of the $(P_1, P_2) = (0, 0)$, when the arguments are restricted to a finite field in terms of modular forms defined at the hyperboloid $H^3 \times CP_2 \subset M^4 \times CP_2$. The geometric variant of the Galois group as a group mapping different roots for a polynomial pair (P_1, P_2) identifiable as regions of the space-time surface (minimal surface) would be in terms of holomorphisms of H.

3. Space-time surfaces inherit the structure of the function field from the members of the pairs identifiable as numbers. To obtain a number field analogous to rationals, one must however restrict the product to a subspace of rational functions (R, Q) with coefficients in some extension E of rationals, Q fixed, and define the product as $(R_1, Q) * (R_2, Q)) = (R_1 R_2, Q)$. This asymmetry conforms also with the interpretation of $Q = 0$ as a definition of the correspondence between the wistor sphere of M^4 and CP_2. One can generalize the notions of integer, prime, rational number and extensions of rationals, and transcendental so that they apply to space-time surfaces identified as numbers: one can multiply, divide, sum and subtract them. Obviously, the definition generalizes to algebraic and analytic functions and produces analogs of extensions of rationals and complex numbers.

   The interpretation of space-time surfaces as numbers leads to a general construction recipe for quantum states in terms of a geometric analog $(P_1, Q) * (P_2, Q) = (P_1 P_2, Q)$ of a tensor product with the property that the product corresponds to a union of the surfaces $(P_1, Q) = (0, 0)$ and $(P_2, Q) = 0$. The asymmetry between P_i and Q has a physical interpretation. The interactions would be associated with stable intersection points of these surfaces and a deformation making $P_1 P_2$ irreducible replacing them with wormhole contacts is suggestive. One could say that TGD is exactly solvable at both classical space-time level and quantum level. The geometric Langlands correspondence extends to a trinity between number theory, geometry and physics.

In the following I will discuss number theoretic and geometric LC in the way as I see them in the TGD framework. My view is not of a professional mathematician but of physicist and TGD dictates my views to a high degree. I have discussed LC several times already earlier [K5, K6] [L13, L30].

2 Number theoretic LC and TGD

In TGD both number theoretic and geometric LC are expected to play a role. Number theoretic LC is naturally associated with the adelic physics [L1, L2] involving number theoretic discretizations of the space-time surface in some extension of rationals. All p-adic number fields are involved.

This requires theoretical universality, which is guaranteed if the definition of space-time surfaces is such that it makes sense in all number fields. In holography=holomorphy vision, the definition of the space-time sheet as a root for a pair of polynomials, with coefficients in some extension E of rationals, guarantees this. In number theoretic vision one must have discretization by assuming that the roots are in rationals or even integers of E. One obtains a hierarchy of adeles labelled by E, forming an evolutionary hierarchy.

In the TGD framework, also a modification of the notion of adele can be considered [L24, L30]. p-Adic number fields and their extensions can be glued together along integers, which can be regarded as simultaneously belonging to several p-adic number fields since the p-adic integers n in the intersection of the p-adic number fields can be expressed as a power series of an integer m divisible by the primes considered. Systems characterized by different p-adic primes, identifiable as ramified primes for a polynomial, must interact, and the modified view of adeles allows us to describe this interaction at the level of discretization.

2.1 How could the number theoretic LC relate to TGD?

Some background is required in order to understand how the number theoretic Langlands involving finite fields and modular forms in Langlands group relate to TGD.

2.1.1 Number theoretical universality and holography=holomorphy principle

Number theoretical universality and holography=holomorphy principle [L30, L28] are central in TGD.

1. In TGD number theoretical universality becomes a basic physical principle [L30]. The number theoretical view and geometric view of physics are proposed to be equivalent. This M^8-H correspondence can be seen as a generalization of momentum-position duality of quantum mechanics required by the replacement of point-like particle with 3-surface and is analogous to the LC .

   An intuitive motivation for this proposal is that in free field theories, the field equations at the level of momentum space reduce to algebraic equations, typically mass shell conditions. At the level of space-time, one has partial differential equations describing the propagation of fields.

2. This view is realized in terms of holography=holomorphy correspondence implying that space-time surfaces are analogs of Bohr orbits [L30, L28]. The field equations characterizing the space-time surfaces reduce to equations, which are a non-linear generalization of massless field equations.

   There is a completely general solution ansatz for which the field equations reduce to minimal surface equations for any general coordinate invariant action principle constructed in terms of the induced geometry. At the level of H, the field equations reduce to purely algebraic equations involving a contraction of two tensor quantities, which are of different types as complex tensors and therefore vanish identically [L30]. Minimal surface equations fail only at the singularities, which have a dimension lower than 4 [L32, L19]. At the level of M^8, identifiable as octonions with number theoretical inner product $Re(o_1o_2)$, associative holography allows to construct the 4-surfaces in terms of 3-D holographic data [L26]. M^8-H duality allows an alternative manner to construct these 4-surfaces.

3. M^4 metric has Minkowskian signature and for the space-time surface the regions outside elementary particles (partonic orbits) are Minkowskian. For Minkowski signature one must generalize the notion of complex structure to Hamilton-Jacobi structure, as I call it [L20]. Hamilton-Jacobi structure is a fusion of ordinary complex structure and hypercomplex structure. Hypercomplex numbers are very much like complex numbers but do not form a number field since the norm vanishes for light-like hypercomplex numbers. Hypercomplex coordinate u as counterpart of complex coordinate z corresponds to a coordinate for which coordinate lines are light-like. Its conjugate v corresponds to a dual light-like coordinate.

4. In the general case, the space-time surfaces can be identified as roots of 2 analytic functions f_1 and f_2 of the 4 generalized complex coordinates of $H = M^4 \times CP_2$. One of them is a hypercomplex coordinate of M^4 and the remaining 3 are ordinary complex coordinates. The coefficients of the analytic functions can be also assumed to belong to an algebraic extension E of rationals and this gives rise to a hierarchy consisting of discrete subspaces of the "world of classical worlds" (WCW).

5. One can also consider a hierarchy of polynomial pairs (P_1, P_2) with coefficients in E. In the simplest case the coefficients are ordinary rationals or equivalently integers. One obtains a

hierarchy of solutions of field equations labelled by the extension and by the degrees of the polynomials and by the extensions of rationals that they define. This hierarchy is identified as an evolutionary hierarchy [L1, L2]. The higher the complexity of the space-time surfaces, the higher the evolutionary level of the system.

2.1.2 Number theoretic discretization gives cognitive representations

In the number theoretic vision [L30], the number theoretic discretization is a second aspect of TGD. It applies to space-time surfaces and, at a more abstract level, to the WCW coordinates defining moduli characterizing the space-time surfaces.

1. One can say that the most concrete cognitive representations correspond to number theoretic discretizations of the space-time surface. The first question concerns the number of roots $(P_1, P_2) = (0, 0)$ for a given prime p characterizing a p-adic number field, in particular the ramified prime of a related polynomial. Can one estimate the number of roots for any finite field in turn defining the number of solutions in corresponding p-adic number field? This is the key question.

2. From the example discussed in Frenkel's interview, I learned how one can get a rough idea of this number at least in some cases. This argument generalizes the well-known counting argument for the number of roots of two polynomials of two real variables. In this case the number of solutions is the number of variables minus the number of conditions. For polynomials defined in a finite field, the number of points (x, y) in a finite field is p^2. The Diophantine equation gives p conditions. Therefore the rough guess for the number of solutions is $p^2/p = p$ (in the discrete situation, subtraction is replaced by division). Already this is a fantastic piece of information.

3. Could it be possible to get the exact number # of integer solutions modulo p for any p-adic prime? LC in the case of the Diophantine equations considered in the interview comes to rescue. The modular function assignable to the equations codes in terms of its Taylor coefficients the deviations of the $\# - p$ from the rough guess for all primes p except a finite number of exceptional ones! Note however that the Diophantine equation possesses a modular symmetry.

2.1.3 Could the number theoretical LC generalize from dimension 2 to dimension 4?

Could the number theoretical LC for polynomials of two arguments generalize from dimension 2 to dimension 4. holography=holomorphy vision suggests a very concrete formulation of this generalization.

Consider first the statement of the problem in the TGD context.

1. Cognitive representation as a discretization of the space-time surface would consist of the points of the space-time with coordinates in the extension E of rationals defining the coefficient field of P_1 and P_2. These points would be rational numbers as ratios of E integers. One should be able to count the numbers of the solutions to the equations $(P_1, P_2) = (0, 0)$ for which the $H = M^4 \times CP_2$ coordinates have values in E. A weaker condition is that only M^4 coordinates belong to E. Could a generalization of LC make this possible?

2. The 2-dimensional elliptic surfaces (analogs of string world sheets) reside in C^2, which has 2-complex dimensions. Space-time surfaces have 1 hypercomplex dimension and 1 complex dimension and have a generalized complex structure, Hamilton-Jacobi structure [L20]. Space-time surface has 2 generalized complex coordinates (u, w). u is a hypercomplex coordinate with light-like coordinate curves and w is a complex coordinate.

   The 2-complex dimensional space C^2 with complex coordinates (x, y) is replaced with the embedding space $H = M^4 \times CP_2$ having 4 generalized complex coordinates (one of them is hypercomplex) so that the polynomials P_1 and P_2 depend on 4 generalized complex coordinates (u, z_1, z_2, z_3).

3. An elliptic surface is a root of a single polynomial of degree 3 depending on (say) 2 complex variables (z_1, z_2) in 2-complex dimensional space C^2. In TGD, space-time surfaces are defined as roots of two polynomial equations $(P_1, P_2) = (0, 0)$.

4. The natural generalization of the 2-dimensional (in real sense) hyperbolic space H^2 (remember Escher) is the hyperbolic 3-space H^3. This corresponds to the mass shell as momentum space or light-cone proper time $a = constant$ surface in Minkowski space M^4 (actually inside a causal diamond cd). What is remarkable is that the Lorentz group $SL(2, C)$ and its modular subgroups act in H^3 and have the same interpretation as in special relativity.

Could one find a modular form in the product $H^3 \times CP_2$, where H^3 us hyperbolic 3-space coding for the numbers of the solutions to the number theoretical discretization conditions for the space-time surface for any pair $(P_1, P_2)$of polynomials? If this were possible, it would have enormous practical value. This might be possible except under very special conditions probably related to the ramified primes of the polynomials involved.

1. The counterparts of the holographic boundary conditions at selected hyperboloids $a = a_k$ should satisfy modular symmetries. Is this implied by the Bohr orbit property for the space-time surface [L30]? This would mean that the space-time surfaces are analogous to H^3 counterparts of discrete plane waves.

2. The rough estimate for the number of solutions of the cubic equation with modular invariance generalizes. In number theoretic discretization, the extension of rationals corresponds to a finite field $G(p, k)$ for some k. The number of elements in the finite field $G(p, k)$ is p^k. One has a p-adic discretization of the space-time surface using points for which the preferred coordinates are algebraic integers in an extension of p-adic numbers induced by the extension of rationals. These points are approximated with points of finite field $G(p, k)$. The generalized complex dimension of H is $D = 4$. Therefore one has p^{4k} points in the discretization. The two vanishing conditions $(P_1, P_2) = (0, 0)$ give p^{2k} conditions so that the rough guess for the number of the solutions is p^{2k}. Already this is a fantastic result, which was new to me.

3. One might hope that if modular invariance is realized at the hyperboloids $H^3(k)$ defining the boundary data, then a modular form assignable to the pair (P_1, P_2) could give the number of roots in $G(p, k)$ by coding the correction $\# - p^{2k}$ by some coefficients in its Taylor expansion. This would in turn give the numbers of the p-adic roots since they can be generated once the solution in finite field $G(p, k)$ is known.

2.2 How to identify the Galois group in the number theoretic picture?

The physical interpretation would be that the Galois group extends the group algebra of ${}^LG$, where G is $SL(2, C)$ acting at H^3. The Galois group brings in additional degrees of freedom: a kind of Galois spin. There are two interpretations of the Galois group.

1. The simplest option is that K corresponds to an extension E of rationals appearing as the coefficient field of the polynomials or even as the field for the Taylor coefficients of analytic functions considered so that the number theoretic Galois group is naturally associated with it.

   In this case one cannot however identify the ramified primes since they require a definition of the extension in terms of the roots of a polynomial. The Galois group would not characterize the polynomial P_1 or P_2 but the field extension E. In this case, one would consider the discretization allowing only E-rational points.

   The group G could be the Lorentz group $SL(2, C)$ represented in the space of modular forms defined in H^3. The complex dual group ${}^LG$ would have the same Lie-algebra as $SL(2, C)$ and would be a semidirect product of ${}^LG^0 \ltimes Gal(E/Q)$.

2. In the TGD framework, one can also consider the identification of the Galois group as that for the extension of E defined by the space-time surface itself, if in the case that it is well-defined. If holography=holomorphy vision is accepted, H has 4 generalized complex coordinates.

Suppose that 2 of the complex coordinates are E rationals. One can solve the 2 remaining generalized complex coordinates from the conditions $(P_1, P_2) = (0, 0)$.

The hypercomplex coordinate of M^4, which has light-like coordinate curves and is real and therefore does not allow complex values, is not a good candidate to be one of these 2 coordinates. This leaves only two options under consideration. The two complex coordinates ξ^1, ξ^2 of CP_2 would make sense when the space-time surface has 4-D CP_2 projection and the induced metric is Euclidean.

The second option corresponds to the complex coordinate ξ^1 or ξ^2 of CP_2 and to the complex coordinate w of M^4 assignable to its Hamilton-Jacobi structure [L20]. The values of, say, ξ^1 and w are obtained by solving $P_1 = 0$ and $P_2 = 0$. One could require that w is E rational and ξ^1 in the extension of E or vice versa. One can assign to both options a Galois group and an extension of E.

3. This picture would conform with the fact that in TGD two kinds of effective Planck constants appear naturally. The first kind of effective Planck constant, identified tentatively as the dimension of extension of E, can be assigned with the space-time surface as a covering of M^4 and would be relatively small since one does not expect that the space-time has very many sheets with the same M^4 projection. The other effective Plank constant corresponds to the space-time surface as covering CP_2 and could be the counterpart for say periodic field pattern having as a TGD counterpart a bundle of monopole flux tubes in H. The number of the flux tubes would correspond to the degree of the polynomials considered and could be very large [L3, L9, L10, L17].

2.3 Objections

The best way to sharpen a hypothesis is to invent objections against it.

1. According to Frenkel, there are exceptional primes for which the coding fails. A good guess is that they correspond to the ramified primes for a cubic polynomial in the case considered. Indeed if the other coordinate of the cubic polynomials is integer, the polynomial reduces to a cubic polynomial with integer coefficients and one can assign to it roots and ramified primes.

   In TGD, the p-adic primes characterizing space-time surfaces are identified as the ramified primes assignable to the polynomials defining the space-time surface. If the problematic primes are ramified primes, the coding would fail for the physically interesting primes! The blessing in disguise is that the number of ramified primes is finite.

2. Modular invariance characterizes the example considered by Frenkel. Modular group acting as symmetries is analogous to lattice translations: torus topology indeed reflects this. Modular invariance implies that the numbers of solutions to a given cubic equation are coded by a particular modular invariant function. Could the TGD analog of this condition, stating that the solutions are analogous to plane waves, be too strong? One might of course hope that by the notion of holography meaning that space-time surfaces are analogs of Bohr orbits requires this condition or that it corresponds to "good" representations of $SL(2, C)$ at quantum level.

Suppose that a modular form for, say 3-D hyperbolic space H^3 (mass shell $a = a_k$), invariant under a modular group identified as discrete subgroup of $SL(2, Z)$ (or $SL(2, Z_E)$), codes for the numbers of say solutions to $(P_1, P_2) = (0, 0)$ belonging to an extension E of rationals appearing as coefficients of P_i. This would mean that all 4 generalized complex coordinates belong to E.

1. One might expect that the modular group acts as symmetries of the space-time surface in the same way as the modular group for elliptic surfaces. This could mean periodicity analogous to that for planewave solutions to field equations. Could the TGD counterparts of plane wave solutions of free field theories correspond to generalized elliptic functions of H^3 for which 4-D translational invariance is replaced with 3-D modular invariance at H^3. Inside the space-time surface translational (or scaling-) invariance with respect to a is probably too much to require but holography=holomorphy principle allows the continuation of the boundary data

to the space-time interior so that it is not needed. The modular invariance would reduce to the modular invariance of the holographic data at some surfaces H^3 with constant value of the light-cone proper time.

One would obtain the H^3 analogs of 3-D plane waves in E^3. In free field theory in M^4, E^3 plane waves can be continued to 4-D plane waves. In TGD, holography allows to continue the counterparts of the E^3 plane waves in H^3 to the interior of the space-time surface.

Could it be that modular invariance is a physically sensical restriction.

1. One cannot have a linear superposition of the H^3 plane waves since TGD is not a free field theory. This restriction holds true for the so-called massless extremals [K1] for which the linear superposition holds only for modes propagating in the same direction: one has an analog of a laser beam. The pulse shape is preserved in the propagation since there is no dispersion.

2. Linear superposition would correspond in TGD to a union of space-time surfaces having the same M^4 projection: a test particle touching the space-time surfaces in the union would experience the sum of the induced gauge fields and gravitational field associated with different space-time sheets.

3. How strong conditions does the modular symmetry pose on the polynomials (P_1, P_2)? Does modularity mean that only a very restricted number of polynomials allows this coding? Or could all solutions of field equations reduce to H^3 analogs of plane waves for some set of $a = constant$ hyperboloids H^3?

   This form of holography I have indeed proposed. This would mean a nonlinear generalization of Fourier analysis from $E^3 \subset M^4$ to the level of the space-time surface with E^3 replaced with the intersections of these special subset of hyperboloids. The intersections of the space-time sheet with $H^3(a_k)$ would represent the Fourier mode and their superposition would correspond to a set theoretic union.

4. I have proposed that the tessellations of H^3 are fundamental in TGD and characterize the solutions of the field equations as symmetries of the boundary data at a finite set of hyperbolic 3-surfaces identifiable as light-cone proper time $a = a_k$ surface. The equations for the minimal surfaces are not completely deterministic and non-determinism would be localized at these hyperboloids. At these hyperboloids the data would satisfy the modular symmetry so that the intersection with $H^3(a_k)$ would reduce to a union of fundamental domains of the modular subgroup, which are analogous to lattice cells in E^3.

   Are space-time surfaces without modular symmetry for any $H^3(a)$ impossible or are they such that the representations of $SL(2, C)$ are not "good"? In the case of H^2 this poses extremely powerful conditions on the polynomials considedere and the same is expected in the case of H^3.

3 Geometric LC and TGD

Also geometric LC would emerge with the holography=holomorphy vision [L30, L31, L28]. In this case, number fields are replaced with function fields defined by generalized holomorphic functions in $H = M^4 \times CP_2$. The pairs (f_1, g_1) of polynomials with coefficient field E as extension of rationals define space-time surfaces as their roots. The pairs (f_1, g), g fixed, form a function field with respect to product $(f_1, g) * (f_2, g) = (f_1 f_2, g)$ so that space-time surfaces as roots $(f_1, g) = (0, 0)$ behave like numbers and form an analog of a number field. Note that here also rational functions can be allowed since their poles do not appear as points of the space-time surface. This also applies to general analytic functions with Taylor coefficients in E.

This asymmetric option is physically favored and the product of space-time surfaces has an interpretation as a geometric analog of a tensor product giving rise to a two-particle state. Many-particle states are obtained by performing arithmetic operations for the space-time surfaces.

3.1 Space-time surfaces as numbers

One can endow the polynomial pairs defining the space-time surfaces with a structure of a function field, which is inherited by the roots so that also space-time surfaces can be regarded as numbers.

1. One can define basic arithmetic operations (multiplication, division, sum and subtraction) for the space-time surfaces defined by the conditions $(P_1, Q) = (0, 0)$ and $(P_2, Q) = (0, 0)$. The product of the space-time surfaces would be defined by the conditions $(P_1P_2, Q) = (0, 0)$. The generalization to the other arithmetic operations is obvious. There are two roots $(P_1, Q) = (0, 0)$ and $(P_2, Q) = (0, 0)$ that correspond to the space-time surfaces having interpretation as free particle states. However, due to the dimensions of space-time surfaces and embedded space, these surfaces have a discrete set of intersection points and the physical intuition suggests that these singularities give rise to interactions between the particles.

   The fixing of Q has interpretation as a condition fixing the relationship between the space-time images of twistor spheres of M^4 and CP_2 and defines cosmological constant as a dynamical scale dependent parameter.

   One can wonder whether one could combine the generalized analytic function f_1 and f_2 to a quaternion valued function $f_1 + Kf_2$, where K is a quaternionic unit? This is formally possible but the product would be non-commutative and would not be a generalized analytic function anymore. Also the nice physical interpretation of the product of space-time surfaces is lost.

2. One can multiply the defining polynomials by functions which do not vanish in the region defined by the space-time surface. This region corresponds naturally to the causal diamond cd of M^4. In this case the space-time surface as a root is not affected. This is analogous to the multiplication of a polynomial with a polynomial (or even rational function), which vanishes nowhere in the range considered. This multiplication corresponds to a multiplication of an ordinary polynomial with a rational number. One can identify elementary particle-like space-time surfaces as those associated with irreducible polynomials P_i and Q. It turns out that the so-called blow-up could eliminate the intersection points as singularities by replacing the P_1P_2 with an irreducible polynomial.

3.2 Is there a relationship to the hierarchy of infinite primes?

How does this picture relate to the notion of infinite-primes [K13]? Infinite primes can be said to correspond to a hierarchy of second quantizations of a supersymmetric arithmetic theory in which single boson- and fermion states at a given level correspond to infinite primes as many-particle states of the previous level. At the lowest level they correspond to ordinary primes having a possible interpretation as momenta.

1. At a given level Dirac vacuum X corresponds to the product of all primes, which is infinite as a real number but finite as a p-adic number for any prime p.

2. The construction of infinite primes as analogs of free particles is analogous to the construction of many particle states in second quantization. The simplest states are of the form $P = mX + n$, where m and n are integers chosen in such a way that P is indivisible by any finite prime.

3. Also infinite primes, analogous to bound states, are possible and are realized as irreducible polynomials $P(X)$ (no decomposition to a product of monomials with integer coefficients) with coefficients, which are (possibly infinite) primes of the previous level. At level n, these polynomials can be regarded as polynomials of n formal variables $X_1, ...X_n$.

4. At the lowest non-trivial level the roots of P correspond to integers and possibly complex algebraic numbers. At the next level, one has polynomials of two variables and roots correspond to 1-D curves. Next level gives 2-D surfaces, third level gives 3-D surfaces and the fourth level 4-D surfaces as roots. This construction generalizes from the case of rational integers to integers in extensions of rationals.

The proposal has been that this hierarchy corresponds to the scale hierarchy of space-time sheets with sheets having particle interpretation. The challenge is to see whether this hierarchy could somehow relate to the hierarchy defined by the pairs (P, Q) of polynomials.

$P = 0$ and $Q = 0$ surfaces define the analogs of 6-D twistor spaces of M^4 and CP_2 having the same space-time surface X^4 as a base space and twistor sphere as a fiber [L30, L31, L28]. The intersection of these two twistor spaces gives X^4. H has 4 generalized complex coordinates so that a 4-level hierarchy for which X can be complex is suggestive.

There are two problems.

1. The first problem is that the hierarchy of infinite primes involves polynomials rather than pairs of them. It however turns that product of space-time surfaces is induced by the product of $(P_1, Q)*(P_2, Q) = (P_1P_2, Q)$ so that Q is fixed for a given hierarchy. Infinite integers would correspond to $(\prod_i P_i, Q)$, where P_i is irreducible and has an interpretation as an infinite integers. Also physical arguments support this view.

2. The second problem is that the higher levels of the hierarchy seem to be absent? The geometric interpretation of the infinite hierarchy of infinite primes encourages the question whether one could obtain also the higher levels of the hierarchy in the holography= holomorphy vision in some way?

   It seems that one must consider H^n instead of H at the n^{th} level of the hierarchy. At the level of the spinor structure this means n^{th} tensor power of the spinor space of H. This conforms with the idea of repeated second quantization and with the ideat the infinite primes define an abstraction hierarchy of statements about statements about.... Note also that in condensed matter physics, the n-particle system is defined in terms of particles in $(E^3)^n$. This could be replaced by H^n, where n is analogous to the number of particles and to the level of the hierarchy.

   Could 4-D space-time surface in H^n correspond to the roots of polynomial-tuples $(P_1, P_2, ...P_k)$ of $k = 4n-4$ polynomials depending on the $4n$ generalized complex coordinates of H^n? The roots satisfy $k = 4n - 4$ conditions eliminating this number of degrees of freedom so that the conditions define a 2-complex-dimensional surface of H^n identifiable as the space-time surface. Its projections to the n Cartesian factors of H^n would define 4-D surfaces, which could represent the geometric view of a particular particle-like entity about the entire state.

   Could this description give a purely geometric description for the interactions of the particles? Or could it give a description for the hierarchy of space-time sheets topologically condensed at larger space-time sheets? The mathematical description of this hierarchy is still far from quantitative. Hierarchy requires an ordering. In the case of infinite primes there is a natural ordering involved. Does this generalize? At n^{th} level one has polynomials of n:th variable having as its coefficient polynomials of $(n-1)^{th}$ level. The ordering is partially determined by requiring that a given level has the highest possible polynomial degree. At each level this selects a finite number of alternatives for the coordinate in question.

3.3 About various passive symmetries

General coordinate invariance creates an apparent tension between number theoretic and geometric vision since general coordinate transformations do not respect polynomiality. Also the number theoretic notions such as the notion of effective Planck constant and ramified primes should be formulated in a general coordinate invariant manner.

3.3.1 General coordinate invariance and number theoretical vision

The technical problem relates to general coordinate invariance, in particular, to the possibility to perform generalized holomorphic coordinate changes, with holomorphy required inside the causal diamond CD or at least in a smaller region containing the space-time surfaces involved. In this kind of transformation the polynomials are mapped to more general functions. Physically the roots of polynomials, whose arguments are replaced with functions of new coordinates, are not affected.

It would be highly desirable to have coordinates in which one can identify uniquely space-time surfaces as roots of function pairs (f_1, f_2) of different kinds forming a hierarchy, which would consist of hierarchies of polynomial pairs with coefficients in various extensions E of rationals, analytic functions with Taylor coefficients (or Laurent coefficients) for various choices of E and finally arbitrary complex coefficients. Note that also rational functions P/Q can be allowed and in this case the poles of Q would define roots. In the case of CP_2 coordinates the poles would correspond to finite points.

Some preferred coordinates defining this hierarchy uniquely should exist. The physical intuition serves as a guideline in the identification of these preferred complex coordinates. The light-like hypercomplex coordinate u and the complex coordinate w associated with the Hamilton-Jacobi structure of M^4 and complex Eguchi-Hanson coordinates ξ^i of CP_2 are indeed natural candidates for the preferred coordinates.

What about the situation at the level of M^8? It would not be too surprising if the 4-surfaces as M^8-H duals of the space-time surfaces could also be defined as roots of a function pair, call it (F_1, F_2). In this case, preferred coordinates are provided by the linear coordinates in which the octonionic arithmetics has its standard form. This allows only the octonionic automorphism group G_2 as coordinate changes. M^8-H duality suggests very close connections between the pairs (f_1, f_2) and (F_1, F_2).

3.3.2 Generalized holomorphies and multiplication by non-vanishing holomorphic functions as gauge symmetries

Generalized holomorphies act as analogs of conformal symmetries, which gives rise to gauge conditions. It is however essential that the corresponding infinitesimal transformations integrate to bijections. If this does not take place, a breaking of conformal symmetry takes place. TGD indeed suggests a hierarchy of conformal symmetry breakings in which sub-algebra A_n of conformal half-algebra with non-negative weights and its commutator $[A_n, A]$ with the entire algebra acts as a gauge algebras and the subalgebra with conformal weights below n acts as dynamical symmetries. This picture is expected to generalize when the functions of a single complex variable are replaced with 4 generalized complex variables.

The multiplication of (f_1, f_2) in an element-wise manner by a function pair (Q_1, Q_2) leaves the space-time surface unaffected if the functions Q_i are non-vanishing inside CD. The requirement that the functions f_i do not decompose in this way, boiling down to irreducibility in the case of polynomials, allows a gauge fixing in this case.

3.4 How to define the geometric analogs for rationals and their extensions?

How do the notions of rationals and extension of rationals generalize? Does the generalized Galois group leave the generalized rationals invariant? A natural approach is provided by zero energy ontology (ZEO) in which the causal diamonds $CD = cd \times CP_2$ play a key role as analogs of perceptive fields of a conscious entity, the observer.

1. Pairs (Q_1, Q_2) of rational functions with Q_2 fixed, of rational functions with coefficients in E would define the counterpart for rationals. If Q_i are non-vanishing inside CDm the element-wise multiplication of $(P_1, Q : 1)$ to give (Q_1P_1, Q_2P_2) does not give rise to new roots inside CD. This corresponds to gauge invariance and the irreducibility constraints drops these polynomial pairs from consideration. Q_i have no have no roots inside CD so that the generalized Galois group permuting the space-time regions representing various roots acts on them trivially. Multiplication by these pairs acts like a gauge transformation.

2. The irreducible polynomials which do not decompose to polynomials are the natural basis for P_i. Note that polynomials of prime degree have no decomposition as a product of polynomials of lower order. This definition is however too restrictive.

3. Different roots for polynomial pairs (P_1, P_2) having roots inside CD correspond to space-time regions defining the analog for an extension of rationals. The roots give rise to analogs of monomials $x - r_n$ as monomials $(z_1 - R_{n,1}, z_2 - R_{n,2})$, where $R_{n,i}$ are algebraic functions as analogs or roots of ordinary polynomials. P_1 and P_2 can be expressed as local products of these monomials. Galois groups realized as generalized homomorphisms of H must map these regions to each other.

3.4.1 Roots of polynomials for space-time surfaces

What are the counterparts of the roots of polynomials as space-time surfaces? What is the counterpart of discriminant, which allows to assign to a polynomial a seet of ramified primes as rational primes. What is the space-time counterpart of rational prime?

1. Ordinary polynomials factorize to a product of monomials $x - r_n$ with roots which are the roots of polynomials. In the recent case the corresponding decomposition in the case of $P_1 = 0$ given by a polynomial having as its only root the space-region correspond to a given root of P_1 having its coefficients in the base field but with the root as algebraic number with the solution of some of the 4 complex variables appearing in P_1 and solved in terms of others as an algebraic function with coefficients depending on the point of the region defined. One would have $z_k - r_k(\{z_l, l \neq k\}$, where the algebraic function r_k gives z_k locally as a solution of $P_1 = 0$.

   Root as an algebraic number is replaced with algebraic function and defines the root as a space-time region. One can say that the surface decomposes to a union of regions defined by the roots of monomials, which join along their boundaries at which two roots co-incide.

2. There are preferred choices for the choice of z_k. The natural condition is that the degree of P_1 as a polynomial of z_k is maximal. The cusp catastrophe $dV(x)/dx = x^3 + 2ax + b = 0$ illustrates the situation: behavior variable x is preferred choice for the counterpart of z_k: the roots for the parameters (a, b) are rational functions because a and b appear linearly.

   In the simplest situation one has pairs (P, Q), where P and Q are irreducible. The 4-D roots for the pair are intersections of the 6-D roots of P and Q. One can substitute to Q $z_{k_1} = r_k(\{z_l, l \neq k\}$ and solve some coordinate z_{k_2}, $k_1 \neq k_2$ as an algebraic function of the other remaining parameter like coordinates.

3.4.2 Generalizing the notions of discriminant and of ramified prime

Do the notions of discriminant and ramified prime as rational prime generalize?

1. Consider also now $P_1 = 0$ as a simpler example. One can decompose P_1 to monomials which are algebraic functions $z_k = r_k$. Since these functions are elements in an extensions of the function field of rational polynomials, one can also form the differences $z_i - z_j$ and define the discriminant as the local product $\prod_{i \neq j}(z_i - z_j)$. For the ordinary polynomials the discriminant decomposes to primes of the coefficient field, say rationals.

   In this case the discriminant should decompose to analogs of primes as rational polynomials. Irreducible rational polynomials of prime degree p are primes in the sense that theyt cannot be decomposed into polynomials of lower degree without leaving the coefficient field. There the ramified primes could correspond to the factors of the discriminant with prime degree.

2. An interesting question is how the prime degrees appearing of the polynomial prime factors of P_1 relate to the ramified primes assignable to the roots of P_1 when restricted to physical special 2-surfaces X^2 fixed to some degree by the condition that $P_{1|X^2}$ has coefficients in some base field E as extension of rational for a suitable chosen complex coordinate for X^2 most naturally one of the generalized complex coordinates used. $P_1 = 0$ and $P_2 = 0$ give 2 conditions giving the space-time surface X^4.

3.5 How could the polynomial of a single variable determining ordinary ramified primes and effective Planck constant emerge?

Ordinary rational primes are of special physical interest in TGD. How could these rational primes, or more generally, E-rational primes emerge? Somehow one should be able to identify a polynomial giving ramified primes as roots and ramified primes. The outcome should be consistent with the recent physical view about ramified primes and hierarchy of Planck constants. One can consider a formal algebraic approach and geometric approach relying strongly on the existing physical picture. In the following I will describe different approaches to the problem but cannot must admit that the problem is far from being fully understood.

3.5.1 Purely algebraic approach

Consider first the algebraic approach.

1. Is the polynomial pair (P_1, P_2) enough or are two additional polynomials (P_3, P_4) needed so that the one would obtain a hierarchy consisting of 6-surfaces as roots of P_i having interpretation as analogs of twistor space of the space-time surface X^4 as their interaction, 2-surface X^2 and a discrete set of points. Since one coordinate is a light-like hypercomplex coordinate. Can one select (P_3, P_4) freely or are they determined by the pair (P_1, P_2)? This would be the case if the surfaces X^2 and the roots of a polynomial at it are determined by some geometric and physical conditions.

2. The 2-surfaces X^2 might actually correspond to their metrically 2-D light-like orbits X^3. Could the partonic orbit correspond to intersection of two roots of (P_1, P_2) as regions X^2 (or X^3). In the case of partonic orbits X^3, these roots could correspond to Euclidean and Minkowskian space-time regions and at their interface the induced metric would be degenerate and effectively 2-D. Therefore the partonic orbits are represented as roots $(P_1, P_2, P_3) = (0, 0, 0)$. Note that the coefficients of P_3 depend on the root of $(P_1, P_2) = (0, 0)$ so that each root corresponds to a different P_3.

3. The fact that partonic orbits are determined by the condition $(P_1, P_2) = 0$ alone, suggests that P^3 cannot be chosen freely in this case. On the other hand, the light-likeness of the partonic orbits might mean non-determinism (for CP_2 type extremals as vacuum extremals of Kähler action, M^4 projection is an arbitrary light-like curve).

   The conditions $P_1 = 0$ and $P_2 = 0$ give z_{k_1} and z_{k_2} as algebraic functions of 2 generalized complex coordinates (w, ξ) at the space-time surface. When one substitutes z_{k_1} and z_{k_2} as to the third polynomial $P_3(z_{k_1}, z_{k_2}, w, \xi)$ as a function of (w, ξ), one should obtain a polynomial of (w, ξ) with coefficients in E. This is a highly non-trivial condition. The roots of P_3 should define partonic orbits X^3.

One wants to assign p-adic primes as ramified primes to the ordinary roots of P_4 restricted to a given X^3 and having form depending on it so that each partonic orbit could correspond to its own p-adic prime as some ramified prime. Algebraic approach is the first option to be considered.

1. The roots should correspond to the roots of a polynomial P_4 of 4 generalized complex coordinates subject to the condition $(P_1, P_2, P_3) = (0, 0, 0)$ defining X^2 or X^3 so that one would have $(P_1, P_2, P_3, P_4) = (0, 0, 0, 0)$.

   The roots could correspond geometrically to light-like curves at metrically 2-D partonic orbits, that is, fermion lines to which one wants to assign ramified primes as p-adic primes. Also now one can ask whether it is possible to P_4 freely as polynomials having coefficients in E does (P_1, P_2) pair fix P_4 completely? Can one identify a physical condition fixing the light-like partonic orbits of fermions? Fermion lines should correspond to the intersections of string world sheets as 2-D singularities of X^4 with X^3. Could this condition fix P_4 so that it is determined by the pair (P_1, P_2).

2. If P_4 is a polynomial of 4 generalized complex coordinates with coefficients in E, it should remain such at the partonic orbit. If 3 generalized complex coordinates are E-rational constants at the partonic orbit, then $P_4(z)$ has coefficients in E and and its roots are in an extension of E and one can assign ramified primes to the roots of P_4. E-rational constancy conditions are very powerful. Note that analogous conditions are encountered for the elliptic curves: in this case the rational points form a lattice due to the underlying symmetry of the elliptic curves.

3.5.2 Geometric approach

The purely algebraic approach looks rather complex. Could geometric approach allow to improve the understanding?

1. The conditions defining light-like partonic orbits ($\sqrt{g_4} = 0$) and the conditions defining light-like fermion lines as intersections of string world sheets as singularities with partonic orbits could be the geometric conditions. This suggests that one should find a manifestly general coordinate invariant formulation, where the polynomial form emerges only in preferred coordinates and ramified primes emerge naturally.

2. Concerning the identification of ramified primes, one could also start from the situation in which the n roots at X^2 are assumed to correspond to the intersections of string world sheets and partonic orbits. One can introduce a complex coordinate z. The points should be algebraic points for this choice of coordinate and therefore expressible as roots of a polynomial P of degree n with coefficients in E. This condition gives very powerful conditions on the coordinate z. If some choice of z is found, one can write the polynomial as a product of monomials defined by the roots and can calculate the discriminant and the ramified primes associated with it.

   How unique is this complex coordinate? The root differences are only scaled under linear modular transformations so that the spectrum of ramified primes is preserved. If these linear transformations define the allowed global holomorphies, the preferred complex coordinate at X^2 is rather unique.

3.5.3 How could the algebraic and geometric approaches relate?

Geometric Langlands correspondence suggests that one should be able to express the geometric view in terms of algebraic conditions.

1. $M^8 - H$ duality [L26, L30] requires that the quaternionic normal space of the surface Y^4 in M^8 should contain a 2-D commutative sub-space at each point. This implies that X^4 contains 2-D surface X^2 having possible several components. This surface should be determined by an additional condition. One can consider surfaces $P_3(.., u) = 0$ or condition $P_4(..., v) = 0$.

   A surface $P_3(.., u) = 0$ would be topologically 3-D, hence the notation X^3. It would be metrically 2-D and some condition should select 2-D surface X_1^2 by fixing the allowed values of u. X_1^2 can have several components corresponding to different partonic orbits and also for a given partonic orbits the number of components could be larger than one. The failure of the generalized holomorphy essential for the identification of vertices as singularities as analogs of poles would occur for these special values of u.

2. The condition $P_4(..., v) = 0$ would naturally define string world sheets X_2^2 with a Minkowskian signature of the induced metric. These surfaces would be 2-D since v appears in P_4 instead of u. These surfaces are transversal to the partonic 2-surfaces X_1^2 and would naturally correspond to co-commutative 2-surfaces. Fermion lines would correspond to the intersections of X^3 and X_2^2.

   The condition $\sqrt{g_4} = 0$ at the partonic orbit makes the 4-D tangent space of X^4 effectively 2-D. This could correspond to the local degeneration of the metric dimension of the string world sheet to 0. The intersection one would be a light-like fermion line rather than a single point.

3. The application of both $P_3 = 0$ and $P_4 = 0$ gives a discrete set of preferred points perhaps identifiable as points with the special values of u and v and the complex coordinate of X_1^2 defining the counterparts of vertices and corners of the fermion line as well as the intersection points of the string world sheets with the partonic 2-surfaces. If these points are roots of a polynomial P, this polynomial could define ramified primes, extension of E, and h_{eff}.

3.5.4 Is it really possible to formulate all geometric statements as statements of algebraic geometry

The TGD view of the geometric Langlands correspondence states that there is a correspondence between the algebraic, essentially linguistic view of physics and the geometric view of physics relying on vision. This leads to a kind of language game. The highly non-trivial challenge is to find whether the geometric picture can be formulated using the language of algebraic geometry involving generalized complex variables of which one is hypercomplex and real.

First of all, one must find out whether the known algebraically universal extremals appearing for practically any conceivable action, deduced by geometric and symmetry arguments, have a simple algebraic description as the roots $(P, Q) = (0, 0)$ where P and Q are analytic functions of generalized complex coordinates of $H = M^4 \times CP_2$. This is not at all obvious. One should carefully check whether CP_2 type extremals, cosmic strings and monopole flux tubes, and massless extremals allow this kind of formulation.

Inequalities are part of geometric description and involve in an essential manner the notion of distance. The representation of topological boundaries gives rise to inequalities. In TGD a long standing question is whether one should allow boundaries and whether the boundary conditions guaranteeing conservation laws indeed allow space-time boundaries. For instance, could one eliminate CP_2 type extremals defining wormhole contacts glued to the Minkowskian background and leaving partonic orbits as boundaries [L14].

1. The problem is that well-ordering required by inequalities characterizes only real numbers: the notion of inequality is not algebraically universal. Inequalities have no natural place in pure algebraic geometry involving complex numbers or p-adic numbers. In TGD, the natural variables are generalized complex coordinates and inequalities cannot be represented for the complex numbers using only complex analytic functions.

   In TGD, the light-like hypercomplex coordinate u is however an exception. u is real and inequalities make sense for it. For instance, the segment $u_1 < u < u_2$ can be defined in the semialgebraic context and the simplest situation corresponds to a position dependent time interval $x - u_1 \leq t \leq x + u_2$ or propagating pulse. The real part $Re(w)$ of the complex coordinate w of the space-time surface defining the analog of the real axis in complex analysis would be a second coordinate of this kind and could be assigned to the partonic 2-surface.

2. Also in the p-adic topology well-ordering is absent and inequalities would be represented in terms of norm but this is not a notion of algebraic geometry. Only the discrete subsets of p-adic numbers defined by powers of p are well-ordered and inequalities can be defined for them. The hierarchy of discretizations as cognitive representations defined by extensions of rationals could however allow to overcome this problem by reducing them to inequalities.

The notion of semi-algebraic geometry makes it possible to represent these observations formally.

1. In semi-algebraic geometry inequalities are allowed in the real case but do not make sense for complex and p-adic numbers. In TGD, semialgebraic geometry would make sense for the regions of space-time surface for which the generalized complex coordinates of H or space-time surface are real.

   All inequalities should be formulated for the real sub-manifolds, which for ordinary complex 4-manifolds are 2-D. This is the case now. String world sheets parameterized by light-like coordinates u and v, would be naturally 2-D surfaces of this kind but the coordinate v does not appear as the argument of the functions (P, Q). Only the inequalities relating to u seem to make sense.

2. Hamilton-Jacobi structure [L20] means a slicing of M^4 by pairs of strings world sheets and partonic 2-surfaces and would allow to generalize this representation to the interior of the space-time surface. Could the inequalities related to the geometry of preferred extremals implied by holography=holomorphy correspondence reduce to this kind of inequalities? The two real coordinates u and $x = Re(w)$ could have interpretation as local choices of light-like direction and polarization direction and inequalities in this sense would be consistent with the notion of semialgebraic geometry.

   An interesting question is whether symplectic structure, which is basic element of the WCW geometry and can be seen as a companion of the generalized complex structure, could correspond to the decomposition of the complex space-time coordinate as $w = P + iQ$ and hypercomplex coordinate as (u, v) such that (P, Q) and (u, v) define canonically conjugate coordinate pairs is consistent with the Hamilton-Jacobi structure. Note that the two real coordinates u and $x = Re(w)$ could have interpretation as local choices of light-like direction and polarization direction and inequalities in this sense would be consistent with the notion of semialgebraic geometry.

Could one get rid of inequalities altogether by a suitable choices of the real coordinate variants (u, x)? There is indeed a well-known trick allowing to get rid of an inequalities representable in the form $t \geq 0$ by a change of the coordinate variable as a replacement $t \rightarrow T = t^2$. Only the points with $t \geq 0$ are allowed by mere reality conditions. This trick might work to inequalities involving u and x.

3.5.5 What kind of singularities space-time surfaces can have?

Space-time surfaces as algebraic surfaces can have various kinds of singularities. In particular cusp-like singularities as kinds of peaks are possible. The earlier out-of-date proposal for the realization of $M^8 - H$ duality led to the proposal [L21] that wormhole contacts identified as deformations of CP_2 to CP_2 type extremals with light-like curve as a M^4 projection, emerge as what algebraic geometers call blow-ups. A blow-up replacing a point with CP_2 type extremal could take place for the peak-like singularities. In this case however two Minkowskian space-time regions connected by the CP_2 type extremal would be in question.

Could space-time surfaces have stable self-intersections? In the generic situation this is the case since the sum of the dimensions of two intersecting space-time reasons give $4 + 4 = 8$ so that the intersection consists of a discrete point set. The situation is not so simple in the recent case. The reason is that by generalized holomorphy the light-like coordinate v is effectively eliminated and H is effectively $7 - D$. For the same reason, the space-time surface itself is effectively 3-D. The dimension of self-intersection is formally $3 + 3 - 7 = -1$ so that no stable self-intersections should exist. The condition $\sqrt{g_4} = 0$ at the partonic orbit worsens the situation since one has $2 + 2 - 6 = -2$.

This is the case also for the intersections of surfaces $P_1(..., u), Q) = (0, 0)$ and $P_2(..., u), Q) = (0, 0)$. However, one can also consider two space-time surfaces with different light-like coordinates u_1 and u_2 such that the Hamilton-Jacobi structures [L20] are not equivalent and this is rather natural for many-particle systems. For instance, for $u_1 = u$ and $u_2 = v$, the intersection consists of a discrete set of points. Also in this case blow-up might allow to eliminate or smooth out the singularity.

This intuition is supported by a 1+1-D analogy. The intersection cannot be removed by an infinitesimal deformation. One could cut small 1-D segments around both points and reconnect the resulting pieces to get two disjoint curves.

The intersections of CP_2 type extremal and Minkowski space-time regions provide a physically highly interesting example. The recent view relies on physical intuition and it is of interest to see whether it can be understood in terms of a deformation removing an intersection point.

1. The simplest CP_2 type extremal is the canonically embedded CP_2 in H. It has constant M^4 coordinates and intersects any space-time surface whose projection contains this point. This situation is definitely non-physical.

2. In the case of Kähler action [K1], one can deform the simplest CP_2 type extremal such that its M^4 projection becomes a light-like curve. If the action contains volume term, CP_2 type extremals can consist of pieces, which are light-like geodesics of M^4. The metric and Kähler form are not affected.

   This surface has no space-like boundary and as such cannot be glued along boundary to a larger space-time surface with Minkowskian signature. It however has homologically non-trivial 3-D ends. The surface is analogous to a general relativistic wormhole leaving the background space-time surface and returning back at a later time. It can also return to another space-time surface. The homologically non-trivial 2-surfaces at the ends behave like monopole charges of opposite sign.

3. What about the dual monopole fluxes at the light-like boundaries of CD? Dual monopole fluxes are topological and by definition fluxes of the Kähler form over 2-surfaces, whose M^4

projection has Minkowskian signature (1,-1). These quantized dual fluxes are encountered in superconductivity. They do not correspond to the electric flux over a space-like surface.

4. One could argue that neither free magnetic charges nor their dual counterparts are physically acceptable. The mathematical reason would be that boundary conditions cannot be satisfied. The dual monopole flux should go somewhere at the boundary of CD. One could add a second roughly parallel Minkowskian space-time sheet and allow the flux flow to this space-time sheet through a wormhole contact at the boundaries of CD. One would obtain a closed dual flux loop in time direction with Euclidean wormhole contacts connecting the two sheets. A creation of fermion-antifermion pairs could create this kind of time-like loop beginning at point-like vertex [L19, L31].

   Interestingly, one could consider closed polygons having as edges CP_2 type extremals with M^4 projections with light-like geodesics meeting at the corners. This kind of light-like polygons appear in the twistor Grassmann approach to gauge theories.

5. One can consider also a situation in which one cuts a piece from the deformed extremal so that the partonic orbit, having boundaries only at its ends of CD, gets a light-like 3-D boundary connecting the ends of CD. This boundary can have non-vanishing magnetic flux (over space-like surface) only if there is a second similar object with an opposite magnetic flux. This partonic orbit can be glued to the background Minkowskian space-time surface. Appropriate boundary conditions are satisfied at the boundary guaranteeing the conservation of charges. One can say that at the partonic orbit the space-time becomes Euclidean.

   The partonic orbit is analogous to a magnetic monopole. One can however direct the flux to a parallel Minkowskian space-time sheet, where it returns back and a closed monopole flux loop is formed. One can say that the space-time surface turns to the direction of CP_2 and meets the parallel Minkowskian space-time sheet. This would be a basic building brick of a general TGD based model for elementary particles.

6. There is however an outwards Kähler magnetic flux at both space-time sheets. If one cannot tolerate monopoles, one must have wormhole contact glued to the same pair of space-time surfaces. A monopole flux would connect the boundaries of the wormhole contact. This gives the recent model for elementary particles as closed monopole flux loops connecting two parallel Minkowskian space-time sheets.

These considerations raise again a long standing question concerning classical electromagnetic charges as electric fluxes. Although this question does not relate directly to the recent topic, it deserves a discussion. The self-duality of Kähler form of CP_2 raises the possibility that the magnetic and electric Kähler fluxes are identical for CP_2 type extrenals. Classical electromagnetic charge receives two contributions coming from Kähler form and the vectorial part of spinor connection of CP_2 [L22].

1. For leptons the simplest assumption is that only the Kähler contribution is non-vanishing for charged leptons L: the contributions would be (-1,0). If neutrinos are Kähler magnetic monopoles, one would have (-1,1) (also (0,0) can be considered).

2. For quarks the two contributions should explain fractional em charges. (ν_L, L) and (U, D) have the same weak isospin but quarks and leptons correspond to different H-hiralities. This suggests that the anti-leptonic charges would be shifted by -1/3 to get the charges of quarks: the contributions to the fluxes would be (1,-1/3) for U and (1,-4/3) for D (also (0,-1/3)can be considered).

3.6 How could various effective Planck constants emerge?

The phenomenology of TGD leads to the proposal that ordinary Planck constant is replaced with a hierarchy of effective Planck constants having interpretation in terms of the dimension of an algebraic extension determined by some polynomial characterizing the particle [L30, L31]. Nottale's proposal [E1] [K12] for the gravitational Planck constant, which characterizes a pair of masses, generalizes in the TGD framework [L3, L10, L9]. The generalization also applies to the electromagnetic, weak and color interactions [L17]. One should understand both kinds of effective Planck constants and also the corresponding ramified primes defining p-adic length scales [?, L7, L15].

3.6.1 What about the effective Planck constant associated with a partonic orbit?

What is the physical interpretation of the dimension $n = h_{eff}/h_0$ of the extension of E defining the effective Planck constant associated with $P_4(w)$ defining points of the partonic surface as its roots? What about the corresponding ramified primes? The first thing to notice is that one could interpret $P_4(w)$ as a restriction of a polynomial $P_4(w,u)$ to say $u = 0$. This polynomial would determine a slicing of the space-time surface as a union light-like partonic orbits defined by the roots $P_4(w,u)$ as u varies.

The fermion lines of the partonic 2-surface could be identified as the light-like boundaries of string world sheets, which mediate various interactions between different partonic orbits.

One can assign an effective Planck constant h_{gr}, h_{em}, etc assignable to gravitational, electromagnetic, weak or strong interactions. They are proportional to the product of charges assignable to ends of the string at different partonic orbits unlike the ordinary Planck constant. How could they emerge from the proposed picture.

A natural assignment of h_{gr}, h_{em}, etc would be to the string world sheets and actually to the ends of string worlds sheets or strings if u if string world sheet corresponds to a region of (u,v) plane. Could they correspond to the roots of the polynomial $P_4(w_n,u)$ determining the positions of the ends of the string at the two partonic orbits involved. The roots would give the values of u at the ends of the string world sheet and the $P_4(w_n,u)$ would characterized the interaction. The roots $P_4(w, u = constant)$ would in turn parametrize the light-like intersections of the string world sheets at the partonic orbit.

Q characterizes the sector of WCW and therefore naturally the background physics including classical gravitational and electric fields and also the length scale dependent cosmological constant as a dynamical parameter. P would in turn characterize particles. Therefore, the roots and ramified primes for R would reflect both the properties of the particle and those of the background. This could explain why the gravitational *resp.* electric Planck constants are proportional masses *resp.* charges of the particle and larger system.

3.6.2 What about the effective Planck constant characterizing single fermion states only?

Besides electric and gravitational Planck constants characterizing the interactions with background fields, there are also effective Planck constants analogous to h and characterizing single fermion states. These ramified primes are important in p-adic mass calculations. How to identify them?

As already found, the points of the partonic 2-surfaces identified as self-intersection points depend on Q characterizing the background and are not strict single particle characteristics. One could of course argue that this conforms with the basic view of the Higgs mechanism. One can however ask, whether a single particle h_{eff} could have some other identification?

1. Could one identify h_{eff} as roots of a polynomial $P_5(v)$ varying along the light-like curves $u = constant$ along the boundary of the string world sheet at a given partonic orbit. The appearance of v as an argument of P_5 obviously means a violation of the generalized conformal invariance. If one assumes the proposed preferred coordinates, the polynomial $P_5(v)$ having coefficients in E would be determined by the geometry, that is by the knowledge of the singular points.

   The dimension $n = h_{eff}/h_0$ of the extension of E associated with P_5 would be a single particle characteristic and have a natural interpretation as the effective Planck constant h_{eff} assuming rather small values as generalization of the ordinary Planck constant.

2. The roots could now correspond to singularities at which the light-like-curve has a corner. Also the vertices for the creation of a fermion pair could correspond to such corners. The physical interpretation would be as a change of light-like momentum located at the corner. Also the interpretation as analog of Brownian motion makes sense and ordinary Brownian motion might relate to these corners.

3. Also a connection with the breaking of conformal invariance is suggestive. If the Kähler action were the sole action, M^4 projection of the CP_2 type extremal, now vacuum exteremal, would be an arbitrary light-like curve and Virasoro conditions would express this property. When the volume term is present, one has a discretized version of this picture, implying the breaking of conformal invariance, which reflects the presence of the volume term in the action. The average velocity is reduced by the change of the direction of motion at the vertices and this can be seen as a signature for massivation describable in terms of p-adic thermodynamics.

4. TGD indeed leads to the picture in which the corners correspond to fermionic vertices at which also the generalized holomorphy fails so that the trace of the second fundamental form is non-vanishing. At the corners, the fermion suffers an instantaneous generalized acceleration. The CP_2 part of the second fundamental form behaves like a Higgs field group theoretically. One can also say that the analog of the Higgs mechanism takes place only at the fermionic vertices.

5. The sequence of the roots v_n of P_5 defines a sequence of values of light-cone proper time a_n, which would naturally correspond to the weak failure of the classical determinism for the dynamics of the space-time surface. This classical determinism would correspond to the quantal non-determinism of "small" state function reductions (SSFRs), whose sequence defines the flow of consciousness in the TGD inspired theory of consciousness.

   This sequence augments the sequence of repeated measurements of the same observable with the measurement of cognitive observables associated with the classical non-determinism and gives rise to a generalization of the Zeno effect: only the states at the passive boundary of CD would remain unaffected in SSFRs.

   This non-determinism would also naturally correspond to p-adic non-determinism inherent to the p-adic dynamics as a correlate for cognition. The degree of P_5 would correspond to the maximal number of very special moments of geometric time for a system in question. At these moments of geometric time the contents of consciousness would be replaced by a new one and remain constant until the next SSFR.

6. String world sheets appear as singularities of the space-time surfaces and since the light-like coordinate v defining the hypercomplex coordinate is constant along the boundary of the string world sheet, at least in piecewise way, the string world sheet itself must correspond to a singularity defined by a polynomial generalizing the condition defining the light-like partonic orbit. By the generalized holomorphy of (f_1, f_2), there is no dependence on the coordinate v, so that one indeed obtains string world sheet locally when one fixes the points of the partonic 2-surface to a root of a polynomial P_4 characterizing a string world sheet.

The objection against this proposal is that it requires that the polynomial defining the singular points as its roots does not depend on the state of the particle. If the singular points are associated with the corners of the light-like curve at which momentum direction changes, this does not seem plausible.

3.7 The counterpart of the Galois group in the geometric LC

Recently I considered a view of the Galois group, which conforms with the geometric Langlands in which the roots of a polynomial are replaced with space-time regions as roots of the pair (P_1, P_2).

1. The ordinary Galois group is assigned with the roots of P defining an extension of E. The Galois group acts as a permutation group of the roots of a polynomial of a polynomial P with integer coefficients. Besides this it acts trivially in k if K is an extension of k.

2. For a pair (P_1, P_2) the roots $(P_1, P_2) = (0, 0)$ correspond to regions of a connected 4-surface. One should generalize the notion of Galois group so that it permutes various roots of $(P_1, P_2) = (0, 0)$ as regions of the 4-D surface.

   Could generalized holomorphic transformation represent the action of the generalized Galois group on the space-time surface as flows permuting the regions representing the roots [L30].

This transformation would not only map the roots as space-time regions to each other but also respect the local root property. This might pose restrictions to the Galois group in the sense that the full permutation group would not be allowed.

3. If this flow reduces to isometries, then the action must reduced to that of a discrete subgroup of $SO(3)$ or $SL(2,C)$ and one obtains that the allowed Galois groups correspond to the hierarchy of discrete subgroups of $SO(3)$ associated with inclusions of hyper-finite factors [L30] and with McKay correspondence [A9, A8, A7, A5, A4] [L4].

4. What about the analogy for the condition that the Galois group leaves the field E invariant? The natural identification is that the counterparts of the field E are pairs of polynomials (or rational functions, or even analytic functions), which are non-vanishing inside CD. The notion of root indeed generalizes also to analytic functions so that the notion of the geometric Galois group is number-theoretically universal.

3.8 The identification of the geometric Langlands group

Just as in the case of the number theoretical LC, the geometric Langlands group would correspond to the semidirect product of ${}^LSL(2,C)^o$ with a Galois group, which would be now the geometric variant of the Galois group. $SL(2,C)$ and its subgroup $SL(2,Z_E)$ would act on the selected discrete set hyperboloids $H^3(a_k)$.

An additional hypothesis, giving hopes for obtaining the numbers of the numbers of p-adic roots $(P_1,P_2)=(0,0)$, is that the Bohr orbitology forces a modular invariance in the sense that the boundary data of holography are analogous to plane waves with a definite discretized 3-momentum in the sense that a discrete subgroup of $SL(2,Z_E)$ defines a periodic tessellations of the H^3 projection of the space-time surface defining boundary data of the holography. The plane waves would correspond to modular forms in the hyperboloid H^3 covariant under $SL(2,Z_E)$.

Also in the CP_2 degrees of freedom analog of modular invariance might hold true for a discrete subgroup of CP_2 so that the 3-surface in CP_2 degrees of freedom would be an analog of Platonic solid. This would conform with the quantum classical correspondence suggested by the Bohr orbitology and suggest that space-time surfaces reflect the quantum numbers of the fermionic quantum states associated with them.

3.9 Master formula for the construction of quantum states using the interpretation of space-time surfaces as numbers

The exact solution of field equations of TGD in terms of holography=holomorphy vision and the recent progress in the understanding of the TGD view of Langlands correspondence allows to propose an explicit recipe, a kind of a master formula, for the construction of states describing the interaction in terms of generalized holomorphic algebraic geometry.

3.9.1 Do space-time surfaces have the structure of a number field?

While writing this article, I was inspired by the idea that the space-time surfaces indeed have a structure of a number field, induced by the structure of the function field formed by the analytic functions with respect to the four generalized complex coordinates of $H=M^4\times CP_2$ (one of the coordinates is hypercomplex light-like coordinate). Function fields are indeed central in the geometric Langlands correspondence.

1. This function field also has a hierarchical structure. There are hierarchies of polymials of various degrees and also rational functions with coefficient fields in different extensions of rationals. Analytical functions for which the Taylor coefficients are in extensions of rationals in the expansion is the next step. At the ultimate limit one has algebraic numbers as coefficients. Also transcendental extensions can be thought of and in this way one eventually ends up with complex numbers.

2. For $H=M^4\times CP_2$, this would correspond to the lowest level of the hierarchy of infinite primes but the Cartesian powers of $H=M^4\times CP_2$ correspond to the higher levels in the

hierarchy of infinite primes. Again, this hierarchy is be analogous to the hierarchy used in the description of condensed matter, $3N$-dimensional spaces, N number of particles.

3. One can consider two options for the product of space-time surfaces. The product could be induced by **a)** the product for the first members of the pairs (f_1, f) and (f_2, f) as (f_1f_2, f) with f fixed or **b)** the component-wise product $(f_1, g_1) * (f_2, g_2) = (f_1f_2, g_1g_2)$.

   For option **a)** function field structure induces a structure of number field in the space of space-time surfaces. For option **b)** there are problems. Pairs of form $(f_1, 0)$ and $(0, f_2)$ can be produced in summation. The corresponding surface would be a 6-dimensional analog of brane. The pair $(f_1, f_2) = (0, 0)$ has the entire H as a representative. Can one accept this? The inverse in these cases does not exist for function fields except formally if $f = \infty$ is accepted as a constant function. It does not have a space-time surface as a representative. It turns out that **a)** is also physically favored.

In zero energy ontology (ZEO), quantum states corresponds to spinor fields of WCW, which consists of space-time surfaces satisfying holography and therefore being analogous to Bohr orbits, and also having interpretation as elements of number field so that one can multiply them. WCW spinor fields assign to a given space-time surface a pair of fermionic Fock states at its 3-D ends located at the opposite light-like boundaries of the causal diamond CD. Could one multiply two WCW spinor fields so that the space-time surfaces appearing as their arguments are multiplied

$$X_1^4 \cup X_2^4 \to X_1^4 * X_2^4 \ ,$$

and the tensor product of the fermionic states at the boundaries of CD is formed. This would give

$$\Psi(X_1^4) \otimes \Psi(X_2^4)(X_1^4 \cup X_2^4) \to \Psi(X_1^4) \otimes \Psi(X_2^4)(X_1^4 * X_2^4) \ .$$

Here $X_1^4 * X_2^4$ would be the product of surfaces induced by the function algebra and the product of fermion states would be tensor product. Could Gods compyte using spacetime surfaces as numbers and could our arithmetics be a shadow on the wall of the cave.

So: could a believer of TGD dream conclude that these meta-levels and perhaps even mathematical thinking could be described within the framework of the mathematics offered by the infinite dimensional number field formed by the space-time surfaces. This quite a lot more complicated than binary math with a cutoff of the order of 10^{38}!

3.9.2 The product of the space-time surfaces as a geometric counterpart of the tensor product?

The product and sum of the space-time surfaces are the natural candidates for the describing interactions of particles. What could the product of space-time surfaces mean concretely? The physical intuition suggests the first guess. Could it correspond to a creation of an interacting pair of 3-D particles identified as 4-D Bohr orbits? The product would be the geometric counterpart of a tensor product, but perhaps also involve interaction. If so, this product could provide a geometric and algebraic description of the interactions. What would one get?

Let us restrict the consideration to option **a)**, which looks more promising both physically and mathematically. One considers polynomials (P_i, Q), and forms only the products of P_i assuming P_i and Q irreducible. Space-time surfaces obtained by multiplying only P_i in the product $(P_1, Q) * (P_2, Q) = (P_1P_2, Q)$ would have a number field structure in a rather strict sense. The product would correspond to the union of the surfaces $(P_1, Q) = 0$ and $(P_2, Q) = 0$ and analogous to a state of two free particles without interaction.

This option would put P and Q in asymmetric positions. This would conform with the idea that the condition $P_1 = 0$ defines the analog of a 6-D twistor space, and the space-time surface X^4 is the intersection of the analogs of the twistor bundles of M^4 and CP_2, i.e., its base space. As already proposed, the condition $Q = 0$ could define an identification for the counterparts of twistor spheres of M^4 and CP_2. This map would define cosmological constant for the analog of dimensional reduction of the product of twistor spaces to the induced twistor space of the space-time surface.

The product of the space-time surfaces produces an almost free two-particle state in topological and geometric sense. Is the discrete set of intersection points enough to describe the interactions?

This is possible in the fermionic sector since these points appear as arguments of fermionic n-point functions.

3.9.3 The sum of the space-time surfaces as a geometric counterpart of the tensor product?

The basic restriction to the sum of the space-time surfaces is the condition that the space-time surfaces appearing as summands allow a common Hamilton-Jacobi structure [L20] in M^4 degrees of freedom in turn inducing it for the space-time surfaces. The summed space-time surfaces must have a common hypercomplex coordinate with light-like coordinate curves and a common complex coordinate. For the product this is not required.

1. Could the sum of the space-time surfaces $(f_1, g) = (0, 0)$ and $(f_2, g) = (0, 0)$ defined as a root of $(f_1, g) + (f_2, g) = (f_1 + f_2, g))$ define a topologically and geometrically non-trivial interaction? If the functions f_1 and f_2 have interiors of causal diamonds CD_1 and CD_2 with different tips as supports (does the complex analyticity allow this?) and CD_1 and CD_2 are located within a larger CD then both f_1 and f_2 are nonvanishing only in the intersection $CD_1 \cap CD_2$.

   Generalized complex analyticity requires a Hamilton-Jacobi structure [L20] inside CD. It must have a common hypercomplex coordinate and complex M^4 coordinate inside CD and therefore inside $CD_1 \cap CD_2$ and also inside CD_1 and CD_2? Suppose that this condition can be satisfied.

   Outside $CD_1 \cap CD_2$ either f_1 and f_2 is identically vanishing and one has $f_1 = 0$ and $f_2 = 0$ as disjoint roots representing incoming particles in topological sense. In the intersection $CD_1 \cap CD_2$ $f_1 + f_2 = 0$ represents a root having interpretation as interaction. f_i "interfere" in this region and this interference is consistent with relativistic causality.

   One could also assign to the sum a tensor product in fermionic degrees of freedom and define n-point functions and restrict their arguments to the self-intersection points of the intersection region $CD_1 \cap CD_2$. One could also say that the sum represents $z = x + y$ in such a way that both summands and sum are realized geometrically.

At this moment it is unclear whether both product and sum or only product or some could be assigned with topological particle interactions. From the number theoretic point of perspective one would expect that both are involved.

3.9.4 How to treat the intersections of the space-time surfaces?

In the generic situation the 4-D surfaces $(P_1, Q) = 0$ and $(P_2, Q) = 0$ have a discrete set of intersection points in H: could the intersection points allow the description of the interactions between the particles?

1. Assume a set of n irreducible polynomial pairs (P_i, Q) as analogs of elementary particles. The product is $(\prod_i P_i, Q)$ and gives rise to n space-time regions X_i^4 defined by the conditions $(P_i, Q) = (0, 0)$.

2. In the generic situation, the intersection of two 4-D regions X_i^4 of 8-D H of the surface $(P_1 P_2, Q = (0, 0)$ would consist of discrete points. This applies also to self-intersections as interactions of sub-regions of a given region. Some of the intersections are just tangential touching points, which can be eliminated by an infinitesimally small deformation (in plane two curves can touch tangentially).

   In the recent case however the generalized holomorphy means that M^4 and X^4 are effectively bundles of light-like curves parameterized by a light-like dual coordinate v so that conditions defining the self intersection are effectively 3-D surfaces in effectively 7-D H. Self-intersections are not stable. Stability requires that the coordinates u_1 and u_2 are different for the two 4-surfaces. For $u_1 \neq u_2$, the intersection would consist of a discrete set of stable intersection points. Different coordinates u_i suggest interpretation as different states of motion.

At the partonic orbits defining boundaries of wormhole contacts, the $v = constant$ lines have a natural interpretation as fermion lines at the partonic orbits defining boundaries of wormhole contacts as blow-ups.

3. If u_1 and u_2 are different then also Q and say P_2 corresponds to different u and the 6-surfaces $P_2 = 0$ and $Q = 0$ have a 2-D self-intersection given rise to a discrete self-intersection of the 4-surface X^4_2. These self-intersection points could correspond to internal fermionic vertices assignable to analog of self-interactions in QFT picture.

What should one do with intersection points.

1. If the scattering amplitudes involve fermionic n-point functions with arguments restricted to the intersection points, there is no need to eliminate the intersection points. This would be the simplest option and bring in the interactions in a minimal possible way. This would also allow unique purely topological identification of the vertices of the n-point function and make the dimensions of the space-time and embedding space unique.

2. The second possibility is that the intersection points of the space-time surfaces are eliminated by a small deformations of the space-time surface involved. The obvious objection is that this deformation is highly non-unique. In any case, the polynomials P_1 and P_2 should be modified so that the product P_1P_2 becomes irreducible. What conditions can one pose on these deformations?

3.9.5 Could it make sense to eliminate the intersection points?

The second option is that the singularities are eliminated or smoothed out. The blow-ups [L21], mentioned already earlier in this article, could take place for genuine intersection points. I have proposed that blow-ups to eliminate the singularities for which the 4-surface Y^4 in M^8 has a peak so that CP_2 point assignable to the normal space is highly non-unique and can span a 3-D subset of CP_2. This would give rise to the CP_2 type extremal. The blow-ups would correspond to wormhole contacts, identified as deformations of CP_2 to CP_2 type extremals with light-like curve as a M^4 projection and connecting two space-time surfaces.

Could the possible deformation of the product polynomial P_1P_2 to an irreducible polynomial involve a blowup? This deformation should not dramatically affect the polynomials involved nor the corresponding space-time surfaces outside the blow-up. The intuitive picture is that the space-time surfaces are deformed so that the intersection point is replaced by wormhole contact connecting the deformed space-time surfaces.

Consider two Minkowskian regions of the space-time surface.

1. How to eliminate the intersection point as singularity?

    (a) One can think of is that the irreducible intersection point corresponds to point-like Kähler monopoles, which are ends of infinitely short and thin monopole flux tubes connecting the two space-time regions. This can be deformed to a wormhole contact connecting two Minkowskian space-time sheets.

    The wormhole contact option would mean that for two Minkowskian regions intersecting at point, the intersection point is replaced by a wormhole contact represented as a deformation of CP_2 to a CP_2 type extremal having in the simplest situation a light-like projection.

    (b) One can also consider the possibility that the wormhole contact is replaced by a monopole flux tube with a Minkowskian induced metric connecting the two space-time sheets involved. This could apply to self-intersections and mean that self-intersection points are replaced by monopole flux tubes.

    (c) One can also perform a blow-up for a Minkowskian string world sheet associated with the orbit of a monopole flux tube. Now the wormhole contact corresponds to homologically non-trivial geodesic sphere of CP_2

2. The number of wormhole contacts or monopole flux tubes would be $\sum_{i\neq j} n_{ij}$, where n_{ij} is the number of nontrivial self intersections for X^4_i and X^4_j. Therefore the process would generate quite a large number of partonic objects. That all these regions can be otherwise disjoint does not conform with the naive particle interpretation.

3. Physical intuition suggests that for interacting particles, which do not form a bound state, the product reduces near the passive boundary of the CD (initial state) of the causal diamond CD to a union of the surfaces with no intersections and thus without wormhole contacts or monopole flux tubes. The surfaces $(P_i, Q) = (0, 0)$ and $(P_j, Q) = (0, 0)$. If this condition is not true, the interpretation would be as a bound state. The TGD view of nuclei, atoms, and molecules assume that particles forming the bound state are indeed connected by monopole flux tubes [L18].

4. One should also understand particle creation. What about the following? One can multiply a given polynomial pair (P, Q) with a polynomial pair (P_1, Q) to give $(P,_1 P_2, Q)$. If P_1 does not vanish inside the CD the spacetime surface does not change. CD however increases during the sequence of "small" state function reductions (SSFRs) and it can happen that P_1 can develop roots inside the CD and a new space-time region emerges.

3.9.6 What about the product of spinors fields?

The WCW spinor field assigns multifermion states to the 3-D ends of a given spacetime surface at the boundaries of the CD. If one can define what happens to the multifermion states associated with the zero energy states in the interaction, then one has a universal construction for the states of WCW as spinor fields of WCW providing a precise description of interactions analogous to an exact solution of an interacting quantum field theory. At the geometric level, the product of the surfaces corresponds to the interaction. At the fermion level, essentially the ordinary tensor product of the multifermion states should correspond to this interaction.

Under what conditions does this vision work for fermionic states as WCW spinors, identified in ZEO as pairs of the many-fermion states at the 3-surfaces at the boundaries of the CD? It is obvious that the definition of the fermion state should be universal in the sense that at the fundamental level the fermion state is defined without saying anything about space-time surfaces involved.

Induction is basic principle of TGD and the induction of spinor fields indeed conforms with this idea. The basic building bricks of WCW spinor fields are second quantized spinor fields of H restricted to the 3-surfaces defining the ends of the space-time surfaces at the boundaries of CD. Therefore the multifermion states are restrictions of the multifermion states of H to the spacetime surfaces. The Fourier components (in the general sense) for the second quantized spinor field ψ of H (not WCW!) and its conjugate $\overline{\psi}$ would only be confined to the ends of X^4 at the light-like boundaries of CD.

The oscillator algebra of H spinor fields makes it possib le to calculate all fermionic propagators and fermionic parts of N-point functions reduce to free fermionic field theory in H but arguments restricted to the space-time surfaces. The dynamics of the formally classical spinor fields of WCW would very concretely be a "shadow" of the dynamics of the second quantized spinor fields of H. One would have a free fermionic field theory in H induced to space-time surfaces!

In this way, one could construct multiparticle states containing an arbitrary number of particles. The construction of quantum spaces would reduce to a multiplication in the number field formed by space-time surfaces, accompanied by fermionic tensor product!

3.9.7 Geometric Langlands duality extends to a trinity involving also physics

The master formula for TGD allowing construction of quantum states using the interpretation of space-time surfaces as numbers realizes the analog of geometric Langlands duality and generalizes it to a trinity. Geometric Langlands correspondence assigns to a pair of elements of a function field, which is a number theoretic object, a geometric object as algebraic surface having interpretation also as a Riemann surface with Kähler structure, twistor structure and spinor structure. This extends the number-theory-algebraic geometry duality to trinity and physics becomes the third part of a trinity.

1. The most high level form of number theory corresponds to function fields, which are infinite-D structured. In TGD, the pairs (f_1, f_2) of two functions of generalized complex coordinates of $H = M^4 \times CP_2$ define a linear space and the functions f_i are elements of a function field. This is the number theoretic side of the Langlands geometric duality.

2. A function pair, whose root $(f_1, f_2) = (0, 0)$ defines a space-time surface in H and induces the number field structure of the function field to the space of space-time surfaces, "world of classical worlds" (WCW). Basic arithmetic operations of the number field apply to the component functions f_i and induce corresponding operations for space-time surfaces in WCW. The notion of induction, which is the basic principle of TGD, is central also here. It is missing from standard physics and also string models.

3. The root as a space-time surface obeys holography =holomorphy principle and is a minimal surface (as classical representation of generalized massless particle and massless field equations) and represents the geometry side of the geometric Langlands duality. This connection represents geometric Langlands duality in TGD. Riemannian geometries restricted to algebraic geometries is what makes the geometric Langlands duality possible.

   It is still unclear whether the choice of the classical action defining space-time surfaces and producing, apart from singularities, a minimal surface as an outcome, is only analogous to a choice of the coordinates and whether the recent choice (volume action + Kähler action) is only the most convenient choice. If so, the laws of physics boil down to a completely action independent form, that is to the construction of quantum states induced by the products for space-time surfaces regarded as generalized numbers.

4. Space-time surfaces as minimal surfaces with generalized complex structure and are extremals for any variational principle constructible in terms of the induced geometry since extremal property reduces to the generalized complex structure. The action makes itself visible only at the singularities.

5. Langlands geometric duality becomes actually a trinity: number theory $\leftrightarrow$ geometry $\leftrightarrow$ physics. The number theory $\leftrightarrow$ geometry part of this trinity duality corresponds to Langlands geometric duality. The geometry$\leftrightarrow$ physics part is the TGD counterpart of Einstein's equations identifying geometry and physics.

4 About the relationship between geometric and number theoretic counterparts of WCW

At the continuum limit, the WCW geometry is determined by a Kähler function identifiable in terms of the action defining the space-time surface. The preferred extremals satisfying holography=holomorphy principle are minimals surfaces which apart from singularities are independent of action.

For the twistor lift the action is naturally the 6-D counterpart of Kähler action decomposing in a dimensional reduction to 4-D Kähler action plus dynamical volume term with dynamical but quantized cosmological constant. This condition uniquely fixes the choice of H [A6]. The dimensional reduction is forced by the condition that the space-time surface can be interpreted as a base space of a twistor bundle. The analogs of wistor bundles of M^4 and CP_2 are represented as 6-surfaces in H and space-time surface is identifiable as an intersection of these. The 6-D variant of α_K appeas as the fundamental coupling parameter. Cosmological constant would be dynamically generated in the dimensional reduction needed to achieve induced twistor bundle structure.

The action defining K might contain additional dynamical terms characterizing the quantum state rather than the vacuum functional. This would make possible also the breaking of various symmetries of the vacuum functional. The action is not visible in interior dynamics since the 4-D interior dynamics is universal and same for any general coordinate invariant action constructible in terms of induced geometry. Action makes itself visible only via the action exponential and at the level of singularities defining the vertices. The action exponential disappears from the scattering amplitudes if only superpositions of 4-surfaces with the same value of K are allowed. This happens quite generally also in perturbative quantum field theories. Here essential role is played also by the equality of the Gaussian determined associated with the functional integral and the metric determinant of WCW Kähler metric.

4.1 Does number theoretic discretization mapping real WCW to its real counterpart involve canonical identification?

Number theoretic discretization means that the coefficients of polynomials P_1 and P_2 (or analytic functions) are numbers in extension E of rationals. This implies a discretization of the real variant WCW. Maximization of the Kähler function defines an even stronger discretization of WCW.

What about the situation in the p-adic sector. Real and p-adic views of continuity differ dramatically for rationals. The direct identification of the E-value Taylor coefficients of the polynomials possibly appearing in rational functions and in analytic functions is therefore not consistent with the idea that cognitive representations are approximately continuous.

Continuity is guaranteed if the finite E rational valued coefficients of the polynomials P and Q appearing in $R = P/Q$ in the real sector are mapped by canonical identification $I : \sum x_n \to x_n p^{-n}$ to the p-adic sector. Canonical identification appears also in p-adic thermodynamics [L15]. Here the map of reals to p-adics is 1 to 2 the real number involves an infinite number of negative powers of p. For finite rationals it is 1-to-1. The map from p-adics to reals is 2-1 for finite rationals. The E valued coefficients are E-rationals so that the 1-1 property is achieved.

The identification of the p-adic primes a ramified prime of the discriminant of extension of E is problematic since it is difficult to decide which ramified prime to select. A democratic form of the canonical identification for nteger n would invert only the ramified primes p in the prime decomposition of n.

The assumption that the Taylor coefficients of analytic function belong to E defines the discretization of WCW in the sector associated with E. The p-adic counterpart of WCW could consist of p-adic analogs of the surfaces with coefficients obtained by the proposed form of the canonical identification. The ramified primes would be determined by the exponent of Kähler function if it corresponds to a power of the discriminant of the polynomial defining the ramified primes.

4.2 The relationship between Kähler function and discriminant

The number theoretical discretization provides additional understanding of the WCW and its number theoretical counterpart.

1. The assumption that the Taylor coefficients of analytic function belong to E defines the discretization of WCW in the sector associated with E. The maxima of the quantity which is product of the vacuum functional with degeneracy factor $N(K)$ counting the number of space-time surfaces with the same value of $exp(K)$ would naturally select some preferred points in the discretization of WCW.

2. Also the degeneracy $d(K)$ for a given value of Kähler function matters. The $d(K)exp(-K) = d(K)f(D)$ is analogous the exponential $exp(-F/T)$ of free energy $E-TS$, $d(K)$ corresponds to the contribution of entropy and would correspond to number of space-time surfaces with the same value of discriminant D.

   Each partonic orbit is expected to contribute a discriminant so that one would have the product of powers of disminants for $f(D) = D^k$, where is an integer. $k = 1$ is a good candidate. In continuum context maximization of the Kähler function is well-understood and could have a straightforward generalization to the number theoretic situation. Obviously, the expression of K in terms of D poses powerful additional conditions and means a powerful tool of physical intuition.

3. The degeneracy factor $d(K)$ serves as a measure for quantum criticality. In the continuum case the number of zero modes serves as a measure for the criticality. $K = constant$ sets in WCW corresponding to maxima of $exp(-F)$ define the valleys of the spin glass landscape. Highly degenerate valleys are separated by mountains. Quantum tunnelling between valleys

of the landscape is possible and one can define ultrametric topology involving various p-adic number fields in the landscape.

4.3 The notion of spin glass energy landscape

The TGD Universe is quantum critical, which means that the system has a very large number of almost degenerate ground states which in TGD means the same or almost the same value of Kähler function. Therefore spin glass energy energy landscape is a natural analog for the p-adic counterpart of WCW [L8].

4.3.1 Ultrametric distance as counterpart of Riemann metric in the adelic spin glass energy landscape

The notion of integral and therefore also the notion of distance fail to be defined in the adelic sector [L2, L1]. The solution of the problem is the replacement of the discretized real variant of WCW as a spin glass energy landscape having ultrametric topology defined by ultrametric distance function. In this topology would decomposes to p-adic topologies glued together along p-adic integers having expansion in powers of integer divisible by the the p-adic primes considered.

1. Usually one defines paths in the spin glass energy landscape between the bottoms of the valleys and tops of mountains corresponding to the minima and maxima of $f(D)$. In TGD the discretizaion of WCW defines the spin glass energy landscape.

   Spin glass energy landscape has ultrametric topology defined by an ultrametric distance function. Consider two valleys A and B with the same value of $D(A) = D(B)$. Note that $D(A)$ would correspond to the exponent of Kähler function. Minimax principle defines ultrametric distance function.

   In the spin glass energy landscape, the distance $d(A, B)$ along a given path connecting A and B is defined by the maximum of $exp(K) = f(D)$. D the discriminant, along this path. The distance $D(A, B)$ is the minimum of this distance in the set of connecting paths $\gamma_{A,B}$. The p-adic primes possibly defining the possible p-adic topologies correspond to a ramified prime p appearing in D. If D for a given surface at the path is not divisible by the ramified prime, the value of D is large. Therefore paths for which the discriminant has p as a divisor are favoured by the minimization. The further minimization with respect to p, would select ramified prime.

2. The assumption $D(A) = D(B)$ can be criticized as too restrictive. One could only assume that some ramified primes are shared by $D(A)$ and $D(B)$. If one drops even this assumption that distance becomes rather large since B corresponds to a very high mountain from the point of view of A.

   This implies that the adelic variant of the discretized WCW decomposes into a union of regions which can be described theoretically in terms of topologies of extensions of the p-adic numbers induced by the extension of E and defined by a polynomial P of single variable defined at partonic 2-surfaces. Valleys of the spin glass landscape with the same value of D have the same depth but can have a large real distance also the ultrametric distance can be small. There exists a large number of polynomials with the same value of D and these can have a large distance since even their degrees can differ.

3. The ratio of the D for the intermediate state and initial state defines the distance between the degenerate values with the same D. Height of the mountain. Is the p-adic prime a ramified prime for the initial state.

4. In quantum description, tunnelling could replace the classical path between valleys of the spin glass landscape. The larger the ultrametric distance, the lower the tunnelling probability. Minimax principle or its dual. The exponent of K must have minima, maxima and saddle points. Description of the tunnelling is in ZEO and in terms of D and ramified primes dividing it. Initial and final state have the same D. Small changes of ramified primes in the intermediate state or even new ramified primes so that the Kähler function changes. Perhaps a new prime appears in discriminant during the tunnelling.

In annealing system ends up to the bottom of the deepest valley: small energy boosts allow to kick up the system from a local valley if it gets stuck there. Note that in number theoretic evolution D increases and $exp(K)$ also and ramified primes appear.

4.4 How do the coupling parameters emerge in the transition from the number theoretical discretization to the QFT limit?

The key question is the relation of α_K related to the number theoretic discretization. A fixed value of α_K poses urealistically strong conditions on the value of the Kähler function for the maxima. It would seem that α_K must be interpreted as a mere normalization factor making possible the p-adic existence of $exp(K)$ for the ramified primes dividing D, necessary for the number theory-geometry duality.

The hypothesis $exp(K/\alpha_K) = D^k$ gives $\alpha_K = K/kln(D)$. Logarithmic dependence of α_K on the ramified primes would make sense. This expression would also determine the value of electroweak U(1) coupling g_K number theoretically. Neither g_K nor other weak couplings appear as coupling parametries in Dirac equation. In QFT picture this would mean that they are absorbed the gauge potential so that only YM action contains the couplings as $1/g^2$ factors. Same applies to other gauge couplings. They can appear only at the QFT limit of TGD.

A more detailed picture would be as follows.

1. The construction of quantum states as products of space-time surfaces interpreted as numbers. In principle it gives coupling parameters as predictions. Transition to QFT limit brings in coupling constants. Quantum states are replaced by quantum fields.

2. Classical action involves only α_K and cosmological constant Λ and holography=holomorphy ansatz predicts Λ as a dynamical parameter. Induced gauge potentials appear at vertices but gauge couplings are absorbed to them.

3. α_K appears as a mere normalization factor of the action, being analogous to temperature. The only function of the α_K could be the scaling of the exponent of Kähler function to a function of D, perhaps power of D. This scaling factor should determine U(1) coupling strength at QFT limit and other coupling should be proportional to it.

4. The gauge couplings could emerge in the following way at the QFT limit. Induced gauge potentials are replaced with quantum fields. If the commutators of operators creating bosons as fermion antifermion pairs are proportional to $n^2\alpha_K$, n a numerical parameter. Commutators must be normalized in the standard way. This forces the renormalization of gauge potential by factor $1/ng_K$ and means that coupling constants appear in the couplings of fermions to the QFT counterparts of the induced gauge potentials.

5 TGD view of computationalism and its physical realization

The TGD view of space-time surfaces as numbers [L27] provides the background for the following considerations.

5.1 The replacement of the static universe with a Universe continuously recreating itself

It seems to me that the problems of computationalism emerge from a single ontological assumption: the "system", be it Universe in some sense or God, is fixed. In quantum TGD this is not the case. The Quantum Universe, which could be seen as a counterpart for God, is continually recreating itself and this means the unavoidable increase of algebraic complexity since the dimensions associated with extensions of rationals defining space-time regions unavoidably increase. This in turn implies evolution.

In zero energy ontology (ZEO) "small" state function reductions (SSFRs) [K15], whose sequence generalizes Zeno effect, which has no effect on physical state. SSFRs have and their sequence gives rise to conscious entities, selves. This makes possible memory [L29]: the outcome of SSFR has classical information about the initial state and also about the transition. Therefore the Universe remembers and learns consciously: one can talk about Akashic records.

This dynamical view of the Universe recreating itself and becoming more intelligent by learning about what it was before the previous SSFR is very different from the view of the Universe as a Turing machine or Universal Computer. These notions are static notions (Universe "out there") and computation is based on integers. In the TGD view one obtains an entire hierarchy of computationalisms based on the hierarchy of extensions of rationals. Even transcendental extension can be considered. TGD Universe as a counterpart of the Turing machine is also conscious and has free will.

5.2 A generalization of the number concept

Also the notion of number generalizes from the set N of integers to the set of space-time surfaces, the "World of Classical Worlds" (WCW) [K4, K2, K14, L23, K10, K3]. The TGD view of geometric Langlands duality means that space-time surfaces can be multiplied and summed and form an algebra. This algebra decomposes to a union of number fields with product,division, sum and subtraction. One can identify space-time surfaces forming analogs for hierarchies of algebraic integers, algebraic rationals, etc... So that the mathematics performed by Quantum Platonia is considerably more complex than counting by 5+5 fingers!

These structures are defined by the corresponding structures for function algebras and fields defined in terms of analytic functions of 8 generalized complex coordinates of $H = M^4 \times CP_2$. One of the coordinates is a hypercomplex coordinate with light-like coordinate curves.

1. In TGD space-time surfaces are numbers [L27]. Their dynamics is almost deterministic (at singularities the determinism fails and this forces us to take space-time surfaces instead of 3-surfaces as basic objects). The space-time surface as an almost deterministic time evolution is analogous to a proof of a theorem. The assumptions correspond to the initial state 3-surface and the outcome of the theorem to the final 3-surface. Second interpretation is as analogs of deterministic computer programs. Space-time surface as a proof of a theorem is analogous to its own Gödel number as a generalized number.

2. Cognition always requires a discretization and the space of space-time surfaces ("world of classical worlds", WCW) allows a hierarchy of discretizations. The Taylor coefficients of the two analytic functions defining space-time belong to some extension of rationals forming a hierarchy. Therefore a given space-time surface corresponds to a discrete set of integers/rationals in an extension so that also WCW is discretized. For polynomials and rational functions this set is discrete. This gives a hierarchy. At the level of the space-time surface an analogous discretization in terms of an extension of rationals takes place.

3. Gödel number for a given theorem as almost deterministic time evolution of 3-surface would be parametrized by the Taylor coefficients in a given extension of rationals. Polynomials are simplest analytic functions and irreducible polynomials define polynomial primes having no decomposition to polynomials of a lower degree. They might be seen as counterparts of axioms.

4. One can form analogs of integers as products of polynomials inducing products of space-time surfaces. The space-time surfaces are unions for the space-time surfaces defined by the factors but an important point is that they have a discrete set of intersection points. Fermionic n-point functions defining scattering amplitudes are defined in terms of these intersection points and give a quantum physical realization giving information of the quantum superpositions of space-time surfaces as quantum theorems.

5.3 Could space-time surfaces replaced as integers replace ordinary integers in computationalism?

It is interesting to play with the idea that space-time surfaces as numbers, in particular integers, could define counterparts of integers in ordinary computationalism and metamathematics.

What might be the counterpart for the possibility to represent theorems as integers deduced using logic and for the Gödel numbering for theorems by integers?

1. In TGD space-time surfaces are numbers. Their dynamics is almost deterministic (at singularities the determinism fails and this forces us to take space-time surfaces instead of 3-surfaces as basic objects). The space-time surface as an almost deterministic time evolution is analogous to a proof of a theorem. The assumptions correspond to the initial state 3-surface and the outcome of the theorem to the final 3-surface. The second interpretation is as analogs of deterministic computer programs. Space-time surface as a proof of a theorem is analogous to its own Gödel number as a generalized number.

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5.4 Adeles and Gödel numbering

Adeles in TGD sense [L1, L2, L24] inspire another interesting development generalizing the Gödelian view of metamathematics.

1. p-Adic number fields are labelled by primes and finite fields induced by their extensions. One can organize the p-adic number fields to adele and the same applies to their extensions so that one has an infinite hierarchy of algebraic extensions of the rational adele. TGD brings something new to this picture.

2. Two p-adic number fields for which elements are power series in powers of p_1 *resp.* p_2 with coefficients smaller than p_1 *resp.* p_2, have common elements for which expansions are in powers of integers $n(k_1, k_2) = p_1^{k_1} \times p_2^{k_2}$, $k_1 > 0, k_2 > 0$ [L24, L30]. This generalizes to the intersection of $p_1, p_2, ...p_n$. One can decompose adeles for a union of p-adic number fields which are glued together along these kinds of subsets. This decomposition is general in the description of interactions between p-adic sectors of adeles. Interactions are localized to these intersections.

3. Mathematical cognition would be based on p-adic numbers. Could one think that ordinary integers should be replaced with the adelic integers for which the p_i:th factor would consist of p-adic integers of type p_i. These integers are not well-ordered so that the one cannot well-order theorems/programs/etc... as in Gödel numbering. The number of p-adic integers is much larger than natural numbers since the pinery expansion can contain an infinite number of terms and one can map p-adic integers to real numbers by what I call canonical identification. Besides this one has fusion of various p-adic number fields.

An interesting question is how this changes the Gödelian views about metamathematics. It is interesting to play with the idea that space-time surfaces as numbers, in particular generalized integers, could define counterparts of integers in ordinary computationalism and metamathematics.

5.5 Numbering of theorems by space-time surfaces?

What might be the counterpart for the possibility to represent theorems as integers deduced using logic and for the Gödel numbering for theorems by integers?

1. In TGD space-time surfaces are numbers. Their dynamics is almost deterministic (at singularities the determinism fails and this forces us to take 4-D space-time surfaces instead of 3-surfaces as basic objects). The space-time surface as an almost deterministic time evolution is analogous to a proof of a theorem. The assumptions correspond to the initial state 3-surface and the outcome of the theorem to the final 3-surface. The second interpretation is as an analog of a deterministic computer program. The third interpretation as a biological function. Space-time surface as a proof of a theorem is analogous to its own Gödel number, but now as a generalized number. One can define the notions of prime , integer , rational and transcendental for the space-time surfaces.

   The counterparts of primes, determined by pairs of irreducible polynomials, could be seen as axioms. The product operation for space-time surfaces generates unions of space-time surfaces with a discrete set of intersection points, which appear as arguments of fermionic n-point functions allowing to define fermionic scattering amplitudes. Also other arithmetic operations are possible.

2. Also functional composition, essential in computationalism, is possible. One can take any pair $(h_1(z), h_2(z))$ of analytic functions of a complex coordinate z and form a functional composites $h_1 \circ f_1$ or $h_2 \circ f_2$. One can also iterate with respect to h_1 or h_2. As a special case on can take $h_1 = h_2 = h$. This would make it possible to realize recursion, essential in computationalism. The iteration increases the degree of the polynomial and therefore also the number of roots with an exponential rate so that the complexity of the space-time surface increases. The iteration leads also to fractals. An interesting question is how this could relate to biological evolution.

   Also self-referentiality becomes possible: one could identify z as one of the genuinely complex H coordinates and perform parameter dependent iteration for z as argument of f_1 using $z \to f_1(z, ...)$ with parameters defined by other H coordinates.

3. One can consider even more general maps $(f_1, f_2) \to G(f_1, f_2) = (g_1(f_1, f_2), g_2(f_1, f_2))$ and iterate also them. The special case $(g_1, g_2) = (g_1(f_1), g_2(f_2))$ gives iterations of functions g_i of a single complex variable appearing in the construction of Mandelbrot and Julia fractals.

Extensions of rationals, Galois groups, and ramified primes assignable to polynomials of a single complex variable are central in the number theoretic vision. It is not however completely clear how they should emerge from the holography= holomorphy vision.

1. If the functions $g_i \equiv P_i$ are polynomials, which vanish at the origin $(0, 0)$ (this is not a necessary condition), the surfaces $(f_1, f_2) = (0, 0)$ are roots of $(P_1(f_1, f_2), P_2(f_1, f_2)) = (0, 0)$. Besides these roots, there are roots for which (f_1, f_2) does not vanish. One can solve the roots $f_2 = h(f_1)$ from $g_2(f_1, f_2) = 0$ and substitute to $P_1(f_1, f_2) = 0$ to get $P_1(f_1, h(f_1)) \equiv P_1 \circ H(f_1)) = 0$. The values of $H(f_1)$ are roots of P_1 and are algebraic numbers if the coefficients of P_1 are in an extension of rationals. One can assign to the roots discriminant, ramified primes, and Galois group. This is just what the phenomenological number theoretical picture requires.

2. In the earliest approach to $M^8 - H$ duality summarized in [L5, L6, L26] polynomials P of a single complex coordinate played a key role. Although this approach was a failure, it added to the number theoretic vision Galois groups and ramified primes as prime factors of the discriminant D, identified as p-adic primes in p-adic mass calculations. Note that in the general case the ramified primes are primes of algebraic extensions of rationals: the simplest case corresponds to Gaussian primes and Gaussian Mersenne primes indeed appear in the applications of TGD [K7, K8].

   The problem was to assign a Galois group and ramified primes to the space-time surfaces as 4-D roots of $(f_1, f_2) = (0, 0)$. One can indeed define the counterpart of the Galois group defined as analytic flows permuting various 4-D roots of $(f_1, f_2) = (0, 0)$ [L27].

Since the roots are 4-D surfaces, it is far from clear whether there exists a definition of discriminant as an analog for the product of root differences. Also it is unclear what the notion of ramified prime could mean. However, the ordinary Galois group plays a key role in the number theoretic vision: can one identify it? An possible identification of the ordinary Galois group and ramified primes would by the assignment to maps defined by $(f_1, f_2) \rightarrow (P_1(f_1, f_2), P_2(f_1, f_2))$ would be in terms of $(P_1(f_1, f_2), P_2(f_1, f_2)) = (0, 0)$ giving the roots $P_1(f_1, h(f_1)) = 0$ as values of $h(f_1)$. The roots belong to extension rationals even when f_i are arbitrary analytic functions of H coordinates but correspond geometrically to 4-surfaces.

3. The earlier proposal is that the ordinary Galois group can be assigned to the partonic 2-surfaces so that points of the partonic 2-surface as roots of a polynomial give rise to the Galois group and ramified primes. The most elegant way to realize this is to introduce 4 polynomials (P_1, P_2, P_3, P_4). The roots of (P_1, P_2, P_3) allow to solve the 3 complex coordinates as a function of the hypercomplex coordinate u. This surface can be identified as a string world sheet.

   The additional condition $P_4(u) = 0$ gives roots which are algebraic numbers if the coefficients of P_4 are in an extension of rationals. Note that only real roots are allowed.

   The interpretation of the roots for u would be as singularities of the space-time surface located at the partonic 2-surfaces where the minimal surface property fails and the trace of the second fundamental form diverges. These points would correspond to vertices for the creation of a fermion pair and would represent defects of the standard smooth structure giving rise to an exotic smooth structure [L31, L11, L32].

Cognition always requires a discretization.

1. The space of space-time surfaces ("world of classical worlds", WCW) allows a hierarchy of discretizations. The Taylor coefficients of the two analytic functions f_1, f_2 defining space-time belong to some extension E of rationals forming a hierarchy. Therefore a given space-time surface corresponds to a discrete set of integers/rationals in an extension of rationals so that also WCW is discretized for given E. For polynomials and rational functions this set is discrete. This gives a hierarchy. At the level of the space-time surface an analogous discretization in terms of E takes place.

2. Gödel number for a given theorem as almost deterministic time evolution of 3-surface would be parametrized by the Taylor coefficients in a given extension of rationals. Polynomials are simplest analytic functions and irreducible polynomials define polynomial primes having no decomposition to polynomials of a lower degree. Polynomial primes might be seen as counterparts of axioms. General analytic functions are analogous to transcendentals.

3. One can form analogs of integers as products of polynomials inducing products of space-time surfaces as their roots. The space-time surfaces are unions for the space-time surfaces defined by the factors but an important point is that they have a discrete set of intersection points. Fermionic n-point functions defining scattering amplitudes are defined in terms of these intersection points and give a quantum physical realization giving information of the quantum superpositions of space-time surfaces as quantum theorems.

6 A more detailed view about the TGD counterpart of Langlands correspondence

The Quanta Magazine article (see this) related to Langlands correspondence and involving concepts like elliptic curves, modular functions, and Galois groups served as an inspiration for these comments. Andrew Wiles in his proof of Fermat's Last Theorem used a relationship between elliptic curves and modular forms. Wiles proved that certain kinds of elliptic curves are modular in the sense that they correspond to a unique modular form. Later it was proved that this is true for all elliptic surfaces. Later the result was generalized to real quadratic extensions of rationals by 3 mathematicians involving Samir Siksek and now by Caraiani and Newton for the imaginary quadratic extensions.

Could this correspondence be proved for all algebraic extensions of rationals? And what about higher order polynomials of two variables? Complex elliptic curves, defined as roots of third order polynomials of two complex variables, are defined in 2-D space with two complex dimensions have the special feature that they allow a 2-D discrete translations as symmetries: in other words, they are periodic for a suitable chosen complex coordinate. I have talked about this from TGD point of view in [L34]. Is the 1-1 correspondence with modular forms possible only for elliptic curves having these symmetries?

How are the Galois groups related to this? Indian mathematical genius Ramanujan realized that modular forms seem to be associated with so-called Galois representations. The Galois group would be the so- called absolute Galois group of the number field involved with the representation. Very roughly, they could be seen as representations of a Lie group which extends the Galois group. Also elliptic curves are associated with Galois representations. This suggests that the Galois representations connect elliptic curves, objects of algebraic geometry and modular forms, which correspond to group representations. These observations led to Langlands program which roughly states a correspondence between geometry and number theory.

The Galois group is indeed involved with Langlands duality. If the Lie group G is defined over field k (in the recent case extension of rationals), the Langlands dual ${}^LG$ of G is an extension of the absolute Galois group of k by a complex Lie group (see this. The representation of the absolute Galois group is finite-dimensional, which suggests that it reduces to a Galois group for a finite-dimensional extension of rationals. Therefore the effective Galois group used can be larger than the Galois group of extension of rationals. ${}^LG$ has the same Lie algebra as G.

In the following, I will consider the situation from a highly speculative view point provided by TGD. In TGD, geometric and number theoretic visions of physics are complementary: $M^8 - H$ duality in which M^8 is analogous to 8-D momentum space associated with 8-D $H = M^4 \times CP_2$ is a formulation for this duality and makes Galois groups and their generalizations dynamic symmetries in the TGD framework [L26]. This complementarity is analogous to momentum position duality of quantum theory and implied by the replacement of a point-like particle with 3-surface, whose Bohr orbit defines space-time surface.

At a very abstract level this view is analogous to Langlands correspondence [L27]. The recent view of TGD involving an exact algebraic solution of field equations based on holography= holomorphy vision allows to formulate the analog Langlands correspondence in 4-D context rather precisely. This requires a generalization of the notion of Galois group from 2-D situation to 4-D situation: there are 2 generalizations and both are required.

1. The first generalization realizes Galois group elements, not as automorphisms of a number field, but as analytic flows in $H = M^4 \times CP_2$ permuting different regions of the space-time surface identified as roots for a pair $f = (f_1, f_2)$ of pairs $f = (f_1, f_2) : H \to C^2$, $i = 1, 2$. The functions f_i are analytic functions of one hypercomplex and 3 complex coordinates of H.

2. Second realization is for the spectrum generating algebra defined by the functional compositions $g \circ f$, where $g : C^2 \to C^2$ is analytic function of 2 complex variables. The interpretation is as a cognitive hierarchy of function of functions of .... and the pairs (f_1, f_2) which do not allow a composition of form $f = g \circ h$ correspond to elementary function and to the lowest levels of this hierarchy, kind of elementary particles of cognition. Also the pairs g can be expressed as composites of elementary functions.

   If g_1 and g_2 are polynomials with coefficients in field E identified as an extension of rationals, one can assign to $g \circ f$ root a set of pairs (r_1, r_2) as roots $f_1, f_2) = (r_1, r_2)$ and r_i are algebraic numbers defining disjoint space-time surfaces. One can assign to the set of root pairs the analog of the Galois group as automorphisms of the algebraic extension of the field E appearing as the coefficient field of (f_1, f_2) and (g_1, g_2). This hierarchy leads to the idea that physics could be seen as analog of formal system appearing in Gödel's theorems and that the hierarchy of functional composites could correspond to a hierarchy of meta levels in mathematical cognition [L33].

6.1 Two Galois groups

In TGD it is possible to define two generalizations of the Galois group: I call them internal and external Galois groups. Both notions of the Galois group are needed.

6.1.1 Internal Galois group

The 4-D Galois group, the internal Galois group, is assumed to permute the regions of a single connected component of the space-time surface realized as roots of the pair (f_1, f_2) defining the space-time surface. The internal Galois group would act as analytic flows of H transforming the regions as roots to ech other so that the action is analogous to that of a braid group.

1. It is easy to see that the space-time surface in general consists of several disjoint regions if (f_1, f_2) is expressible as the composite $(f_1, f_2) = (g_1(h_1, h_2), g_2(h_1, h_2))$. In this case the space-time surface is union of disjoint surfaces $h_i = r_i$, where r_i correspond to the roots of g_i. The permutations of the roots for a connected component of the space-time surface would realized as analogs of braidings.

2. The space-time regions identified as roots of (f_1, f_2) for a single connected component would have string world sheets as interfaces having hypercomplex time coordinates u, v. Suppose that there are n string world sheets. The number of string world sheets/folds can be larger than n. If folds are between any pair i, j are present then the number of folds cannot be larger than $(n-1)n$: in this case all pairs i, j would have two folds. Circle is a simple example: it has 2 sheets and 2-folds: 1,2 and 2,1.

   Since the M^4 complex coordinates w and roots as its values labelling the string world sheets are in general complex, one can say that the fold is complexified. For a cusp (see this) the two folds can be ordered. Fold would now involve a string world sheet and cusp would combine two folds. At the vertex of the cusp where 3 roots co-incide, two folds would disappear. This suggests that the string world sheets connect at their ends associated with the disappearing folds and form a single string world sheet.

3. Catastrophe theory suggests that all catastrophes and hence also the space-time surfaces can be constructed from complexified cusps. The folds, which appear on a cloth, can be ordered. If so, folds between roots $i, i+1$ and $i-1, i$ are possible and would come from a single cusp but folds with $|i-j \geq 1$ would not be possible. This could give rise to the ordering of the roots w_i. Does this mean that the Galois group is cyclic?

4. This brings in mind twistor amplitudes and planar diagrams, which correspond to Feynman diagrams with no crossing lines and therefore embeddable in plane. Non-planar Feynman diagrams are a problem of the twistor Grasmannian approach [B3, B2] since they have no twistorial representation. The Feynman diagrams with crossing lines can be embedded in the plane with holes, whose boundaries are connected by cylinders as kinds of wormholes. In string models, the corresponding diagrams involve this kind of wormholes. This suggests that if the 2-D projection of the space-time sheets with constant values of hypercomplex coordinates has a topology with a genus g larger than 0, the space-time surface contains wormholes connecting roots with $|i-j \geq 1$. In this case also the generalized Galois group is non-Abelian. Wormhole contacts defining Euclidean regions (CP_2 type extremals) could be such connections.

   To include wormhole contacts as connectors of the Minkowskian space-time sheets, one should allow besides the Minkowskian folds also the presence of the Euclidean CP_2 type extremals with a light-like M^4 curve, possibly geodesic, as M^4 projection. For these Euclidean regions the string world sheet would reduce to this curve since the second hypercomplex coordinate would be constant.

The internal Galois group could relate to the TGD view of topological qubits [L35].

1. The quantum-mechanical transfer of fermions between regions corresponding to roots of (f_1, f_2) does not require a continuous path. Classical transfer requires a path going through a fold at which the two roots as space-time regions meet. Fold corresponds to a boundary of

a string world sheet identified as fermion line. Folds are labelled by the values of the complex coordinate w having interpretation as roots.

2. There is a direct analogy to the case of condensed matter majorana fermions suggested to define topological qubits. For a Majorana fermion two branches of the Fermi surface touch each other at point and the energy difference for the branches is zero at this point. Majorana fermions are assigned with these points and they would be located at the ends of a wire [L35]. In the TGD framework the folds would correspond to the seats of topological qubits.

6.1.2 Outer Galois group

The element-wise multiplication of the function for pairs (f_1, f_2) is essential for the identification of the outer Galois group and gives an algebra, which is enough for identifying the Galois group as group of automorphisms for the algebraic extension of rationals involved. Outer Galois group permutes the roots of g, which are algebraic numbers in the extension of E and label the disjoint components of the spacetime surfaces. These two Galois groups commute and the outer Galois group relates to the internal Galois group in the same way as the Galois group of an extension of rationals to the Galois group of complex rations generated by complex conjugation.

The outer Galois group is natural for the TGD realization of the Langlands duality, discussed from the TGD point of view in [L27].

1. A simpler version of the outer Galois group is associated with dynamical complex analytic symmetries $g : C \to C$: $(f_1, f_2) \to (g_1 \circ f_1, f_2)$. Here g_1 does not have a parametric dependence on f_2. The outer Galois group relates to each other *disjoint* space-time surfaces. When g reduces to map $g : C \to C$, one can assign to it an ordinary Galois group relating to each other the disjoint roots of $g \circ f$ realized as disjoint 4-surfaces $(f_1, f_2) = (r_1, 0)$.

2. The notion of outer Galois group generalizes to the general situation $g = (g_1, g_2)$. Also now the roots of $g \circ f$ are disjoint space-time surfaces representing roots as pairs of algebraic numbers $(f_1, f_2) = (r_{i,1}, r_{i,2})$. Is it possible to assign to the roots the analog of the Galois group?

   This group should act as a group of automorphisms of some algebraic structure. This structure cannot be a field but algebra structure is enough. These arithmetic operations would be component-wise sum $(a, b) + (c + d) = (a + c, b + d)$ and componentwise multiplication $(a, b) * (c, d) = (ac, bd)$. The basic algebra would correspond to the points of $(x, y) \in E^2$ or rationals and the extension would be generated by the pairs $(f_1, f_2) = (r_{i,1}, r_{i,2})$. This structure has an automorphism group and would serve as a Galois group. The dimension of the extension of E^2 could define the value of the effective Planck constant.

   Also the notion of discriminant can be generalized to a pair (D_1, D_2) of discriminants using the component-wise product for the differences of root pairs. Could D_i be decomposed to a product of powers of algebraic primes of the extension of E^2?

3. In [L27] the idea that the space-time surfaces can be regarded as numbers was discussed. For a given g, one can indeed construct polynomials having any for algebraic numbers in the extension F of E defined by g. g itself can be represented in terms of its n roots $r_i = (r_{i,1}, r_{i,2})$, $i = 1, n$ represented as space-time surfaces as a product $\prod_i (f_1 - r_{i,1}, f_2 - r_{i,2})$ of pairs of monomials. One can generalize this construction by replacing the pairs $(r_{i,1}, r_{i,2})$ with any pair of algebraic numbers in F. Therefore all algebraic numbers in F can be represented as space-time surfaces. Also the sets formed by numbers in F can be represented as unions of the corresponding space-time surfaces.

6.2 Symmetries and dynamical symmetries

It is good to look first at the action of g on f in more detail.

1. In the simplest situation f_i and g_i are polynomials with coefficients in E. The functional composition $f \to g \circ f$ increases algebraic complexity if the degree of g is higher than 1. The interpretation would be as a spectrum generating or dynamical symmetry. If the degree of g

is 1, the complexity does not increase and a natural interpretation would be as an ordinary symmetry.

2. f_i and g_i can also be rational functions. Since the roots of f_i correspond to the roots of the polynomial P defining the numerator of $R = P/Q$, Q does not affect the roots as space-time surfaces.

   In the case g_i the situation is different. In the case of Möbius transformations one has $g(z) = (az+b)/(cz+d)$. For instance, the Möbius transformation transformation $g(z) = 1/z$ acting on the polynomial $f(z) = \sum_{k=0}^{n} p_k z^k +$ gives $f = z^{-n}\sum(p_k^{(} n-k) = z^{-n}(p_n + ...p_0 z^n$. Degree is not reduced for $p_0 \neq 0$. Quite generally, the condition $p \neq 0$ would exclude the roots $z = 0$, which can be regarded as trivial. The action on $R = P/Q$ gives $z^{-n+m}(p_n + ... + p_0 z^n)/(q_m + ... + q_0 z^m)$. The roots are also now the roots of the numerator and the situation is almost the same as in the first. The factor z^{-n+m} can give additional root $f = 0$ for $m - n > 0$. Note that negative powers do not give roots.

6.2.1 Prime polynomials and complexity hierarchy

The polynomials (P_1, P_2) and also the rational functions $(g_1 = P_1/Q_1, g_2 = P_2/Q_2)$ form a well-defined complexity hierarchy.

1. In the general case, the space-time surfaces $(P_1, P_2) = (0, 0)$ have several disjoint components. This is the case if (f_1, f_2) is a composite function of form $f = g(h)$: in other words one has $(f_1, f_2) = (g_1(h_1, h_2), g_2(h_1, h_2))$. The space-time surfaces correspond to roots $h_i = r_i$, which are disjoint.

   To avoid disjoint union of space-time surfaces f_i must be a prime polynomial with respect to functional composition. For the polynomials of a single variable, this is the case if the degree of the polynomial is prime but this is not a necessary condition for primeness. As already found, this condition generalizes to the polynomials of 3 complex variables considered in the recent case.

   Space-time surfaces of these kinds are excellent candidates for fundamental objects and the polynomial in question would have prime degree with respect to each of the 3 complex coordinates of H: this would make 3, presumably small primes. The composites formed of maps g and of these fundamental function pairs f would define cognitive representations of the surface defined by f as kind of statements about statements. An interesting question is whether these surfaces could correspond to elementary particles.

2. There is also a natural measure of complexity as the number of maps g, which correspond to prime polynomials with $g_i = P_i/Q_i$ appearing in the functional composite with a pair of prime polynomials (f_1, f_2). Here the prime polynomials P_i must have degree higher than 1 in order to increase the complexity. In this case, 2 primes would characterize the prime polynomial P_i.

What could be the physical interpretation of the prime polynomials (f_1, f_2) and (g_1, g_2), in particular (g_1, Id) and how it relates to the p-adic length scale hypothesis [L15]?

1. Probably the primes as orders of prime polynomials do not correspond to very large p-adic primes ($M_{127} = 2^{127} - 1$ for electron) assigned in p-adic mass calculations to elementary particles and tentatively identified as ramified primes [L15] appearing as divisors of the discriminant of a polynomials define as the product of root differences, which could correspond to that for $g = (g_1, Id)$.

2. p-Adic length scale hypothesis states that the physically preferred p-adic primes correspond to powers $p \simeq 2^k$. Also powers $p \simeq q^k$ of other small primes q can be considered [K9] and there is empirical evidence of time scales coming as powers of $q = 3$ [?, ?]. For Mersenne primes $M_n = 2^n - 1$, n is prime and this inspires the question whether k could be prime quite generally. The proposal has been that the p and k would correspond to a very large and small p-adic length scale. Could the 3 primes characterizing the prime polynomials f_i correspond to the small primes q and could the ramified primes $p \simeq 2^k$ be associated with the polynomials obtained to theire iterated functional composites?

Could small-p p-adicity make sense and could the p-adic length scale hypothesis relate small-p p-adicity and large-p p-acidity?

1. Could the p-adic length scale hypothesis in its basic form reflect 2-adicity at the fundamental level or could it reflect that $p = 2$ is the degree for the lowest prime polynomials, certainly the most primitive cognitive level. Or could it reflect both?

2. Could $p \simeq 2^k$ emerge when the action of a polynomial g_1 of degree 2 with respect to say the complex coordinate w of M^4 on polynomial Q is iterated functionally: $Q \rightarrow PcircQ \rightarrow P \circ ...P \circ Q$ and give $n = 2^k$ disjoint space-time surfaces as representations of the roots. For $p = 2$ the iteration is the procedure giving rise to Mandelbrot fractals and Julia sets. Electrons would correspond to objects with 127 iterations and cognitive hierarchy with 127 levels! Could $p = M_{127}$ be a ramified prime associated with $P \circ ... \circ P$.

   If this is the case, $p \simeq 2^k$ and k would tell about cognitive abilities of an electron and not so much about the system characterized by the function pair (f_1, f_2) at the bottom. Could the 2^k disjoint space-time surfaces correspond to a representation of $p \simeq 2^k$ binary numbers represented as disjoint space-time surfaces realizing binary mathematics at the level of space-time surfaces? This representation brings in mind the totally discontinuous compact-open p-adic topology. Cognition indeed decomposes the perceptive field into objects.

3. This generalizes to a prediction of hierarchies $p \simeq q^k$, where q is a small prime as compared to p and identifiable as the prime order of a prime polynomial with respect to, say, variable w.

A highly interesting observation is that the numbers allowing expansions in powers of an integer n having powers of primes belonging to some set can be regarded as p-adic integers for all these primes. One might say that these numbers belong to an intersection of these number fields. This could allow gluing of p-adic factors of adeles to single continuous structure. This suggests the possibility of multi-p p-adicity. The discriminant D of a polynomial defined as root differences can be expressed as a product of powers of so called ramified primes and the question is which of them is physically selected and why. Could multi-p p-adicity prevail that the expansions of physical quantities are in powers of D. I have also proposed that D, or its suitable power, is the number theoretical counterpart for the exponent of Kähler function as vacuum functional.

6.2.2 Witt vectors and Witt polynomials and the representation of p-adic numbers as space-time surfaces

We have had very inspiring discussions with Robert Paster, who advocates the importance of universal Witt vectors and Witt polynomials (see this) in the modelling of the brain, have been very inspiring. As the special case Witt vectors code for p-adic number fields. Witt polynomials are characterized by their roots, and the TGD vidw about space-time surfaces both as generalized numbers and representations of ordinary numbers, inspires the idea how the roots of Witt polynomials could be represented as space-time surfaces in the TGD framework. This would give a representation of p-adic numbers as space-time surfaces.

Could the *prime* polynomial pairs $(g_1, g_2) : C^2 \rightarrow C^2$ and $(f_1, f_2) : H = M^4 \times CP_2 \rightarrow C^2$ (perpaps states of pure, non-reflective awareness) characterized by small primes give rise to p-adic numbers represented in terms of space-time surfaces such that these primes could correspond to p-adic primes?

1. Universal Witt vectors and polynomials can be assigned to any commutative ring R. Witt vectors X_n define sequences of elements of R and Universal Witt polynomials $W_n(X_n)$ define a sequence of polynomials of order n. In the general case the Witt polynomial can be written as $W_n = \sum_{d|n} dX^{n/d}$, where d is a divisor of n, with 1 and n included. Clearly W_n characterizes the number theoretical anatomy of n.

2. The roots of W_n characterize W_n it and just for fun one can ask whether also W_ns could determine space-time surfaces as a representation for the anatomy of integer n and n itself. The roots of W_n would define a set of disjoint surfaces. One can ask whether this kind of representation of polynomials in terms of their roots in turn represented in terms of space-time surfaces is a universal feature of mathematical cognition.

3. Cognition would really create worlds! In Finland we have Kalevala as a national epic and it roughly says that things were discovered by first discovering the word describing the thing. Something similar appears in the Bible: "In the beginning was the Word, and the Word was with God, and the Word was God. Word is world!

Also p-adic numbers could be represented in terms of space-time surfaces.

1. UWVs are defined for any Abelian ring, not only ordinary numbers. For instance, the function pairs $(f_1, f_2) : M^4 \rightarrow C^2$ define space-time surfaces as their roots form an Abelian ring with respect to element-wise sum and multiplication. One could therefore consider the n:th powers of (f_1^n, f_2^n) of (f_1, f_2). The roots of the polynomial W_n contain $(f_1, f_2) = 0$ but the remaining roots of W_n are different and correspond to pairs of algebraic numbers $(f_1, f_2) = (r_1, r_2)$. W_n gives rise to n disjoint space-time surfaces as its roots.

2. The function pairs $g = (g_1, g_2) : C^2 \rightarrow C^2$ define by the iteration reflective hierarchies thoughts about... about prime thoughts defined by prime pairs (f_1, f_2), which do not allow further functional composition to $g \circ f$ Also the pairs g allow prime pairs and their algebra Witt vectors and corresponding Witt polynomials. The roots for (g_1, g_2), which are functional primes and its powers (g_1^n, g_2^n) correspond to discrete points.

3. Could the Witt polynomials relate to the abstraction hierarchy in which the degrees of polynomials appear as powers of primes associated with the prime function pair (f_1, f_2) or (g_1, g_2). For a general Witt polynomial W_n, the degree increases very slowly with n rather than exponentially. Special conditions on the values of n are required.

   However, if one restricts the consideration to p-adic numbers, the values of n come as powers p^k, in the same way as in the case of functional iteration, and a good candidate for p could be the prime associated with a prime polynomial g or prime polynomial f.

   However, in the general case 3 primes characterize the pair f as a prime and 2 primes characterize the prime pair g as prime with respect to functional composition. Does this mean multi-p p-adicity so that the multi-p-adic integers would be a power series of an integer n divisible by several primes and belong to the intersection of several p-adic number fields. The elements of the p-adic number field would be represented as p-adic Witt vectors represented in terms of Witt polynomials represented in terms of their roots. Thoughts as elements of p-adic number fields would represent unions of space-time surfaces!

It seems that the small primes could indeed define p-adic primes and p-adic topologies. As a matter of fact, the roots of W_n are disjoint conforms with the idea about total disconnectedness of the p-adic topology in which all sets are both open and closed. The roots of W_n define discriminant D, decomposing to the powers of ramified primes identified as large p-adic primes in p-adic mass calculations [L15].

Could this give a generalization of the p-adic length scale hypothesis $p \simeq q^k$, such that q is a small prime assignable to pair (f_1, f_2), which is prime with respect to functional composition, and p is the ramified prime of W_{q^k}. Note that the spectrum of ramified primes would depend on the power q^k. One should be able to show that the spectrum contains $p \simeq 2^k$. Large ramified primes are indeed possible since the degree of W_{q^k} increases exponentially with k and therefore also the number of roots as its degree q^k. For electrons one would have $p = M_{127} = 2^{127} - 1$ and $k = 127$. The coefficients of W_{q^k} are powers q^i, $i \leq k$.

6.2.3 About the identification of the Lie groups appearing in Langlands duality?

Transformations (g_1, g_2) acting as symmetries should not increase the complexity and therefore should preserve the degree of the numerator or perhaps decrease it. Several alternatives can be considered.

1. If it is required that the polynomials f_i remain polynomials, then $SL(2, C)$ that acts on (f_1, f_2) like in spinors is a natural alternative. A possible interpretation is as a Lorentz group or alternatively as a group $SL(2, C)$ assignable to the Virasoro algebra.

2. The allowance of rational transformations conforms with the notion of modular group representations. If they are are allowed and if one requires that there is no mixing of f_1 and f_2 as mildly suggested by the element-wise product for (f_1, f_2), the group reduces to $SL(2,C) \times SL(2,C)$. $SL(2,C)$ consists of Möbius transformations $z-->(az+b)/(cz+d)$ (see this)

   $SL(2,C)$ has a rich spectrum of subgroups and the modular representations are invariant under some discrete subgroup of $SL(2,C)$. The modular group corresponds to $SL(2,Z)$ which has various discrete subgroups leaving modular forms invariant. There is an entire hierarchy of subgroups associated with the algebraic extensions of Z and in this case the matrix elements would be algebraic integers. Now the integers for subgroup $SL(2,Z)$ would be replaced with the algebraic integers for E appearing as the coefficients of f_i and g_i.

3. If one allows the mixing of f_i, Möbius group is replaced with group $SL(3,C)$. What is interesting is that $SL(3,C)$ contains $SU(3)$ as a subgroup acting as isometries of CP_2. A second interesting observation is that also $SL(3,C)$ allows McKay correspondence in which the finite subgroups of $SU(2)$ are replaced by finite subgroups of $SU(3)$ [L12]. This is highly desirable in the TGD framework since $SU(3)$ acts as isometries of CP_2. An interesting question is whether the McKay correspondence holds true for $SL(n,C)$, $n>3$.

Where should the Lie group for the analogs of Möbius transformations act? It is not natural to require that a discrete subgroup would leave the space-time surface invariant. The most natural option is that the action takes place in the "world of classical worlds" (WCW) formed by the generalized Bohr orbits satisfying holography= holomorphy principle. The counterparts of modular forms could correspond to WCW spinor fields invariant under the appropriate discrete subgroup of the generalized Möbius group.

6.2.4 Physical interpretation of the generalized modular group and spectrum generating group

One can consider several physical interpretations for the generalized modular group and dynamical spectrum generating algebra formed by the maps $g: C^2 \to C^2$.

1. Is the interpretation of $SL(2,C)$ as a Lorentz group reasonable? The McKay correspondence would refer to finite subgroups of $SU(2)$. This interpretation is not necessary since the Lorentz group and Poincare group act in the moduli space of causal diamonds (CDs). The discrete subgroups of $SU(2)$ appearing in Mac-Kay correspondence act in C as modular transformations.

2. Could $SL(3,C)$ refer to $SU(3)$. It is known that $SL(3,C)$ allows the generalization of Mac-Kay correspondence to the finite subgroups of $SU(3)$. $SU(2)$ can be identified as a rotation group and a subgroup of the color group.

   Does this pose an interpretational problem? I have encountered a similar problem earlier in the twistorialization [L26]: the twistor spheres of M^4 twistor space and CP_2 twistor space are identified and this strongly suggests a close correspondence with the representations of rotation group and weak gauge group, the holonomy group $U(2)$ of CP_2, which is identifiable as a subgroup of $SU(3)$. The quark and lepton doublets are indeed spin and isospin doublets and this would allow us to realize this kind of correlation. In the recent formulation of the twistorialization without explicit introduction of the twistor spaces of M^4 and CP_2, the twistor spheres appear also as spheres embedded to the spacetime surfaces in H. Could the identification of these two $SU(2)$ subgroups be a part of the same story?

3. $SL(2,C)$ could also correspond to the sub-algebra of the Virasoro algebra of the string models. $SL(3,C)$ would naturally generalize this algebra to a 4-D situation. A generalization of Super Virasoro algebra involving two variables occurs naturally in TGD. The gauge conditions satisfied for the Super Virasoro algebra and associated Kac Moody type algebras are essential in string models. A possible interpretation of the Super Virasoro algebra in terms of infinitesimal analytic transformations which have interpretation as general coordinate transformations so that although they do not respect the degree of the polynomial they do not change the physics.

In the TGD framework, a breaking of superconformal invariance is assumed to occur. The half-algebras associated with these algebras allow an infinite fractal hierarchy of sub-algebras isomorphic to the entire algebra and super-conformal symmetry can break down to this kind of sub-algebra [L23]. Therefore algebra generators with finite conformal weight below some maximum value would not act anymore as gauge symmetries but tranform to dynamical symmetries. In the recent case, these generators could correspond to maps g, which correspond to polynomial or rational functions with degree below some maximum value.

4. $SL(3,C)$ would naturally generalize this algebra to a 4-D situation and define the extension of Virasoro algebra two the case of two complex variables. This would be natural because the string world sheets are replaced by spacetime surfaces.

Also the representations of the analogs of Super Virasoro and Super Kac-Moody algebras (in particular super-symplectic algebra) are essential in TGD [L23]. A natural expectation is that they are also generalized modular representations and therefore involve the outer Galois group associated with the space-time surfaces at the various levels of the hierarchy defined by the maps g. This would conform with the view that the outer Galois group acts as physical symmetry group in the TGD Universe. I have earlier developed this view in detail in the construction of quantum states. The original identification of the Galois group was not however quite correct.

6.2.5 Langlands duality for the representations of the Lorentz group

In TGD, the modular forms defined in the hyperbolic space H^3 are especially interesting. Lorentz group acts on both. The earlier proposal is that modular forms can be generalized to H^3 as an analog of mass shell or Lorentz invariant cosmic time=constant hyperboloid. The discrete subgroup of $SL(2,C)$ as a symmetry group would define tessellations of H^3: this is a rather strong assumption.

Lorentz group and its discrete subgroups act on H^3 or possibly on the light-cone boundary at which the holographic data resides. Generalized modular forms could be also assigned with WCW spinor fields. The counterpart of the Galois group would be the same as in the above proposal. This picture applies also to color symmetries. This would give rise to the analogs of lattice waves in E^3. The holographic data invariant under a discrete subgroup would define tessellations as analogs of the lattices in E^3 [L16]. One application is a proposal of a universal realization of genetic code based on completely exceptional tessellation of H^3 involves instead of single Platonic solid the three Platonic solids with triangular faces. Also applications in cosmological scales are possible and there is some empirical evidence that stars could be assigned to a tessellation of H^3 [L25].

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