

The idea of  $\mathbb{M}^8 - \mathbb{H}$  duality has progressed through frustratingly many twists and turns and I have discussed several variants of  $\mathbb{M}^8 - \mathbb{H}$  duality. There are 2 options, call them  $\mathbb{T}$  and  $\mathbb{N}$ : either the local tangent space  $\mathbb{T}$  or normal space  $\mathbb{N}$  of  $\mathbb{Y}^4 \subset \mathbb{M}^8$  is quaternionic and contains a complex subspace  $\mathbb{C}$ . This makes it possible to map  $\mathbb{Y}^4 \subset \mathbb{M}^8$  to the space-time surface  $\mathbb{X}^4 \subset \mathbb{H} = \mathbb{M}^4 \times \mathbb{CP}_2$ . Which of them or possibly both? Any integrable distribution of quaternionic normal spaces  $\mathbb{N}$  is allowed whereas for tangent spaces this is not the case. This led to a too hasty rejection of the  $\mathbb{T}$  option.

The second problem relates to the lack of the concrete realization of the  $\mathbb{M}^8 - \mathbb{H}$  duality. Is the  $\mathbb{M}^8 - \mathbb{H}$  duality between 4-D surfaces in  $\mathbb{M}^8$  and space-time surfaces in  $\mathbb{H}$  or is it enough that only the 3-D holographic data in  $\mathbb{H}$  are fixed by  $\mathbb{M}^8 - \mathbb{H}$  duality.

A modification of the original form of the  $\mathbb{M}^8 - \mathbb{H}$  duality formulated in terms of a real octonion analytic functions  $f(o) : \mathbb{O} - \mathbb{O}$  leads to a possible solution of these problems. All the conditions  $f(o) = 0$ ,  $f(o) = 1$  and  $Imf(o) = 0$ , and  $Ref(o) = 0$  are invariant under local  $G_2$  and the local  $G_2$  acts as a dynamical spectrum generating symmetry group since  $f \circ g_2 = g_2 \circ f$  holds true. The task reduces to that of finding 4-surfaces with constant quaternionic normal space  $\mathbb{N}$  or tangent space  $\mathbb{T}$ .  $\mathbb{Y}^4 = \mathbb{E}^4 \subset \mathbb{M}^8$  and  $\mathbb{Y}^4 = \mathbb{M}^4 \subset \mathbb{M}^8$  provide the simplest examples of them. Local  $G_2$  transformations give more general surfaces  $\mathbb{Y}^4$ .

The roots of  $Im(f)(o) = 0$  *resp.*  $Re(f)(o) = 0$  are unions  $\cup_{o_0} \mathbb{S}^6(o_0)$  of 6-spheres, where  $o_0$  is octonionic real coordinate  $o_0$ . The 3-surface  $\mathbb{Y}^3 = \mathbb{S}^6(o_0) \cap \mathbb{E}^4(o_0) = \mathbb{S}^3(o_0)$  defines holographic data for  $\mathbb{Y}^4 \subset \mathbb{E}^4(o_0)$  as its boundary. The union  $\mathbb{Y}^3 = \cup_{o_0} \mathbb{S}^6(o_0) \cap \mathbb{M}^4(o_0) = \cup_{o_0} \mathbb{S}^2(o_0)$  in turn defines holographic data for  $\mathbb{Y}^4 \subset \mathbb{M}^4(o_0)$  as its boundary. Therefore both the  $\mathbb{N}$  - and  $\mathbb{T}$  option can be realized.

One can choose the function  $f(o)$  to be an analytic function of a hypercomplex coordinate of  $\mathbb{M}^4$  and 3 complex coordinates of  $\mathbb{M}^8$ . The natural conjecture is that the image  $\mathbb{X}^4$  of  $\mathbb{Y}^4$  has the same property and satisfies holography = holomorphy principle.

The simultaneous roots of  $Im(f)(o) = 0$  *resp.*  $Re(f)(o) = 0$  are 6-spheres with fixed value of  $o_0$  and the radius  $r_7$  of  $\mathbb{S}^6(o_0)$ . Two 4-surfaces  $\mathbb{Y}_1^4$  and  $\mathbb{Y}_2^4$ , both of type  $\mathbb{N}$  or  $\mathbb{T}$ , and satisfying  $Im(f)(o) = 0$  *resp.*  $Re(f)(o) = 0$  along  $\mathbb{S}^3(o_0)$  or  $\mathbb{S}^2(o_0)$ . This makes it possible to build Feynman diagram-like structures with lines which have Minkowskian or Euclidean number theoretic metric signatures. At the vertices smoothness is violated and this supports the view that they give rise to exotic smooth structures as defects of the standard smooth structure.