Classical Part of Twistor Story

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Abstract

Twistor Grassmannian formalism has made a breakthrough in $\mathcal{N} = 4$ supersymmetric gauge theories and the Yangian symmetry suggests that much more than mere technical breakthrough is in question. Twistors seem to be tailor made for TGD but it seems that the generalization of twistor structure to that for 8-D imbedding space $H = M^4 \times CP_2$ is necessary. M^4 (and S^4 as its Euclidian counterpart) and CP_2 are indeed unique in the sense that they are the only 4-D spaces allowing twistor space with Kähler structure.

The Cartesian product of twistor spaces $P_3 = SU(2,2)/SU(2,1) \times U(1)$ and F_3 defines twistor space for the imbedding space H and one can ask whether this generalized twistor structure could allow to understand both quantum TGD and classical TGD defined by the extremals of Kähler action. In the following I summarize the background and develop a proposal for how to construct extremals of Kähler action in terms of the generalized twistor structure. One ends up with a scenario in which space-time surfaces are lifted to twistor spaces by adding CP_1 fiber so that the twistor spaces give an alternative representation for generalized Feynman diagrams.

There is also a very closely analogy with superstring models. Twistor spaces replace Calabi-Yau manifolds and the modification recipe for Calabi-Yau manifolds by removal of singularities can be applied to remove self-intersections of twistor spaces and mirror symmetry emerges naturally. The overall important implication is that the methods of algebraic geometry used in super-string theories should apply in TGD framework.

The physical interpretation is totally different in TGD. The landscape is replaced with twistor spaces of space-time surfaces having interpretation as generalized Feynman diagrams and twistor spaces as sub-manifolds of $P_3 \times F_3$ replace Witten's twistor strings.

The classical view about twistorialization of TGD makes possible a more detailed formulation of the previous ideas about the relationship between TGD and Witten's theory and twistor Grassmann approach. Furthermore, one ends up to a formulation of the scattering amplitudes in terms of Yangian of the super-symplectic algebra relying on the idea that scattering amplitudes are sequences consisting of algebraic operations (product and co-product) having interpretation as vertices in the Yangian extension of super-symplectic algebra. These sequences connect given initial and final states and having minimal length. One can say that Universe performs calculations.

1 Introduction

Twistor Grassmannian formalism has made a breakthrough in $\mathcal{N} = 4$ supersymmetric gauge theories and the Yangian symmetry suggests that much more than mere technical breakthrough is in question. Twistors seem to be tailor made for TGD but it seems that the generalization of twistor structure to that for 8-D imbedding space $H = M^4 \times CP_2$ is necessary. M^4 (and S^4 as its Euclidian counterpart) and CP_2 are indeed unique in the sense that they are the only 4-D spaces allowing twistor space with Kähler structure.

The Cartesian product of twistor spaces $P_3 = SU(2,2)/SU(2,1) \times U(1)$ and F_3 defines twistor space for the imbedding space H and one can ask whether this generalized twistor structure could allow to understand both quantum TGD [K14, K15, K21] and classical TGD [K13] defined by the extremals of Kähler action.

In the following I summarize first the basic results and problems of the twistor approach. After that I describe some of the mathematical background and develop a proposal for how to construct extremals of Kähler action in terms of the generalized twistor structure. One ends up with a scenario in which space-time surfaces are lifted to twistor spaces by adding CP_1 fiber so that the twistor spaces give an alternative representation for generalized Feynman diagrams having as lines space-time surfaces with Euclidian signature of induced metric and having wormhole contacts as basic building bricks.

There is also a very close analogy with superstring models. Twistor spaces replace Calabi-Yau manifolds [A1, A9] and the modification recipe for Calabi-Yau manifolds by removal of singularities can be applied to remove self-intersections of twistor spaces and mirror symmetry [B3]emerges naturally. The overall important implication is that the methods of algebraic geometry used in super-string theories should apply in TGD framework.

The physical interpretation is totally different in TGD. Twistor space has space-time as base-space rather than forming with it Cartesian factors of a 10-D space-time. The Calabi-Yau landscape is replaced with the space of twistor spaces of space-time surfaces having interpretation as generalized Feynman diagrams and twistor spaces as sub-manifolds of $P_3 \times F_3$ replace Witten's twistor strings [B8]. The space of twistor spaces is the lift of the "world of classical worlds" (WCW) by adding the CP_1 fiber to the space-time surfaces so that the analog of landscape has beautiful geometrization.

The classical view about twistorialization of TGD makes possible a more detailed formulation of the previous ideas about the relationship between TGD and Witten's theory and twistor Grassmann approach.

1. The notion of quaternion analyticity extending the notion of ordinary analyticity to 4-D context is highly attractive but has remained one of the long-standing ideas difficult to take quite seriously but equally difficult to throw to paper basked. Four-manifolds possess almost quaternion structure. In twistor space context the formulation of quaternion analyticity becomes possible and relies on an old notion of tri-holomorphy about which I had not been aware earlier. The natural formulation for the preferred extremal property is as a condition stating that various charges associated with generalized conformal algebras vanish for preferred extremals. This leads to ask whether Euclidian space-time regions could be quaternion-Kähler manifolds for which twistor spaces are so called Fano spaces. In Minkowskian regions so called Hamilton-Jacobi property would apply.

2. The generalization of Witten's twistor theory to TGD framework is a natural challenge and the 2-surfaces studied defining scattering amplitudes in Witten's theory could correspond to partonic 2-surfaces identified as algebraic surfaces characterized by degree and genus. Besides this also string world sheets are needed. String worlds have 1-D lines at the light-like orbits of partonic 2-surfaces as their boundaries serving as carriers of fermions. This leads to a rather detailed generalization of Witten's approach using the generalization of twistors to 8-D context.

3. The generalization of the twistor Grassmannian approach to 8-D context is second fascinating challenge. If one requires that the basic formulas relating twistors and four-momentum generalize one must consider the situation in tangent space M^8 of imbedding space ($M^8 - H$ duality) and replace the usual sigma matrices having interpretation in terms of complexified quaternions with octonionic sigma matrices.

   The condition that octonionic spinors are are equivalent with ordinary spinors has strong consequences. Induced spinors must be localized to 2-D string world sheets, which are (co-)commutative sub-manifolds of (co-)quaternionic space-time surface. Also the gauge fields should vanish since they induce a breaking of associativity even for quaternionic and complex surface so that CP_2 projection of string world sheet must be 1-D. If one requires also the vanishing of gauge potentials, the projection is geodesic circle of CP_2 so that string world sheets are restricted to Minkowskian space-time regions. Although the theory would be free in fermionic degrees of freedom, the scattering amplitudes are non-trivial since vertices correspond to partonic 2-surfaces at which partonic orbits are glued together along common ends. The classical light-like 8-momentum associated with the boundaries of string world sheets defines the gravitational dual for 4-D momentum and color quantum numbers associated with imbedding space spinor harmonics. This leads to a more detailed formulation of Equivalence Principle which would reduce to $M^8 - H$ duality basically.

   Number theoretic interpretation of the positivity of Grassmannians is highly suggestive since the canonical identification maps p-adic numbers to non-negative real numbers. A possible generalization is obtained by replacing positive real axis with upper half plane defining hyperbolic space having key role in the theory of Riemann surfaces. The interpretation of scattering amplitudes as representations of permutations generalizes to interpretation as braidings at surfaces formed by the generalized Feynman diagrams having as lines the light-like orbits of partonic surfaces. This because 2-fermion vertex is the only interaction vertex and induced by the non-continuity of the induced Dirac operator at partonic 2-surfaces. OZI rule generalizes and implies an interpretation in terms of braiding consistent with the TGD as almost topological QFT vision. This suggests that non-planar twistor amplitudes are constructible as analogs of knot and braid invariants by a recursive procedure giving as an outcome planar amplitudes.

4. Yangian symmetry is associated with twistor amplitudes and emerges in TGD from completely different idea interpreting scattering amplitudes as representations of algebraic manipulation sequences of minimal length (preferred extremal instead of path integral over space-time surfaces) connecting given initial and final states at boundaries of causal diamond. The algebraic manipulations are carried out in Yangian using product and co-product defining the basic 3-vertices analogous to gauge boson absorption and emission. 3-surface representing elementary particle splits into two or vice versa such that second copy carries quantum numbers of gauge boson or its super counterpart. This would fix the scattering amplitude for given 3-surface and leave only the functional integral over 3-surfaces.

2 Background And Motivations

In the following some background plus basic facts and definitions related to twistor spaces are summarized. Also reasons for why twistor are so relevant for TGD is considered at general level.

2.1 Basic Results And Problems Of Twistor Approach

The author describes both the basic ideas and results of twistor approach as well as the problems.

2.1.1 *Basic results*

There are three deep results of twistor approach besides the impressive results which have emerged after the twistor resolution.

1. Massless fields of arbitrary helicity in 4-D Minkowski space are in 1-1 correspondence with elements of Dolbeault cohomology in the twistor space CP_3. This was already the discovery of Penrose..The connection comes from Penrose transform. The light-like geodesics of M^4 correspond to points of 5-D sub-manifold of CP_3 analogous to light-cone boundary. The points of M^4 correspond to complex lines (Riemann spheres) of the twistor space CP_3: one can imagine that the point of M^4 corresponds to all light-like geodesics emanating from it and thus to a 2-D surface (sphere) of CP_3. Twistor transform represents the value of a massless field at point of M^4 as a weighted average of its values at sphere of CP_3. This correspondence is formulated between open sets of M^4 and of CP_3. This fits very nicely with the needs of TGD since causal diamonds which can be regarded as open sets of M^4 are the basic objects in zero energy ontology (ZEO).

2. Self-dual instantons of non-Abelian gauge theories for SU(n) gauge group are in one-one correspondence with holomorphic rank-N vector bundles in twistor space satisfying some additional conditions. This generalizes the correspondence of Penrose to the non-Abelian case. Instantons are also usually formulated using classical field theory at four-sphere S^4 having Euclidian signature.

3. Non-linear gravitons having self-dual geometry are in one-one correspondence with spaces obtained as complex deformations of twistor space satisfying certain additional conditions. This is a generalization of Penrose's discovery to the gravitational sector.

Complexification of M^4 emerges unavoidably in twistorial approach and Minkowski space identified as a particular real slice of complexified M^4 corresponds to the 5-D subspace of twistor space in which the quadratic form defined by the SU(2,2) invariant metric of the 8-dimensional space giving twistor space as its projectivization vanishes. The quadratic form has also positive and negative values with its sign defining a projective invariant, and this correspond to complex continuations of M^4 in which positive/negative energy parts of fields approach to zero for large values of imaginary part of M^4 time coordinate.

Interestgingly, this complexification of M^4 is also unavoidable in the number theoretic approach to TGD: what one must do is to replace 4-D Minkowski space with a 4-D slice of 8-D complexified quaternions. What is interesting is that real M^4 appears as a projective invariant consisting of light-like projective vectors of C^4 with metric signature (4,4). Equivalently, the points of M^4 represented as linear combinations of sigma matrices define hermitian matrices.

2.1.2 *Basic problems of twistor approach*

The best manner to learn something essential about a new idea is to learn about its problems. Difficulties are often put under the rug but the thesis is however an exception in this respect. It starts directly from the problems of twistor approach. There are two basic challenges.

1. Twistor approach works as such only in the case of Minkowski space. The basic condition for its applicability is that the Weyl tensor is self-dual. For Minkowskian signature this leaves only Minkowski space under consideration. For Euclidian signature the conditions are not quite so restrictive. This looks a fatal restriction if one wants to generalize the result of Penrose to a general space-time geometry. This difficulty is known as "googly" problem.

   According to the thesis MHV construction of tree amplitudes of $\mathcal{N}=4$ SYM based on topological twistor strings in CP_3 meant a breakthrough and one can indeed understand also have analogs of non-self-dual amplitudes. The problem is however that the gravitational theory assignable to topological twistor strings is conformal gravity, which is generally regarded as non-physical. There have been several attempts to construct the on-shell scattering amplitudes of Einstein's gravity theory as subset of amplitudes of conformal gravity and also thesis considers this problem.

2. The construction of quantum theory based on twistor approach represents second challenge. In this respect the development of twistor approach to $\mathcal{N}=4$ SYM meant a revolution and one can indeed construct twistorial scattering amplitudes in M^4.

2.2 Results About Twistors Relevant For TGD

First some background.

1. The twistors originally introduced by Penrose (1967) have made breakthrough during last decade. First came the twistor string theory of Edward Witten [B8] proposed twistor string theory and the work of Nima-Arkani Hamed and collaborators [B10] led to a revolution in the understanding of the scattering amplitudes of scattering amplitudes of gauge theories [B5, B4, B11]. Twistors do not only provide an extremely effective calculational method giving even hopes about explicit formulas for the scattering amplitudes of $\mathcal{N}=4$ supersymmetric gauge theories but also lead to an identification of a new symmetry: Yangian symmetry [A2], [B6, B7], which can be seen as multilocal generalization of local symmetries.

   This approach, if suitably generalized, is tailor-made also for the needs of TGD. This is why I got seriously interested on whether and how the twistor approach in empty Minkowski space M^4 could generalize to the case of $H = M^4 \times CP_2$. The twistor space associated with H should be just the cartesian product of those associated with its Cartesian factors. Can one assign a twistor space with CP_2?

2. First a general result [A6] deserves to be mentioned: any oriented manifold X with Riemann metric allows 6-dimensional twistor space Z as an almost complex space. If this structure is integrable, Z becomes a complex manifold, whose geometry describes the conformal geometry of X. In general relativity framework the problem is that field equations do not imply conformal geometry and twistor Grassmann approach certainly requires conformal structure.

3. One can consider also a stronger condition: what if the twistor space allows also Kähler structure? The twistor space of empty Minkowski space M^4 (and its Euclidian counterpart S^4 is the Minkowskian variant of $P_3 = SU(2,2)/SU(2,1) \times U(1)$ of 3-D complex projective space $CP_3 = SU(4)/SU(3) \times U(1)$ and indeed allows Kähler structure.

   The points of the Euclidian twistor space $CP_3 = SU(4)/SU(3) \times U(1)$ can be represented by any column of the 4×4 matrix representing element of $SU(4)$ with columns differing by phase multiplication being identified. One has four coordinate charts corresponding to four different choices of the column. The points of its Minkowskian variant $CP_{2,1} = SU(2,2)/SU(2,1) \times U(1)$ can be represented in similar manner as U(1) gauge equivalence classes for given column of SU(3,1) matrices, whose rows and columns satisfy orthonormality conditions with respect to the hermitian inner product defined by Minkowskian metric $\epsilon = (1,1,-1,-1)$.

Rather remarkably, there are *no other space-times* with Minkowski signature allowing twistor space with Kähler structure. Does this mean that the empty Minkowski space of special relativity is much more than a limit at which space-time is empty?

This also means a problem for GRT. Twistor space with Kähler structure seems to be needed but general relativity does not allow it. Besides twistor problem GRT also has energy problem: matter makes space-time curved and the conservation laws and even the definition of energy and momentum are lost since the underlying symmetries giving rise to the conservation laws through Noether's theorem are lost. GRT has therefore two bad mathematical problems which might explain why the quantization of GRT fails. This would not be surprising since quantum theory is to high extent representation theory for symmetries and symmetries are lost. Twistors would extend these symmetries to Yangian symmetry but GRT does not allow them.

4. What about twistor structure in CP_2? CP_2 allows complex structure (Weyl tensor is self-dual), Kähler structure plus accompanying symplectic structure, and also quaternion structure. One of the really big personal surprises of the last years has been that CP_2 twistor space indeed allows Kähler structure meaning the existence of antisymmetric tensor representing imaginary unit whose tensor square is the negative of metric in turn representing real unit.

   The article by Nigel Hitchin, a famous mathematical physicist, describes a detailed argument identifying S^4 and CP_2 as the only compact Riemann manifolds allowing Kählerian twistor space [A6]. Hitchin sent his discovery for publication 1979. An amusing co-incidence is that I discovered CP_2 just this year after having worked with S^2 and found that it does not really allow to understand standard model quantum numbers and gauge fields. It is difficult to avoid thinking that maybe synchrony indeed a real phenomenon as TGD inspired theory of consciousness predicts to be possible but its creator cannot quite believe. Brains at different side of globe discover simultaneously something closely related to what some conscious self at the higher level of hierarchy using us as instruments of thinking just as we use nerve cells is intensely pondering.

   Although 4-sphere S^4 allows twistor space with Kähler structure, it does not allow Kähler structure and cannot serve as candidate for S in $H = M^4 \times S$. As a matter of fact, S^4 can be seen as a Wick rotation of M^4 and indeed its twistor space is CP_3.

   In TGD framework a slightly different interpretation suggests itself. The Cartesian products of the intersections of future and past light-cones - causal diamonds (CDs) - with CP_2 - play a key role in ZEO (ZEO) [K1]. Sectors of "world of classical worlds" (WCW) [K10, K6] correspond to 4-surfaces inside $CD \times CP_2$ defining a the region about which conscious observer can gain conscious information: state function reductions - quantum measurements - take place at its light-like boundaries in accordance with holography. To be more precise, wave functions in the moduli space of CDs are involved and in state function reductions come as sequences taking place at a given fixed boundary. This kind of sequence is identifiable as self and give rise to the experience about flow of time. When one replaces Minkowski metric with Euclidian metric, the light-like boundaries of CD are contracted to a point and one obtains topology of 4-sphere S^4.

5. Another really big personal surprise was that there are *no other* compact 4-manifolds with Euclidian signature of metric allowing twistor space with Kähler structure! The imbedding space $H = M^4 \times CP_2$ is not only physically unique since it predicts the quantum number spectrum and classical gauge potentials consistent with standard model but also mathematically unique!

   After this I dared to predict that TGD will be the theory next to GRT since TGD generalizes string model by bringing in 4-D space-time. The reasons are many-fold: TGD is the only known solution to the two big problems of GRT: energy problem and twistor problem. TGD is consistent with standard model physics and leads to a revolution concerning the identification of space-time at microscopic level: at macroscopic level it leads to GRT but explains some of its anomalies for which there is empirical evidence (for instance, the observation that neutrinos arrived from SN1987A at two different speeds different from light velocity [**?**] has natural explanation in terms of many-sheeted space-time). TGD avoids the landscape problem of M-theory and anthropic non-sense. I could continue the list but I think that this is enough.

6. The twistor space of CP_2 is 3-complex dimensional flag manifold $F_3 = SU(3)/U(1) \times U(1)$ having interpretation as the space for the choices of quantization axes for the color hypercharge and isospin. This choice is made in quantum measurement of these quantum numbers and a means localization to single point in F_3. The localization in F_3 could be higher level measurement leading to the choice of quantizations for the measurement of color quantum numbers.

   F_3 is symmetric space meaning that besides being a coset space with $SU(3)$ invariant metric it also has involutions acting as a reflection at geodesics through a point remaining fixed under the involution. As a symmetric space with Fubini-Study metric F_3 is positive constant curvature space having thus positive constant sectional curvatures. This implies Einstein space property. This also conforms with the fact that F_3 is CP_1 bundle over CP_2 as base space (for more details see `http://www.cirget.uqam.ca/~apostolo/papers/AGAG1.pdf`).

   The points of flag manifold $SU(3)/U(1)\times U(1)$ can be represented locally by identifying $SU(3)$ matrices for which columns differ by multiplication from left with exponential $e^{i(aY+bI_3)}$, a and b arbitrary real numbers. This transformation allows what might be called a "gauge choice". For instance, first two elements of the first row can be made real in this manner. These coordinates are not global.

7. Analogous interpretation could make sense for M^4 twistors represented as points of P_3. Twistor corresponds to a light-like line going through some point of M^4 being labelled by 4 position coordinates and 2 direction angles: what higher level quantum measurement could involve a choice of light-like line going through a point of M^4? Could the associated spatial direction specify spin quantization axes? Could the associated time direction specify preferred rest frame? Does the choice of position mean localization in the measurement of position? Do momentum twistors relate to the localization in momentum space? These questions remain fascinating open questions and I hope that they will lead to a considerable progress in the understanding of quantum TGD.

8. It must be added that the twistor space of CP_2 popped up much earlier in a rather unexpected context [K9]: I did not of course realize that it was twistor space. Topologist Barbara Shipman [A5] has proposed a model for the honeybee dance leading to the emergence of F_3. The model led her to propose that quarks and gluons might have something to do with biology. Because of her position and specialization the proposal was forgiven and forgotten by community. TGD however suggests both dark matter hierarchies and p-adic hierarchies of physics [K8, K23]. For dark hierarchies the masses of particles would be the standard ones but the Compton scales would be scaled up by $h_{eff}/h = n$ [K23]. Below the Compton scale one would have effectively massless gauge boson: this could mean free quarks and massless gluons even in cell length scales. For p-adic hierarchy mass scales would be scaled up or down from their standard values depending on the value of the p-adic prime.

2.3 Basic Definitions Related To Twistor Spaces

One can find from web several articles explaining the basic notions related to twistor spaces and Calabi-Yau manifolds. At the first look the notions of twistor as it appears in the writings of physicists and mathematicians don't seem to have much common with each other and it requires effort to build the bridge between these views. The bridge comes from the association of points of Minkowski space with the spheres of twistor space: this clearly corresponds to a bundle projection from the fiber to the base space, now Minkowski space. The connection of the mathematician's formulation with spinors remains still somewhat unclear to me although one can understand CP_1 as projective space associated with spinors with 2 complex components. Minkowski signature poses additional challenges. In the following I try my best to summarize the mathematician's view, which is very natural in classical TGD.

There are many variants of the notion of twistor depending on whether how powerful assumptions one is willing to make. The weakest definition of twistor space is as CP_1 bundle of almost complex structures in the tangent spaces of an orientable 4-manifold. Complex structure at given point means selection of antisymmetric form J whose natural action on vector rotates a vector in the plane defined by it by $\pi/2$ and thus represents the action of imaginary unit. One must perform this kind of choice also in normal plane and the direct sum of the two choices defines the full J. If one chooses J to be self-dual or anti-self-dual (eigenstate of Hodge star operation), one can fix J uniquely. Orientability makes possible the Hodge star operation involving 4-dimensional permutation tensor.

The condition $i^1 = -1$ is translated to the condition that the tensor square of J equals to $J^2 = -g$. The possible choices of J span sphere S^2 defining the fiber of the twistor spaces. This is not quite the complex sphere CP_1, which can be thought of as a projective space of spinors with two complex components. Complexification must be performed in both the tangent space of X^4 and of S^2. Note that in the standard approach to twistors the entire 6-D space is projective space P_3 associated with the C^8 having interpretation in terms of spinors with 4 complex components.

One can introduce almost complex structure also to the twistor space itself by extending the almost complex structure in the 6-D tangent space obtained by a preferred choice of J by identifying it as a point of S^2 and acting in other points of S^2 identified as antisymmetric tensors. If these points are interpreted as imaginary quaternion units, the action is commutator action divided by 2. The existence of quaternion structure of space-time surfaces in the sense as I have proposed in TGD framework might be closely related to the twistor structure.

Twistor structure as bundle of almost complex structures having itself almost complex structure is characterized by a hermitian Kähler form ω defining the almost complex structure of the twistor space. Three basic objects are involved: the hermitian form h, metric g and Kähler form ω satisfying $h = g + i\omega$, $g(X,Y) = \omega(X, JY)$.

In the base space the metric of twistor space is the metric of the base space and in the tangent space of fibre the natural metric in the space of antisymmetric tensors induced by the metric of the base space. Hence the properties of the twistor structure depend on the metric of the base space.

The relationship to the spinors requires clarification. For 2-spinors one has natural Lorentz invariant antisymmetric bilinear form and this seems to be the counterpart for J?

One can consider various additional conditions on the definition of twistor space.

1. Kähler form ω is not closed in general. If it is, it defines symplectic structure and Kähler structure. S^4 and CP_2 are the only compact spaces allowing twistor space with Kähler structure.

2. Almost complex structure is not integrable in general. In the general case integrability requires that each point of space belongs to an open set in which vector fields of type (1, 0) or (0, 1) having basis ∂/∂_{z^k} and $\partial/\partial_{\overline{z}^k}$ expressible as linear combinations of real vector fields with complex coefficients commute to vector fields of same type. This is non-trivial conditions since the leading names for the vector field for the partial derivatives does not yet guarantee these conditions.

   This necessary condition is also enough for integrability as Newlander and Nirenberg have demonstrated. An explicit formulation for the integrability is as the vanishing of Nijenhuis tensor associated with the antisymmetric form J (see (`http://insti.physics.sunysb.edu/conf/simonsworkII/talks/LeBrun.pdf` and `http://en.wikipedia.org/wiki/Almost_complex_manifold#Integrable_almost_complex_structures`). Nijenhuis tensor characterizes Nijenhuis bracket generalizing ordinary Lie bracket of vector fields (for detailed formula see `http://en.wikipedia.org/wiki/FrlicherNijenhuis_bracket`).

3. In the case of twistor spaces there is an alternative formulation for the integrability. Curvature tensor maps in a natural manner 2-forms to 2-forms and one can decompose the Weyl tensor W identified as the traceless part of the curvature tensor to self-dual and anti-self-dual parts W^+ and W^-, whose actions are restricted to self-dual resp. antiself-dual forms (self-dual and anti-self-dual parts correspond to eigenvalue + 1 and -1 under the action of Hodge * operation: for more details see `http://www.math.ucla.edu/~greene/YauTwister(8-9).pdf`). If W^+ or W^- vanishes - in other worlds W is self-dual or anti-self-dual - the assumption that J is self-dual or anti-self-dual guarantees integrability. One says that the metric is anti-self-dual (ASD). Note that the vanishing of Weyl tensor implies local conformal flatness (M^4 and

sphere are obviously conformally flat). One might think that ASD condition guarantees that the parallel translation leaves J invariant.

ASD property has a nice implication: the metric is balanced. In other words one has $d(\omega \wedge \omega) = 2\omega \wedge d\omega = 0$.

4. If the existence of complex structure is taken as a part of definition of twistor structure, one encounters difficulties in general relativity. The failure of spin structure to exist is similar difficulty: for CP_2 one must indeed generalize the spin structure by coupling Kähler gauge potential to the spinors suitably so that one obtains gauge group of electroweak interactions.

5. One could also give up the global existence of complex structure and require symplectic structure globally: this would give $d\omega = 0$. A general result is that hyperbolic 4-manifolds allow symplectic structure and ASD manifolds allow complex structure and hence balanced metric.

2.4 Why Twistor Spaces With Kähler Structure?

I have not yet even tried to answer an obvious question. Why the fact that M^4 and CP_2 have twistor spaces with Kähler structure could be so important that it could fix the entire physics? Let us consider a less general question. Why they would be so important for the classical TGD - exact part of quantum TGD - defined by the extremals of Kähler action [K3] ?

1. Properly generalized conformal symmetries are crucial for the mathematical structure of TGD [K6, K18, K5, K17]. Twistor spaces have almost complex structure and in these two special cases also complex, Kähler, and symplectic structures (note that the integrability of the almost complex structure to complex structure requires the self-duality of the Weyl tensor of the 4-D manifold).

   The Cartesian product $CP_3 \times F_3$ of the two twistor spaces with Kähler structure is expected to be fundamental for TGD. The obvious wishful thought is that this space makes possible the construction of the extremals of Kähler action in terms of holomorphic surfaces defining 6-D twistor sub-spaces of $CP_3 \times F_3$ allowing to circumvent the technical problems due to the signature of M^4 encountered at the level of $M^4 \times CP_2$. It would also make the the magnificent machinery of the algebraic geometry so powerful in string theories a tool of TGD. For years ago I considered the possibility that complex 3-manifolds of $CP_3 \times CP_3$ could have the structure of S^2 fiber space and have space-time surfaces as base space. I did not realize that this spaces could be twistor spaces nor did I realize that CP_2 allows twistor space with Kähler structure so that $CP_3 \times F_3$ is a more plausible choice.

2. Every 4-D orientable Riemann manifold allows a twistor space as 6-D bundle with CP_1 as fiber and possessing almost complex structure. Metric and various gauge potentials are obtained by inducing the corresponding bundle structures. Hence the natural guess is that the twistor structure of space-time surface defined by the induced metric is obtained by induction from that for $CP_3 \times F_3$ by restricting its twistor structure to a 6-D (in real sense) surface of $CP_3 \times F_3$ with a structure of twistor space having at least almost complex structure with CP_1 as a fiber. For the imbedding of the twistor space of space-time this requires the identification of S^2 fibers of CP_3 and F_3. If so then one can indeed identify the base space as 4-D space-time surface in $M^4 \times SCP_2$ using bundle projections in the factors CP_3 and F_3.

3. There might be also a connection to the number theoretic vision about the extremals of Kähler action. At space-time level however complexified quaternions and octonions could allow alternative formulation. I have indeed proposed that space-time surfaces have associative of co-associative meaning that the tangent space or normal space at a given point belongs to quaternionic subspace of complexified octonions.

3 The Identification Of 6-D Twistor Spaces As Sub-Manifolds Of $CP_3 \times F_3$

How to identify the 6-D sub-manifolds with the structure of twistor space? Is this property all that is needed? Can one find a simple solution to this condition? What is the relationship of twistor spaces to the Calabi-Yau manifolds of suyper string models? In the following intuitive considerations of a simple minded physicist. Mathematician could probably make much more interesting comments.

3.1 Conditions For Twistor Spaces As Sub-Manifolds

Consider the conditions that must be satisfied using local trivializations of the twistor spaces. Before continuing let us introduce complex coordinates $z_i = x_i + iy_i$ resp. $w_i = u_i + iv_i$ for CP_3 resp. F_3.

1. 6 conditions are required and they must give rise by bundle projection to 4 conditions relating the coordinates in the Cartesian product of the base spaces of the two bundles involved and thus defining 4-D surface in the Cartesian product of compactified M^4 and CP_2.

2. One has Cartesian product of two fiber spaces with fiber CP_1 giving fiber space with fiber $CP_1^1 \times CP_1^2$. For the 6-D surface the fiber must be CP_1. It seems that one must identify the two spheres CP_1^i. Since holomorphy is essential, holomorphic identification $w_1 = f(z_1)$ or $z_1 = f(w_1)$ is the first guess. A stronger condition is that the function f is meromorphic having thus only finite numbers of poles and zeros of finite order so that a given point of CP_1^i is covered by CP_1^{i+1}. Even stronger and very natural condition is that the identification is bijection so that only Möbius transformations parametrized by SL(2, C) are possible.

3. Could the Möbius transformation $f : CP_1^1 \rightarrow CP_1^2$ depend parametrically on the coordinates z_2, z_3 so that one would have $w_1 = f_1(z_1, z_2, z_3)$, where the complex parameters a, b, c, d $(ad - bc = 1)$ of Möbius transformation depend on z_2 and z_3 holomorphically? Does this mean the analog of local SL(2, C) gauge invariance posing additional conditions? Does this mean that the twistor space as surface is determined up to SL(2, C) gauge transformation?

   What conditions can one pose on the dependence of the parameters a, b, c, d of the Möbius transformation on (z_2, z_3)? The spheres CP_1 defined by the conditions $w_1 = f(z_1, z_2, z_3)$ and $z_1 = g(w_1, w_2, w_3)$ must be identical. Inverting the first condition one obtains $z_1 = f^{-1}(w_1, z_2, z_3)$. If one requires that his allows an expression as $z_1 = g(w_1, w_2, w_3)$, one must assume that z_2 and z_3 can be expressed as holomorphic functions of (w_2, w_3): $z_i = f_i(w_k)$, $i = 2, 3$, $k = 2, 3$. Of course, non-holomorphic correspondence cannot be excluded.

4. Further conditions are obtained by demanding that the known extremals - at least non-vacuum extremals - are allowed. The known extremals [K3] can be classified into CP_2 type vacuum extremals with 1-D light-like curve as M^4 projection, to vacuum extremals with CP_2 projection, which is Lagrangian sub-manifold and thus at most 2-dimensional, to massless extremals with 2-D CP_2 projection such that CP_2 coordinates depend on arbitrary manner on light-like coordinate defining local propagation direction and space-like coordinate defining a local polarization direction, and to string like objects with string world sheet as M^4 projection (minimal surface) and 2-D complex sub-manifold of CP_2 as CP_2 projection, . There are certainly also other extremals such as magnetic flux tubes resulting as deformations of string like objects. Number theoretic vision relying on classical number fields suggest a very general construction based on the notion of associativity of tangent space or co-tangent space.

5. The conditions coming from these extremals reduce to 4 conditions expressible in the holomorphic case in terms of the base space coordinates (z_2, z_3) and (w_2, w_3) and in the more general case in terms of the corresponding real coordinates. It seems that holomorphic ansatz is not consistent with the existence of vacuum extremals, which however give vanishing contribution to transition amplitudes since WCW ("world of classical worlds") metric is completely degenerate for them.

The mere condition that one has CP_1 fiber bundle structure does not force field equations since it leaves the dependence between real coordinates of the base spaces free. Of course, CP_1 bundle structure alone does not imply twistor space structure. One can ask whether non-vacuum extremals could correspond to holomorphic constraints between (z_2, z_3) and (w_2, w_3).

6. The metric of twistor space is not Kähler in the general case. However, if it allows complex structure there is a Hermitian form ω, which defines what is called balanced Kähler form [A8] satisfying $d(\omega \wedge \omega) = 2\omega \wedge d\omega = 0$: ordinary Kähler form satisfying $d\omega = 0$ is special case about this. The natural metric of compact 6-dimensional twistor space is therefore balanced. Clearly, mere CP_1 bundle structure is not enough for the twistor structure. If the the Kähler and symplectic forms are induced from those of $CP_3 \times Y_3$, highly non-trivial conditions are obtained for the imbedding of the twistor space, and one might hope that they are equivalent with those implied by Kähler action at the level of base space.

7. Pessimist could argue that field equations are additional conditions completely independent of the conditions realizing the bundle structure! One cannot exclude this possibility. Mathematician could easily answer the question about whether the proposed CP_1 bundle structure with some added conditions is enough to produce twistor space or not and whether field equations could be the additional condition and realized using the holomorphic ansatz.

3.2 Twistor Spaces By Adding CP_1 Fiber To Space-Time Surfaces

The physical picture behind TGD is the safest starting point in an attempt to gain some idea about what the twistor spaces look like.

1. Canonical imbeddings of M^4 and CP_2 and their disjoint unions are certainly the natural starting point and correspond to canonical imbeddings of CP_3 and F_3 to $CP_3 \times F_3$.

2. Deformations of M^4 correspond to space-time sheets with Minkowskian signature of the induced metric and those of CP_2 to the lines of generalized Feynman diagrams. The simplest deformations of M^4 are vacuum extremals with CP_2 projection which is Lagrangian manifold.

   Massless extremals represent non-vacuum deformations with 2-D CP_2 projection. CP_2 coordinates depend on local light-like direction defining the analog of wave vector and local polarization direction orthogonal to it.

   The simplest deformations of CP_2 are CP_2 type extremals with light-like curve as M^4 projection and have same Kähler form and metric as CP_2. These space-time regions have Euclidian signature of metric and light-like 3-surfaces separating Euclidian and Minkowskian regions define parton orbits.

   String like objects are extremals of type $X^2 \times Y^2$, X^2 minimal surface in M^4 and Y^2 a complex sub-manifold of CP_2. Magnetic flux tubes carrying monopole flux are deformations of these.

   Elementary particles are important piece of picture. They have as building bricks wormhole contacts connecting space-time sheets and the contacts carry monopole flux. This requires at least two wormhole contacts connected by flux tubes with opposite flux at the parallel sheets.

3. Space-time surfaces are constructed using as building bricks space-time sheets, in particular massless exrremals, deformed pieces of CP_2 defining lines of generalized Feynman diagrams as orbits of wormhole contacts, and magnetic flux tubes connecting the lines. Space-time surfaces have in the generic case discrete set of self intersections and it is natural to remove them by connected sum operation. Same applies to twistor spaces as sub-manifolds of $CP_3 \times F_3$ and this leads to a construction analogous to that used to remove singularities of Calabi-Yau spaces [A8].

Physical intuition suggests that it is possible to find twistor spaces associated with the basic building bricks and to lift this engineering procedure to the level of twistor space in the sense that the twistor projections of twistor spaces would give these structure. Lifting would essentially mean assigning CP_1 fiber to the space-time surfaces.

1. Twistor spaces should decompose to regions for which the metric induced from the $CP_3 \times F_3$ metric has different signature. In particular, light-like 5-surfaces should replace the light-like 3-surfaces as causal horizons. The signature of the Hermitian metric of 4-D (in complex sense) twistor space is (1, 1, -1, -1). Minkowskian variant of CP_3 is defined as projective space $SU(2,2)/SU(2,1) \times U(1)$. The causal diamond (CD) (intersection of future and past directed light-cones) is the key geometric object in ZEO (ZEO) and the generalization to the intersection of twistorial light-cones is suggestive.

2. Projective twistor space has regions of positive and negative projective norm, which are 3-D complex manifolds. It has also a 5-dimensional sub-space consisting of null twistors analogous to light-cone and has one null direction in the induced metric. This light-cone has conic singularity analogous to the tip of the light-cone of M^4.

   These conic singularities are important in the mathematical theory of Calabi-You manifolds since topology change of Calabi-Yau manifolds via the elimination of the singularity can be associated with them. The S^2 bundle character implies the structure of S^2 bundle for the base of the singularity (analogous to the base of the ordinary cone).

3. Null twistor space corresponds at the level of M^4 to the light-cone boundary (causal diamond has two light-like boundaries). What about the light-like orbits of partonic 2-surfaces whose light-likeness is due to the presence of CP_2 contribution in the induced metric? For them the determinant of induced 4-metric vanishes so that they are genuine singularities in metric sense. The deformations for the canonical imbeddings of this sub-space (F_3 coordinates constant) leaving its metric degenerate should define the lifts of the light-like orbits of partonic 2-surface. The singularity in this case separates regions of different signature of induced metric.

   It would seem that if partonic 2-surface begins at the boundary of CD, conical singularity is not necessary. On the other hand the vertices of generalized Feynman diagrams are 3-surfaces at which 3-lines of generalized Feynman digram are glued together. This singularity is completely analogous to that of ordinary vertex of Feynman diagram. These singularities should correspond to gluing together 3 deformed F_3 along their ends.

4. These considerations suggest that the construction of twistor spaces is a lift of construction space-time surfaces and generalized Feynman diagrammatics should generalize to the level of twistor spaces. What is added is CP_1 fiber so that the correspondence would rather concrete.

5. For instance, elementary particles consisting of pairs of monopole throats connected buy flux tubes at the two space-time sheets involved should allow lifting to the twistor level. This means double connected sum and this double connected sum should appear also for deformations of F_3 associated with the lines of generalized Feynman diagrams. Lifts for the deformations of magnetic flux tubes to which one can assign CP_3 in turn would connect the two F_3s.

6. A natural conjecture inspired by number theoretic vision is that Minkowskian and Euclidian space-time regions correspond to associative and co-associative space-time regions. At the level of twistor space these two kinds of regions would correspond to deformations of CP_3 and F_3. The signature of the twistor norm would be different in this regions just as the signature of induced metric is different in corresponding space-time regions.

   These two regions of space-time surface should correspond to deformations for disjoint unions of CP_3s and F_3s and multiple connected sum form them should project to multiple connected sum (wormhole contacts with Euclidian signature of induced metric) for deformed CP_3s. Wormhole contacts could have deformed pieces of F_3 as counterparts.

There are interesting questions related to the detailed realization of the twistor spaces of space-time surfaces.

1. In the case of CP_2 J would naturally correspond to the Kähler form of CP_2. Could one identify J for the twistor space associated with space-time surface as the projection of J? For deformations of CP_2 type vacuum extremals the normalization of J would allow to satisfy

the condition $J^2 = -g$. For general extremals this is not possible. Should one be ready to modify the notion of twistor space by allowing this?

2. Or could the associativity/co-associativity condition realized in terms of quaternionicity of the tangent or normal space of the space-time surface guaranteeing the existence of quaternion units solve the problem and J could be identified as a representation of unit quaternion? In this case J would be replaced with vielbein vector and the decomposition 1+3 of the tangent space implied by the quaternion structure allows to use 3-dimensional permutation symbol to assign antisymmetric tensors to the vielbein vectors. Also the triviality of the tangent bundle of 3-D space allowing global choices of the 3 imaginary units could be essential.

3. Does associativity/co-associativity imply twistor space property or could it provide alternative manner to realize this notion? Or could one see quaternionic structure as an extension of almost complex structure. Instead of single J three orthogonal J: s (3 almost complex structures) are introduced and obey the multiplication table of quaternionic units? Instead of S^2 the fiber of the bundle would be $SO(3) = S^3$. This option is not attractive. A manifold with quaternionic tangent space with metric representing the real unit is known as quaternionic Riemann manifold and CP_2 with holonomy U(2) is example of it. A more restrictive condition is that all quaternion units define closed forms: one has quaternion Kähler manifold, which is Ricci flat and has in 4-D case Sp(1)=SU(2) holonomy. (see `http://www.encyclopediaofmath.org/index.php/Quaternionic_structure`).

4. Anti-self-dual property (ASD) of metric guaranteeing the integrability of almost complex structure of the twistor space implies the condition $\omega \wedge d\omega = 0$ for the twistor space. What does this condition mean physically for the twistor spaces associated with the extremals of Kähler action? For the 4-D base space this property is of course identically true. ASD property need of course not be realized.

3.3 Twistor Spaces As Analogs Of Calabi-Yau Spaces Of Super String Models

CP_3 is also a Calabi-Yau manifold in the strong sense that it allows Kähler structure and complex structure. Witten's twistor string theory considers 2-D (in real sense) complex surfaces in twistor space CP_3. This inspires some questions.

1. Could TGD in twistor space formulation be seen as a generalization of this theory?

2. General twistor space is not Calabi-Yau manifold because it does does not have Kähler structure. Do twistor spaces replace Calabi-Yaus in TGD framework?

3. Could twistor spaces be Calabi-Yau manifolds in some weaker sense so that one would have a closer connection with super string models.

Consider the last question.

1. One can indeed define non-Kähler Calabi-Yau manifolds by keeping the hermitian metric and giving up symplectic structure or by keeping the symplectic structure and giving up hermitian metric (almost complex structure is enough). Construction recipes for non-Kähler Calabi-Yau manifold are discussed in [A8]. It is shown that these two manners to give up Kähler structure correspond to duals under so called mirror symmetry [B3] which maps complex and symplectic structures to each other. This construction applies also to the twistor spaces.

2. For the modification giving up symplectic structure, one starts from a smooth Kähler Calabi-Yau 3-fold Y, such as CP_3. One assumes a discrete set of disjoint rational curves diffeomorphic to CP_1. In TGD framework work they would correspond to special fibers of twistor space.

   One has singularities in which some rational curves are contracted to point - in twistorial case the fiber of twistor space would contract to a point - this produces double point singularity which one can visualize as the vertex at which two cones meet (sundial should give an idea about what is involved). One deforms the singularity to a smooth complex manifold. One could interpret this as throwing away the common point and replacing it with connected sum contact: a tube connecting the holes drilled to the vertices of the two cones. In TGD one would talk about wormhole contact.

3. Suppose the topology looks locally like $S^3 \times S^2 \times R_\pm$ near the singularity, such that two copies analogous to the two halves of a cone (sundial) meet at single point defining double point singularity. In the recent case S^2 would correspond to the fiber of the twistor space. S^3 would correspond to 3-surface and $R_\pm$ would correspond to time coordinate in past/future direction. S^3 could be replaced with something else.

   The copies of $S^3 \times S^2$ contract to a point at the common end of R_+ and R_- so that both the based and fiber contracts to a point. Space-time surface would look like the pair of future and past directed light-cones meeting at their tips.

   For the first modification giving up symplectic structure only the fiber S^2 is contracted to a point and $S^2 \times D$ is therefore replaced with the smooth "bottom" of S^3. Instead of sundial one has two balls touching. Drill small holes two the two S^3s and connect them by connected sum contact (wormhole contact). Locally one obtains $S^3 \times S^3$ with k connected sum contacts.

   For the modification giving up Hermitian structure one contracts only S^3 to a point instead of S^2. In this case one has locally two CP_3: s touching (one can think that CP_n is obtained by replacing the points of C^n at infinity with the sphere CP_1). Again one drills holes and connects them by a connected sum contact to get k-connected sum of CP_3.

   For k CP_1s the outcome looks locally like to a k-connected sum of $S^3 \times S^3$ or CP_3 with $k \geq 2$. In the first case one loses symplectic structure and in the second case hermitian structure. The conjecture is that the two manifolds form a mirror pair.

   The general conjecture is that all Calabi-Yau manifolds are obtained using these two modifications. One can ask whether this conjecture could apply also the construction of twistor spaces representable as surfaces in $CP_3 \times F_3$ so that it would give mirror pairs of twistor spaces.

4. This smoothing out procedures isa actually unavoidable in TGD because twistor space is sub-manifold. The 6-D twistor spaces in 12-D $CP_3 \times F_3$ have in the generic case self intersections consisting of discrete points. Since the fibers CP_1 cannot intersect and since the intersection is point, it seems that the fibers must contract to a point. In the similar manner the 4-D base spaces should have local foliation by spheres or some other 3-D objects with contract to a point. One has just the situation described above.

   One can remove these singularities by drilling small holes around the shared point at the two sheets of the twistor space and connected the resulting boundaries by connected sum contact. The preservation of fiber structure might force to perform the process in such a manner that local modification of the topology contracts either the 3-D base (S^3 in previous example or fiber CP_1 to a point.

The interpretation of twistor spaces is of course totally different from the interpretation of Calabi-Yaus in superstring models. The landscape problem of superstring models is avoided and the multiverse of string models is replaced with generalized Feynman diagrams! Different twistor spaces correspond to different space-time surfaces and one can interpret them in terms of generalized Feynman diagrams since bundle projection gives the space-time picture. Mirror symmetry means that there are two different Calabi-Yaus giving the same physics. Also now twistor space for a given space-time surface can have several imbeddings - perhaps mirror pairs define this kind of imbeddings.

To sum up, the construction of space-times as surfaces of H lifted to those of (almost) complex sub-manifolds in $CP_3 \times F_3$ with induced twistor structure shares the spirit of the vision that induction procedure is the key element of classical and quantum TGD. It also gives deep connection with the mathematical methods applied in super string models and these methods should be of direct use in TGD.

3.4 Are Euclidian Regions Of Preferred Extremals Quaternion- Kähler Manifolds?

In blog comments Anonymous gave a link to an article about construction of 4-D quaternion-Kähler metrics with an isometry: they are determined by so called $SU(\infty)$ Toda equation. I tried to see whether quaternion-Kähler manifolds could be relevant for TGD.

From Wikipedia one can learn that QK is characterized by its holonomy, which is a subgroup of $Sp(n) \times Sp(1)$: $Sp(n)$ acts as linear symplectic transformations of $2n$-dimensional space (now real). In 4-D case tangent space contains 3-D sub-manifold identifiable as imaginary quaternions. CP_2 is one example of QK manifold for which the subgroup in question is $SU(2) \times U(1)$ and which has non-vanishing constant curvature: the components of Weyl tensor represent the quaternionic imaginary units. QKs are Einstein manifolds: Einstein tensor is proportional to metric.

What is really interesting from TGD point of view is that twistorial considerations show that one can assign to QK a special kind of twistor space (twistor space in the mildest sense requires only orientability). Wiki tells that if Ricci curvature is positive, this (6-D) twistor space is what is known as projective Fano manifold with a holomorphic contact structure. Fano variety has the nice property that as (complex) line bundle (the twistor space property) it has enough sections to define the imbedding of its base space to a projective variety. Fano variety is also complete: this is algebraic geometric analogy of topological property known as compactness.

3.4.1 *QK manifolds and twistorial formulation of TGD*

How the QKs could relate to the twistorial formulation of TGD?

1. In the twistor formulation of TGD [K17] the space-time surfaces are 4-D base spaces of 6-D twistor spaces in the Cartesian product of 6-D twistor spaces of M^4 and CP_2 - the only twistor spaces with Kähler structure. In TGD framework space-time regions can have either Euclidian or Minkowskian signature of induced metric. The lines of generalized Feynman diagrams are Euclidian.

2. Could the twistor spaces associated with the lines of generalized Feynman diagrams be projective Fano manifolds? Could QK structure characterize Euclidian regions of preferred extremals of Kähler action? Could a generalization to Minkowskian regions exist.

   I have proposed that so called Hamilton-Jacobi structure [K18] characterizes preferred extremals in Minkowskian regions. It could be the natural Minkowskian counterpart for the quaternion Kähler structure, which involves only imaginary quaternions and could make sense also in Minkowski signature. Note that unit sphere of imaginary quaternions defines the sphere serving as fiber of the twistor bundle.

Why it would be natural to have QK that is corresponding twistor space, which is projective contact Fano manifold?

1. QK property looks very strong condition but might be true for the preferred extremals satisfying very strong conditions stating that the classical conformal charges associated with various conformal algebras extending the conformal algebras of string models [K18], [?]. These conditions would be essentially classical gauge conditions stating that strong form of holography implies by strong form of General Coordinate Invariance (GCI) is realized: that is partonic 2-surfaces and their 4-D tangent space data code for quantum physics.

2. Kähler property makes sense for space-time regions of Euclidian signature and would be natural is these regions can be regarded as small deformations of CP_2 type vacuum extremals with light-like M^4 projection and having the same metric and Kähler form as CP_2 itself.

3. Fano property implies that the 4-D Euclidian space-time region representing line of the Feynman diagram can be imbedded as a sub-manifold to complex projective space CP_n. This would allow to use the powerful machinery of projective geometry in TGD framework. This could also be a space-time correlate for the fact that CP_ns emerge in twistor Grassmann approach expected to generalize to TGD framework.

4. CP_2 allows both projective (trivially) and contact (even symplectic) structures. $\delta M^4_+ \times CP_2$ allows contact structure - I call it loosely symplectic structure. Also 3-D light-like orbits of partonic 2-surfaces allow contact structure. Therefore holomorphic contact structure for the twistor space is natural.

5. Both the holomorphic contact structure and projectivity of CP_2 would be inherited if QK property is true. Contact structures at orbits of partonic 2-surfaces would extend to holomorphic contact structures in the Euclidian regions of space-time surface representing lines of generalized Feynman diagrams. Projectivity of Fano space would be also inherited from CP_2 or its twistor space $SU(3)/U(1) \times U(1)$ (flag manifold identifiable as the space of choices for quantization axes of color isospin and hypercharge).

The article considers a situation in which the QK manifold allows an isometry. Could the isometry (or possibly isometries) for QK be seen as a remnant of color symmetry or rotational symmetries of M^4 factor of imbedding space? The only remnant of color symmetry at the level of imbedding space spinors is anomalous color hyper charge (color is like orbital angular momentum and associated with spinor harmonic in CP_2 center of mass degrees of freedom). Could the isometry correspond to anomalous hypercharge?

3.4.2 *How to choose the quaternionic imaginary units for the space-time surface?*

Parallellizability is a very special property of 3-manifolds allowing to choose quaternionic imaginary units: global choice of one of them gives rise to twistor structure.

1. The selection of time coordinate defines a slicing of space-time surface by 3-surfaces. GCI would suggest that a generic slicing gives rise to 3 quaternionic units at each point each 3-surface? The parallelizability of 3-manifolds - a unique property of 3-manifolds - means the possibility to select global coordinate frame as section of the frame bundle: one has 3 sections of tangent bundle whose inner products give rose to the components of the metric (now induced metric) guarantees this. The tri-bein or its dual defined by two-forms obtained by contracting tri-bein vectors with permutation tensor gives the quanternionic imaginary units. The construction depends on 3-metric only and could be carried out also in GRT context. Note however that topology change for 3-manifold might cause some non-trivialities. The metric 2-dimensionality at the light-like orbits of partonic 2-surfaces should not be a problem for a slicing by space-like 3-surfaces. The construction makes sense also for the regions of Minkowskian signature.

2. In fact, any 4-manifold [A10] allows almost quaternionic as the above slicing argument relying on parallelizibility of 3-manifolds strongly suggests.

3. In zero energy ntology (ZEO)- a purely TGD based feature - there are very natural special slicings. The first one is by linear time-like Minkowski coordinate defined by the direction of the line connecting the tips of the causal diamond (CD). Second one is defined by the light-cone proper time associated with either light-cone in the intersection of future and past directed light-cones defining CD. Neither slicing is global as it is easy to see.

3.4.3 *The relationship to quaternionicity conjecture and $M^8 - H$ duality*

One of the basic conjectures of TGD is that preferred extremals consist of quaternionic/ co-quaternionic (associative/co-associative) regions [K16]. Second closely related conjecture is $M^8 - H$ duality allowing to map quaternionic/co-quaternionic surfaces of M^8 to those of $M^4 \times CP_2$. Are these conjectures consistent with QK in Euclidian regions and Hamilton-Jacobi property in Minkowskian regions? Consider first the definition of quaternionic and co-quaternionic space-time regions.

1. Quaternionic/associative space-time region (with Minkowskian signature) is defined in terms of induced octonion structure obtained by projecting octonion units defined by vielbein of $H = M^4 \times CP_2$ to space-time surface and demanding that the 4 projections generate quaternionic sub-algebra at each point of space-time.

If there is also unique complex sub-algebra associated with each point of space-time, one obtains one can assign to the tangent space-of space-time surface a point of CP_2. This allows to realize M^8-H duality [K16] as the number theoretic analog of spontaneous compactification (but involving no compactification) by assigning to a point of $M^4 = M^4 \times CP_2$ a point of $M^4 \times CP_2$. If the image surface is also quaternionic, this assignment makes sense also for space-time surfaces in H so that $M^8 - H$ duality generalizes to $H - H$duality allowing to assign to given preferred extremal a hierarchy of extremals by iterating this assignment. One obtains a category with morphisms identifiable as these duality maps.

2. Co-quaternionic/co-associative structure is conjectured for space-time regions of Euclidian signature and 4-D CP_2 projection. In this case normal space of space-time surface is quaternionic/associative. A multiplication of the basis by preferred unit of basis gives rise to a quaternionic tangent space basis so that one can speak of quaternionic structure also in this case.

3. Quaternionicity in this sense requires unique identification of a preferred time coordinate as imbedding space coordinate and corresponding slicing by 3-surfaces and is possible only in TGD context. The preferred time direction would correspond to real quaternionic unit. Preferred time coordinate implies that quaternionic structure in TGD sense is more specific than the QK structure in Euclidian regions.

4. The basis of induced octonionic imaginary unit allows to identify quaternionic imaginary units linearly related to the corresponding units defined by tri-bein vectors. Note that the multiplication of octonionic units is replaced with multiplication of antisymmetric tensors representing them when one assigns to the quaternionic structure potential QK structure. Quaternionic structure does not require Kähler structure and makes sense for both signatures of the induced metric. Hence a consistency with QK and its possible analog in Minkowskian regions is possible.

5. The selection of the preferred imaginary quaternion unit is necessary for $M^8 - H$ correspondence. This selection would also define the twistor structure. For quaternion-Kähler manifold this unit would be covariantly constant and define Kähler form - maybe as the induced Kähler form.

6. Also in Minkowskian regions twistor structure requires a selection of a preferred imaginary quaternion unit. Could the induced Kähler form define the preferred imaginary unit also now? Is the Hamilton-Jacobi structure consistent with this?

   Hamilton-Jacobi structure involves a selection of 2-D complex plane at each point of space-time surface. Could induced Kähler magnetic form for each 3-slice define this plane? It is not necessary to require that 3-D Kähler form is covariantly constant for Minkowskian regions. Indeed, massless extremals representing analogs of photons are characterized by local polarization and momentum direction and carry time-dependent Kähler-electric and -magnetic fields. One can however ask whether monopole flux tubes carry covariantly constant Kähler magnetic field: they are indeed deformations of what I call cosmic strings [K3, K7] for which this condition holds true?

3.5 Could Quaternion Analyticity Make Sense For The Preferred Extremals?

The 4-D generalization of conformal invariance suggests strongly that the notion of analytic function generalizes somehow. The obvious ideas coming in mind are appropriately defined quaternionic and octonion analyticity. I have used a considerable amount of time to consider these possibilities but had to give up the idea about octonion analyticity could somehow allow to preferred extemals.

3.5.1 *Basic idea*

One can argue that quaternion analyticity is the more natural option in the sense that the local octonionic imbedding space coordinate (or at least M^8 or E^8 coordinate, which is enough if M^8-H duality holds true) would for preferred extremals be expressible in the form

$$o(q) = u(q) + v(q) \times I \ . \tag{3.1}$$

Here q is quaternion serving as a coordinate of a quaternionic sub-space of octonions, and I is octonion unit belonging to the complement of the quaternionic sub-space, and multiplies $v(q)$ from *right* so that quaternions and qiaternionic differential operators acting from left do not notice these coefficients at all. A stronger condition would be that the coefficients are real. $u(q)$ and $v(q)$ would be quaternionic Taylor- of even Laurent series with coefficients multiplying powers of q from right for the same reason.

The signature of M^4 metric is a problem. I have proposed complexification of M^8 and M^4 to get rid of the problem by assuming that the imbedding space corresponds to surfaces in the space M^8 identified as octonions of form $o_8 = Re(o) + iIm(o)$, where o is imaginary part of ordinary octonion and i is commuting imaginary unit. M^4 would correspond to quaternions of form $q_4 = Re(q) + iIm(q)$. What is important is that powers of q_4 and o_8 belong to this sub-space (as follows from the vanishing of cross product term in the square of octonion/quaternion) so that powers of q_4 (o_8) has imaginary part proportional to $Im(q)$ ($Im(o)$)

I ended up to reconsider the idea of quaternion analyticity after having found two very interesting articles discussing the generalization of Cauchy-Riemann equations. The first article [A10] was about so called triholomorphic maps between 4-D almost quaternionic manifolds. The article gave as a reference an article [A7] about quaternionic analogs of Cauchy-Riemann conditions discussed by Fueter long ago (somehow I have managed to miss Fueter's work just like I missed Hitchin's work about twistorial uniqueness of M^4 and CP_2), and also a new linear variant of these conditions, which seems especially interesting from TGD point of view as will be found.

3.5.2 *The first form of Cauchy-Riemann-Fueter conditions*

Cauhy-Riemann-Fueter (CRF) conditions generalize Cauchy-Riemann conditions. These conditions are however not unique. Consider first the translationally invariant form of CRF conditions.

1. The translationally invariant form of CRF conditions is $\partial_{\overline{q}} f = 0$ or explicitly

$$\partial_{\overline{q}} f = (\partial_t - \partial_x I - \partial_y J - \partial_z K) f = 0 \ . \tag{3.2}$$

   This form does not allow quaternionic Taylor series. Note that the Taylor coefficients multiplying powers of the coordinate from right are arbitrary quaternions. What looks pathological is that even linear functions of q fail be solve this condition. What is however interesting that in flat space the equation is equivalent with Dirac equation for a pair of Majorana spinors [A10].

2. The condition allows functions depending on complex coordinate z of some complex-plane only. It also allows functions satisfying two separate analyticity conditions, say

$$\begin{aligned} \partial_{\overline{u}} f &= (\partial_t - \partial_x I) f = 0 \ , \\ \partial_{\overline{v}} f &= -(\partial_y J + \partial_z K) f = -J(\partial_y - \partial_z I) f = 0 \ . \end{aligned} \tag{3.3}$$

   In the latter formula J multiplies from *left*! One has good hopes of obtaining holomorphic functions of two complex coordinates. This might be enough to understand the preferred extremals of Kähler action as quaternion analytic mops.

   There are potential problems due to non-commutativity of $u = t \pm xI$ and $v = yJ \pm zK = (y \pm zI)J$ (note that J multiplies from *right*!) and ∂_u and ∂_v. A prescription for the ordering of the powers u and v in the polynomials of u and v appearing in the double Taylor series seems to be needed. For instance, powers of u can be taken to be at left and v or of a related variable at right.

By the linearity of $\partial_{\overline{v}}$ one can leave $J$to the left and commute only $(\partial_y - \partial_z I)$ through the u-dependent part of the series: this operation is trivial. The condition $\partial_v f = 0$ is satisfied if the polynomials of y and z are polynomials of $y+iz$ multiplied by J from right. The solution ansatz is thus product of Taylor series of monomials $f_{mn} = (x+iy)^m(y+iz)^n J$ with Taylor coefficients a_{mn}, which multiply the monomials from right and are arbitrary quaternions. Note that the monomials $(y+iz)^n$ do not reduce to polynomials of v and that the ordering of these powers is arbitrary. If the coefficients a_{mn} are real f maps 4-D quaternionic region to 2-D region spanned by J and K. Otherwise the image is 4-D.

3. By linearity the solutions obey linear superposition. They can be also multiplied if product is defined as ordered product in such a manner that only the powers $t+ix$ and $y+iz$ are multiplied together at left and coefficients a_{mn} are multiplied together at right. The analogy with quantum non-commutativity is obvious.

4. In Minkowskian signature one must multiply imaginary units I, J, K with an additional commuting imaginary unit i. This would give solutions as powers of (say) $t+ex$, $e = iI$ with $e^2 = 1$ representing imaginary unit of hyper-complex numbers. The natural interpretation would be as algebraic extension which is analogous to the extension of rational number by adding algebraic number, say $\sqrt{2}$ to get algebraically 2-dimensional structure but as real numbers 1-D structure. Only the non-commutativity with J and K distinguishes e from $e = \pm 1$ and if J and K do not appear in the function, one can replace e by ± 1 in $t+ex$ to get just $t \pm x$ appearing as argument for waves propagating with light velocity.

3.5.3 *Second form of CRF conditions*

Second form of CRF conditions proposed in [A7] is tailored in order to realize the almost obvious manner to realize quaternion analyticity.

1. The ingenious idea is to replace preferred quaternionic imaginary unit by a imaginary unit which is in radial direction: $e_r = (xI + yJ + zK)/r$ and require analyticity with respect to the coordinate $t+er$. The solution to the condition is power series in $t + re_r = q$ so that one obtains quaternion analyticity.

2. The excplicit form of the conditions is

$$(\partial_t - e_r\partial_r)f = (\partial_t - \frac{e_r}{r} r\partial_r)f = 0 \quad . \tag{3.4}$$

This form allows both the desired quaternionic Taylor series and ordinary holomorphic functions of complex variable in one of the 3 complex coordinate planes as general solutions.

3. This form of CRF is neither Lorentz invariant nor translationally invariant but remains invariant under simultaneous scalings of t and r and under time translations. Under rotations of either coordinates or of imaginary units the spatial part transforms like vector so that quaternionic automorphism group $SO(3)$ serves as a moduli space for these operators.

4. The interpretation of the latter solutions inspired by ZEO would be that in Minkowskian regions r corresponds to the light-like radial coordinate of the either boundary of CD, which is part of $\delta M^4_\pm$. The radial scaling operator is that assigned with the light-like radial coordinate of the light-cone boundary. A slicing of CD by surfaces parallel to the $\delta M^4_\pm$ is assumed and implies that the line $r = 0$ connecting the tips of CD is in a special role. The line connecting the tips of CD defines coordinate line of time coordinate. The breaking of rotational invariance corresponds to the selection of a preferred quaternion unit defining the twistor structure and preferred complex sub-space.

In regions of Euclidian signature r could correspond to the radial Eguchi-Hanson coordinate of CP_2 and $r = 0$ corresponds to a fixed point of $U(2)$ subgroup under which CP_2 complex coordinates transform linearly.

5. Also in this case one can ask whether solutions depending on two complex local coordinates analogous to those for translationally invariant CRF condition are possible. The remain imaginary units would be associated with the surface of sphere allowing complex structure.

3.5.4 *Generalization of CRF conditions?*

Could the proposed forms of CRF conditions be special cases of much more general CRF conditions as CR conditions are?

1. Ordinary complex analysis suggests that there is an infinite number of choices of the quaternionic coordinates related by the above described quaternion-analytic maps with 4-D images. The form of of the CRF conditions would be different in each of these coordinate systems and would be obtained in a straightforward manner by chain rule.

2. One expects the existence of large number of different quaternion-conformal structures not related by quaternion-analytic transformations analogous to those allowed by higher genus Riemann surfaces and that these conformal equivalence classes of four-manifolds are characterized by a moduli space and the analogs of Teichmueller parameters depending on 3-topology. In TGD framework strong form of holography suggests that these conformal equivalence classes for preferred extremals could reduce to ordinary conformal classes for the partonic 2-surfaces. An attractive possibility is that by conformal gauge symmetries the functional integral over WCW reduces to the integral over the conformal equivalence classes.

3. The quaternion-conformal structures could be characterized by a standard choice of quaternionic coordinates reducing to the choice of a pair of complex coordinates. In these coordinates the general solution to quaternion-analyticity conditions would be of form described for the linear ansatz. The moduli space corresponds to that for complex or hyper-complex structures defined in the space-time region.

3.5.5 *Geometric formulation of the CRF conditions*

The previous naive generalization of CRF conditions treats imaginary units without trying to understand their geometric content. This leads to difficulties when when tries to formulate these conditions for maps between quaternionic and hyper-quaternionic spaces using purely algebraic representation of imaginary units since it is not clear how these units relate to each other.

In [A10] the CRF conditions are formulated in terms of the antisymmetric (1, 1) type tensors representing the imaginary units: they exist for almost quaternionic structure and presumably also for almost hyper-quaternionic structure needed in Minkowskian signature.

The generalization of CRF conditions is proposed in terms of the Jacobian J of the map mapping tangent space TM to TN and antisymmetric tensors J_u and J_u representing the quaternionic imaginary units of N and M. The generalization of CRF conditions reads as

$$J - \sum_u J_u \circ J \circ j_u = 0 \ . \tag{3.5}$$

For $N = M$ it reduces to the translationally invariant algebraic form of the conditions discussed above. These conditions seem to be well-defined also when one maps quaternionic to hyper-quaternionic space or vice versa. These conditions are not unique. One can perform an SO(3) rotation (quaternion automorphism) of the imaginary units mediated by matrix Λ^{uv} to obtain

$$J - \Lambda^{uv} J_u \circ J \circ j_v = 0 \ . \tag{3.6}$$

The matrix Λ can depend on point so that one has a kind of gauge symmetry. The most general triholomorphic map allows the presence of Λ Note that these conditions make sense on any coordinates and complex analytic maps generate new forms of these conditions.

Covariant forms of structure constant tensors reduce to octonionic structure constants and this allows to write the conditions explicitly. The index raising of the second index of the structure constants is however needed using the metrics of M and N. This complicates the situation and spoils linearity: in particular, for surfaces induced metric is needed. Whether local SO(3) rotation can eliminate the dependence on induced metric is an interesting question. Since M^4 imaginary units differ only by multiplication by i, Minkowskian structure constants differ only by sign from the Euclidian ones.

In the octonionic case the geometric generalization of CRF conditions does not seem to make sense. By non-associativity of octonion product it is not possible to have a matrix representation for the matrices so that a faithful representation of octonionic imaginary units as antisymmetric 1-1 forms does not make sense. If this representation exists it it must map octonionic associators to zero. Note however that CRF conditions do not involve products of three octonion units so that they make sense as algebraic conditions at least.

3.5.6 *Does residue calculus generalize?*

CRF conditions allow to generalize Cauchy formula allowing to express value of analytic function in terms of its boundary values [A10]. This would give a concrete realization of the holography in the sense that the physical variables in the interior could be expressed in terms of the data at the light-like partonic orbits and and the ends of the space-time surface. Triholomorphic function satisfies d'Alembert/Laplace equations - in induced metric in TGD framework- so that the maximum modulus principle holds true. The general ansatz for a preferred extremals involving Hamilton-Jacobi structure leads to d'Alembert type equations for preferred extremals [K18].

Could the analog of residue calculus exist? Line integral would become 3-D integral reducing to a sum over poles and possible cuts inside the 3-D contour. The space-like 3-surfaces at the ends of CDs could define natural integration contours, and the freedom to choose contour rather freely would reflect General Coordinate Invariance. A possible choice for the integration contour would be the closed 3-surface defined by the union of space-like surfaces at the ends of CD and by the light-like partonic orbits.

Poles and cuts would be in the interior of the space-time surface. Poles have co-dimension 2 and cuts co-dimension 1. Strong form of holography suggests that partonic 2-surfaces and perhaps also string world sheets serve as candidates for poles. Light-like 3-surfaces (partonic orbits) defining the boundaries between Euclidian and Minkowskian regions are singular objects and could serve as cuts. The discontinuity would be due to the change of the signature of the induced metric. There are CDs inside CDs and one can also consider the possibility that sub-CDs define cuts, which in turn reduce to cuts associated with sub-CDs.

3.5.7 *Could one understand the preferred extremals in terms of quaternion- analyticity?*

Could one understand the preferred extremals in terms of quaternion-analyticity or its possible generalization to an analytic representation for co-quaternionicity expected in space-time regions with Euclidian signature? What is the generalization of the CRF conditions for the counterparts of quaternion-analytic maps from hyper-quaternionic X^4 to quaternionic CP_2 and from quaternionic X^4 to hyper-quaternionic M^4? It has already become clear that this problem can be probably solved by using the the geometric representation for quaternionic imaginary units.

The best thing to do is to look whether this is possible for the known extremals: CP_2 type vacuum extremals, vacuum extremals expressible as graph of map from M^4 to a Lagrangian sub-manifold of CP_2, cosmic strings of form $X^2 \times Y^2 \subset M^4 \times CP_2$ such that X^2 is string world sheet (minimal surface) and Y^2 complex sub-manifold of CP_2. One can also check whether Hamilton-Jacobi structure of M^4 and of Minkowskian space-time regions and complex structure of CP_2 have natural counterparts in the quaternion-analytic framework.

1. Consider first cosmic strings. In this case the quaternionic-analytic map from $X^4 = X^2 \times Y^2$ to $M^4 \times CP_2$ with octonion structure would be map X^4 to 2-D string world sheet in M^2 and

Y^2 to 2-D complex manifold of CP_2. This could be achieved by using the linear variant of CRF condition. The map from X^4 to M^4 would reduce to ordinary hyper-analytic map from X^2 with hyper-complex coordinate to M^4 with hyper-complex coordinates just as in string models. The map from X^4 to CP_2 would reduce to an ordinary analytic map from X^2 with complex coordinates. One would not leave the realm of string models.

2. For the simplest massless extremals (MEs) CP_2 coordinates are arbitrary functions of light-like coordinate $u = k \cdot m$, k constant light-like vector, and of $v = \epsilon \cdot m$, ϵ a polarization vector orthogonal to k. The interpretation as classical counterpart of photon or Bose-Einstein condense of photons is obvious. There are good reasons to expect that this ansatz generalizes by replacing the variables u and v with coordinate along the light-like and space-like coordinate lines of Hamilton-Jacobi structure. The non-geodesic motion of photons with light-velocity and variation of the polarization direction would be due to interactions with the space-time sheet to which it is topologically condensed. Note that light-likeness condition for the coordinate curve gives rise to Virasoro conditions. This observation led long time ago to the idea that 2-D conformal invariance must have a non-trivial generalization to 4-D case.

   Now space-time surface would have naturally M^4 coordinates and the map $M^4 \rightarrow M^4$ would be just identity map satisfying the radial CRF condition. Can one understand CP_2 coordinates in terms of quaternion- analyticity? The dependence of CP_2 coordinates on $u = t - x$ only can be formulated as CFR condition $\partial_{\overline{u}} s^k = 0$ and this could is expected to generalize in the formulation using the geometric representation of quaternionic imaginary units at both sides. The dependence on light-light coordinate u follows from the translationally invariant CRF condition.

   The dependence on the real coordinate v is however problematic since the dependence is naturally on complex coordinate w assignable to the polarization plane of form $z = f(w)$. This would give dependence on 2 transversal coordinates and CP_2 projection would be 3-D rather than 2-D. One can of course ask whether this dependence is actually present for preferred extremals? Could the polarization vector be complex local polarization vector orthogonal to the light-like vector? In quantum theory complex polarization vectors are used of routinely and become oscillator operators in second quantization and in TGD Universe MEs indeed serve as space-time correlates for photons or their BE condensates.

3. Vacuum extremals with Lagrangian manifold as (in the generic case 2-D) CP_2 projection are not expected to be preferred extremals for obvious reasons. One one can however try similar approach. Hyper-quaternionic structure for space-time surface using Hamilton-Jacobi structure is the first guess. CP_2 should allow a quaternionic coordinate decomposing to a pair of complex coordinates such that second complex coordinate is constant for 2-D Lagrangian manifold and second parameterizes it. Any 2-D surface allows complex structure defined by the induced metric so that there are good hopes that these coordinates exist. The quaternion-analytic map would map in the most general case is trivial for both hypercomplex and complex coordinate of M^4 but the quaternionic Taylor coefficients reduce to real numbers to that the image is 2-D.

4. For CP_2 type vacuum extremals the M^4 projection is random light-like curve. Now one expects co-quaternionicity and that quaternion-analyticity is not the correct manner to formulate the situation. "Co-" suggests that instead of expressing surface as graph one should perhaps express it in terms of conditions stating that some quaternionic analytic functions in H are vanish.

   One can fix the coordinates of X^4 to be complex coordinates of CP_2 so that one gets rid of the degeneracy due to the choice of coordinates. M^4 allows hyper-quaternionic coordinates and Hamilton-Jacobi structures define different choices of hyper-quaternionic coordinates. Now the second light- like coordinate would vary along random light-like curves providing slicing of M^4 by 3-D surfaces. Hamilton-Jacobi structure defines at each point a plane $M^2(x)$ fixed by the light-like vector at the point and the 2-D orthogonal plane. In fact 4-D coordinate grid is defined. This local choice must be integrable, which means that one has slicing by 2-D string world sheets and polarization planes orthogonal to them.

The problem is that the mapping of quaternionic CP_2 coordinate to hyper-quaternionic coordinates of M^4 (say $v = 0, w = 0$) in terms of quaternionic analyticity is not easy. "Co-" suggets that, one could formulate light-likeness condition using Hamilton-Jacobi structure as conditions $\overline{w} - constant = 0$ and $v - constant = 0$. Note that one has $\overline{u} = v$.

5. In the naive generalization CRF conditions are linear. Whether this is the case in the formulation using the geometric representation of the imaginary units is not clear since the quaternionic imaginary units depend on the vielbein of the induced 3-metric (note however that the SO(3) gauge rotation appearing in the conditions could allow to compensate for the change of the tensors in small deformations of the spaced-time surface). If linearity is real and not true only for small perturbations, one could have linear superpositions of different types of solutions, which looks strange. Could the superpositions describe perturbations of say cosmic strings and massless extremals?

6. According to [A7] both forms of the algebraic C-R-F conditions generalize to the octonionic situation and right multiplication of powers of octonion by Taylor coefficients plus linearity imply that there are no problems with associativity. This inspires several questions.

   Could octonion analytic maps of imbedding space allow to construct new solutions from the existing ones? Could quaternion analytic maps applied at space-time level act as analogs of holomorphic maps and generalize conformal gauge invariance to 4-D context?

3.5.8 *Do isometry currents of preferred extremals satisfy Frobenius integrability conditions?*

During the preparation of the book I learned that Agostino Prastaro [A3, A4] has done highly interesting work with partial differential equations, also those assignable to geometric variational principles such as Kähler action in TGD. I do not understand the mathematical details but the key idea is a simple and elegant generalization of Thom's cobordism theory, and it is difficult to avoid the idea that the application of Prastaro's idea might provide insights about the preferred extremals, whose identification is now on rather firm basis.

One could also consider a definition of what one might call dynamical homotopy groups as a genuine characteristics of WCW topology. The first prediction is that the values of conserved classical Noether charges correspond to disjoint components of WCW. Could the natural topology in the parameter space of Noether charges zero modes of WCW metric) be p-adic and realize adelic physics at the level of WCW? An analogous conjecture was made on basis of spin glass analogy long time ago. Second surprise is that the only the 6 lowest dynamical homotopy/homology groups of WCW would be non-trivial. The Kähler structure of WCW suggests that only Π_0, Π_2, and Π_4 are non-trivial.

The interpretation of the analog of Π_1 as deformations of generalized Feynman diagrams with elementary cobordism snipping away a loop as a move leaving scattering amplitude invariant conforms with the number theoretic vision about scattering amplitude as a representation for a sequence of algebraic operation can be always reduced to a tree diagram. TGD would be indeed topological QFT: only the dynamical topology would matter.

A further outcome is an ansatz for generalizing the earlier proposal for preferred extremals and stating that non-vanishing conserved isometry currents satisfy Frobenius integrability conditions so that one could assign global coordinate with their flow lines. This ansatz looks very similar to the Cauchy-Riemann-Fuerter conditions stating quaternion analyticity [?].

3.5.9 *Conclusions*

To sum up, connections between different conjectures related to the preferred extremals - $M^8 - H$ duality, Hamilton-Jacobi structure, induced twistor space structure, quaternion-Kähler property and its Minkowskian counterpart, and even quaternion analyticity, are clearly emerging. The underlying reason is strong form of GCI forced by the construction of WCW geometry and implying strong from of holography posing extremely powerful quantization conditions on the extremals of Kähler action in ZEO. Without the conformal gauge conditions the mutual inconsistency of these conjectures looks rather infeasible.

4 Witten's Twistor String Approach And TGD

The twistor Grassmann approach has led to a phenomenal progress in the understanding of the scattering amplitudes of gauge theories, in particular the $\mathcal{N} = 4$ SUSY.

As a non-specialist I have been frustrated about the lack of concrete picture, which would help to see how twistorial amplitudes might generalize to TGD framework. A pleasant surprise in this respect was the proposal of a particle interpretation for the twistor amplitudes by Nima Arkani Hamed et al in the article "Unification of Residues and Grassmannian Dualities" [B12] (see `http://arxiv.org/pdf/0912.4912.pdf`) // In this interpretation incoming particles correspond to spheres CP_1 so that n-particle state corresponds to $(CP_1)^n/Gl(2)$ (the modding by $Gl(2)$ might be seen as a kind of formal generalization of particle identity by replacing permutation group S_2 with $Gl(2)$ of 2×2 matrices). If the number of "wrong" helicities in twistor diagram is k, this space is imbedded to $CP^n_{k-1}/Gl(k)$ as a surface having degree $k-1$ using Veronese map to achieve the imbedding. The imbedding space can be identified as Grassmannian $G(k,n)$. This surface defines the locus of the multiple residue integral defining the twistorial amplitude.

The particle interpretation brings in mind the extension of single particle configuration space E^3 to its Cartesian power E^{3n}/S_n for n-particle system in wave mechanics. This description could make sense when point-like particle is replaced with 3-surface or partonic 2-surface: one would have Cartesian product of WCWs divided my S_n. The generalization might be an excellent idea as far calculations are considered but is not in spirit with the very idea of string models and TGD that many-particle states correspond to unions of 3-surfaces in H (or light-like boundaries of causal diamond (CD) in Zero Energy Ontology (ZEO).

Witten's twistor string theory [B8] is more in spirit with TGD at fundamental level since it is based on the identification of generalization of vertices as 2-surfaces in twistor space.

1. There are several kinds of twistors involved. For massless external particles in eigenstates of momentum and helicity null twistors code the momentum and helicity and are pairs of 2-spinor and its conjugate. More general momenta correspond to two independent 2-spinors.

   One can perform twistor Fourier transform for the conjugate 2-spinor to obtain twistors coding for the points of compactified Minkowski space. Wave functions in this twistor space characterized by massless momentum and helicity appear in the construction of twistor amplitudes. BCFW recursion relation [B4] allows to construct more complex amplitudes assuming that intermediate states are on mass shells massless states with complex momenta.

   One can perform twistor Fourier transformation (there are some technical problems in Minkowski signature) also for the second 2-spinor to get what are called momentum twistors providing in some aspects simpler description of twistor amplitudes. These code for the four-momenta propagating between vertices at which the incoming particles arrive and the differences if two subsequent momenta are equal to massless external momenta.

2. In Witten's theory the interactions of incoming particles correspond to amplitudes in which the twistors appearing as arguments of the twistor space wave functions characterized by momentum and helicity are localized to complex curves X^2 of twistor space CP_3 or its Minkowskian counterpart. This can be seen as a non-local twistor space variant of local interactions in Minkowski space.

   The surfaces X^2 are characterized by their degree d (of the polynomial of complex coordinates defining the algebraic 2-surface) the genus g of the algebraic surface, by the number k of "wrong" (helicity violating) helicities, and by the number of loops of corresponding diagram of SUSY amplitude: one has $d = k - 1 + l$, $g \leq l$. The interaction vertex in twistor space is not anymore completely local but the n particles are at points of the twistorial surface X^2.

In the following a proposal generalizing Witten's approach to TGD is discussed.

1. The fundamental challenge is the generalization of the notion of twistor associated with massless particle to 8-D context, first for $M^4 = M^4 \times E^4$ and then for $H = M^4 \times CP_2$. The notion of twistor space solves this question at geometric level. As far as construction of the TGD variant of Witten's twistor string is considered, this might be quite enough.

2. $M^8 - H$ duality and quantum-classical correspondence however suggest that M^8 twistors might allow tangent space description of four-momentum, spin, color quantum numbers and electroweak numbers and that this is needed. What comes in mind is the identification of fermion lines as light-like geodesics possessing M^8 valued 8-momentum, which would define the long sought gravitational counterparts of four-momentum and color quantum numbers at classical point-particle level. The M^8 part of this four-momentum would be equal to that associated with imbedding space spinor mode characterizing the ground state of super-conformal representation for fundamental fermion.

   Hence one might also think of starting from the 4-D condition relating Minkowski coordinates to twistors and looking what it could mean in the case of M^8. The generalization is indeed possible in $M^8 = M^4 \times E^4$ by its flatness if one replaces gamma matrices with octonionic gamma matrices.

   In the case of $M^4 \times CP_2$ situation is different since for octonionic gamma matrices $SO(1,7)$ is replaced with G_2, and the induced gauge fields have different holonomy structure than for ordinary gamma matrices and octonionic sigma matrides appearing as charge matrices bring in also an additional source of non-associativity. Certainly the notion of the twistor Fourier transform fails since CP_2 Dirac operator cannot be algebraized.

   Algebraic twistorialization however works for the light-like fermion lines at which the ordinary and octonionic representations for the induced Dirac operator are equivalent. One can indeed assign 8-D counterpart of twistor to the particle classically as a representation of light-like hyper-octonionic four-momentum having massive M^4 and CP_2 projections and CP_2 part perhaps having interpretation in terms of classical tangent space representation for color and electroweak quantum numbers at fermionic lines.

   If all induced electroweak gauge fields - rather than only charged ones as assumed hitherto - vanish at string world sheets, the octonionic representation is equivalent with the ordinary one. The CP_2 projection of string world sheet should be 1-dimensional: inside CP_2 type vacuum extremals this is impossible, and one could even consider the possibility that the projection corresponds to CP_2 geodesic circle. This would be enormous technical simplification. What is important that this would not prevent obtaining non-trivial scattering amplitudes at elementary particle level since vertices would correspond to re-arrangement of fermion lines between the generalized lines of Feynman diagram meeting at the vertices (partonic 2-surfaces).

3. In the fermionic sector one is forced to reconsider the notion of the induced spinor field. The modes of the imbedding space spinor field should co-incide in some region of the space-time surface with those of the induced spinor fields. The light-like fermionic lines defined by the boundaries of string world sheets or their ends are the obvious candidates in this respect. String world sheets is perhaps too much to require.

   The only reasonable identification of string world sheet gamma matrices is as induced gamma matrices and super-conformal symmetry requires that the action contains string world sheet area as an additional term just as in string models. String tension would correspond to gravitational constant and its value - that is ratio to the CP_2 radius squared, would be fixed by quantum criticality.

4. The generalization of the Witten's geometric construction of scattering amplitudes relying on the induction of the twistor structure of the imbedding space to that associated with space-time surface looks surprisingly straight-forward and would provide more precise formulation of the notion of generalized Feynman diagrams forcing to correct some wrong details. One of the nice outcomes is that the genus appearing in Witten's formulation naturally corresponds to family replication in TGD framework.

4.1 Basic Ideas About Twistorialization Of TGD

The recent advances in understanding of TGD motive the attempt to look again for how twistor amplitudes could be realized in TGD framework. There have been several highly non-trivial steps of progress leading to a new more profound understanding of basic TGD.

1. $M^4 \times CP_2$ is twistorially unique [K17] in the sense that its factors are the only 4-D geometries allowing twistor space with Kähler structure (M^4 corresponds to S^4 in Euclidian signature) [A6]. The twistor spaces in question are CP_3 for S^4 and its Minkowskian variant for M^4 (I will use P^3 as short hand for both twistor spaces) and the flag manifold $F = SU(3)/U(1) \times U(1)$ parametrizing the choices of quantization axes for color group $SU(3)$ in the case of CP_2. Recall that twistor spaces are S^2 bundles over the base space and that all orientable four-manifolds have twistor space in this sense. Note that space-time surfaces allow always almost quaternionic structure and that that preferred extremals are suggested to be quaternionic [K17].

2. The light-likeness condition for twistors in M^4 is fundamental in the ordinary twistor approach. In 8-D context light-likeness holds in generalized sense for the spinor harmonics of H: the square of the Dirac operator annihilates spinor modes. In the case M^8 one can indeed define twistors by generalizing the standard approach by replacing ordinary gamma matrices with octonionic ones [**?**] For H octonionic and ordinary gamma matrices are equivalent at the fermionic lines defined by the light-like boundaries of string world sheets and at string world sheets if they carry vanishing induced electro-weak gauge fields that is have 1-D CP_2 projection.

3. Twistor spaces emerge in TGD framework as lifts of space-time surfaces to corresponding twistor spaces realized as 6-surfaces in the lift of $M^4 \times CP_2$ to $T(H) = P^3 \times F$ having as base spaces space-time surfaces. This implies that that generalized Feynman diagrams and also generalized twistor diagrams can be lifted to diagrams in T and that the construction of twistor spaces as surfaces of T has very concrete particle interpretation.

   The modes of the imbedding space spinor field defining ground states of the extended conformal algebras for which classical conformal charges vanish at the ends of the space-time surface (this defines gauge conditions realizing strong form of holography [K18]) are lifted to the products of modes of spinor fields in $T(H)$ characterized by four-momentum and helicity in M^4 degrees of freedom and by color quantum numbers and electroweak quantum numbers in F degrees of freedom. Thus twistorialization provides a purely geometric representation of spin and electro-weak spin and it seems that twistorialization allows to a formulation without H-spinors.

   What is especially nice, that twistorialization extends to from spin to include also electroweak spin. These two spins correspond correspond to M^4 and CP_2 helicities for the twistor space amplitude, and are non-local properties of these amplitudes. In TGD framework only twistor amplitudes for which helicities correspond to that for massless fermion and antifermion are possible and by fermion number conservation the numbers of positive and negative helicities are identical and equal to the fermion number (or antifermion number). Separate lepton and baryon number conservation realizing 8-D chiral symmetry implies that M^4 and CP_2 helicities are completely correlated.

   For massless fermions in M^4 sense helicity is opposite for fermion and antifermion and conserved. The contributions of initial and final states to k are same and equal to $k_i = k_f = 2(n(F) - n(\overline{F})$. This means restriction to amplitudes with $k = 2(n(F) - n(\overline{F})$. If fermions are massless only in M^8 sense, chirality mixing occurs and this rule does not hold anymore. This holds true in quark and lepton sector separately.

4. All generalized Feynman graphs defined in terms of Euclidian regions of space-time surface are lifted to twistor spaces [K5]. Incoming particles correspond quantum mechanically to twistor space amplitudes defined by their momenta and helicities and and classically to the entire twistor space of space-time surface as a surface in the twistor space of H. Of course, also the Minkowskian regions have this lift. The vertices of Feynman diagrams correspond to regions of twistor space in which the incoming twistor spaces meet along their 5-D ends having also S^2 bundle structure over space-like 3-surfaces. These space-like 3-surfaces correspond to ends of Euclidian and Minkowskian space-time regions separated from each other by light-like 3-surfaces at which the signature of the metric changes from Minkowskian to Euclidian. These "partonic orbits" have as their ends 2-D partonic surfaces. By strong form of General Coordinate Invariance implying strong of holography, these 2-D partonic surfaces and their

4-D tangent space data should code for quantum physics. Their lifts to twistor space are 4-D S^2 bundles having partonic 2-surface X^2 as base.

5. The well-definedness of em charge for the spinor modes demands that they are localized at 2-D string world sheets [K18] and also these world sheets are lifted to sub-spaces of twistor space of space-time surface. If one demands that octonionic Dirac operator makes sense at string world sheets, they must carry vanishing induced electro-weak gauge fields and string world sheets could be minimal surfaces in $M^4 \times S^1$, $S^1 \subset CP_2$ a geodesic circle.

   The boundaries of string world sheets at partonic orbits define light-like curves identifiable as carriers of fermion number and they define the analogs of lines of Feynman diagrams in ordinary sense. The only purely fermionic vertices are 2-fermion vertices at the partonic 2-surfaces at which the end of space-time surface meet. As already explained, the string world sheets can be seen as correlates for the correlations between fermion vertices at different wormhole throats giving rise to the counterpart of bosonic propagator in quantum field theories.

   The localization of spinor fields to 2-D string world sheets corresponds to the localization of twistor amplitudes to their 4-D lifts, which are S^2 bundles and the boundaries of string world sheets are lifted to 3-D twistorial lifts of fermion lines. Clearly, the localization of spinors to string world sheets would be absolutely essential for the emergence of twistor description.

6. All elementary particles are many particle bound states of massless fundamental fermions: the non-collinearity (and possible complex character) of massless momenta explains massivation. The fundamental fermions are localized at wormhole throats defining the light-like orbits of partonic 2-surfaces. Throats are associated with wormhole contacts connecting two space-time sheets. Stability of the contact is guaranteed by non-vanishing monopole magnetic flux through it and this requires the presence of second wormhole contact so that a closed magnetic flux tube carrying monopole flux and involving the two space-time sheets is formed. The net fermionic quantum numbers of the second throat correspond to particle's quantum numbers and above weak scale the weak isospins of the throats sum up to zero.

7. Fermionic 2-vertex is the only *local* many-fermion vertex [K5] being analogous to a mass insertion. The non-triviality of fundamental 4-fermion vertex is due to classical interactions between fermions at opposite throats of worm-hole. The non-triviality of the theory involves also the analog of OZI mechanism: the fermionic lines inside partonic orbits are redistributed in vertices. Lines can also turn around in time direction which corresponds to creation or annihilation of a pair. 3-particle vertices are obtained only in topological sense as 3 space-time surfaces are glued together at their ends. The interaction between fermions at different wormhole throats is described in terms of string world sheets.

8. The earlier proposal was that the fermions in the internal fermion lines are massless in M^4 sense but have non-physical helicity so that the algebraic M^4 Dirac operator emerging from the residue integration over internal four-momentum does not annihilate the state at the end of the propagator line. Now the algebraic induced Dirac operator defines the propagator at fermion lines. Should one assume generalization of non-physical helicity also now?

9. All this stuff must be lifted to twistorial level and one expects that the lift to S^2 bundle allows an alternative description of fermions and spinor structure so that one can speak of induced twistor structure instead of induced spinor structure. This approach allows also a realization of M^4 conformal symmetries in terms of globally well-defined linear transformations so that it might be that twistorialization is not a mere reformulation but provides a profound unification of bosonic and fermionic degrees of freedom.

4.2 The Emergence Of The Fundamental 4-Fermion Vertex And Of Boson Exchanges

The emergence of the fundamental 4-fermion vertex and of boson exchanges deserves a more detailed discussion.

1. I have proposed that the discontinuity of the Dirac operator at partonic two-surface (corner of fermion line) defines both the fermion boson vertex and TGD analog of mass insertion (not scalar but imbedding space vector) giving rise to mass parameter having interpretation as Higgs vacuum expectation and various fermionic mixing parameters at QFT limit of TGD obtained by approximating many-sheeted space-time of TGD with the single sheeted region of M^4 such that gravitational field and gauge potentials are obtained as sums of those associated with the sheets.

2. Non-trivial scattering requires also correlations between fermions at different partonic 2-surfaces. Both partonic 2-surfaces and string world sheets are needed to describe these correlations. Therefore the string world sheets and partonic 2-surfaces cannot be dual: both are needed and this means deviation from Witten's theory. Fermion vertex corresponds to a "corner" of a fermion line at partonic 2-surface at which generalized 4-D lines of Feynman diagram meet and light-like fermion line changes to space-like one. String world sheet with its corners at partonic 2-surfaces (wormhole throats) describes the momentum exchange between fermions. The space-like string curve connecting two wormhole throats serves as the analog of the exchanged gauge boson.

3. Two kinds of 4-fermion amplitudes can be considered depending on whether the string connects throats of single wormhole contact (CP_2 scale) or of two wormhole contacts (p-adic length scale - typically of order elementary particle Compton length). If string worlds sheets have 1-D CP_2 projection, only Minkowskian string world sheets are possible. The exchange in Compton scale should be assignable to the TGD counterpart of gauge boson exchange and the fundamental 4-fermion amplitude should correspond to single wormhole contact: string need not to be involved now. Interaction is basically classical interaction assignable to single wormhole contact generalizing the point like vertex.

4. The possible TGD counterparts of BCFW recursion relations [B4] should use the twistorial representations of fundamental 4-fermion scattering amplitude as seeds. Yangian invariance poses very strong conditions on the form of these amplitudes and the exchange of massless bosons is suggestive for the general form of amplitude.

   The 4-fermion amplitude assignable to two wormhole throats defines the analog of gauge boson exchange and is expressible as fusion of two fundamental 4-fermion amplitudes such that the 8-momenta assignable to the fermion and anti-fermion at the opposite throats of exchanged wormhole contact are complex by BCFW shift acting on them to make the exchanged momenta massless but complex. This entity could be called fundamental boson (not elementary particle).

5. Can one assume that the fundamental 4-fermion amplitude allows a purely formal composition to a product of $F\overline{F}B_v$ amplitudes, B_v a purely fictive boson? Two 8-momenta at both $F\overline{F}B_v$ vertices must be complex so that at least two external fermionic momenta must be complex. These external momenta are naturally associated with the throats of the a wormhole contact defining virtual fundamental boson. Rather remarkably, without the assumption about product representation one would have general four-fermion vertex rather than boson exchange. Hence gauge theory structure is not put in by hand but emerges.

4.3 What About SUSY In TGD?

Extended super-conformal symmetry is crucial for TGD and extends to quaternion-super-conformal symmetry giving excellent hopes about calculability of the theory. $\mathcal{N} = 4$ space-time supersymmetry is in the key role in the approach of Witten and others.

In TGD framework space-time SUSY could be present as an approximate symmetry.

1. The many fermion states at partonic surfaces are created by oscillator operators of fermionic Clifford algebra having also interpretation as a supersymmetric algebra but in principle having $\mathcal{N} = \infty$. This SUSY is broken since the generators of SUSY carry four-momentum.

2. More concrete picture would be that various SUSY multiplets correspond to collinear many-fermion states at the same wormhole throat. By fermionic statistics only the collinear states

for which internal quantum numbers are different are realized: other states should have antisymmetric wave function in spatial degrees of freedom implying wiggling in CP_2 scale so that the mass of the state would be very high. In both quark and lepton sectors one would have $\mathcal{N} = 4$ SUSY so that one would have the analog $\mathcal{N} = \forall$ SUSY (color is not spin-like quantum number in TGD).

At the level of diagrammatics single line would be replaced with "line bundle" representing the fermions making the many-fermion state at the light-like orbit of the partonic 2-surface. The fusion of neighboring collinear multifermion stats in the twistor diagrams could correspond to the process in which partonic 2-surfaces and single and many-fermion states fuse.

3. Right handed neutrino modes, which are not covariantly constant, are also localized at the fermionic lines and defines the least broken $\mathcal{N} = 2$ SUSY. The covariantly constant mode seems to be a pure gauge degree of freedom since it carriers no quantum numbers and the SUSY norm associated with it vanishes. The breaking would be smallest for $\mathcal{N} = 2$ variant assignable to right-handed neutrino having no weak and color interactions with other particles but whose mixing with left-handed neutrino already induces SUSY breaking.

Why this SUSY has not been observed? One can consider two scenarios [K20].

1. The first scenario relies on the absence of weak and color interactions: one can argue that the bound states of fermions with right-handed neutrino are highly unstable since only gravitational interaction so that sparticle decays very rapidly to particle and right-handed or left-handed neutrino. By Uncertainty Principle this makes sparticle very massive, maybe having mass of order CP_2 mass. Neutrino mixing caused by the mixing of M^4 and CP_2 gamma matrices in induced gamma matrices is the weak point of this argument.

2. The mixing of left and right-handed neutrinos could be characterized by the p-adic mass scale of neutrinos and be long. Sparticles would have same p-adic mass scale as particles and would be dark having non-standard value of Planck constant $h_{eff} = n \times h$: this would scale up the lifetime by factor n and correlate with breaking of conformal symmetry assignable to the mixing [K20].

What implications the approximate SUSY would have for scattering amplitudes?

1. $k = 2(n(F) - n(\overline{F})$ condition reduces the number of amplitudes dramatically if the fermions are massless in M^4 sense but the presence of weak iso-spin implies that the number of amplitudes is 2^n as in non-supersymmetric gauge theories. One however expects broken SUSY with generators consisting of fermionic oscillator operators at partonic 2-surfaces with symmetry breaking taking place only at the level of physical particles identifiable as many particle bound states of massless (in 8-D sense) particles. This motivates the guess that the formal $F\overline{F}B_v$ amplitudes defining fundamental 4-fermion vertex are expressible as those associated with $\mathcal{N} = 4$ SUSY in quark and lepton sectors respectively. This would reduce the number of independent amplitudes to just one.

2. Since SUSY and its breaking emerge automatically in TGD framework, super-space can provide a useful technical tool but is not fundamental.

   Side note: The number of external fermions is always even suggesting that the super-conformal anomalies plaguing the amplitudes with odd n (`http://arxiv.org/pdf/0903.2083v3.pdf`) [B15] are absent.

4.4 What Does One Really Mean With The Induction Of Imbedding Space Spinors?

The induction of spinor structure is a central notion of TGD but its detailed definition has remained somewhat obscure. The attempt to generalize Witten's approach has made it clear that the mere restriction of spinor fields to space-time surfaces is not enough and that one must understand in detail the correspondence between the modes of imbedding space spinor fields and those of induced spinor fields.

Even the identification of space-time gamma matrices is far from obvious at string world sheets.

1. The simplest notion of the space-time gamma matrices is as projections of imbedding space gamma matrices to the space-time surface - induced gamma matrices. If one assumes that induced spinor fields are defined at the entire space-time surfaces, this notion fails to be consistent with fermionic super-conformal symmetry unless one replaces Kähler action by space-time volume. This option is certainly unphysical.

2. The notion of Kähler-Dirac matrices in the interior of space as gamma matrices defined as contractions of canonical momentum densities of Kähler with imbedding space gamma matrices allows to have conformal super-symmetry with fermionic super charges assignable to the modes of the induced spinor field. Also Chern-Simons action could define gamma matrices in the same manner at the light-like 3-surfaces between Minkowskian and Euclidian space-time regions and at space-like 3-surfaces at the ends of space-time surface. Chern-Simons-Dirac matrices would involve only CP_2 gamma matrices.

It is however not quite clear whether the spinor fields in the interior of space-time surface are needed at all in the twistorial approach and they seem to be only an un-necessary complication. For instance, their modes would have well-defined electromagnetic charge only when induced W gauge fields vanish, which implies that CP_2 projection is 2-dimensional. This forces to consider very seriously the possibility that induced spinor fields reside at string world sheets only (their ends are at partonic 2-surfaces). This option supported also by strong form of holography and number theoretic universality.

What about the space-time gamma matrices at string world sheets and their boundaries?

1. The first option would be reduction of Kähler-Dirac gamma matrices by requiring that they are parallel to the string world sheets. This however poses additional conditions besides the vanishing of W fields already restricting the dimension to two in the generic case. The conditions state that the imbedding space 1-forms defined by the canonical momentum densities of Kähler action involve only 2 linearly independent ones and that they are proportional to imbedding space coordinate gradients: this gives Frobenius conditions. These conditions look first too strong but one can also think that one fixes first string world sheets, partonic 2-surfaces, and perhaps also their light-like orbits, requires that the normal components of canonical momentum currents at string world sheets vanish, and deduces space-time surface from this data. This would be nothing but strong form of holography!

   For this option the string world sheets could emerge in the sense that it would be possible to express Kähler action as an area of string world sheet in the effective metric defined by the anticommutator of K-D gamma matrices appearing also in the expressions for the matrix elements of WCW metric. Gravitational constant would be a prediction of the theory.

2. Second possibility is to use induced gamma matrices automatically parallel to the string world sheet so that no additional conditions would result. This would also conform with the ordinary view about string world sheets and spinors.

   Supersymmetry would require the addition of the area of string world sheet to the action defining Kähler function in Euclidian regions and its counterpart in Minkowskian regions. This would bring in gravitational constant, which otherwise remains a prediction. Quantum criticality could fix the ratio of $\hbar G/R^2$ (R is CP_2 radius). The vanishing of induced weak gauge fields requires that string world sheets have 1-D CP_2 projection and are thus restricted to Minkowskian regions with at most 3-D CP_2 projection. Even stronger condition would be that string world sheets are minimal surfaces in $M^4 \times S^1$, S^1 a geodesic sphere of CP_2.

   There are however grave objections. The presence of a dimensional parameter G as fundamental coupling parameter does not encourage hopes about the renomalizibility of the theory. The idea that strings connecting partonic 2-surfaces gives rise to the formation of gravitationally bound states is suggested by AdS/CFT correspondence. The problem is that the string tension defined by gravitational constant is so large that only Planck length sized bound states are feasible. Even the replacement $\hbar \rightarrow \hbar_{eff}$ fails to allow gravitationally bound states with length scale of order Schwartschild radius. For the K-D option the string tension behaves like $1/\hbar^2$ and there are no problems in this respect.

At this moment my feeling is that the first option - essentially the original view - is the correct one. The short belief that the second option is the correct choice was a sidetrack, which however helped to become convinced that the original vision is indeed correct, and to understand the general mechanism for the formation of bound states in terms of strings connection partonic 2-surfaces (in the earlier picture I talked about magnetic flux tubes carrying monopole flux: the views are equivalent).

Both options have the following nice features.

1. Induced gammas at the light-like string boundaries would be light-like. Massless Dirac equation would assign to spinors at these lines a light-like space-time four-momentum and twistorialize it. This four-momentum would be essentially the tangent vector of the light-like curve and would not have a constant direction. Light-likeness for it means light-likeness in 8-D sense since light-like curves in H correspond to non-like momenta in M^4. Both M^4 mass squared and CP_2 mass would be conserved. Even four-momentum could be conserved if M^4 projection of stringy curve is geodesic line of M^4.

2. A new connection with Equivalence Principle (EP) would emerge. One could interpret the induced four-momentum as gravitational four-momentum which would be massless in 4-D sense and correspond to inertial 8-momentum. EP wold state in the weakest form that only the M^4 masses associated with the two momenta are identical. Stronger condition would be that that the Minkowski parts of the two momenta co-incide at the ends of fermion lines at partonic 2-surfaces. Even stronger condition is that the 8-momentum is 8-momentum is conserved along fermion line. This is certainly consistent with the ordinary view about Feynman graphs. This is guaranteed if the light-like curve is light-like geodesic of imbedding space.

The induction of spinor fields has also remained somewhat imprecise notion. It how seems that quantum-classical correspondence forces a unique picture.

1. Does the induced spinor field co-incide with imbedding space spinor harmonic in some domain? This domain would certainly include the ends of fermionic lines at partonic 2-surfaces associated with the incoming particles and vertices. Could it include also the boundaries of string world sheets and perhaps also the string world sheets? The Kähler-Dirac equation certainly does not allow this for entire space-time surface.

2. Strong form of holography suggest that the light-like momenta for the Dirac equation at the ends of light-like string boundaries has interpretation as 8-D light-like momentum has M^4 projection equal to that of H spinor-harmonic. The mass squared of M^4 momentum would be same as the CP_2 momentum squared in both senses. Unless the gravitational four-momentum assignable to the induced Dirac operato r is conserved one cannot pose stronger condition.

3. If the induced spinor mode equals to imbedding space-spinor mode also at fermion line, the light like momentum is conserved. The fermion line would be also light-like geodesic of the imbedding space so that twistor polygons would have very concrete interpretation. This condition would be clearly analogous to the conditions in Witten's twistor string theory. A stronger condition would be that the mode of the imbedding space spinor field co-incides with induced spinor field at the string world sheet.

4. p-Adic mass calculations require that the massive excitations of imbedding space spinor fields with CP_2 mass scale are involved. The thermodynamics could be for fermion line, wormhole throat carrying possible several fermions, or wormhole contact carrying fermion at both throats. In the case of fermions physical intuition suggests that p-adic thermodynamics must be associated with single fermionic line. The massive excitations would correspond to light-like geodesics of the imbedding space.

To minimize confusion I must confess that until recently I have considered a different options for the momenta associated with fermionic lines.

1. The action of the Kähler-Dirac operator on the induced spinor field at the fermionic line equals to that of 4-D Dirac operator $p^k\gamma_k$ for a massless momentum identified as M^4 momentum [K5].

   Now the action reduces to that of 8-D massless algebraic Dirac operator for light-like string boundaries in the case of induced gamma matrices. Explicit calculation shows that in case of K-D gamma matrices and for light-like string boundaries it can happen that the 8-momentum of the mode can be tachyonic. Intriguingly, p-adic mass calculations require a tachyonic ground state?

2. For this option the helicities for virtual fermions were assumed to be non-physical in order to get non-vanishing fermion lines by residue integration: momentum integration for Dirac operator would replace Dirac propagators with Dirac operators. This would be the counterpart for the disappearance of bosonic propagators in residue integration.

3. This option has problems: quantum classical correspondence is not realized satisfactorily and the interpretation of p-adic thermodynamics is problematic.

4.5 About The Twistorial Description Of Light-Likeness In 8-D Sense Using Octonionic Spinors

The twistor approach to TGD [K17] require that the expression of light-likeness of M^4 momenta in terms of twistors generalizes to 8-D case. The light-likeness condition for twistors states that the 2×2 matrix representing M^4 momentum annihilates a 2-spinor defining the second half of the twistor. The determinant of the matrix reduces to momentum squared and its vanishing implies the light-likeness. This should be generalized to a situation in one has M^4 and CP_2 twistor which are not light-like separately but light-likeness in 8-D sense holds true.

4.5.1 *The case of* $M^8 = M^4 \times E^4$

$M^8 - H$ duality [K16] suggests that it might be useful to consider first the twistorialiation of 8-D light-likeness first the simpler case of M^8 for which CP_2 corresponds to E^4. It turns out that octonionic representation of gamma matrices provide the most promising formulation.

In order to obtain quadratic dispersion relation, one must have 2×2 matrix unless the determinant for the 4×4 matrix reduces to the square of the generalized light-likeness condition.

1. The first approach relies on the observation that the 2×2 matrices characterizing four-momenta can be regarded as hyper-quaternions with imaginary units multiplied by a commuting imaginary unit. Why not identify space-like sigma matrices with hyper-octonion units?

2. The square of hyper-octonionic norm would be defined as the determinant of 4×4 matrix and reduce to the square of hyper-octonionic momentum. The light-likeness for pairs formed by M^4 and E^4 momenta would make sense.

3. One can generalize the sigma matrices representing hyper-quaternion units so that they become the 8 hyper-octonion units. Hyper-octonionic representation of gamma matrices exists ($\gamma_0 = \sigma_z \times 1$, $\gamma_k = \sigma_y \times I_k$) but the octonionic sigma matrices represented by octonions span the Lie algebra of G_2 rather than that of $SO(1,7)$. This dramatically modifies the physical picture and brings in also an additional source of non-associativity. Fortunately, the flatness of M^8 saves the situation.

4. One obtains the square of $p^2 = 0$ condition from the massless octonionic Dirac equation as vanishing of the determinant much like in the 4-D case. Since the spinor connection is flat for M^8 the hyper-octonionic generalization indeed works.

This is not the only possibility that I have by-passingly considered [K5].

1. Is it enough to allow the four-momentum to be complex? One would still have 2×2 matrix and vanishing of complex momentum squared meaning that the squares of real and imaginary parts are same (light-likeness in 8-D sense) and that real and imaginary parts are orthogonal to each other. Could E^4 momentum correspond to the imaginary part of four-momentum?

2. The signature causes the first problem: M^8 must be replaced with complexified Minkowski space M^4_c for to make sense but this is not an attractive idea although M^4_c appears as subspace of complexified octonions.

3. For the extremals of Kähler action Euclidian regions (wormhole contacts identifiable as deformations of CP_2 type vacuum extremals) give imaginary contribution to the four-momentum. Massless complex momenta and also color quantum numbers appear also in the standard twistor approach. Also this suggest that complexification occurs also in 8-D situation and is not the solution of the problem.

4.5.2 *The case of* $M^8 = M^4 \times CP_2$

What about twistorialization in the case of $M^4 \times CP_2$? The introduction of wave functions in the twistor space of CP_2 seems to be enough to generalize Witten's construction to TGD framework and that algebraic variant of twistors might be needed only to realize quantum classical correspondence. It should correspond to tangent space counterpart of the induced twistor structure of space-time surface, which should reduce effectively to 4-D one by quaternionicity of the space-time surface.

1. For $H = M^4 \times CP_2$ the spinor connection of CP_2 is not trivial and the G_2 sigma matrices are proportional to M^4 sigma matrices and act in the normal space of CP_2 and to M^4 parts of octonionic imbedding space spinors, which brings in mind co-associativity. The octonionic charge matrices are also an additional potential source of non-associativity even when one has associativity for gamma matrices.

   Therefore the octonionic representation of gamma matrices in entire H cannot be physical. It is however equivalent with ordinary one at the boundaries of string world sheets, where induced gauge fields vanish. Gauge potentials are in general non-vanishing but can be gauge transformed away. Here one must be of course cautious since it can happen that gauge fields vanish but gauge potentials cannot be gauge transformed to zero globally: topological quantum field theories represent basic example of this.

2. Clearly, the vanishing of the induced gauge fields is needed to obtain equivalence with ordinary induced Dirac equation. Therefore also string world sheets in Minkowskian regions should have 1-D CP_2 projection rather than only having vanishing W fields if one requires that octonionic representation is equivalent with the ordinary one. For CP_2 type vacuum extremals electroweak charge matrices correspond to quaternions, and one might hope that one can avoid problems due to non-associativity in the octonionic Dirac equation. Unless this is the case, one must assume that string world sheets are restricted to Minkowskian regions. Also imbedding space spinors can be regarded as octonionic (possibly quaternionic or co-quaternionic at space-time surfaces): this might force vanishing 1-D CP_2 projection.

   (a) Induced spinor fields would be localized at 2-surfaces at which they have no interaction with weak gauge fields: of course, also this is an interaction albeit very implicit one! This would not prevent the construction of non-trivial electroweak scattering amplitudes since boson emission vertices are essentially due to re-groupings of fermions and based on topology change.

   (b) One could even consider the possibility that the projection of string world sheet to CP_2corresponds to CP_2 geodesic circle so that one could assign light-like 8-momentum to entire string world sheet, which would be minimal surface in $M^4 \times S^1$. This would mean enormous technical simplification in the structure of the theory. Whether the spinor harmonics of imbedding space with well-defined M^4 and color quantum numbers can co-incide with the solutions of the induced Dirac operator at string world sheets defined by minimal surfaces remains an open problem.

   (c) String world sheets cannot be present inside wormhole contacts which have 4-D CP_2 projection so that string world sheets cannot carry vanishing induced gauge fields.

4.6 How To Generalize Witten's Twistor String Theory To TGD Framework?

The challenge is to lift the geometric description of particle like aspects of twistorial amplitudes involving only algebraic curves (2-surfaces) in twistor space to TGD framework.

1. External particles correspond to the lifts of H-spinor harmonics to spinor harmonics in the twistor space and are labeled by four-momentum, helicity, color, and weak helicity (isospin) so that there should be no need to included fermions explicitly. The twistorial wave functions would be superpositions of the eigenstates of helicity operator which would become a non-local property in twistor space. Light-likeness would hold true in 8-D sense for spinor harmonics as well as for the corresponding twistorial harmonics.

2. The surfaces X^2 in Witten's theory would be replaced with the lifts of partonic 2-surfaces X^2 to 4-D surfaces with bundle structure with X^2 as base and S^2 as fiber. S^2 would be non-dynamical. Whether X^2 or its lift to 4-surface allows identification as algebraic surface is not quite clear but it seems that X^2 could be the relevant object identifiable as surface of the base space of $T(X^4)$. If X^2 is the basic object one would have the additional constraint (not present in Witten's theory) that it belongs to the base space X^4. The genus of the lift of X^2 would be that of its base space X^2. One obtains a union of partonic 2-surfaces rather than single surface and lines connecting them as boundaries of string world sheets.

   The n points of given X^2 would correspond to the ends of boundaries of string world sheets at the partonic 2-surface X^2 carrying fermion number so that the helicities of twistorial spinor modes would be highly fixed unless M^4 chiralities mix making fermions massive in M^4 sense. This picture is in accordance with the fact that the lines of fundamental fermions should correspond to QFT limit of TGD.

3. In TGD genus g of the partonic 2-surface labels various fermion families and $g < 3$ holds true for physical fermions. The explanation could be that Z^2 acts as global conformal symmetry (hyper-ellipticity) for $g < 3$ surfaces irrespective of their conformal moduli but for $g > 3$ only in for special moduli. Physically for $g > 2$ the additional handles would make the partonic 2-surface to behave like many-particle state of free particles defined by the handles.

   This assumption suggests that assigns to the partonic surface what I have called modular invariant elementary particle vacuum functional (EVPF) in modular degrees of freedom such that for a particle characterized by genus g one has $l \geq g$ and $l > g$ amplitudes are possible because the EPVF leaks partially to higher genera [K4]. This would also induce a mixing of boundary topologies explaining CKM mixing and its leptonic counterpart. In this framework it would be perhaps more appropriate to define the number of loops as $l_1 = l - g$.

   A more precise picture is as follows. Elementary particles have actually four wormhole throats corresponding to the two wormhole contacts. In the case of fermions the wormhole throat carrying the electroweak quantum numbers would have minimum value g of genus characterized by the fermion family. Furthermore, the universality of the standard model physics requires that the couplings of elementary fermions to gauge bosons do not depend on genus. This is the case if one has quantum superposition of the wormhole contacts carrying the quantum numbers of observed gauge bosons at their opposite throats over the three lowest genera $g = 0, 1, 2$ with identical coefficients. This meas SU(3) singlets for the dynamical SU(3) associated with genus degeneracy. Also their exotic variants - say octets - are in principle possible.

4. This description is not complete although already twistor string description involves integration over the conformal moduli of the partonic 2-surface. One must integrate over the "world of classical worlds" (WCW) -roughly over the generalized Feynman diagrams and their complements consisting of Minkowskian and Euclidian regions. TGD as almost topological QFT reduces this integration to that of the boundaries of space-time regions.

   By quaternion conformal invariance [K17] this functional integral could reduce to ordinary integration over the quaternionic-conformal moduli space of space-time surfaces for which the moduli space of partonic 2-surfaces should be contained (note that strong form of holography suggests that only the modular invariants associated with the tangent space data should enter the description). One might hope that twistor space approach allows an elegant description of the moduli assignable to the tangent space data.

4.7 Yangian Symmetry

One of the victories of the twistor Grassmannian approach is the discovery of Yangian symmetry [A2], [B7, B11], [K17], whose variant associated with extended super-conformal symmetries is expected to be in key role in TGD.

1. The very nature of the residue integral implies that the complex surface serving as the locus of the integrand of the twistor amplitude is highly non-unique. Indeed, the Yangian symmetry [K17] acting as multi-local symmetry and implying dual of ordinary conformal invariance acting on momentum twistors, has been found to reduce to diffeomorphisms of $G(k,n)$ respecting positivity property of the decomposition of $G(k,n)$ to polyhedrons. It is quite possible that this symmetry corresponds to quaternion conformal symmetries in TGD framework.

2. Positivity of a given regions means parameterization by non-negative coordinates in TGD framework a possible interpretation is based on the observation that canonical identification mapping reals to p-adic number and vice versa is well-defined only for non-negative real numbers. Number theoretical Universality of spinor amplitudes so that they make sense in all number fields, would therefore be implied.

3. Could the crucial Yangian invariance generalizing the extended conformal invariance of TGD could reduce to the diffeomorphisms of the extended twistor space $T(H)$ respecting positivity. In the case of CP_2 all coordinates could be regarded as angle coordinates and be replaced by phase factors coding for the angles which do not make sense p-adically. The number theoretical existence of phase factors in p-adic case is guaranteed if they belong to some algebraic extension of rationals and thus also p-adics containing these phases as roots of unity. This implies discretization of CP_2.

   ZEO allows to reduce the consideration to causal diamond CD defined as an intersection of future and past directed light-cones and having two light-like boundaries. CD is indeed a natural counterpart for S^4. One could use as coordinates light-cone proper time a invariant under Lorentz transformations of either boundary of CD, hyperbolic angle η and two spherical angles (θ, ϕ). The angle variables allow representation in terms of finite algebraic extension. The coordinate a is naturally non-negative and would correspond to positivity. The diffeomorphisms perhaps realizing Yangian symmetry would respect causality in the sense that they do not lead outside CD.

   Quaternionic conformal symmetries the boundaries of $CD \times CP_2$ continued to the interior by translation of the light-cones serve as a good candidates for the diffeomorphisms in question since they do not change the value of the Minkowski time coordinate associated with the line connecting the tips of CD.

4.8 Does BCFW Recursion Have Counterpart In TGD?

Could BCFW recursion for tree diagrams and its generalization to diagrams with loops have a generalization in TGD framework? Could the possible TGD counterpart of BCFW recursion have a representation at the level of the TGD twistor space allowing interpretation in terms of geometry of partonic 2-surfaces and associated string world sheets? Supersymmetry is essential ingredient in obtaining this formula and the proposed SUSY realized at the level of amplitudes and broken at the level of states gives hopes for it. One could however worry about the fact that spinors are Dirac spinors in TGD framework and that Majorana property might be essential element.

4.8.1 *How to produce Yangian invariants*

Nima Arkani-Hamed et al [B11] (`http://arxiv.org/pdf/1008.2958v2.pdf`) describe in detail various manners to form Yangian invariants defining the singular parts of the integrands of the amplitudes allowing to construct the full amplitudes. The following is only a rough sketch about what is involved using particle picture and I cannot claim of having understood the details.

1. One can *add* particle ($(k,n) \rightarrow (k+1,n+1)$) to the amplitude by deforming the momentum twistors of two neighboring particles in a manner depending on the momentum twistor of the added particle. One inserts the new particle between n-1:th and 1st particle, modifies their momentum twistors without changing their four-momenta, and multiplying the resulting amplitude by a twistor invariant known as $[n-2,n-1,n,1,2]$ so that there is dependence on the added n:th momentum twistor.

2. One can *remove* particle ($(k,n) \rightarrow (k-1,n-1)$) by contour integrating over the momentum twistor variable of one particle.

3. One can *fuse* invariants simply by multiplying them.

4. One can *merge* invariants by identifying momentum twistors appearing in the two invariants. The integration over the common twistor leads to an elimination of particle.

5. One can form a *BCFW bridge* between n_1+1-particle invariant and n_2+1-particle invariant to get $n = n_1+n_2$-particle invariant using the operations listed. One starts with the *fusion* giving the product $I_1(1,...,n_1,I)I_2(n_1+1,..n,I)$ of Yangian invariants followed by *addition* of n_1+1 to I_1 between n_1 and I and 1 to I_2 between I and n_1+1 (see the first item for details). After that follows the *merging* of lines labelled by I next to n_1 in I_1 and the predecessor of n_1+1 in I_2 reducing particle number by one unit and followed by residue integration over Z_I reducing particle number further by one unit so that the resulting amplitude is n-particle amplitude.

6. One can perform *entangled removal* of two particles. One could remove them one-by-one by independent contour integrations but one can also perform the contour integrations in such a manner that one first integrates over two twistors at the same complex line and then over the lines: this operation adds to n-particle amplitude loop.

4.8.2 *BCFW recursion formula*

BCFW recursion formula allows to express n-particle amplitudes with l loops in terms of amplitudes with amplitudes having at most $l-1$ loops. The basic philosophy is that singularities serve as data allowing to deduce the full integrands of the amplitudes by generalized unitarity and other kinds of arguments.

Consider first the arguments behind the BCFW formula.

1. BCFW formula is derived by performing the canonical momentum twistor deformation $Z_n \rightarrow z_n + zZ_{n-1}$, multiplying by $1/z$ and performing integration along small curve around origin so that one obtains original amplitude from the residue inside the curve. One obtains also and alternative of the residue integral expression as sum of residues from its complement. The singularities emerge by residue integral from poles of scattering amplitudes and eliminate two lines so that the recursion formula for n-particle amplitude can involve at most $n+2$-particle amplitudes.

   It seems that one must combine all n-particle amplitudes to form a single entity defining the full amplitude. I do not quite understand what how this is done. In ZEO zero energy state involving different particle numbers for the final state and expressible in terms of S-matrix (actually its generalization to what I call M-matrix) might allow to understand this.

2. In the general formula for the BCFW bridge of the "left" and "right" amplitudes one has $n_L + n_R = n+2$, $k_L + k_R = k-1$, and $l_L + l_R = l$.

3. The singularities are easy to understand in the case of tree amplitudes: they emerge from the poles of the conformally invariant quantities in the denominators of amplitudes. Physically this means that the sum of the momenta for a subset of particles corresponds to a complex pole (BCFW deformation makes two neighboring momenta complex). Hence one obtains sum over products of $j+1$-particle amplitudes BCFW bridged with $n-j$-particle amplitude to give n-particle amplitude by the merging process.

4. This is not all that is needed since the diagrams could be reduced to products of 1 loop 3-particle amplitudes which vanish by the triviality of coupling constant evolution in $\mathcal{N}=4$ SUSY. Loop amplitudes serving as a kind of source in the recursion relation save the situation. There is indeed also a second set of poles coming from loop amplitudes.

   One-loop case is the simplest one. One begins from $n+2$ particle amplitude with $l-1$ loops. At momentum space level the momenta the neighboring particles have opposite light-like momenta: one of the particles is not scattered at all. This is called forward limit. This limit suffers from collinear divergences in a generic gauge theory but in supersymmetric theories the limit is well-defined. This forward limit defines also a Yangian invariant at the level of twistor space. It can be regarded as being obtained by entangled removal of two particles combined with merge operation of two additional particles. This operation leads from $(n+2, l-1)$ amplitude to (n, l) amplitude.

4.8.3 *Does BCFW formula make sense in TGD framework?*

In TGD framework the four-fermion amplitude but restricted so that two outgoing particles have (in general) complex massless 8-momenta is the basic building brick. This changes the character of BCFW recursion relations although the four-fermion vertex effectively reduces to $F\overline{F}B$ vertex with boson identified as wormhole contact carrying fermion and antifermion at its throats.

The fundamental 4-fermion vertices assignable to wormhole contact could be formally expressed in terms of the product of two $F\overline{F}B_v$ vertices (MHV expression), where B_v is purely formal gauge boson, using the analog of MHV expression and taking into account that the second $F\overline{F}$ pair is associated with wormhole contact analogous to exchanged gauge boson.

If the fermions at fermion lines of the same partonic 2-surface can be assumed to be collinear and thus to form single coherent particle like unit, the description as superspace amplitude seems appropriate. Consequently, the effective $F\overline{F}B_v$ vertices could be assumed to have supersymmetry defined by the fermionic oscillator operator algebra at the partonic 2-surface (Clifford algebra). A good approximation is to restrict this algebra to that generating various spinor components of imbedding space spinors so that $\mathcal{N}=4$ SUSY is obtained in leptonic and quark sector. Together these give rise to $\mathcal{N}=8$ SUSY at the level of vertices broken however at the level of states.

Side note: The number of external fermions is always even suggesting that the super-conformal anomalies plaguing the SUSY amplitudes with odd n (`http://arxiv.org/pdf/0903.2083v3.pdf`) [B15] are absent in TGD: this would be basically due to the decomposition of gauge bosons to fermion pairs.

The leading singularities of scattering amplitudes would naturally correspond to the boundaries of the moduli space for the unions of partonic 2-surfaces and string world sheets.

1. The tree contribution to the gauge boson scattering amplitudes with $k=0,1$ vanish as found by Parke and Taylor who also found the simple twistorial form for the $k=2$ case (`http://en.wikipedia.org/wiki/MHV_amplitudes`). In TGD framework, where lowest amplitude is 4-fermion amplitude, this situation is not encountered. According to Wikipedia article the so called CSW rules inspired by Witten's twistor theory have a problem due to the vanishing of $++-$ vertex which is not MHV form unless one changes the definition of what it is to be "wrong helicity". $++-$ is needed to construct $++++$ amplitude at one loop which does not vanish in YM theory. In SUSY it however vanishes.

   In TGD framework one does not encounter these problems since 4-fermion amplitudes are the basic building bricks. Fermion number conservation and the assumption that helicities do not mix (light-likeness in M^4 rather than only M^8-sense) implies $k = 2(n(F) - n(\overline{F})$.

   In the general formula for the BCFW bridge of the "left" and "right" amplitudes one has $n_L + n_R = n+2$, $k_L + k_R = k-1$. If the TGD counterpart of the bridge eliminates two antifermions with the same "wrong" helicity -1/2, and one indeed has $k_L + k_R = k-1$ if fermions have well-defined M^4 helicity rather than being in superposition in completely correlated M^4 and CP_2 helicities.

2. In string theory loops correspond to handles of a string world sheet. Now one has partonic 2-surfaces and string world sheets and both can in principle have handles. The condition

$l \geq g$ of Witten's theory suggests that $l - g$ defines the handle number for string world sheet so that l is the total number of handles.

The identification of loop number as the genus of partonic 2-surface is second alternative: one would have $l = g$ and string world sheets would not contain handles. This might be forced by the Minkowskian signature of the induced metric at string world sheet. The signature of the induced metric would be presumably Euclidian in some region of string world sheet since the M^4 projection of either homology generator assignable with the handle would presumably define time loop in M^4 since the derivative of M^4 time coordinate with respect to string world sheet time should vanish at the turning points for M^4 time. Minimal surface property might eliminate Euclidian regions of the string world sheet. In any case, the area of string world sheet would become complex.

3. In the moduli space of partonic 2-surfaces first kind of leading singularities could correspond to pinches formed as n partonic 2-surfaces decomposes to two 2-surfaces having at least single common point so that moduli space factors into a Cartesian product. This kind of singularities could serve as counterparts for the merge singularities appearing in the BCFW bridging of amplitudes. The numbers of loops must be additive and this is consistent with both interpretations for l.

4. What about forward limit? One particle should go through without scattering and is eliminated by entangled removal. In ZEO one can ask whether there is also quantum entanglement between the positive and negative energy parts of this single particle state and state function reduction does not occur. The addition of particle and merging it with another one could correspond to a situation in which two points of partonic 2-surface touch. This means addition of one handle so that loop number l increases.

   It seems that analytically the loop is added by the entangled removal but at the level of partonic surface it is added by the merging. Also now both $l > g$ and $l = g$ options make sense.

4.9 Possible Connections Of TGD Approach With The Twistor Grassmannian Approach

For a non-specialist lacking the technical skills, the work related to twistors is a garden of mysteries and there are a lot of questions to be answered: most of them of course trivial for the specialist. The basic questions are following.

How the twistor string approach of Witten and its possible TGD generalization relate to the approach involving residue integration over projective sub-manifolds of Grassmannians $G(k, n)$?

1. In [B12] Nima et al argue that one can transform Grassmannian representation to twistor string representation for tree amplitudes. The integration over $G(k, n)$ translates to integration over the moduli space of complex curves of degree $d = k - 1 + l$, $l \geq g$ is the number of loops. The moduli correspond to complex coefficients of the polynomial of degree d and they form naturally a projective space since an overall scaling of coefficients does not change the surfaces. One can expect also in the general case that moduli space of the partonic 2-surfaces can be represented as a projective sub-manifold of some projective space. Loop corrections would correspond to the inclusion of higher degree surfaces.

2. This connection gives hopes for understanding the integration contours in $G(k, n)$ at deeper level in terms of the moduli spaces of partonic 2-surfaces possibly restricted by conformal gauge conditions.

Below I try to understand and relate the work of Nima Arkani Hamed et al with twistor Grassmannian approach to TGD.

4.9.1 *The notion of positive Grassmannian*

The notion of positive Grassmannian is one of the central notions introduced by Nima et al.

1. The claim is that the sub-spaces of the real Grassmannian $G(k,n)$ contributing to the amplitudes for $++--$ signature are such that the determinants of the $k \times k$ minors associated with ordered columns of the $k \times n$ matrix C representing point of $G(k,n)$ are positive. To be precise, the signs of all minors are positive or negative simultaneously: only the ratios of the determinants defining projective invariants are positive.

2. At the boundaries of positive regions some of the determinants vanish. Some k-volumes degenerate to a lower-dimensional volume. Boundaries are responsible for the leading singularities of the scattering amplitudes and the integration measure associated with $G(k,n)$ has a logarithmic singularity at the boundaries. These boundaries would naturally correspond to the boundaries of the moduli space for the partonic 2-surfaces. Here also string world sheets could contribute to singularities.

3. This condition has a partial generalization to the complex case: the determinants whose ratios serve as projectively invariant coordinates are non-vanishing. A possible further manner to generalize this condition would be that the determinants have positive real part so that apart from rotation by $\pi/2$ they would reside in the upper half plane of complex plane. Upper half plane is the hyperbolic space playing key role in complex analysis and in the theory of hyperbolic 2-manifolds for which it serves as universal covering space by a finite discrete subgroup of Lorentz group $SL(2,C)$. The upper half-plane having a deep meaning in the theory of Riemann surfaces might play also a key role in the moduli spaces of partonic 2-surfaces. The projective space would be based - not on projectivization of C^n but that of H^n, H the upper half plane.

Could positivity have some even deeper meaning?

1. In TGD framework the number theoretical universality of amplitudes suggests this. Canonical identification maps $\sum x_n p^n \rightarrow \sum x_n p^{-n}$ p-adic number to non-negative reals. p-Adicization is possible for angle variables by replacing them by discrete phases, which are roots of unity. For non-angle like variables, which are non-negative one uses some variant of canonical identification involving cutoffs [K22]. The positivity should hold true for all structures involved, the $G(2,n)$ points defined by the twistors characterizing momenta and helicities of particles (actually pairs of orthogonal planes defined by twistors and their conjugates), the moduli space of partonic 2-surfaces, etc...

2. p-Adicization requires discretization of phases replacing angles so that they come as roots of unity associated with the algebraic extension used. The p-adic valued counterpart of Riemann or Lebesque integral does not make sense p-adically. Residue integrals can however allow to define p-adic integrals by analytic continuation of the integral and discretization of the phase factor along the integration contour does not matter (not however the p-adically troublesome factor 2π!).

3. TGD suggests that the generalization of positive real projectively invariant coordinates to complex coordinates of the hyperbolic space representable as upper half plane, or equivalently as unit disk obtained from the upper half plane by exponential mapping $w = exp(iz)$: positive coordinate α would correspond to the radial coordinate for the unit disk (Poincare hyperbolic disk appearing in Escher's paintings). The real measure $d\alpha/\alpha$ would correspond to $dz = dw/w$ restricted to a radial line from origin to the boundary of the unit disk. This integral should correspond to integral over a closed contour in complex case. This is the case if the integrand is discontinuity over a radial cut and equivalent with an integral over curve including also the boundary of the unit disk. This integral would reduce to the sum of the residues of poles inside the unit disk.

4.9.2 *The notion of amplituhedron*

The notion of amplituhedron is the latest step of progress in the twistor Grassmann approach [B2, B1]. What is so remarkable, is the simplicity of the expressions for all-loop amplitudes and the fact that positivity implies locality and unitarity for $\mathcal{N} = 4$ SUSY.

Consider first tree amplitudes with general value of k.

1. The notion of amplituhedron relies on the mapping of $G(k,n)$ to $G_+(k,k+m)$ $n \geq k+m$. $G_+(k,k+m)$ is positive Grassmannian characterized by the condition that all $k \times k$- minors $k \times (k+m)$ matrix representing the point of $G_+(k,k+m)$ are non-non-negative and vanish at the boundaries $G_+(k,k+m)$. The value of m is $m=4$ and follows from the conditions that amplitudes come out correctly. The constraint $Y = C \cdot Z$, where Y corresponds to point of $G_+(k,k+4)$ and Z to the point of $G(k,n)$ performs this mapping, which is clearly many-to one. One can decompose $G_+(k,k+4)$ to positive regions intersecting only along their common boundary portions. The decomposition of a convex polygon in plane represent the basic example of this kind of decomposition.

2. Each decomposition defines a sum of contributions to the scattering amplitudes involving integration of a projectively invariant volume form over the positive region in question. The form has a logarithmic singularity at the boundaries of the integration region but spurious singularities cancel so that only the contribution of the genuine boundary of $G_+(k,k+4)$ remains. There are additional delta function constraints fixing the integral completely in real case.

3. In complex case one has residue integral. The proposed generalization to the complex case is by analytic continuation. TGD inspired proposal is that the positivity condition in the real case is generalized to the condition that the positive coordinates are replaced by complex coordinates of hyperbolic space representable as upper half plane or equivalently as the unit disk obtained from upper half plane by exponential mapping $w = exp(iz)$. The measure $d\alpha/\alpha$ would correspond to $dz = dw/w$. If taken over boundary circle labelled by discrete phase factors $exp(i\phi)$ given by roots of unity the integral would be numerically a discrete Riemann sum making no sense p-adically but residue theorem could allow to avoid the discretizaton and to define the p-adic variant of the integral by analytic continuation. These conditions would be completely general conditions on various projectively invariant moduli involved.

4. One must extend the bosonic twistors Z_a of external particles by adding k coordinates. Somewhat surprisingly, these coordinates are anticommutative super-coordinates expressible as linear combinations of fermionic parts of super-twistor using coefficients, which are also Grassmann numbers. Integrating over these one ends up with the standard expression of the amplitude using canonical integration measure for the regions in the decomposition of amplituhedron.

   What looks to me intriguing is that there is only super-integration involved over the additional k degrees of freedom. In Witten's approach $k-1$ corresponds to the minimum degree of the polynomial defining the string world sheet representing tree diagram. In TGD framework $k+1$ (rather than $k-1$) could correspond to the minimum degree of partonic 2-surface. In TGD approximate SUSY would correspond to Grassmann algebra of fermionic oscillator operators defined by the spinor basis for imbedding space spinors. The interpretation could be that each fermion whose helicity differs from that allowed by light-likeness in M^4 sense (this requires non-vanishing M^4 mass), contributes $\Delta k = 1$ to the degree of corresponding partonic 2-surface. Since the partonic 2-surface is common for all particles, one must have $d = k+1$ at least. The k-fold super integration would be basically integral over the moduli characterizing the polynomials of degree k realizing quantum classical correspondence in fermionic degrees of freedom.

BFCW recursion formula involves also loop amplitudes for which amplituhedron provides also a very nice representation.

1. The basic operation is the addition of a loop to get (n,k,l) amplitude from $(n+2,k,l-1)$ amplitude. That 2 particles must be removed for each loop is not obvious in $\mathcal{N}=4$ SUSY but follows from the condition that positivity of the integration domain is preserved. This procedure removes from $(n+2,k,l-1)$-amplitude 2 particles with opposite four-momenta so that (n,k,l) amplitude is obtained. In the case of L-loops one extends $G(k,n)$ by adding its "complement" as a Cartesian factor $G(n-k,n)$ and imbeds to $G(n-k,n)$ 2-plane for each loop. Positivity conditions can be generalized so that they apply to $(k+2l)\times(k+2l)$-minors associated with matrices having as rows $0 \leq l \leq L$ ordered D_{i_k}:s and of C. The general

expressions of the loop contributions are of the same form as for tree contributions: only the number of integration variables is $4 \times (k + L)$.

2. As already explained, in TGD framework the addition of loop would correspond to the formation of a handle to the partonic surface by fusing two points of partonic 2-surface and thus creating a surface intermediate between topologies with g and g+1 handles. g would correspond to the genus characterizing fermion family and one would have $L \geq g$. In elementary particle wave functionals loop [K4] contributions would correspond to higher genus contributions $l_1 = l - g > 0$ with basic contribution coming from genus g. For scattering amplitudes loop contributions would involve the change of the genus of the incoming wormhole throat so that they correspond to singular surfaces at the boundaries of their moduli space identifiable as loop corrections. $l_1 = l - g > 0$ would represent the number of pinches of the partonic 2-surface.

4.9.3 *What about non-planar amplitudes?*

Non-planar Feynman diagrams have remained a challenge for the twistor approach. The problem is simple: there is no canonical ordering of the extrenal particles and the loop integrand involving tricky shifts in integrations to get finite outcome is not unique and well-defined so that twistor Grasmann approach encounters difficulties.

Recently Nima Arkani-Hamed et al have considered also non-planar MHV diagrams [B13] (having minimal number of "wrong" helicities) of N=4 SUSY, and shown that they can be reduced to non-planar diagrams for different permutations of vertices of planar diagrams ordered naturally. There are several integration regions identified as positive Grassmannians corresponding to different orderings of the external lines inducing non-planarity. This does not however hold true generally.

At the QFT limit the crossings of lines emerges purely combinatorially since Feynman diagrams are purely combinatorial objects with the ordering of vertices determining the topological properties of the diagram. Non-planar diagrams correspond to diagrams, which do not allow crossing-free imbedding to plane but require higher genus surface to get rid of crossings.

1. The number of the vertices of the diagram and identification of lines connecting them determines the diagram as a graph. This defines also in TGD framework Feynman diagram like structure as a graph for the fermion lines and should be behind non-planarity in QFT sense.

2. Could 2-D Feynman graphs exists also at geometric rather than only combinatorial level? Octonionization at imbedding space level requires identification of preferred $M^2 \subset M^4$ defining a preferred hyper-complex sub-space. Could the projection of the Fermion lines defined concrete geometric representation of Feynman diagrams?

3. Despite their purely combinatorial character Feynman diagrams are analogous to knots and braids. For years ago [K11] I proposed the generalization of the construction of knot invariants in which one gradually eliminates the crossings of the knot projection to end up with a trivial knot is highly suggestive as a procedure for constructing the amplitudes associated with the non-planar diagrams. The outcome should be a collection of planar diagrams calculable using twistor Grassmannian methods. Scattering amplitudes could be seen as analogs of knot invariants. The reduction of MHV diagrams to planar diagrams could be an example of this procedure.

One can imagine also analogs of non-planarity, which are geometric and topological rather than combinatorial and not visible at the QFT limit of TGD.

1. The fermion lines representing boundaries of string world sheets at the light-like orbits of partonic 2-surfaces can get braided. The same can happen also for the string boundaries at space-like 3-surfaces at the ends of the space-time surface. The projections of these braids to partonic 2-surfaces are analogs of non-planar diagrams. If the fermion lines at single wormhole throat are regarded effectively as a line representing one member of super-multiplet, this kind of braiding remains below the resolution used and cannot correspond to the braiding at QFT limit.

2. 2-knotting and 2-braiding are possible for partonic 2-surfaces and string world sheets as 2-surfaces in 4-D space-time surfaces and have no counterpart at QFT limit.

4.10 Permutations, Braidings, And Amplitudes

In [B9] Nima Arkani-Hamed demonstrates that various twistorially represented on-mass-shell amplitudes (allowing light-like complex momenta) constructible by taking products of the 3-particle amplitude and its conjugate can be assigned with unique permutations of the incoming lines. The article describes the graphical representation of the amplitudes and its generalization. For 3-particle amplitudes, which correspond to $++-$ and $+--$ twistor amplitudes, the corresponding permutations are cyclic permutations, which are inverses of each other. One actually introduces double cover for the labels of the particles and speaks of decorated permutations meaning that permutation is always a right shift in the integer and in the range $[1, 2\times n]$.

4.10.1 *Amplitudes as representation of permutations*

It is shown that for on mass shell twistor amplitudes the definition using on-mass-shell 3-vertices as building bricks is highly reducible: there are two moves for squares defining 4-particle sub-amplitudes allowing to reduce the graph to a simpler one. The first ove is topologically like the s-t duality of the old-fashioned string models and second one corresponds to the transformation black $\leftrightarrow$ white for a square sub-diagram with lines of same color at the ends of the two diagonals and built from 3-vertices.

One can define the permutation characterizing the general on mass shell amplitude by a simple rule. Start from an external particle a and go through the graph turning in in white (black) vertex to left (right). Eventually this leads to a vertex containing an external particle and identified as the image $P(a)$ of the a in the permutation. If permutations are taken as right shifts, one ends up with double covering of permutation group with $2\times n!$ elements - decorated permutations. In this manner one can assign to any any line of the diagram two lines. This brings in mind 2-D integrable theories where scattering reduces to braiding and also topological QFTs where braiding defines the unitary S-matrix. In TGD parton lines involve braidings of the fermion lines so that an assignment of permutation also to vertex would be rather nice.

BCFW bridge has an interpretation as a transposition of two neighboring vertices affecting the lines of the permutation defining the diagram. One can construct all permutations as products of transpositions and therefore by building BCFW bridges. BCFW bridge can be constructed also between disjoint diagrams as done in the BCFW recursion formula.

Can one generalize this picture in TGD framework? There are several questions to be answered.

(a) What should one assume about the states at the light-like boundaries of string world sheets? What is the precise meaning of the supersymmetry: is it dynamical or gauge symmetry or both?

(b) What does one mean with particle: partonic 2-surface or boundary line of string world sheet? How the fundamental vertices are identified: 4 incoming boundaries of string world sheets or 3 incoming partonic orbits or are both aspects involved?

(c) How the 8-D generalization of twistors bringing in second helicity and doubling the M^4 helicity states assignable to fermions does affect the situation?

(d) Does the crucial right-left rule relying heavily on the possibility of only 2 3-particle vertices generalize? Does M^4 massivation imply more than 2 3-particle vertices implying many-to-one correspondence between on-mass-shell diagrams and permutations? Or should one generalize the right-left rule in TGD framework?

4.10.2 *Fermion lines for fermions massless in 8-D sense*

What does one mean with particle line at the level of fermions?

(a) How the addition of CP_2 helicity and complete correlation between M^4 and CP_2 chiralities does affect the rules of $\mathcal{N}=4$ SUSY? Chiral invariance in 8-D sense guarantees fermion number conservation for quarks and leptons separately and means conservation of the product of M^4 and CP_2 chiralities for 2-fermion vertices. Hence only M^4 chirality need to be considered. M^4 massivation allows more 4-fermion vertices than $\mathcal{N}=4$ SUSY.

(b) One can assign to a given partonic orbit several lines as boundaries of string world sheets connecting the orbit to other partonic orbits. Supersymmetry could be understoond in two manners.

   i. The fermions generating the state of super-multiplet correspond to boundaries of different string world sheets which need not connect the string world sheet to same partonic orbit. This SUSY is dynamical and broken. The breaking is mildest breaking for line groups connected by string world sheets to same partonic orbit. Right handed neutrinos generated the least broken $\mathcal{N}=2$ SUSY.

   ii. Also single line carrying several fermions would provide realization of generalized SUSY since the multi-fermion state would be characterized by single 8-momentum and helicity. One would have $\mathcal{N}=4$ SUSY for quarks and leptons separately and $\mathcal{N}=8$ if both quarks and leptons are allowed. Conserved total for quark and antiquarks and leptons and antileptons characterize the lines as well.
   What would be the propagator associated with many-fermion line? The first guess is that it is just a tensor power of single fermion propagator applied to the tensor power of single fermion states at the end of the line. This gives power of $1/p^{2n}$ to the denominator, which suggests that residue integral in momentum space gives zero unless one as just single fermion state unless the vertices give compensating powers of p. The reduction of fermion number to 0 or 1 would simplify the diagrammatics enormously and one would have only 0 or 1 fermions per given string boundary line. Multi-fermion lines would represent gauge degrees of freedom and SUSY would be realized as gauge invariance. This view about SUSY clearly gives the simplest picture, which is also consistent with the earlier one, and will be assumed in the sequel

(c) The multiline containing n fermion oscillator operators can transform by chirality mixing in 2^n manners at 4-fermion vertex so that there is quite a large number of options for incoming lines with n_i fermions.

(d) In 4-D Dirac equation light-likeness implies a complete correlation between fermion number and chirality. In 8-D case light-likeness should imply the same: now chirality correspond to fermion number. Does this mean that one must assume just superposition of different M^4 chiralities at the fermion lines as 8-D Dirac equation requires. Or should one assume that virtual fermions at the end of the line have wrong chirality so that massless Dirac operator does not annihilate them?

4.10.3 *Fundamental vertices*

One can consider two candidates for fundamental vertices depending on whether one identifies the lines of Feynman diagram as fermion lines or as light-like orbits of partonic 2-surfaces. The latter vertices reduces microscopically to the fermionic 4-vertices.

(a) If many-fermion lines are identified as fundamental lines, 4-fermion vertex is the fundamental vertex assignable to single wormhole contact in the topological vertex defined by common partonic 2-surface at the ends of incoming light-like 3-surfaces. The discontinuity is what makes the vertex non-trivial.

(b) In the vertices generalization of OZI rule applies for many-fermion lines since there are no higher vertices at this level and interactions are mediated by classical induced gauge fields and chirality mixing. Classical induced gauge fields vanish if CP_2 projection is 1-dimensional for string world sheets and even gauge potentials vanish if the projection is to geodesic circle. Hence only the chirality mixing due to the mixing of M^4 and

CP_2 gamma matrices is possible and changes the fermionic M^4 chiralities. This would dictate what vertices are possible.

(c) The possibility of two helicity states for fermions suggests that the number of amplitudes is considerably larger than in $\mathcal{N} = 4$ SUSY. One would have 5 independent fermion amplitudes and at each 4-fermion vertex one should be able to choose between 3 options if the right-left rule generalizes. Hence the number of amplitudes is larger than the number of permutations possibly obtained using a generalization of right-left rule to right-middle-left rule.

(d) Note however that for massless particles in M^4 sense the reduction of helicity combinations for the fermion and antifermion making virtual gauge boson happens. The fermion and antifermion at the opposite wormhole throats have parallel four-momenta in good approximation. In M^4 they would have opposite chiralities and opposite helicities so that the boson would be M^4 scalar. No vector bosons would be obtained in this manner.

In 8-D context it is possible to have also vector bosons since the M^4 chiralities can be same for fermion and anti-fermion. The bosons are however massive, and even photon is predicted to have small mass given by p-adic thermodynamics [K12]. Massivation brings in also the M^4 helicity 0 state. Only if zero helicity state is absent, the fundamental four-fermion vertex vanishes for $++++$ and $----$ combinations and one extend the right-left rule to right-middle-left rule. There is however no good reason for he reduction in the number of 4-fermion amplitudes to take place.

4.10.4 *Partonic surfaces as 3-vertices*

At space-time level one could identify vertices as partonic 2-surfaces.

(a) At space-time level the fundamental vertices are 3-particle vertices with particle identified as wormhole contact carrying many-fermion states at both wormhole throats. Each line of BCFW diagram would be doubled. This brings in mind the representation of permutations and leads to ask whether this representation could be re-interpreted in TGD framework. For this option the generalization of the decomposition of diagram to 3-particle vertices is very natural. If the states at throats consist of bound states of fermions as SUSY suggests, one could characterize them by total 8-momentum and helicity in good approximation. Both helicities would be however possible also for fermions by chirality mixing.

(b) A genuine decomposition to 3-vertices and lines connecting them takes place if two of the fermions reside at opposite throats of wormhole contact identified as fundamental gauge boson (physical elementary particles involve two wormhole contacts).

The 3-vertex can be seen as fundamental and 4-fermion vertex becomes its microscopic representation. Since the 3-vertices are at fermion level 4-vertices their number is greater than two and there is no hope about the generalization of right-left rule.

4.10.5 *OZI rule implies correspondence between permutations and amplitudes*

The realization of the permutation in the same manner as for $\mathcal{N} = 4$ amplitudes does not work in TGD. OZI rule following from the absence of 4-fermion vertices however implies much simpler and physically quite a concrete manner to define the permutation for external fermion lines and also generalizes it to include braidings along partonic orbits.

(a) Already $\mathcal{N} = 4$ approach assumes decorated permutations meaning that each external fermion has effectively two states corresponding to labels k and $k + n$ (permutations are shifts to the right). For decorated permutations the number of external states is effectively 2^n and the number of decorated permutations is $2 \times n!$. The number of different helicity configurations in TGD framework is 2^n for incoming fermions at the vertex defined by the partonic 2-surface. By looking the values of these numbers for lowest integers one finds $2n \geq 2^n$: for $n = 2$ the equation is saturated. The inequality $log(n!) > nlog(n)/e) + 1$ (see Wikipedia) gives

$$\frac{log(2n!)}{log(2^n)} \geq \frac{log(2) + 1 + nlog(n/e)}{nlog(2)} = log(n/e)/log(2) + O(1/n)$$

so that the desired inequality holds for all interesting values of n.

(b) If OZI rule holds true, the permutation has very natural physical definition. One just follows the fermion line which must eventually end up to some external fermion since the only fermion vertex is 2-fermion vertex. The helicity flip would map $k \rightarrow k + n$ or vice versa.

(c) The labelling of diagrams by permutations generalizes to the case of diagrams involving partonic surfaces at the boundaries of causal diamond containing the external fermions and the partonic 2-surfaces in the interior of CD identified as vertices. Permutations generalize to braidings since also the braidings along the light-like partonic 2-surfaces are allowed. A quite concrete generalization of the analogs of braid diagrams in integrable 2-D theories emerges.

(d) BCFW bridge would be completely analogous to the fundamental braiding operation permuting two neighboring braid strands. The almost reduction to braid theory - apart from the presence of vertices conforms with the vision about reduction of TGD to almost topological QFT.

To sum up, the simplest option assumes SUSY as both gauge symmetry and broken dynamical symmetry. The gauge symmetry relates string boundaries with different fermion numbers and only fermion number 0 or 1 gives rise to a non-vanishing outcome in the residue integration and one obtains the picture used hitherto. If OZI rule applies, the decorated permutation symmetry generalizes to include braidings at the parton orbits and $k \rightarrow k \pm n$ corresponds to a helicity flip for a fermion going through the 4-vertex. OZI rules follows from the absence of non-linearities in Dirac action and means that 4-fermion vertices in the usual sense making theory non-renormalizable are absent. Theory is essentially free field theory in fermionic degrees of freedom and interactions in the sense of QFT are transformed to non-trivial topology of space-time surfaces.

3. If one can approximate space-time sheets by maps from M^4 to CP_2, one expects General Relativity and QFT description to be good approximations. GRT space-time is obtained by replacing space-time sheets with single sheet - a piece of slightly deformed Minkowski space but without assupmtion about imbedding to H. Induced classical gravitational field and gauge fields are sums of those associated with the sheets. The generalized Feynman diagrams with lines at various sheets and going also between sheets are projected to single piece of M^4. Many-sheetedness makes 1-homology non-trivial and implies analog of braiding, which should be however invisible at QFT limit.

A concrete manner to eliminate line crossing in non-planar amplitude to get nearer to non-planar amplitude could proceed roughly as follows. This is of course a pure guess motivated only by topological considerations. Professional might kill it in few seconds.

1. If the lines carry no quantum numbers, reconnection allows to eliminate the crossings. Consider the crossing line pair connecting AB in the initial state to CD in final state. The crossing lines are AD and BC. Reconnection can take place in two manners: AD and BC transform either to AB and CD or to AC and BD: neither line pair has crossing. The final state of the braid would be quantum superposition of the resulting more planar braids.

2. The crossed lines however carry different quantum numbers in the generic situation: for instance, they can be fermionic and bosonic. In this particular case the reconnection does not make sense since a line carrying fermion number would transform to a line carrying boson.

   In TGD framework all lines are fermion lines at fundamental level but the constraint due to different quantum numbers still remains and it is easy to see that mere reconnection is not enough. Fermion number conservation allows only one of the two alternatives to be realized. Conservation of quantum numbers forces to restrict gives an additional constraint which for simplest non-planar diagram with two crossed fermion lines forces the quantum numbers of fermions to be identical.

It seems also more natural to consider pairs of wormhole contacts defining elementary particles as "lines" in turn consisting of fermion lines. Yangian symmetry allows to develop a more detailed view about what this decomposition could mean.

Quantum number conservation demands that reconnection is followed by a formation of an additional internal line connecting the non-crossing lines obtained by reconnection. The additional line representing a quantum number exchange between the resulting non-crossing lines would guarantee the conservation of quantum numbers. This would bring in two additional vertices and one additional internal line. This would be enough to reduce planarity. The repeated application of this transformation should produced a sum of non-planar diagrams.

3. What could go wrong with this proposal? In the case of gauge theory the order of diagram increases by g^2 since two new vertices are generated. Should a multiplication by $1/g^2$ accompany this process? Or is this observation enough to kill the hypothesis in gauge theory framework? In TGD framework the situation is not understood well enough to say anything. Certainly the critical value of α_K implies that one cannot regard it as a free parameter and cannot treat the contributions from various orders as independent ones.

5 Could The Universe Be Doing Yangian Arithmetics?

One of the old TGD inspired really crazy ideas about scattering amplitudes is that Universe is doing some sort of arithmetics so that scattering amplitude are representations for computational sequences of minimum length. The idea is so crazy that I have even given up its original form, which led to an attempt to assimilate the basic ideas about bi-algebras, quantum groups [K2], Yangians [K17], and related exotic things. The work with twistor Grassmannian approach inspired a reconsideration of the original idea seriously with the idea that super-symplectic Yangian could define the arithmetics. I try to describe the background, motivation, and the ensuing reckless speculations in the following.

5.1 Do Scattering Amplitudes Represent Quantal Algebraic Manipulations?

I seems that tensor product $\otimes$ and direct sum $\oplus$ - very much analogous to product and sum but defined between Hilbert spaces rather than numbers - are naturally associated with the basic vertices of TGD. I have written about this a highly speculative chapter - both mathematically and physically [K19]. The chapter [K2] is a remnant of earlier similar speculations.

1. In $\otimes$ vertex 3-surface splits to two 3-surfaces meaning that the 2 "incoming" 4-surfaces meet at single common 3-surface and become the outgoing 3-surface: 3 lines of Feynman diagram meeting at their ends. This has a lower-dimensional shadow realized for partonic 2-surfaces. This topological 3-particle vertex would be higher-D variant of 3-vertex for Feynman diagrams.

2. The second vertex is trouser vertex for strings generalized so that it applies to 3-surfaces. It does not represent particle decay as in string models but the branching of the particle wave function so that particle can be said to propagate along two different paths simultaneously. In double slit experiment this would occur for the photon space-time sheets.

3. The idea is that Universe is doing arithmetics of some kind in the sense that particle 3-vertex in the above topological sense represents either multiplication or its time-reversal co-multiplication.

The product, call it $\circ$, can be something very general, say algebraic operation assignable to some algebraic structure. The algebraic structure could be almost anything: a random list of structures popping into mind consists of group, Lie-algebra, super-conformal algebra quantum algebra, Yangian, etc.... The algebraic operation $\circ$ can be group multiplication, Lie-bracket, its generalization to super-algebra level, etc...). Tensor product and thus linear (Hilbert) spaces are involved always, and in product operation tensor product $\otimes$ is replaced with $\circ$.

1. The product $A_k \otimes A_l \rightarrow C = A_k \circ A_l$ is analogous to a particle reaction in which particles A_k and A_l fuse to particle $A_k \otimes A_l \rightarrow C = A_k \circ A_l$. One can say that $\otimes$ between reactants is transformed to $\circ$ in the particle reaction: kind of bound state is formed.

2. There are very many pairs A_k, A_l giving the same product C just as given integer can be divided in many manners to a product of two integers if it is not prime. This of course suggests that elementary particles are primes of the algebra if this notion is defined for it! One can use some basis for the algebra and in this basis one has $C = A_k \circ A_l = f_{klm} A_m$, f_{klm} are the structure constants of the algebra and satisfy constraints. For instance, associativity $A(BC) = (AB)C$ is a constraint making the life of algebraist more tolerable and is almost routinely assumed.

   For instance, in the number theoretic approach to TGD associativity is proposed to serve as fundamental law of physics and allows to identify space-time surfaces as 4-surfaces with associative (quaternionic) tangent space or normal space at each point of octonionic imbedding space $M^4 \times CP_2$. Lie algebras are not associative but Jacobi-identities following from the associativity of Lie group product replace associativity.

3. Co-product can be said to be time reversal of the algebraic operation $\circ$. Co-product can be defined as $C = A_k \rightarrow \sum_{lm} f_k^{lm} A_l \otimes A_m$, where f_k^{lm} are the structure constants of the algebra. The outcome is quantum superposition of final states, which can fuse to C (the "reaction" $A_k \otimes A_l \rightarrow C = A_k \circ A_l$ is possible). One can say that $\circ$ is replaced with $\otimes$: bound state decays to a superposition of all pairs, which can form the bound states by product vertex.

There are motivations for representing scattering amplitudes as sequences of algebraic operations performed for the incoming set of particles leading to an outgoing set of particles with particles identified as algebraic objects acting on vacuum state. The outcome would be analogous to Feynman diagrams but only the diagram with minimal length to which a preferred extremal can be assigned is needed. Larger ones must be equivalent with it.

The question is whether it could be indeed possible to characterize particle reactions as computations involving transformation of tensor products to products in vertices and co-products to tensor products in co-vertices (time reversals of the vertices). A couple of examples gives some idea about what is involved.

1. The simplest operations would preserve particle number and to just permute the particles: the permutation generalizes to a braiding and the scattering matrix would be basically unitary braiding matrix utilized in topological quantum computation.

2. A more complex situation occurs, when the number of particles is preserved but quantum numbers for the final state are not same as for the initial state so that particles must interact. This requires both product and co-product vertices. For instance, $A_k \otimes A_l \rightarrow f_{kl}^m A_m$ followed by $A_m \rightarrow f_m^{rs} A_r \otimes A_s$ giving $A_k \rightarrow f_{kl}^m f_m^{rs} A_r \otimes A_s$ representing 2-particle scattering. State function reduction in the final state can select any pair $A_r \otimes A_s$ in the final state. This reaction is characterized by the ordinary tree diagram in which two lines fuse to single line and defuse back to two lines. Note also that there is a non-deterministic element involved. A given final state can be achieved from a given initial state after large enough number of trials. The analogy with problem solving and mathematical theorem proving is obvious. If the interpretation is correct, Universe would be problem solver and theorem prover!

3. More complex reactions affect also the particle number. 3-vertex and its co-vertex are the simplest examples and generate more complex particle number changing vertices. For instance, on twistor Grassmann approach on can construct all diagrams using two 3-vertices. This encourages the restriction to 3-vertice (recall that fermions have only 2-vertices)

4. Intuitively it is clear that the final collection of algebraic objects can be reached by a large - maybe infinite - number of ways. It seems also clear that there is the shortest manner to end up to the final state from a given initial state. Of course, it can happen that there is no

way to achieve it! For instance, if $\circ$ corresponds to group multiplication the co-vertex can lead only to a pair of particles for which the product of final state group elements equals to the initial state group element.

5. Quantum theorists of course worry about unitarity. How can avoid the situation in which the product gives zero if the outcome is element of linear space. Somehow the product should be such that this can be avoided. For instance, if product is Lie-algebra commutator, Cartan algebra would give zero as outcome.

5.2 Generalized Feynman Diagram As Shortest Possible Algebraic Manipulation Connecting Initial And Final Algebraic Objects

There is a strong motivation for the interpretation of generalized Feynman diagrams as shortest possible algebraic operations connecting initial and final states. The reason is that in TGD one does not have path integral over all possible space-time surfaces connecting the 3-surfaces at the ends of CD. Rather, one has in the optimal situation a space-time surface unique apart from conformal gauge degeneracy connecting the 3-surfaces at the ends of CD (they can have disjoint components).

Path integral is replaced with integral over 3-surfaces. There is therefore only single minimal generalized Feynman diagram (or twistor diagram, or whatever is the appropriate term). It would be nice if this diagram had interpretation as the shortest possible computation leading from the initial state to the final state specified by 3-surfaces and basically fermionic states at them. This would of course simplify enormously the theory and the connection to the twistor Grassmann approach is very suggestive. A further motivation comes from the observation that the state basis created by the fermionic Clifford algebra has an interpretation in terms of Boolean quantum logic and that in ZEO the fermionic states would have interpretation as analogs of Boolean statements $A \to B$.

To see whether and how this idea could be realized in TGD framework, let us try to find counterparts for the basic operations $\otimes$ and $\circ$ and identify the algebra involved. Consider first the basic geometric objects.

1. Tensor product could correspond geometrically to two disjoint 3-surfaces representing 3-particles. Partonic 2-surfaces associated with a given 3-surface represent second possibility. The splitting of a partonic 2-surface to two could be the geometric counterpart for co-product.

2. Partonic 2-surfaces are however connected to each other and possibly even to themselves by strings. It seems that partonic 2-surface cannot be the basic unit. Indeed, elementary particles are identified as pairs of wormhole throats (partonic 2-surfaces) with magnetic monopole flux flowing from throat to another at first space-time sheet, then through throat to another sheet, then back along second sheet to the lower throat of the first contact and then back to the thirst throat. This unit seems to be the natural basic object to consider. The flux tubes at both sheets are accompanied by fermionic strings. Whether also wormhole throats contain strings so that one would have single closed string rather than two open ones, is an open question.

3. The connecting strings give rise to the formation of gravitationally bound states and the hierarchy of Planck constants is crucially involved. For elementary particle there are just two wormhole contacts each involving two wormhole throats connected by wormhole contact. Wormhole throats are connected by one or more strings, which define space-like boundaries of corresponding string world sheets at the boundaries of CD. These strings are responsible for the formation of bound states, even macroscopic gravitational bound states.

5.3 Does Super-Symplectic Yangian Define The Arithmetics?

Super-symplectic Yangian would be a reasonable guess for the algebra involved.

1. The 2-local generators of Yangian would be of form $T_1^A = f^A_{BC} T^B \otimes T^C$, where f^A_{BC} are the structure constants of the super-symplectic algebra. n-local generators would be obtained

by iterating this rule. Note that the generator T_1^A creates an entangled state of T^B and T^C with f_{BC}^A the entanglement coefficients. T_n^A is entangled state of T^B and T_{n-1}^C with the same coefficients. A kind replication of T_{n-1}^A is clearly involved, and the fundamental replication is that of T^A. Note that one can start from any irreducible representation with well defined symplectic quantum numbers and form similar hierarchy by using T^A and the representation as a starting point.

That the hierarchy T_n^A and hierarchies irreducible representations would define a hierarchy of states associated with the partonic 2-surface is a highly non-trivial and powerful hypothesis about the formation of many-fermion bound states inside partonic 2-surfaces.

2. The charges T^A correspond to fermionic and bosonic super-symplectic generators. The geometric counterpart for the replication at the lowest level could correspond to a fermionic/bosonic string carrying super-symplectic generator splitting to fermionic/bosonic string and a string carrying bosonic symplectic generator T^A. This splitting of string brings in mind the basic gauge boson-gauge boson or gauge boson-fermion vertex.

   The vision about emission of virtual particle suggests that the entire wormhole contact pair replicates. Second wormhole throat would carry the string corresponding to T^A assignable to gauge boson naturally. T^A should involve pairs of fermionic creation and annihilation operators as well as fermionic and anti-fermionic creation operator (and annihilation operators) as in quantum field theory.

3. Bosonic emergence suggests that bosonic generators are constructed from fermion pairs with fermion and anti-fermion at opposite wormhole throats: this would allow to avoid the problems with the singular character of purely local fermion current. Fermionic and anti-fermionic string would reside at opposite space-time sheets and the whole structure would correspond to a closed magnetic tube carrying monopole flux. Fermions would correspond to superpositions of states in which string is located at either half of the closed flux tube.

4. The basic arithmetic operation in co-vertex would be co-multiplication transforming T_n^A to $T_{n+1}^A = f_{BC}^A T_n^B \otimes T^C$. In vertex the transformation of T_{n+1}^A to T_n^A would take place. The interpretations would be as emission/absorption of gauge boson. One must include also emission of fermion and this means replacement of T^A with corresponding fermionic generators F^A, so that the fermion number of the second part of the state is reduced by one unit. Particle reactions would be more than mere braidings and re-grouping of fermions and anti-fermions inside partonic 2-surfaces, which can split.

5. Inside the light-like orbits of the partonic 2-surfaces there is also a braiding affecting the M-matrix. The arithmetics involved would be therefore essentially that of measuring and "co-measuring" symplectic charges.

   Generalized Feynman diagrams (preferred extremals) connecting given 3-surfaces and many-fermion states (bosons are counted as fermion-anti-fermion states) would have a minimum number of vertices and co-vertices. The splitting of string lines implies creation of pairs of fermion lines. Whether regroupings are part of the story is not quite clear. In any case, without the replication of 3-surfaces it would not be possible to understand processes like e-e scattering by photon exchange in the proposed picture.

It is easy to hear the comments of the skeptic listener in the back row.

1. The attribute "minimal" - , which could translate to minimal value of Kähler function - is dangerous. It might be very difficult to determine what the minimal diagram is - consider only travelling salesman problem or the task of finding the shortest proof of theorem. It would be much nicer to have simple calculational rules.

   The original proposal might help here. The generalization of string model duality was in question. It stated that that it is possible to move the positions of the vertices of the diagrams just as one does to transform s-channel resonances to t-channel exchange. All loops of generalized diagrams could be be eliminated by transforming the to tadpoles and snipped away so that only tree diagrams would be left. The variants of the diagram were identified as different continuation paths between different paths connecting sectors of WCW corresponding to different 3-topologies. Each step in the continuation procedure would involve product or co-product defining what continuation between two sectors means for WCW spinors. The continuations between two states require some minimal number of steps. If this is true, all computations connecting identical states are also physically equivalent. The value of the vacuum functional be same for all of them. This looks very natural.

That the Kähler action should be same for all computational sequences connecting the same initial and final states looks strange but might be understood in terms of the vacuum degeneracy of Kähler action.

2. QFT perturbation theory requires that should have superposition of computations/continuations. What could the superposition of QFT diagrams correspond to in TGD framework?

   Could it correspond to a superposition of generators of the Yangian creating the physical state? After all, already quantum computer perform superpositions of computations. The fermionic state would not be the simplest one that one can imagine. Could AdS/CFT analogy allow to identify the vacuum state as a superposition of multi-string states so that single super-symplectic generator would be replaced with a superposition of its Yangian counterparts with same total quantum numbers but with a varying number of strings? The weight of a given superposition would be given by the total effective string world sheet area. The sum of diagrams would emerge from this superposition and would basically correspond to functional integration in WCW using exponent of Kähler action as weight. The stringy functional integral ("functional" if also wormhole contacts contain string portion, otherwise path integral) would give the perturbation theory around given string world sheet. One would have effective reduction of string theory.

5.4 How Does This Relate To The Ordinary Perturbation Theory?

One can of course worry about how to understand the basic results of the usual perturbation theory in this picture. How does one obtain a perturbation theory in powers of coupling constant, what does running coupling constant mean, etc...? I have already discussed how the superposition of diagrams could be understood in the new picture.

1. The QFT picture with running coupling constant is expected at QFT limit, when many-sheeted space-time is replaced with a slightly curved region of M^4 and gravitational field and gauge potentials are identified as sums of the deviations of induced metric from M^4 metric and classical induced gauge potentials associated with the sheets of the many-sheeted space-time. The running coupling constant would be due to the dependence of the size scale of CD, and p-adic coupling constant evolution would be behind the continuous one.

2. The notion of running coupling constant is very physical concept and should have a description also at the fundamental level and be due to a finite computational resolution, which indeed has very concrete description in terms of Noether charges of super-symplectic Yangian creating the states at the ends of space-time surface at the boundaries of CD. The space-time surface and the diagram associated with a given pair of 3-surfaces and stringy Noether charges associated with them can be characterized by a complexity measured in terms of the number of vertices (3-surface at which three 3-surfaces meet).

   For instance, 3-particle scattering can be possible only by using the simplest 3-vertex defined by product or co-product for pairs of 3-surfaces. In the generic case one has more complex diagram and what looks first 3-particle vertex has complex substructure rather than being simple product or co-product.

3. Complexity seems to have two separate aspects: the complexities of the positive and negative parts of zero energy state as many-fermion states and the complexity of associated 3-surfaces. The generalization of AdS/CFT however suggests that once the string world sheets and partonic 2-surfaces appearing in the diagram have been fixed, the space-time surface itself is fixed. The principle also suggests that the fixing partonic 2-surface and the strings connecting them at the boundaries of CD fixes the 3-surface apart from the action of sub-algebra of

Yangian acting as gauge algebra (vanishing classical Noether charges). If one can determine the minimal sequence of allowed algebraic operation of Yangian connecting initial and final fermion states, one knows the minimum number of vertices and therefore the topological structure of the connecting minimal space-time surface.

4. In QFT spirit one could describe the finite measurement resolution by introducing effective 3-point vertex, which is need not be product/co-produce anymore. 3-point scattering amplitudes in general involve microscopic algebraic structure involving several vertices. One can however give up the nice algebraic interpretation and just talk about effective 3-vertex for practical purposes. Just as the QFT vertex described by running coupling constant decomposes to sum of diagrams, product/co-product in TGD could be replaced with effective product/co-product expressible as a longer computation. This would imply coupling constant evolution.

   Fermion lines could however remain as such since they are massless in 8-D sense and mass renormalization does not make sense.

   Similar practical simplification could be done the initial and final states to get rid of superposition of the Yangian generators with different numbers of strings ("cloud of virtual particles"). This would correspond to wave function renormalization.

5. The number of vertices and wormhole contact orbits serves as a measure for the complexity of the diagram. Since fermion lines are associated with wormhole throats assignable with wormhole contacts identifiable as deformations CP_2 type vacuum extremals, one expects that the exponent of the Kähler function defining vacuum functional is in the first approximation the total CP_2 volume of wormhole contacts giving a measure for the importance of the contribution in functional integral. If it converges very rapidly only Gaussian approximation around maximum is needed.

6. Convergence depends on how large the fraction of volume of CP_2 is associated with a given wormhole contact. The volume is proportional to the length of the wormhole contact orbit. One expects exponential convergence with the number of fermion lines and their lengths for long lines. For short distances the exponential damping is small so that diagrams with microscopic structure of diagrams are needed and are possible. This looks like adding small scale details to the algebraic manipulations.

7. One must be of course be very cautious in making conclusions. The presence of $1/\alpha_K \propto h_{eff}$ in the exponent of Kähler function would suggest that for large values of h_{eff} only the 3-surfaces with smallest possible number of wormhole contact orbits contribute. On the other hand, the generalization of AdS/CFT duality suggests that Kähler action reducible to area of string world sheet in the effective metric defined by canonical momentum currents of Kähler action behaves as $\alpha_K^2 \propto 1/h_{eff}^2$. What does this mean?

To sum up, the identification of vertex as a product or co-product in Yangian looks highly promising approach. The Nother charges of the super-symplectic Yangian are associated with strings and are either linear or bilinear in the fermion field. The fermion fields associated with the partonic 2-surface defining the vertex are contracted with fermion fields associated with other partonic 2-surfaces using the same rule as in Wick expansion in quantum field theories. The contraction gives fermion propagator for each leg pair associated with two vertices. Vertex factor is proportional to the contraction of spinor modes with the operators defining the Noether charge or super charge and essentially Kähler-Dirac gamma matrix and the representation of the action of the symplectic generator on fermion realizable in terms of sigma matrices. This is very much like the corresponding expression in gauge theories but with gauge algebra replaced with symplectic algebra. The possibility of contractions of creation and annihilation operator for fermion lines associated with opposite wormhole throats at the same partonic 2-surface (for Noether charge bilinear in fermion field) gives bosonic exchanges as lines in which the fermion lines turns in time direction: otherwise only regroupings of fermions would take place.

5.5 This Was Not The Whole Story Yet

The proposed amplitude represents only the value of WCW spinor field for single pair of 3-surfaces at the opposite boundaries of given CD. Hence Yangian construction does not tell the whole story.

1. Yangian algebra would give only the vertices of the scattering amplitudes. On basis of previous considerations, one expects that each fermion line carries propagator defined by 8-momentum. The structure would resemble that of super-symmetric YM theory. Fermionic propagators should emerge from summing over intermediate fermion states in various vertices and one would have integrations over virtual momenta which are carried as residue integrations in twistor Grassmann approach. 8-D counterpart of twistorialization would apply.

2. Super-symplectic Yangian would give the scattering amplitudes for single space-time surface and the purely group theoretical form of these amplitudes gives hopes about the independence of the scattering amplitude on the pair of 3-surfaces at the ends of CD near the maximum of Kähler function. This is perhaps too much to hope except approximately but if true, the integration over WCW would give only exponent of Kähler action since metric and poorly defined Gaussian and determinants would cancel by the basic properties of Kähler metric. Exponent would give a non-analytic dependence on α_K.

   The Yangian supercharges are proportional to $1/\alpha_K$ since covariant Kähler-Dirac gamma matrices are proportional to canonical momentum currents of Kähler action and thus to $1/\alpha_K$. Perturbation theory in powers of $\alpha_K = g_K^2/4\pi\hbar_{eff}$ is possible after factorizing out the exponent of vacuum functional at the maximum of Kähler function and the factors $1/\alpha_K$ multiplying super-symplectic charges.

   The additional complication is that the characteristics of preferred extremals contributing significantly to the scattering amplitudes are expected to depend on the value of α_K by quantum interference effects. Kähler action is proportional to $1/\alpha_K$. The analogy of AdS/CFT correspondence states the expressibility of Kähler function in terms of string area in the effective metric defined by the anti-commutators of K-D matrices. Interference effects eliminate string length for which the area action has a value considerably larger than one so that the string length and thus also the minimal size of CD containing it scales as h_{eff}. Quantum interference effects therefore give an additional dependence of Yangian super-charges on h_{eff} leading to a perturbative expansion in powers of α_K although the basic expression for scattering amplitude would not suggest this.

3. Non-planar diagrams of quantum field theories should have natural counterpart and linking and knotting for braids defines it naturally. This suggests that the amplitudes can be interpreted as generalizations of braid diagrams defining braid invariants: braid strands would appear as legs of 3-vertices representing product and co-product. Amplitudes could be constructed as generalized braid invariants transforming recursively braided tree diagram to an un-braided diagram using same operations as for braids. In [**?**] I considered a possible breaking of associativity occurring in weak sense for conformal field theories and was led to the vision that there is a fractal hierarchy of braids such that braid strands themselves correspond to braids. This hierarchy would define an operad with subgroups of permutation group in key role. Hence it seems that various approaches to the construction of amplitudes converge.

6 Appendix: Some Mathematical Details About Grasmannian Formalism

In the following I try to summarize my amateurish understanding about the mathematical structure behind the Grassmann integral approach. The representation summarizes what I have gathered from the articles of Arkani-Hamed and collaborators [B10, B11]. These articles are rather sketchy and the article of Bullimore provides additional details [B14] related to soft factors. The article of Mason and Skinner provides excellent introduction to super-twistors [B7] and dual super-conformal invariance. I apologize for unavoidable errors.

Before continuing a brief summary about the history leading to the articles of Arkani-Hamed and others is in order. This summary covers only those aspects which I am at least somewhat familiar with and leaves out many topics about existence which I am only half-conscious.

1. It is convenient to start by summarizing the basic facts about bi-spinors and their conjugates allowing to express massless momenta as $p^{aa'} = \lambda_a \tilde{\lambda}_{a'}$ with $\tilde{\lambda}$ defined as complex conjugate of λ and having opposite chirality. When λ is scaled by a complex number $\tilde{\lambda}$ suffers an opposite scaling. The bi-spinors allow the definition of various inner products

$$\begin{aligned} \langle \lambda, \mu \rangle &= \epsilon_{ab} \lambda^a \mu^b \ , \\ \left[\tilde{\lambda}, \tilde{\mu} \right] &= \epsilon_{a'b'} \lambda^{a'} \mu^{b'} \ , \\ p \cdot q &= \langle \lambda, \mu \rangle \left[\tilde{\lambda}, \tilde{\mu} \right] \ , \ (q_{aa'} = \mu_a \tilde{\mu}_{a'}) \ . \end{aligned} \tag{6.1}$$

   If the particle has spin one can assign it a positive or negative helicity $h = \pm 1$. Positive helicity can be represented by introducing artitrary negative (positive) helicity bispinor μ_a ($\mu_{a'}$) not parallel to λ_a ($\mu_{a'}$) so that one can write for the polarization vector

$$\begin{aligned} \epsilon_{aa'} &= \frac{\mu_a \tilde{\lambda}_{a'}}{\langle \mu, \lambda \rangle} \ , \ \text{positive helicity} \ , \\ \epsilon_{aa'} &= \frac{\lambda_a \tilde{\mu}_{a'}}{\left[\tilde{\mu}, \tilde{\lambda} \right]} \ , \ \text{negative helicity} \ . \end{aligned} \tag{6.2}$$

   In the case of momentum twistors the μ part is determined by different criterion to be discussed later.

2. Tree amplitudes are considered and it is convenient to drop the group theory factor $Tr(T_1 T_2 \cdots T_n)$. The starting point is the observation that tree amplitude for which more than $n-2$ gluons have the same helicity vanish. MHV amplitudes have exactly $n-2$ gluons of same helicity-taken by a convention to be negative- have extremely simple form in terms of the spinors and reads as

$$A_n = \frac{\langle \lambda_x, \lambda_y \rangle^4}{\prod_{i=1}^{n} \langle \lambda_i, \lambda_{i+1} \rangle} \tag{6.3}$$

   When the sign of the helicities is changed $\langle .. \rangle$ is replaced with $[..]$.

3. The article of Witten [B8] proposed that twistor approach could be formulated as a twistor string theory with string world sheets “living” in 6-dimensional CP_3 possessing Calabi-Yau structure and defining twistor space. In this article Witten introduced what is known as half Fourier transform allowing to transform momentum integrals over light-cone to twistor integrals. This operation makes sense only in space-time signature $(2,2)$. Witten also demonstrated that maximal helicity violating (MHV) twistor amplitudes (two gluons with negative helicity) with n particles with $k+2$ negative helicities and l loops correspond in this approach to holomorphic 2-surfaces defined by polynomials defined by polynomials of degree $D = k - 1 + l$, where the genus of the surface satisfies $g \leq l$. AdS/CFT duality provides a second stringy approach to $\mathcal{N} = 4$ theory allowing to understand the scattering amplitudes in terms of Wilson loops with light-like edges: about this I have nothing to say. In any case, the generalization of twistor string theory to TGD context is highly attractive idea and will be considered later.

4. In the article [B5] Cachazo, Svrcek, and Witten propose the analog of Feynman diagrammatics in which MHV amplitudes can be used as analogs of vertices and ordinary $1/P^2$ propagator

as propagator to construct tree diagrams with arbitrary number of negative helicity gluons. This approach is not symmetric with respect to the change of the sign of helicities since the amplitudes with two positive helicities are constructed as tree diagrams. The construction is non-trivial because one must analytically continue the on mass shell tree amplitudes to off mass shell momenta. The problem is how to assign a twistor to these momenta. This is achieved by introducing an arbitrary twistor $\eta^{a'}$ and defining λ_a as $\lambda_a = p_{aa'}\eta^{a'}$. This works for both massless and massive case. It however leads to a loss of the manifest Lorentz invariance. The paper however argues and the later paper [B4, B4] shows rigorously that the loss is only apparent. In this paper also BCFW recursion formula is introduced allowing to construct tree amplitudes recursively by starting from vertices with 2 negative helicity gluons. Also the notion which has become known as BCFW bridge representing the massless exchange in these diagrams is introduced. The tree amplitudes are not tree amplitudes in gauge theory sense where correspond to leading singularities for which 4 or more lines of the loop are massless and therefore collinear. What is important that the very simple MHV amplitudes become the building blocks of more complex amplitudes.

5. The nex step in the progress was the attempt to understand how the loop corrections could be taken into account in the construction BCFW formula. The calculation of loop contributions to the tree amplitudes revealed the existence of dual super-conformal symmetry which was found to be possessed also by BCFW tree amplitudes besides conformal symmetry. Together these symmetries generate infinite-dimensional Yangian symmetry [B7].

6. The basic vision of Arkani-Hamed and collaborators is that the scattering amplitudes of $\mathcal{N}=4$ SYM are constructible in terms of leading order singularities of loop diagrams. These singularities are obtained by putting maximum number of momenta propagating in the lines of the loop on mass shell. The non-leading singularities would be induced by the leading singularities by putting smaller number of momenta on mass shell are dictated by these terms. A related idea serving as a starting point in [B10] is that one can define loop integrals as residue integrals in momentum space. If I have understood correctly, this means that one an imagine the possibility that the loop integral reduces to a lower dimensional integral for on mass shell particles in the loops: this would resemble the approach to loop integrals based on unitarity and analyticity. In twistor approach these momentum integrals defined as residue integrals transform to residue integrals in twistor space with twistors representing massless particles. The basic discovery is that one can construct leading order singularities for n particle scattering amplitude with $k+2$ negative helicities as Yangian invariants $Y_{n,k}$ for momentum twistors and invariants constructed from them by canonical operations changing n and k. The correspondence $k=l$ does not hold true for the more general amplitudes anymore.

6.1 Yangian Algebra And Its Super Counterpart

The article of Witten [B6] gives a nice discussion of the Yangian algebra and its super counterpart. Here only basic formulas can be listed and the formulas relevant to the super-conformal case are given.

6.1.1 Yangian algebra

Yangian algebra $Y(G)$ is associative Hopf algebra. The elements of Yangian algebra are labelled by non-negative integers so that there is a close analogy with the algebra spanned by the generators of Virasoro algebra with non-negative conformal weight. The Yangian symmetry algebra is defined by the following relations for the generators labeled by integers $n=0$ and $n=1$. The first half of these relations discussed in very clear manner in [B6] follows uniquely from the fact that adjoint representation of the Lie algebra is in question

$$\left[J^A, J^B\right] = f_C^{AB} J^C \quad , \quad \left[J^A, J^{(1)B}\right] = f_C^{AB} J^{(1)C} \quad . \tag{6.4}$$

Besides this Serre relations are satisfied. These have more complex and read as

$$
\begin{aligned}
\left[J^{(1)A},\left[J^{(1)B},J^{C}\right]\right]+\left[J^{(1)B},\left[J^{(1)C},J^{A}\right]\right]+\left[J^{(1)C},\left[J^{(1)A},J^{B}\right]\right]& \\
=\frac{1}{24}f^{ADK}f^{BEL}f^{CFM}f_{KLM}\{J_D,J_E,J_F\}\ ,& \\
\left[\left[J^{(1)A},J^{(1)B}\right],\left[J^{C},J^{(1)D}\right]\right]+\left[\left[J^{(1)C},J^{(1)D}\right],\left[J^{A},J^{(1)B}\right]\right]& \\
=\frac{1}{24}f^{AGL}f^{BEM}f_K^{CD}& \\
+f^{CGL}f^{DEM}f_K^{AB})f^{KFN}f_{LMN}\{J_G,J_E,J_F\}\ .&
\end{aligned}
\tag{6.5}
$$

The indices of the Lie algebra generators are raised by invariant, non-degenerate metric tensor g_{AB} or g^{AB}. $\{A,B,C\}$ denotes the symmetrized product of three generators.

Repeated commutators allow to generate the entire algebra whose elements are labeled by non-negative integer n. The generators obtain in this manner are n-local operators arising in $(n-1)$-commutator of $J^{(1)}$: s. For $SU(2)$ the Serre relations are trivial. For other cases the first Serre relation implies the second one so the relations are redundant. Why Witten includes it is for the purposed of demonstrating the conditions for the existence of Yangians associated with discrete one-dimensional lattices (Yangians exists also for continuum one-dimensional index).

Discrete one-dimensional lattice provides under certain consistency conditions a representation for the Yangian algebra. One assumes that each lattice point allows a representation R of J^A so that one has $J^A=\sum_i J_i^A$ acting on the infinite tensor power of the representation considered. The expressions for the generators J^{1A} are given as

$$
J^{(1)A} \quad = \quad f_{BC}^{A}\sum_{i<j}J_i^BJ_j^C \ . \tag{6.6}
$$

This formula gives the generators in the case of conformal algebra. This representation exists if the adjoint representation of G appears only one in the decomposition of $R\otimes R$. This is the case for SU(N) if R is the fundamental representation or is the representation of by k^{th} rank completely antisymmetric tensors.

This discussion does not apply as such to $\mathcal{N}=4$ case the number of lattice points is finite and corresponds to the number of external particles so that cyclic boundary conditions are needed guarantee that the number of lattice points reduces effectively to a finite number. Note that the Yangian in color degrees of freedom does not exist for SU(N) SYM.

As noticed, Yangian algebra is a Hopf algebra and therefore allows co-product. The co-product Δ is given by

$$
\begin{aligned}
\Delta(J^A) &= J^A\otimes 1+1\otimes J^A \ ,\\
\Delta(J^{(1)A}) &= J^{(1)A}\otimes 1+1\otimes J^{(1)A}+f_{BC}^{A}J^B\otimes J^C\ per,
\end{aligned}
\tag{6.7}
$$

Δ allows to imbed Lie algebra to the tensor product in non-trivial manner and the non-triviality comes from the addition of the dual generator to the trivial co-product. In the case that the single spin representation of $J^{(1)A}$ is trivial, the co-product gives just the expression of the dual generator using the ordinary generators as a non-local generator. This is assumed in the recent case and also for the generators of the conformal Yangian.

6.1.2 Super-Yangian

Also the Yangian extensions of Lie super-algebras make sense. From the point of physics especially interesting Lie super-algebras are $SU(m|m)$ and $U(m|m)$. The reason is that $PSU(2,2|4)$ (P refers to “projective”) acting as super-conformal symmetries of $\mathcal{N}=4$ SYM and this super group is a real form of $PSU(4|4)$. The main point of interest is whether this algebra allows Yangian representation and Witten demonstrated that this is indeed the case [B6].

These algebras are Z_2 graded and decompose to bosonic and fermionic parts which in general correspond to n- and m-dimensional representations of $U(n)$. The representation associated with the fermionic part dictates the commutation relations between bosonic and fermionic generators. The anti-commutator of fermionic generators can contain besides identity also bosonic generators if the symmetrized tensor product in question contains adjoint representation. This is the case if fermions are in the fundamental representation and its conjugate. For $SU(3)$ the symmetrize tensor product of adjoint representations contains adjoint (the completely symmetric structure constants d_{abc}) and this might have some relevance for the super $SU(3)$ symmetry.

The elements of these algebras in the matrix representation (no Grassmann parameters involved) can be written in the form

$$x = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \ .$$

a and d representing the bosonic part of the algebra are $n \times n$ matrices and $m \times m$ matrices corresponding to the dimensions of bosonic and fermionic representations. b and c are fermionic matrices are $n \times m$ and $m \times n$ matrices, whose anti-commutator is the direct sum of $n \times n$ and $n \times n$ matrices. For $n = m$ bosonic generators transform like Lie algebra generators of $SU(n) \times SU(n)$ whereas fermionic generators transform like $n \otimes \overline{n} \oplus \overline{n} \otimes n$ under $SU(n) \times SU(n)$. Supertrace is defined as $Str(x) = Tr(a) - Tr(b)$. The vanishing of Str defines $SU(n|m)$. For $n \neq m$ the super trace condition removes identity matrix and $PU(n|m)$ and $SU(n|m)$ are same. That this does not happen for $n = m$ is an important delicacy since this case corresponds to $\mathcal{N} = 4$ SYM. If any two matrices differing by an additive scalar are identified (projective scaling as now physical effect) one obtains $PSU(n|n)$ and this is what one is interested in.

Witten shows that the condition that adjoint is contained only once in the tensor product $R \otimes \overline{R}$ holds true for the physically interesting representations of $PSU(2,2|4)$ so that the generalization of the bilinear formula can be used to define the generators of $J^{(1)A}$ of super Yangian of $PU(2,2|4)$. The defining formula for the generators of the Super Yangian reads as

$$\begin{aligned} J_C^{(1)} &= g_{CC'} J^{(1)C'} = g_{CC'} f^{C'}_{AB} \sum_{i<j} J_i^A J_j^B \\ &= g_{CC'} f^{C'}_{AB} g^{AA'} g^{BB'} \sum_{i<j} J^i_{A'} J^j_{B'} \ . \end{aligned} \tag{6.8}$$

Here $g_{AB} = Str(J_A J_B)$ is the metric defined by super trace and distinguishes between $PSU(4|4)$ and $PSU(2,2|4)$. In this formula both generators and super generators appear.

6.1.3 Generators of super-conformal Yangian symmetries

The explicit formula for the generators of super-conformal Yangian symmetries in terms of ordinary twistors is given by

$$\begin{aligned} j_B^A &= \sum_{i=1}^{n} Z_i^A \partial_{Z_i^B} \ , \\ j_B^{(1)A} &= \sum_{i<j} (-1)^C \left[Z_i^A \partial_{Z_j^C} Z_j^C \partial_{Z_j^B} \right] \ . \end{aligned} \tag{6.9}$$

This formula follows from completely general formulas for the Yangian algebra discussed above and allowing to express the dual generators $j_N^{(1)}$ as quadratic expression of j_N involving structures constants. In this rather sketchy formula twistors are ordinary twistors. Note however that in the recent case the lattice is replaced with its finite cutoff corresponding to the external particles of the scattering amplitude. This probably corresponds to the assumption that for the representations considered only finite number of lattice points correspond to non-trivial quantum numbers or to cyclic symmetry of the representations.

In the expression for the amplitudes the action of transformations is on the delta functions and by partial integration one finds that a total divergence results. This is easy to see for the linear generators but not so for the quadratic generators of the dual super-conformal symmetries. A similar formula but with j_B^A and $j_B^{(1)A}$ interchanged applies in the representation of the amplitudes as Grassmann integrals using ordinary twistors. The verification of the generalization of Serre formula is also straightforward.

6.2 Twistors And Momentum Twistors And Super-Symmetrization

In [B7] the basics of twistor geometry are summarized. Despite this it is perhaps good to collect the basic formulas here.

6.2.1 Conformally compactified Minkowski space

Conformally compactified Minkowski space can be described as $SO(2,4)$ invariant (Klein) quadric

$$T^2 + V^2 - W^2 - X^2 - Y^2 - Z^2 = 0 \ . \tag{6.10}$$

The coordinates (T,V,W,X,Y,Z) define homogenous coordinates for the real projective space RP^5. One can introduce the projective coordinates $X_{\alpha\beta} = -X_{\beta\alpha}$ through the formulas

$$\begin{array}{lll} X_{01} = W - V \ , & X_{02} = Y + iX \ , & X_{03} = \frac{i}{\sqrt{2}}T - Z \ , \\ X_{12} = -\frac{i}{\sqrt{2}}(T+Z) \ , & X_{13} = Y - iX \ , & X_{23} = \frac{1}{2}(V+W) \ . \end{array} \tag{6.11}$$

The motivation is that the equations for the quadric defining the conformally compactified Minkowski space can be written in a form which is manifestly conformally invariant:

$$\epsilon^{\alpha\beta\gamma\delta}X_{\alpha\beta}X_{\gamma\delta} \ = \ 0 \ per. \tag{6.12}$$

The points of the conformally compactified Minkowski space are null separated if and only if the condition

$$\epsilon^{\alpha\beta\gamma\delta}X_{\alpha\beta}Y_{\gamma\delta} = 0 \tag{6.13}$$

holds true.

6.2.2 Correspondence with twistors and infinity twistor

One ends up with the correspondence with twistors by noticing that the condition is equivalent with the possibility to expression $X_{\alpha\beta}$ as

$$X_{\alpha\beta} = A_{[\alpha}B_{\beta]} \ , \tag{6.14}$$

where brackets refer to antisymmetrization. The complex vectors A and B define a point in twistor space and are defined only modulo scaling and therefore define a point of twistor space CP_3 defining a covering of 6-D Minkowski space with metric signature $(2,4)$. This corresponds to the fact that the Lie algebras of $SO(2,4)$ and $SU(2,2)$ are identical. Therefore the points of conformally compactified Minkowski space correspond to lines of the twistor space defining spheres CP_1 in CP_3.

One can introduce a preferred scale for the projective coordinates by introducing what is called infinity twistor (actually a pair of twistors is in question) defined by

$$I_{\alpha\beta} = \begin{pmatrix} \epsilon^{A'B'} & 0 \\ 0 & 0 \end{pmatrix} \ . \tag{6.15}$$

Infinity twistor represents the projective line for which only the coordinate X_{01} is non-vanishing and chosen to have value $X_{01} = 1$.

One can define the contravariant form of the infinite twistor as

$$I^{\alpha\beta} = \epsilon^{\alpha\beta\gamma\delta} I_{\gamma\delta} = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon^{AB} \end{pmatrix} \quad . \tag{6.16}$$

Infinity twistor defines a representative for the conformal equivalence class of metrics at the Klein quadric and one can express Minkowski distance as

$$(x-y)^2 = \frac{X^{\alpha\beta} Y_{\alpha\beta}}{I_{\alpha\beta} X^{\alpha\beta} I_{\mu\nu} Y^{\mu\nu}} \quad . \tag{6.17}$$

Note that the metric is necessary only in the denominator. In twistor notation the distance can be expressed as

$$(x-y)^2 = \frac{\epsilon(A,B,C,D)}{\langle AB \rangle \langle CD \rangle} \quad . \tag{6.18}$$

Infinite twistor $I_{\alpha\beta}$ and its contravariant counterpart project the twistor to its primed and unprimed parts usually denoted by $\mu^{A'}$ and λ^A and defined spinors with opposite chiralities.

6.2.3 Relationship between points of M^4 and twistors

In the coordinates obtained by putting $X_{01} = 1$ the relationship between space-time coordinates $x^{AA'}$ and $X^{\alpha\beta}$ is

$$X_{\alpha\beta} = \begin{pmatrix} -\frac{1}{2}\epsilon^{A'B'} x^2 & -ix^{A'}_{B} \\ ix^{B'}_{A} & \epsilon_{A,B} \end{pmatrix} \quad , \quad X^{\alpha\beta} = \begin{pmatrix} \epsilon_{A'B}{}' x^2 & -ix_{A'}^{B} \\ ix^{A}_{B'} & -\frac{1}{2}\epsilon^{AB} x^2 \end{pmatrix} \quad , \tag{6.19}$$

If the point of Minkowski space represents a line defined by twistors (μ_U, λ_U) and (μ_V, λ_V), one has

$$x^{AC'} = i\frac{(\mu_V \lambda_U - \mu_U \lambda_V)^{AC'}}{\langle UV \rangle} \tag{6.20}$$

The twistor μ for a given point of Minkowski space in turn is obtained from λ by the twistor formula by

$$\mu^{A'} = -ix^{AA'} \lambda_A \quad . \tag{6.21}$$

6.2.4 Generalization to the super-symmetric case

This formalism has a straightforward generalization to the super-symmetric case. CP_3 is replaced with $CP_{3|4}$ so that Grassmann parameters have four components. At the level of coordinates this means the replacement $[W_I] = [W_\alpha, \chi_\alpha]$. Twistor formula generalizes to

$$\mu^{A'} = -ix^{AA'} \lambda_A \quad , \quad \chi_\alpha = \theta^A_\alpha \lambda_A \quad . \tag{6.22}$$

The relationship between the coordinates of chiral super-space and super-twistors generalizes to

$$(x, \theta) = \left(i\frac{(\mu_V \lambda_U - \mu_U \lambda_V)}{\langle UV \rangle}, \frac{(\chi_V \lambda_U - \chi_U \lambda_V)}{\langle UV \rangle} \right) \tag{6.23}$$

The above formulas can be applied to super-symmetric variants of momentum twistors to deduce the relationship between region momenta x assigned with edges of polygons and twistors assigned with the ends of the light-like edges. The explicit formulas are represented in [B7]. The geometric picture is following. The twistors at the ends of the edge define the twistor pair representing the region momentum as a line in twistor space and the intersection of the twistor lines assigned with the region momenta define twistor representing the external momenta of the graph in the intersection of the edges.

6.2.5 Basic kinematics for momentum twistors

The super-symmetrization involves replacement of multiplets with super-multiplets

$$\Phi(\lambda, \tilde{\lambda}, \eta) = G^{+}(\lambda, \tilde{\lambda}) + \eta_i \Gamma^a \lambda, \tilde{\lambda}) + \cdots + \epsilon_{abcd}\eta^a\eta^b\eta^c\eta^d G^{-}(\lambda, \tilde{\lambda}) \ . \tag{6.24}$$

Momentum twistors are dual to ordinary twistors and were introduced by Hodges. The light-like momentum of external particle a is expressed in terms of the vertices of the closed polygon defining the twistor diagram as

$$p_i^{\mu} = x_i^{\mu} - x_{i+1}^{\mu} = \lambda_i \tilde{\lambda}_i \ \ , \ \ \theta_i - \theta_{i+1} = \lambda_i \eta_i \ . \tag{6.25}$$

One can say that massless momenta have a conserved super-part given by $\lambda_i\eta_i$. The dual of the super-conformal group acts on the region momenta exactly as the ordinary conformal group acts on space-time and one can construct twistor space for dua region momenta.

Super-momentum conservation gives the constraints

$$\sum p_i = 0 \ \ , \ \ \sum \lambda_i \eta_i = 0 \ . \tag{6.26}$$

The twistor diagrams correspond to polygons with edges with lines carrying region momenta and external massless momenta emitted at the vertices.

This formula is invariant under overall shift of the region momenta x_a^{μ}. A natural interpretation for x_a^{μ} is as the momentum entering to the the vertex where p_a is emitted. Overall shift would have interpretation as a shift in the loop momentum. x_a^{μ} in the dual coordinate space is associated with the line $Z_{a-1}Z_a$ in the momentum twistor space. The lines $Z_{a-1}Z_a$ and Z_aZ_{a+1} intersect at Z_a representing a light-like momentum vector p_a^{μ}.

The brackets $\langle abcd\rangle \equiv \epsilon_{IJKL}Z_a^IZ_b^JZ_c^KZ_d^L$ define fundamental bosonic conformal invariants appearing in the tree amplitudes as basic building blocks. Note that Z_a define points of 4-D complex twistor space to be distinguished from the projective twistor space CP_3. Z_a define projective coordinates for CP_3 and one of the four complex components of Z_a is redundant and one can take $Z_a^0 = 1$ without a loss of generality.

6.3 Brief Summary Of The Work Of Arkani-Hamed And Collaborators

The following comments are an attempt to summarize my far from complete understanding about what is involved with the representation as contour integrals. After that I shall describe in more detail my impressions about what has been done.

6.3.1 Limitations of the approach

Consider first the limitations of the approach.

1. The basis idea is that the representation for tree amplitudes generalizes to loop amplitudes. On other words, the amplitude defined as a sum of Yangian invariants expressed in terms of Grassmann integrals represents the sum of loops up to some maximum loop number. The problem is here that shifts of the loop momenta are essential in the UV regularization procedure. Fixing the coordinates $x_1, \cdots, x_n$ having interpretation as momenta associated with lines in the dual coordinate space allows to eliminate the non-uniqueness due to the common shift of these coordinates.

2. It is not however not possible to identify loop momentum as a loop momentum common to different loop integrals unless one restricts to planar loops. Non-planar diagrams are obtained from a planar diagram by permuting the coordinates x_i but this means that the unique coordinate assignment is lost. Therefore the representation of loop integrands as Grassmann integrals makes sense only for planar diagrams. From TGD point of view one could argue that this is one good reason for restricting the loops so that they are for on mass shell particles with non-parallel on mass shell four-momenta and possibly different sign of energies for given wormhole contact representing virtual particle.

3. IR regularization is needed even in $\mathcal{N} = 4$ for SYM given by "moving out on the Coulomb branch theory" so that IR singularities remain the problem of the theory.

6.3.2 What has been done?

The article proposes a generalization of the BCFW recursion relation for tree diagrams of $\mathcal{N} = 4$ for SYM so that it applies to planar diagrams with a summation over an arbitrary number of loops.

1. The basic goal of the article is to generalize the recursion relations of tree amplitudes so that they would apply to loop amplitudes. The key idea is following. One can formally represent loop integrand as a contour integral in complex plane whose coordinate parameterizes the deformations $Z_n \rightarrow Z_n + \epsilon Z_{n-1}$ and re-interpret the integral as a contour integral with oppositely oriented contour surrounding the rest of the complex plane which can be imagined also as being mapped to Riemann sphere. What happens only the poles which correspond to lower number of loops contribute this integral. One obtains a recursion relation with respect to loop number. This recursion seems to be the counterpart for the recursive construction of the loops corrections in terms of absorptive parts of amplitudes with smaller number of loop using unitarity and analyticity.

2. The basic challenge is to deduce the Grassmann integrands as Yangian invariants. From these one can deduce loop integrals by integration over the four momenta associated with the lines of the polygonal graph identifiable as the dual coordinate variables x_a. The integration over loop momenta can induce infrared divergences breaking Yangian symmetry. The big idea here is that the operations described above allow to construct loop amplitudes from the Yangian invariants defining tree amplitudes for a larger number of particles by removing external particles by fusing them to form propagator lines and by using the BCFW bridge to fuse lower-dimensional invariants. Hence the usual iterative procedure (bottom-up) used to construct scattering amplitudes is replaced with a recursive procedure (top-down). Of course, once lower amplitudes has been constructed they can be used to construct amplitudes with higher particle number.

3. The first guess is that the recursion formula involves the same lower order contributions as in the case of tree amplitudes. These contributions have interpretation as factorization of channels involving single particle intermediate states. This would however allow to reduce loop amplitudes to 3-particle loop amplitudes which vanish in $\mathcal{N} = 4$ SYM by the vanishing of coupling constant renormalization. The additional contribution is necessary and corresponds to a source term identifiable as a "forward limit" of lower loop integrand. These terms are obtained by taking an amplitude with two additional particles with opposite four-momenta and forming a state in which these particles are entangled with respect to momentum and other quantum numbers. Entanglement means integral over the massless momenta on one hand. The insertion brings in two momenta x_a and x_b and one can imagine that the loop is represented by a branching of propagator line. The line representing the entanglement of the massless states with massless momentum define the second branch of the loop. One can of course ask whether only massless momentum in the second branch. A possible interpretation is that this state is expressible by unitarity in terms of the integral over light-like momentum.

4. The recursion formula for the loop amplitude $M_{n,k,l}$ involves two terms when one neglects the possibility that particles can also suffer trivial scattering (cluster decomposition). This term basically corresponds to the Yangian invariance of n arguments identified as Yangian invariant of $n-1$ arguments with the same value of k.

(a) The first term corresponds to single particle exchange between particle groups obtained by splitting the polygon at two vertices and corresponds to the so called BCFW bridge for tree diagrams. There is a summation over different splittings as well as a sum over loop numbers and dimensions k for the Grassmann planes. The helicities in the two groups are opposite.

(b) Second term is obtained from an amplitude obtained by adding of two massless particles with opposite momenta and corresponds to $n+2, k+1, l-1$. The integration over the light-like momentum together with other operations implies the reduction $n+2 \rightarrow n$. Note that the recursion indeed converges. Certainly the allowance of added zero energy states with a finite number of particles is necessary for the convergence of the procedure.

6.4 The General Form Of Grassmannian Integrals

If the recursion formula proposed in [B11] is correct, the calculations reduce to the construction of N^kMHV (super) amplitudes. MHV refers to maximal helicity violating amplitudes with 2 negative helicity gluons. For N^kMHV amplitude the number of negative helicities is by definition $k+2$ [B10]. Note that the total right handed R-charge assignable to 4 super-coordinates η_i of negative helicity gluons can be identified as $R = 4k$. BCFW recursion formula [B4, B4] allows to construct from MHV amplitudes with arbitrary number of negative helicities.

The basic object of study are the leading singularities of color-stripped n-particle N^kMHV amplitudes. The discovery is that these singularities are expressible in terms Yangian invariants $Y_{n,k}(Z_1, \cdots, Z_n)$, where Z_i are momentum super-twistors. These invariants are defined by residue integrals over the compact $nk-1$-dimensional complex space $G(n,k) = U(n)/U(k) \times U(n-k)$ of k-planes of complex n-dimensional space. n is the number of external massless particles, k is the number negative helicity gluons in the case of N^kMHV amplitudes, and Z_a, $i = 1, \cdots, n$ denotes the projective 4-coordinate of the super-variant $CP^{3|4}$ of the momentum twistor space CP_3 assigned to the massless external particles is following. $Gl(n)$ acts as linear transformations in the n-fold Cartesian power of twistor space. Yangian invariant $Y_{n,k}$ is a function of twistor variables Z^a having values in super-variant $CP_{3|3}$ of momentum twistor space CP_3 assigned to the massless external particles being simple algebraic functions of the external momenta.

It is also possible to define N^kMHV amplitudes in terms of Yangian invariants $L_{n,k+2}(W_1, \cdots, W_n)$ by using ordinary twistors W_a and identical defining formula. The two invariants are related by the formula $L_{n,k+2}(W_1, \cdots, W_n) = M^{tree}_{MHV} \times Y_{n,k}(Z_1, \cdots, Z_n)$. Here M^{tree}_{MHV} is the tree contribution to the maximally helicity violating amplitude for the scattering of n particles: recall that these amplitudes contain two negative helicity gluons whereas the amplitudes containing a smaller number of them vanish [B5]. One can speak of a factorization to a product of n-particle amplitudes with $k-2$ and 2 negative helicities as the origin of the duality. The equivalence between the descriptions based on ordinary and momentum twistors states the dual conformal invariance of the amplitudes implying Yangian symmetry. It has been conjectured that Grassmannian integrals generate all Yangian invariants.

The formulas for the Grassmann integrals for twistors and momentum twistors appearing in the expressions of N^kMHV amplitudes are given by following expressions.

1. The integrals $L_{n,k}(W_1, \cdots, W_n)$ associated with $N^{k-2}MHV$ amplitudes in the description based on ordinary twistors correspond to k negative helicities and are given by

$$\begin{aligned} L_{n,k}(W_1, \cdots, W_n) &= \frac{1}{Vol(GL(2))} \int \frac{d^{k\times n}C_{\alpha a}}{(1\cdots k)(2\cdots k+1)\cdots(n1\cdots k-1)} \times \\ &\times \prod_{\alpha=1}^{k} d^{4|4}Y_\alpha \prod_{i=1}^{n} \delta^{4|4}(W_i - C_{\alpha i}Y_\alpha) \ . \end{aligned} \tag{6.27}$$

Here $C_{\alpha a}$ denote the $n \times k$ coordinates used to parametrize the points of $G_{k,n}$.

2. The integrals $Y_{n,k}(W_1, \cdots, W_n)$ associated with N^kMHV amplitudes in the description based on momentum twistors are defined as

$$Y_{n,k}(Z_1,\cdots,Z_n) \quad = \quad \frac{1}{Vol(GL(k))}\times\int\frac{d^{k\times n}C_{\alpha a}}{(1\cdots k)(2\cdots k+1)\cdots(n1\cdots k-1)}\times\prod_{\alpha=1}^{k}\delta^{4|4}(C_{\alpha a}Z_a)\ . \tag{6.28}$$

The possibility to select $Z_a^0=1$ implies $\sum_k C_{\alpha k}=0$ allowing to eliminate $C_{\alpha n}$ so that the actual number of coordinates Grassman coordinates is $nk-1$. As already noticed, $L_{n,k+2}(W_1,\cdots,W_n)=M_{MHV}^{tree}\times Y_{n,k}(Z_1,\cdots,Z_n)$. Momentum twistors are obviously calculationally easier since the value of k is smaller by two units.

The $4k$ delta functions reduce the number of integration variables of contour integrals from nk to $(n-4)k$ in the bosonic sector (the definition of delta functions involves some delicacies not discussed here). The n quantities $(m,\cdots m+k)$ are $k\times k$-determinants defined by subsequent columns from m to $m+k-1$ of the $k\times n$ matrix defined by the coordinates $C_{\alpha a}$ and correspond geometrically to the k-volumes of the k-dimensional parallel-pipeds defined by these column vectors. The fact that the scalings of twistor space coordinates Z_a can be compensated by scalings of $C_{\alpha a}$ deforming integration contour but leaving the residue integral invariant so that the integral depends on projective twistor coordinates only.

Since the integrand is a rational function, a multi-dimensional residue calculus allows to deduce the values of these integrals as residues associated with the poles of the integrand in a recursive manner. The poles correspond to the zeros of the $k\times k$ determinants appearing in the integrand or equivalently to singular lower-dimensional parallel-pipeds. It can be shown that local residues are determined by $(k-2)(n-k-2)$ conditions on the determinants in both cases. The value of the integral depends on the explicit choice of the integration contour for each variable $C_{\alpha a}$ left when delta functions are taken into account. The condition that a correct form of tree amplitudes is obtained fixes the choice of the integration contours.

For the ordinary twistors W the residues correspond to projective configurations in CP_{k-1}, or more precisely in the space $CP_{k-1}^n/Gl(k)$, which is $(k-1)n-k^2$-dimensional space defining the support for the residues integral. $Gl(k)$ relates to each other different complex coordinate frames for k-plane and since the choice of frame does not affect the plane itself, one has $Gl(k)$ gauge symmetry as well as the dual $Gl(n-k)$ gauge symmetry.

CP_{k-1} comes from the fact that C_{α_k} are projective coordinates: the amplitudes are indeed invariant under the scalings $W_i\rightarrow t_iW_i$, $C_{\alpha i}\rightarrow tC_{\alpha i}$. The coset space structure comes from the fact that $Gl(k)$ is a symmetry of the integrand acting as $C_{\alpha i}\rightarrow\Lambda_\alpha^{\ \beta}C_{\beta_i}$. This analog of gauge symmetry allows to fix k arbitarily chosen frame vectors $C_{\alpha i}$ to orthogonal unit vectors. For instance, one can have $C_{\alpha i}=\delta_{\alpha_i}$ for $\alpha=i\in 1,\cdots,k$. This choice is discussed in detail in [B10]. The reduction to CP_{k-1} implies the reduction of the support of the integral to line in the case of MHV amplitudes and to plane in the case of NMHV as one sees from the expression $d\mu=\prod_\alpha d^{4|4}Y_\alpha\prod_{i=1}^n\delta^{4|4}(W_i-C_{\alpha i}Y_\alpha)$. For $(i_1,\cdots,i_k)=0$ the vectors $i_1,..i_k$ belong to $k-2$-dimensional plane of CP_{k-1}. In the case of $NMHV$ (N^2MHV) amplitudes this translates at the level of twistors to the condition that the corresponding twistors $\{i_1,i_2,i_3\}$ ($\{i_1,i_2,i_3,i_4\}$) are collinear (in the same plane) in twistor space. This can be understood from the fact that the delta functions in $d\mu$ allow to express W_i in terms of $k-1$ Y_α: s in this case.

The action of conformal transformations in twistor space reduces to the linear action of $SU(2,2)$ leaving invariant Hermitian sesquilinear form of signature $(2,2)$. Therefore the conformal invariance of the Grassmannian integral and its dual variant follows from the possibility to perform a compensating coordinate change for $C_{\alpha a}$ and from the fact that residue integral is invariant under small deformations of the integration contour. The above described relationship between representations based on twistors and momentum twistors implies the full Yangian invariance.

6.5 Canonical Operations For Yangian Invariants

General l-loop amplitudes can be constructed from the basic Yangian invariants defined by N^kMHV amplitudes by various operations respecting Yangian invariance apart from possible IR anomalies. There are several operations that one can perform for Yangian invariants $Y_{n,k}$ and all these operations appear in the recursion formula for planar all loop amplitudes. These operations are described in [B11] much better than I could do it so that I will not go to any details. It is possible to add and remove particles, to fuse two Yangian invariants, to merge particles, and to construct from two Yangian invariants a higher invariant containing so called BCFW bridge representing single particle exchange using only twistorial methods.

6.5.1 Inverse soft factors

Inverse soft factors add to the diagram a massless collinear particles between particles a and b and by definition one has

$$O_{n+1}(a,c,b,\cdots) = \frac{\langle ab\rangle}{\langle ac\rangle\langle cb\rangle}O_n(a'b') \ . \tag{6.29}$$

At the limit when the momentum of the added particle vanishes both sides approach the original amplitude. The right-handed spinors and Grassmann parameters are shifted

$$\begin{array}{ll} \tilde{\lambda}'_a = \tilde{\lambda}_a + \frac{\langle cb\rangle}{\langle ab\rangle}\tilde{\lambda}_c \ , & \tilde{\lambda}'_b = \tilde{\lambda}_b + \frac{\langle ca\rangle}{\langle ba\rangle}\tilde{\lambda}_c \ , \\ \eta'_a = \eta_a + \frac{\langle cb\rangle}{\langle ab\rangle}\eta_c \ , & \eta'_b = \eta_b + \frac{\langle ca\rangle}{\langle ba\rangle}\eta_c \ . \end{array} \tag{6.30}$$

There are two kinds of inverse soft factors.

1. The addition of particle leaving the value k of negative helicity gluons unchanged means just the re-interpretation

$$Y'_{n,k}(Z_1,\cdots,Z_{n-1},Z_n) = Y_{n-1,k}(Z_1,\cdots,Z_{n-1}) \tag{6.31}$$

without actual dependence on Z_n. There is however a dependence on the momentum of the added particle since the relationship between momenta and momentum twistors is modified by the addition obtained by applying the basic rules relating region super momenta and momentum twistors (light-like momentum determines λ_i and twistor equations for x_i and λ_i,η_i determine (μ_i,χ_i)) is expressible assigned to the external particles [B14]. Modifications are needed only for the new vertex and its neighbors.

2. The addition of a particle increasing k with single unit is a more complex operation which can be understood in terms of a residue of $Y_{n,k}$ proportional to $Y_{n-1,k-1}$ and Yangian invariant $[z_1\cdots z_5]$ with five arguments constructed from basic Yangian invariants with four arguments. The relationship between the amplitudes is now

$$Y'_{n,k}(..,Z_{n-1}Z_n,Z_1\cdots) = [n-2\ n-1\ n\ 1\ 2]\times Y_{n-1,k-1}(\cdots\hat{Z}_{n-1},\hat{Z}_1,\cdots) \ . \tag{6.32}$$

Here

$$[abcde] = \frac{\delta^{0|4}(\eta_a\langle bcde\rangle + cyclic)}{\langle abcd\rangle\langle bcde\rangle\langle cdea\rangle\langle deab\rangle\langle eabc\rangle} \ . \tag{6.33}$$

denoted also by $R(a,b,c,d,e)$ is the fundamental R-invariant appearing in one loop corrections of MHV amplitudes and will appears also in the recursion formulas. $\langle abcd\rangle$ is the fundamental super-conformal invariant associated with four super twistors defined in terms of the permutation symbol.

$\hat{Z}_{n-1},\hat{Z}_1$ are deformed momentum twistor variables. The deformation is determined from the relationship between external momenta, region momenta and momentum twistor variables. $\hat{Z}^1$ is the intersection $\hat{Z}^1=(n-2\ n-1\ 2)\cap(12)$ of the the line (12) with the plane $(n-2\ n-1\ 2)$ and $\hat{Z}^{n-1}$ the intersection $\hat{Z}^1=(12n)\cap(n-2\ n-1)$ of the the line $(n-2\ n-1)$ with the plane

$(12n)$. The interpretation for the intersections at the level of ordinary Feynman diagrams is in terms of the collinearity of the four-momenta involved with the underlying box diagram with parallel on mass shell particles. These result from unitarity conditions obtained by putting maximal number of loop momenta on mass shell to give the leading singularities.

The explicit expressions for the momenta are

$$\begin{aligned} \hat{Z}^1 &\equiv (n-2\ n-1\ 2) \cap (12) Z_1 = \langle 2\ n-2\ n-1\ n\rangle + Z_2 \langle n-2\ n-1\ n\ 1\rangle \ , \\ \hat{Z}^{n-1} &\equiv (12n) \cap (n-2\ n-1) = Z_{n-2}\langle n-2\ n-1\ n\ 2\rangle + Z_{n-1}\langle n\ 1\ 2\ n-2\rangle \ . \end{aligned} \tag{6.34}$$

These intersections also appear in the expressions defining the recursion formula.

6.5.2 Removal of particles and merge operation

Particles can be also removed. The first manner to remove particle is by integrating over the twistor variable characterizing the particle. This reduces k by one unit. Merge operation preserves the number of loops but removes a particle particle by identifying the twistor variables of neighboring particles. This operation corresponds to an integral over on mass shell loop momentum at the level of tree diagrams and by Witten's half Fourier transform can be transformed to twistor integral.

The product

$$Y'(Z_1, \cdots Z_n) = Y_1(Z_1, \cdots Z_m) \times Y_2(Z_{m+1}, \cdots Z_n) \tag{6.35}$$

of two Yangian invariants is again a Yangian invariant. This is not quite trivial since the dependence of region momenta and momentum twistors on the momenta of external particles makes the operation non-trivial.

Merge operation allows to construct more interesting invariants from the products of Yangian invariants. One begins from a product of Yangian invariants (Yangian invariant trivially) represented cyclically as points of circle and identifies the last twistor argument of given invariant with the first twistor argument of the next invariant and performs integrals over the momentum twistor variables appearing twice. The soft k-increasing and preserving operations can be described also in terms of this operation for Yangian invariants such that the second invariant corresponds to 3-vertex. The cyclic merge operation applied to four MHV amplitudes gives NMHV amplitudes associated with on mass shell momenta in box diagrams. By applying similar operation to NMHV amplitudes and MHV amplitudes one obtains 2-loop amplitudes. In [B11] examples about these operations are described.

6.5.3 BCFW bridge

BCFW bridge allows to build general tree diagrams from MHV tree diagrams [B4, B4] and recursion formula of [B11] generalizes this to arbitrary diagrams. At the level of Feynman diagrams it corresponds to a box diagram containing general diagrams labeled by L and R and MHV and $\overline{MHV}$ 3-vertices ($\overline{MHV}$ 3-vertex allows expression in terms of MHV diagrams) with the lines of the box on mass shell so that the three momenta emanating from the vertices are parallel and give rise to a one-loop leading singularity.

At the level of Feynman diagrams BCFW bridge corresponds to so called "two-mass hard" leading singularities associated with box diagrams with light-like momenta at the four lines of the diagram [B10]. The motivation for the study of these diagrams comes from the hypothesis the leading order singularities obtained by putting as many particles as possible on mass shell contain the data needed to construct scattering amplitudes of $\mathcal{N}=4$ SYM completely. This representation of the leading singularities generalizes to arbitrary loops. The recent article is a continuation of this program to planar amplitudes.

Also BCFW bridge allows an interpretation as a particular kind fusion for Yang invariants and involves all the basic operations. One starts from the amplitudes $Y^L_{n_L,k_L}$ and $Y^R_{n_R,k_R}$ and constructs an amplitude $Y'_{n_L+n_R,k_L+k_R+1}$ representing the amplitude which would correspond to a generalization of the MHV diagrams with the two tree diagrams connected by the MHV propagator (BCFW bridge) replaced with arbitrary loop diagrams. Particle "1" *resp.* "j+1" is added by the soft k-increasing factor to Y_{n_L+1,k_L+1} *resp.* Y_{n_R+1,k_R+1} giving amplitude with $n+2$ particles and with k-charge equal to k_L+k_R+2. The subsequent operations must reduce k-charge by one unit. First repeated "1" and "j+1" are identified with their copies by k conserving merge operation, and after that one performs an integral over the twistor variable Z^I associated with the internal line obtained and reducing k by one unit. The soft k-increasing factors bring in the invariants $[n-1\ n\ 1\ I\ j+2]$ associated with Y_L and $[1\ I\ j+1\ j\ j-1]$ associated with Y_R. The integration contour is chosen so that it selects the pole defined by $\angle n-1\ n\ 1\ I\rangle$ in the denominator of $[n-1\ n\ 1\ I\ j+2]$ and the pole defined by $\langle 1\ I\ j+1\ j\rangle$ in the denominator of $[1\ I\ j+1\ j\ j-1]$.

The explicit expression for the BCFW bridge is very simple:

$$\begin{aligned}(Y_L\otimes_{BCFW}Y_R)(1,\cdots,n) &= [n-1\ n\ 1\ j\ j+1]\times Y_R(1,\cdots,j,I)Y_L(I,j+1,\cdots,n-1,\hat{n})\ ,\\ \hat{n} &= (n-1\ n)\cap(j\ j+1\ 1)\ ,\ \ I=(j\ j+1)\cap(n-1\ n\ 1)\ .\end{aligned}\tag{6.36}$$

6.5.4 Single cuts and forward limit

Forward limit operation is used to increase the number of loops by one unit. The physical picture is that one starts from say 1-loop amplitude and cuts one line by assigning to the pieces of the line opposite light-like momenta having interpretation as incoming and outgoing particles. The resulting amplitude is called forward limit. The only reasonable interpretation seems to be that the loop integration is expressed by unitarity as forward limit meaning cutting of the line carrying the loop momentum. This operation can be expressed in a manifestly Yangian invariant way as entangled removal of two particles with the merge operation meaning the replacement $Z_n\rightarrow Z_{n-1}$. Particle $n+1$ is added adjacent to A,B as a k-increasing inverse soft factor and then A and B are removed by entangled integration, and after this merge operation identifies $n+1$ and 1.

Forward limit is crucial for the existence of loops and for Yangian invariants it corresponds to the poles arising from $\langle(AB)_qZ_n(z)Z_1)\rangle$ the integration contour Z_n+zZ_{n-1} around Z_b in the basic formula $M=\oint(dz/z)M_n$ leading to the recursion formula. A and B denote the momentum twistors associated with opposite light-like momenta. In the generalized unitarity conditions the singularity corresponds to the cutting of line between particles n and 1 with momenta q and $-q$, summing over the multiplet of stats running around the loop. Between particles n_2 and 1 one has particles $n-1$, n with momenta $q,-q$. $q=x_1-x_n=-x_n+x_{n-1}$ giving $x_1=x_{n-1}$. Light-likeness of q means that the lines $(71)=(76)$ and (15) intersect. At the forward limit giving rise to the pole Z_6 and Z_7 approach to the intersection point $(76)\cap(15)$. In a generic gauge theories the forward limits are ill-defined but in super-symmetric gauge theories situation changes.

The corresponding Yangian operation removes two external particles with opposite four-momenta and involves integration over two twistor variables Z_a and Z_b and gives rise to the following expression

$$\int_{GL(2)}Y(\cdots,Z_n,Z_A,Z_B,Z_1,\cdots)\ .\tag{6.37}$$

The integration over $GL(2)$ corresponds to integration over twistor variables associated Z_A and Z_B. This operation allows addition of a loop to a given amplitude. The line Z_aZ_b represents loop momentum on one hand and the dual x-coordinate identified as momentum propagating along the line on the other hand.

The integration over these variables is equivalent to an integration over loop momentum as the explicit calculation of [B11] (see pages 12-13) demonstrates. If the integration contours are products in the product of twistor spaces associated with a and b the and gives lower order Yangian invariant as answer. It is however also possible to choose the integration contour to be entangled in the sense that it cannot be reduced to a product of integration contours in the Cartesian product of twistor spaces. In this case the integration gives a loop integral. In the removal operation Yangian invariance can be broken by IR singularities associated with the integration contour and the procedure does not produce genuine Yangian invariant always.

What is highly interesting from TGD point of view is that this integral can be expressed as a contour integral over $CP_1 \times CP_1$ combined with integral over loop momentum. If TGD vision about generalized Feynman graps in zero energy ontology is correct, the loop momentum integral is discretized to an an integral over discrete mass shells and perhaps also to a sum over discretized momenta and one can therefore avoid IR singularities.

6.6 Explicit Formula For The Recursion Relation

Recall that the recursion formula is obtained by considering super-symmetric momentum-twistor deformation $Z_n \rightarrow Z_n + zZ_{n-1}$ and by integrating over z to get the identity

$$M_{n,k,l} = \oint \frac{dz}{z} \hat{M}_{n,k,l}(z) \ . \tag{6.38}$$

This integral equals to integral with reversed integration contour enclosing the exterior of the contour. The challenge is to deduce the residues contributing to the residue integral and the claim of [B11] is that these residues reduce to simple basic types.

1. The first residue corresponds to a pole at infinity and reduces the particle number by one giving a contribution $M_{n-1,k,l}(1,\cdots,n-1)$ to $M_{n,k,l}(1,\cdots,n-1,n)$. This is not totally trivial since the twistor variables are related to momenta in different manner for the two amplitudes. This gives the first contribution to the right hand side of the formula below.

2. Second pole corresponds to the vanishing of $\langle Z_n(z) Z_1 Z_j Z_{j+1}\rangle$ and corresponds to the factorization of channels. This gives the second BCFW contribution to the right hand side of the formula below. These terms are however not enough since the recursion formula would imply the reduction to expressions involving only loop corrections to 3-loop vertex which vanish in $\mathcal{N}=4$ SYM.

3. The third kind of pole results when $\langle (AB)_q Z_n(z) Z_1\rangle$ vanishes in momentum twistor space. $(AB)_q$ denotes the line in momentum twistor space associated with q: th loop variable.

The explicit formula for the recursion relation yielding planar all loop amplitudes is obtained by putting all these pieces together and reads as

$$\begin{aligned} M_{n,k,l}(1,\cdots,n) &= M_{n-1,k,l}(1,\cdots,n-1) \\ &+ \sum_{n_L,k_L,l_L;j} [j\ j+1\ n-1\ n\ 1]\, M^R_{n_R,k_R,l_R}(1,\cdots,j,I_j) \times M^L_{n_L,k_L,l_L}(I_j,j+1,\cdots,\hat{n}_j) \\ &+ \int_{GL(2)} [AB\ n-1\ n\ 1]\, M_{n+2,k+1,n,k-1}(1,\cdots,\hat{n}_{AB},\hat{A},B) \ , \\ n_L &+ n_R = n+2 \ , \ \ k_L+k_R = k-1 \ , \ \ l_R+l_L = l \ . \end{aligned} \tag{6.39}$$

The momentum super-twistors are given by

$$\begin{aligned} &\hat{n}_j = (n-1\ n)\cap(j\ j+1\ 1) \ , \quad I_j = (j\ j+1\ 1)\cap(n-1\ n\ 1) \ , \\ &\hat{n}_{AB} = (n-1\ n)\cap(AB\ 1) \ , \quad \hat{A} = (AB)\cap(n-1\ n\ 1) \ . \end{aligned} \tag{6.40}$$

The index l labels loops in $n+2$-particle amplitude and the expression is fully symmetrized with equal weight for all loop integration variables $(AB)_l$. A and B are removed by entangled integration meaning that $GL(2)$ contour is chosen to encircle points where both points A, B on the line (AB) are located at the intersection of the line (AB) with the plane $(n-1\ n\ 1)$. GL(2) integral can be done purely algebraically in terms of residues.

In [B11] and [B14] explicit calculations for N^kMHV amplitudes are carried out to make the formulas more concrete. For N^1MHV amplitudes second line of the formula vanishes and the integrals are rather simple since the determinants are 1×1 determinants.

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