Riemann Hypothesis and Physics

M. Pitkänen,

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Email: matpitka@luukku.com.
http://tgdtheory.com/public_html/.
Recent postal address: Karkinkatu 3 I 3, 00360, Karkkila, Finland.

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Abstract

Riemann hypothesis states that the nontrivial zeros of Riemann Zeta function lie on the critical line $Re(s) = 1/2$. Since Riemann zeta function allows a formal interpretation as thermodynamical partition function for a quantum field theoretical system consisting of bosons labeled by primes, it is interesting to look Riemann hypothesis from the perspective of physics. The complex value of temperature is not however consistent with thermodynamics. In zero energy ontology one obtains quantum theory as a square root of thermodynamics and this objection can be circumvented and a nice argument allowing to interpret RH physically emerges.

Conformal invariance leads to a beautiful generalization of Hilbert-Polya conjecture allowing to interpret RH in terms of coherent states rather than energy eigenstates of a Hamiltonian. In zero energy ontology the interpretation is that the coherent states in question represent Bose-Einstein condensation at criticality. Zeros of zeta correspond to coherent states orthogonal to the coherent state characterized by $s = 0$, which has finite norm, and therefore does not represent Bose-Einstein condensation.

Quantum TGD and also TGD inspired theory of consciousness provide additional view points to the hypothesis and suggests sharpening of Riemann hypothesis, detailed strategies of proof of the sharpened hypothesis, and heuristic arguments for why the hypothesis is true. These considerations are however highly speculative and are represented at the end of the chapter.

1. *Super-conformal invariance and generalization of Hilbert-Polya hypothesis*

Super-conformal invariance inspires a strategy for proving the Riemann hypothesis. The vanishing of the Riemann Zeta reduces to an orthogonality condition for the eigenfunctions of a non-Hermitian operator D^+ having the zeros of Riemann Zeta as its eigenvalues. The construction of D^+ is inspired by the conviction that Riemann Zeta is associated with a physical system allowing super-conformal transformations as its symmetries and second quantization in terms of the representations of the super-conformal algebra. The eigenfunctions of D^+ are analogous to coherent states of a harmonic oscillator and in general they are not orthogonal to each other. The states orthogonal to a vacuum state (having a negative norm squared) correspond to the zeros of Riemann Zeta. The physical states having a positive norm squared correspond to the zeros of Riemann Zeta at the critical line. Riemann hypothesis follows both from the hermiticity and positive definiteness of the metric in the space of states corresponding to the zeros of ζ. Also conformal symmetry in appropriate sense implies Riemann hypothesis and after one year from the discovery of the basic idea it became clear that one can actually construct a rigorous twenty line long analytic proof for the Riemann hypothesis using a standard argument from Lie group theory.

2. *Zero energy ontology and RH*

A further approach to RH is based on zero energy ontology and is consistent with the approach based on the notion of coherent state. The postulate that all zero energy states for Riemann system are zeros of zeta and critical in the sense being non-normalizable (Bose-Einstein condensation) combined with the fact that $s = 1$ is the only pole of ζ implies that the all zeros of ζ correspond to $Re(s) = 1/2$ so that RH follows from purely physical assumptions. The behavior at $s = 1$ would be an essential element of the argument. The interpretation as a zero energy counterpart of a coherent state seems to makes sense also now. Note that in ZEO coherent state property is in accordance with energy conservation. In the case of coherent states of Cooper pairs same applies to fermion number conservation. With this interpretation the condition would state orthogonality with respect to the coherent zero energy state characterized by $s = 0$, which has finite norm and does not represent Bose-Einstein condensation. This would give a connection for the proposal for the strategy for proving Riemann Hypothesis by replacing eigenstates of energy with coherent states.

3. *Miscellaneous ideas*

During years I have also considered several ideas about Riemann hypothesis which I would not call miscellaneous. I have moved them to the end of the chapter because of the highly speculative nature.

1 Introduction

Riemann hypothesis states that the nontrivial zeros of Riemann Zeta function lie on the critical line $Re(s) = 1/2$. Since Riemann zeta function allows a formal interpretation as thermodynamical partition function for a quantum field theoretical system consisting of bosons labeled by primes, it is interesting to look Riemann hypothesis from the perspective of physics. The complex value of temperature is not however consistent with thermodynamics. In zero energy ontology one obtains quantum theory as a square root of thermodynamics and this objection can be circumvented and a nice argument allowing to interpret RH physically emerges.

Conformal invariance leads to a beautiful generalization of Hilbert-Polya conjecture allowing to interpret RH in terms of coherent states rather than energy eigenstates of a Hamiltonian. In zero energy ontology the interpretation is that the coherent states in question represent Bose-Einstein condensation at criticality. Zeros of zeta correspond to coherent states orthogonal to the coherent state characterized by $s = 0$, which has finite norm, and therefore does not represent Bose-Einstein condensation.

Quantum TGD and also TGD inspired theory of consciousness provide additional view points to the hypothesis and suggests sharpening of Riemann hypothesis, detailed strategies of proof of the sharpened hypothesis, and heuristic arguments for why the hypothesis is true. These considerations are however highly speculative and are represented at the end of the chapter.

1.1 Super-Conformal Invariance And Generalization Of Hilbert-Polya Hypothesis

Super-conformal invariance inspires a strategy for proving the Riemann hypothesis. The vanishing of the Riemann Zeta reduces to an orthogonality condition for the eigenfunctions of a non-Hermitian operator D^+ having the zeros of Riemann Zeta as its eigenvalues. The construction of D^+ is inspired by the conviction that Riemann Zeta is associated with a physical system allowing super-conformal transformations as its symmetries and second quantization in terms of the representations of the super-conformal algebra. The eigenfunctions of D^+ are analogous to coherent states of a harmonic oscillator and in general they are not orthogonal to each other. The states orthogonal to a vacuum state (having a negative norm squared) correspond to the zeros of Riemann Zeta. The physical states having a positive norm squared correspond to the zeros of Riemann Zeta at the critical line. Riemann hypothesis follows both from the hermiticity and positive definiteness of the metric in the space of states corresponding to the zeros of ζ. Also conformal symmetry in appropriate sense implies Riemann hypothesis and after one year from the discovery of the basic idea it became clear that one can actually construct a rigorous twenty line long analytic proof for the Riemann hypothesis using a standard argument from Lie group theory.

1.2 Zero Energy Ontology And RH

A further approach to RH is based on zero energy ontology and is consistent with the approach based on the notion of coherent state. The postulate that all zero energy states for Riemann system are zeros of zeta and critical in the sense being non-normalizable (Bose-Einstein condensation) combined with the fact that $s = 1$ is the only pole of ζ implies that the all zeros of ζ correspond to $Re(s) = 1/2$ so that RH follows from purely physical assumptions. The behavior at $s = 1$ would be an essential element of the argument. The interpretation as a zero energy counterpart of a coherent state seems to makes sense also now. Note that in ZEO coherent state property is in accordance with energy conservation. In the case of coherent states of Cooper pairs same applies to fermion number conservation. With this interpretation the condition would state orthogonality with respect to the coherent zero energy state characterized by $s = 0$, which has finite norm and does not represent Bose-Einstein condensation. This would give a connection for the proposal for the strategy for proving Riemann Hypothesis by replacing eigenstates of energy with coherent states.

1.3 Miscellaneous Ideas

During years I have also considered several ideas about Riemann hypothesis which I would not call miscellaneous. I have moved them to the end of the chapter because of the highly speculative nature.

1.3.1 Logarithmic waves for zeros of zeta as complex algebraic numbers?

The idea that the evolution of cognition involves the increase of the dimensions of finite-dimensional extensions of p-adic numbers associated with p-adic space-time sheets emerges naturally in TGD inspired theory of consciousness. A further input that led to a connection with Riemann Zeta was the work of Hardmuth Mueller [B1] suggesting strongly that e and its $p-1$ powers at least should belong to the extensions of p-adics. The basic objects in Mueller's approach are so called logarithmic waves $exp(iklog(u))$ which should exist for $u = n$ for a suitable choice of the scaling momenta k.

Logarithmic waves appear also as the basic building blocks (the terms $n^s = exp(log(n)(Re[s] + iIm[s]))$ in Riemann Zeta. This inspires naturally the hypothesis that also Riemann Zeta function is universal in the sense that it is defined at is zeros $s = 1/2 + iy$ not only for complex numbers but also for all p-adic number fields provided that an appropriate finite-dimensional extensions involving also transcendentals are allowed. This allows in turn to algebraically continue Zeta to any number field. The zeros of Riemann zeta are determined by number theoretical quantization and are thus universal and should appear in the physics of critical systems. The hypothesis $log(p) = \frac{q_1(p)exp[q_2(p)]}{\pi}$ explains the length scale hierarchies based on powers of e, primes p and Golden Mean.

Mueller's logarithmic waves lead also to an elegant concretization of the Hilbert Polya conjecture and to a sharpened form of Riemann hypothesis: the phases q^{-iy} for the zeros of Riemann Zeta belong to a finite-dimensional extension of R_p for any value of primes q and p and any zero $1/2+iy$ of ζ. The question whether the imaginary parts of the Riemann Zeta are linearly independent (as assumed in the previous work) or not is of crucial physical significance. Linear independence implies that the spectrum of the super-symplectic weights is essentially an infinite-dimensional lattice. Otherwise a more complex structure results. The numerical evidence supporting the translational invariance of the correlations for the spectrum of zeros together with p-adic considerations leads to the working hypothesis that for any prime p one can express the spectrum of zeros as the product of a subset of Pythagorean phases and of a fixed subset U of roots of unity. The spectrum of zeros could be expressed as a union over the translates of the same basic spectrum defined by the roots of unity translated by the phase angles associated with a subset of Pythagorean phases: this is consistent with what the spectral correlations strongly suggest. That decompositions defined by different primes p yield the same spectrum would mean a powerful number theoretical symmetry realizing p-adicities at the level of the spectrum of Zeta.

These approaches reflect the evolution of the vision about TGD based physics as a generalized number theory. Two new realizations of the super-conformal algebra result and the second realization has direct application to the modelling of $1/f$ noise. The zeros of ζ would code for the states of an arithmetic quantum field theory coded also by infinite primes: also the hierarchical structure of the many-sheeted space-time would be coded.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://tgdtheory.fi/cmaphtml.html` [L4]. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L5].

2 General Vision

Quantum TGD has inspired several strategies of proof of the Riemann hypothesis. The first strategy is based on the modification of Hilbert Polya hypothesis by requiring that the physical system in question has super-conformal transformations as its symmetries. Second strategy is based on considerations based on TGD inspired quantum theory of cognition and a generalization of the number concept inspired by it. Together with some physical inputs one ends up to a hypothesis that Riemann Zeta is well defined in all number fields near its zeros provided finite-dimensional extensions of p-adic numbers are allowed. This hypothesis generalizes the earlier hypothesis assuming that the extensions are trivial or at most algebraic. Third strategy is based on, what I call, Universality Principle.

There are also strong physical motivations to say something explicit about the spectrum of zeros and here p-adicization program inspires the hypothesis the numbers q^{iy}, q prime, belong to a finite algebraic extension of p-adic number field R_p for every prime p. The findings about the correlations of the spectrum of zeros inspire very concrete hypothesis about the spectrum of zeros as a union of translates of the same basic spectrum and this hypothesis is supported by the physical identification of the zeros of Zeta as super-symplectic conformal weights.

2.1 Generalization Of The Number Concept And Riemann Hypothesis

The hypothesis about p-adic physics as physics of cognition leads to a generalization of the notion of number obtained by gluing reals and various p-adic number fields together along rational numbers common to all of them. This structure is visualizable as a book like structure with pages represented by the number fields and the rim of the book represented by rationals. Even this structure can be generalized by allowing all finite-dimensional extensions of p-adic numbers including also those containing transcendental numbers and performing similar identification. Kind of fractal book might serve as a visualization of this structure.

In TGD inspired theory of consciousness intentions are assumed to correspond to quantum jumps involving the transformation of p-adic space-time sheets to real ones. An intuitive expectation is p-adic and real space-time sheets to each other must have a maximum number of common rational points. The building of idealized model for this transformation leads to the problem of defining functions having Taylor series with rational coefficients and continuable to both real and p-adic functions from a subset of rational numbers (or points of space-time sheet with rational coordinates). In this manner one ends up with the hypothesis that p-adic space-time sheets correspond to finite-dimensional extensions of p-adic numbers, which can involve also transcendental numbers such as e. This leads to a series of number theoretic conjectures.

The idea that the evolution of cognition involves the increase of the dimensions of finite-dimensional extensions of p-adic numbers associated with p-adic space-time sheets emerges naturally in TGD inspired theory of consciousness. A further input that led to a connection with Riemann Zeta was the work of Hardmuth Mueller [B1] suggesting strongly that e and its $p-1$ powers at least should belong to extensions of p-adics. The basic objects in Mueller's approach are so called logarithmic waves $exp(iklog(u))$ which should exist for $u=n$ for a suitable choice of the scaling momenta k.

Logarithmic waves appear also as the basic building blocks (the terms $n^s = exp(log(n)(Re[s]+iIm[s]))$ in Riemann Zeta. This inspires naturally the hypothesis that also Riemann Zeta function is universal in the sense that it is defined at is zeros $s=1/2+iy$ not only for complex numbers but also for all p-adic number fields provided that an appropriate finite-dimensional extensions involving also transcendentals are allowed. This allows in turn to algebraically continue Zeta to any number field. The zeros of Riemann zeta are determined by number theoretical quantization and are thus universal and should appear in the physics of critical systems. A hierarchy of number theoretical conjectures stating that a finite number of iterated logarithms about transcendentals appearing in the extension forms a closed system under the operation of taking logarithms. Mueller's logarithmic waves lead also to an elegant concretization of the Hilbert Polya conjecture and to a sharpened form of Riemann hypothesis: the complex numbers p^{-iy} for the zeros of Riemann Zeta belong to a finite-dimensional extension of R_p for any value of p and any zero $1/2+iy$ of ζ.

2.2 Modified Form Of Hilbert-Polya Hypothesis

Super-conformal invariance inspires a strategy for proving (not a proof of, as was the first over-optimistic belief) the Riemann hypothesis. The vanishing of Riemann Zeta reduces to an orthogonality condition for the eigenfunctions of a non-Hermitian operator D^+ having the zeros of Riemann Zeta as its eigenvalues. The construction of D^+ is inspired by the conviction that Riemann Zeta is associated with a physical system allowing super-conformal transformations as its symmetries and second quantization in terms of the representations of super-conformal algebra. The eigenfunctions of D^+ are analogous to the so called coherent states and in general not orthogonal to each other. The states orthogonal to a vacuum state (having a negative norm squared) correspond to the zeros of Riemann Zeta. The physical states having a positive norm squared correspond to the zeros of Riemann Zeta at the critical line and possibly those having $Re[s] > 1/2$.

A possible proof of the Riemann hypothesis by reductio ad absurdum results if one assumes that the states corresponding to zeros of ζ span a space with a hermitian metric. Riemann hypothesis follows both from the hermiticity and positive definiteness of the metric in the space of states corresponding to the zeros of ζ. Also conformal invariance in appropriate sense implies Riemann hypothesis. Indeed, a rather rigorous proof of Riemann hypothesis results from the observation that certain generator of conformal algebra permutes the two zeros located symmetrically with respect to the critical line. If the action of this generator exponentiates, Riemann hypothesis follows since exponentiation would imply the existence of infinite number of zeros along a line parallel to $Re[s]$-axis. One can formulate this argument rigorously using first order differential equation, and if one forgets all the preceding refined philosophical arguments, one can prove Riemann hypothesis using twenty line long analytic argument! Perhaps Ramajunan could have made this!

As already noticed, the state space metric can be made positive definite provided Riemann hypothesis holds true. Thus the system in question might quite well serve as a concrete physical model for quantum critical systems possessing super-conformal invariance as both dynamical and gauge symmetry.

2.3 Riemann Hypothesis In Zero Energy Ontology

Zeta reduces to a product $\zeta(s) = \prod_p Z_p(s)$ of partition functions $Z_p(s) = 1/[1-p^{-s}]$ over particles labelled by primes p. This relates very closely also to infinite primes and one can talk about Riemann gas with particle momenta/energies given by $log(p)$. s is in general complex number and for the zeros of the zeta one has $s = 1/2 + iy$. The imaginary part y is non-rational number. At $s = 1$ zeta diverges and for $Re(s) \leq 1$ the definition of zeta as product fails. Physicist would interpret this as a phase transition taking place at the critical line $s = 1$ so that one cannot anymore talk about Riemann gas. Should one talk about Riemann liquid? Or - anticipating what follows- about quantum liquid? What the vanishing of zeta could mean physically? Certainly the thermodynamical interpretation as sum of something interpretable as thermodynamical probabilities apart from normalization fails.

The basic problem with this interpretation is that it is only formal since the temperature parameter is complex. How could one overcome this problem?

A possible answer emerged as I read the interview.

1. One could interpret zeta function in the framework of TGD - or rather in zero energy ontology (ZEO) - in terms of square root of thermodynamics! This would make possible the complex analog of temperature. Thermodynamical probabilities would be replaced with probability amplitudes.

2. Thermodynamical probabilities would be replaced with complex probability amplitudes, and Riemann zeta would be the analog of vacuum functional of TGD which is product of exponent of Kähler function - Kähler action for Euclidian regions of space-time surface - and exponent of imaginary Kähler action coming from Minkowskian regions of space-time surface and defining Morse function. In QFT picture taking into account only the Minkowskian regions of space-time would have only the exponent of this Morse function: the problem is that path integral does not exist mathematically. In thermodynamics picture taking into account only the Euclidian regions of space-time one would only the exponent of Kähler function and would lose interference effects fundamental for QFT type systems. In quantum TGD both Kähler and Morse are present. With rather general assumptions the imaginary part and real part of exponent of vacuum functional are proportional to each other and to sum over the values of Chern-Simons action for 3-D wormhole throats and for space-like 3-surfaces at the ends of CD. This is non-trivial.

3. Zeros of zeta would in this case correspond to a situation in which the integral of the vacuum functional over the "world of classical worlds" (WCW) vanishes. The pole of ζ at $s = 1$ would correspond to divergence fo the integral for the modulus squared of Kähler function.

What the vanishing of the zeta could mean if one accepts the interpretation quantum theory as a square root of thermodynamics?

1. What could the infinite value of zeta at $s = 1$ mean? The The interpretation in terms of square root of thermodynamics implied following. In zero energy ontology zeta function function decomposition to $\prod_p Z_p$ corresponds to a product of single particle partition functions for which one can assigns probabilities $p^{-s}/Z_p(s)$ to single particle states. This does not make sense physically for complex values of s.

2. In ZEO one can however assume that the complex number p^{-sn} define the entanglement coefficients for positive and negative energy states with energies $nlog(p)$ and -$nlog(p)$: n bosons with energy log(p) just as for black body radiation. The sum over amplitudes over over all combinations of these states with some bosons labelled by primes p gives Riemann zeta which vanishes at critical line if RH holds.

3. One can also look for the values of thermodynamical probabilities given by $|p^{-ns}|^2 = p{-}n$ at critical line irrespective of zero. The sum over these gives for given p the factor $p/(p-1)$ and the product of all these factors gives $\zeta(1) = \infty$. Thermodynamical partition function diverges. The physical interpretation is in terms of Bose-Einstein condensation.

4. What the vanishing of the trace for the matrix coding for zeros of zeta defined by the amplitudes is physically analogous to the statement $\int \Psi dV = 0$ and is indeed true for many systems such as hydrogen atom. But what this means? Does it say that the zero energy state is orthogonal to vacuum state defined by unit matrix between positive and negative energy states? In any case, zeros and the pole of zeta would be aspects of one and same thing in this interpretation. This is an something genuinely new and an encouraging sign. Note that in TGD based proposal for a strategy for proving Riemann hypothesis, similar condition states that coherent state is orthogonal to "false" tachyonic vacuum.

5. RH would state in this framework that all zeros of ζ correspond to zero energy states for which thermodynamical partition function diverges. Another manner to say this is that the system is critical. (Maximal) Quantum Criticality is indeed the key postulate about TGD Universe and fixes the Kähler coupling strength characterizing the theory uniquely (plus possible other free parameters). Quantum Criticality guarantees that the Universe is maximally complex. Physics as generalized number theory would suggest that also number theory is quantum critical! When the sum over numbers proportional to probabilities diverges, the probabilities are considerably different from zero for infinite number of states. At criticality the presence of fluctuations in all scales implying fractality indeed implies this. A more precise interpretation is in terms of Bose-Eisntein condensation.

6. The postulate that all zero energy states for Riemann system are zeros of zeta and critical in the sense being non-normalizable (Bose-Einstein condensation) combined with the fact that $s = 1$ is the only pole of ζ implies that the all zeros of ζ correspond to $Re(s) = 1/2$ so that RH follows from purely physical assumptions. The behavior at $s = 1$ would be an essential element of the argument. The interpretation as a zero energy counterpart of a coherent state seems to makes sense also now. Note that in ZEO coherent state property is in accordance with energy conservation. In the case of coherent states of Cooper pairs same applies to fermion number conservation. With this interpretation the condition would state orthogonality with respect to the coherent zero energy state characterized by $s = 0$, which has finite norm and does not represent Bose-Einstein condensation. This would give a connection for the proposal for the strategy for proving Riemann Hypothesis by replacing eigenstates of energy with coherent states.

3 Riemann Hypothesis And Super-Conformal Invariance

Hilbert and Polya [A12] conjectured a long time ago that the non-trivial zeroes of Riemann Zeta function could have spectral interpretation in terms of the eigenvalues of a suitable self-adjoint differential operator H such that the eigenvalues of this operator correspond to the imaginary parts of the nontrivial zeros $z = x + iy$ of ζ. One can however consider a variant of this hypothesis stating that the eigenvalue spectrum of a non-hermitian operator D^+ contains the non-trivial zeros of ζ. The eigen states in question are eigen states of an annihilation operator type operator D^+ and analogous to the so called coherent states encountered in quantum physics [A14]. In particular, the eigenfunctions are in general non-orthogonal and this is a quintessential element of the the proposed strategy of proof.

In the following an explicit operator having as its eigenvalues the non-trivial zeros of ζ is constructed.

1. The construction relies crucially on the interpretation of the vanishing of ζ as an orthogonality condition in a hermitian metric which is is a priori more general than Hilbert space inner product.

2. Second basic element is the scaling invariance motivated by the belief that ζ is associated with a physical system which has super-conformal transformations [A13] as its symmetries.

The core elements of the construction are following.

1. All complex numbers are candidates for the eigenvalues of D^+ (formal hermitian conjugate of D) and genuine eigenvalues are selected by the requirement that the condition $D^\dagger = D^+$ holds true in the set of the genuine eigenfunctions. This condition is equivalent with the hermiticity of the metric defined by a function proportional to ζ.

2. The eigenvalues turn out to consist of $z = 0$ and the non-trivial zeros of ζ and only the eigenfunctions corresponding to the zeros with $Re[s] = 1/2$ define a subspace possessing a hermitian metric. The vanishing of ζ tells that the "physical" positive norm eigenfunctions (in general *not* orthogonal to each other), are orthogonal to the "un-physical" negative norm eigenfunction associated with the eigenvalue $z = 0$.

The proof of the Riemann hypothesis by reductio ad absurdum results if one assumes that the space $\mathcal{V}$ spanned by the states corresponding to the zeros of ζ inside the critical strip has a hermitian induced metric. Riemann hypothesis follows also from the requirement that the induced metric in the spaces subspaces $\mathcal{V}_s$ of $\mathcal{V}$ spanned by the states Ψ_s and $\Psi_{1-\overline{s}}$ does not possess negative eigenvalues: this condition is equivalent with the positive definiteness of the metric in $\mathcal{V}$. Conformal invariance in the sense of gauge invariance allows only the states belonging to $\mathcal{V}$. Riemann hypothesis follows also from a restricted form of a dynamical conformal invariance in $\mathcal{V}$. This allows the reduction of the proof to a standard analytic argument used in Lie-group theory.

3.1 Modified Form Of The Hilbert-Polya Conjecture

One can modify the Hilbert-Polya conjecture by assuming scaling invariance and giving up the hermiticity of the Hilbert-Polya operator. This means introduction of the non-hermitian operators D^+ and D which are hermitian conjugates of each other such that D^+ has the nontrivial zeros of ζ as its complex eigenvalues

$$D^+\Psi = z\Psi. \tag{3.1}$$

The counterparts of the so called coherent states [A14] are in question and the eigenfunctions of D^+ are not expected to be orthogonal in general. The following construction is based on the idea that D^+ also allows the eigenvalue $z = 0$ and that the vanishing of ζ at z expresses the orthogonality of the states with eigenvalue $z = x + iy \neq 0$ and the state with eigenvalue $z = 0$ which turns out to have a negative norm.

The trial

$$\begin{aligned} D &= L_0 + V, \quad D^+ = -L_0 + V \\ L_0 &= t\tfrac{d}{dt}, \qquad V = \tfrac{dlog(F)}{d(log(t))} = t\tfrac{dF}{dt}\tfrac{1}{F} \end{aligned} \tag{3.2}$$

is motivated by the requirement of invariance with respect to scalings $t \to \lambda t$ and $F \to \lambda F$. The range of variation for the variable t consists of non-negative real numbers $t \geq 0$. The scaling invariance implying conformal invariance (Virasoro generator L_0 represents scaling which plays a fundamental role in the super-conformal theories [A13]) is motivated by the belief that ζ codes for the physics of a quantum critical system having, not only super-symmetries [A9], but also super-conformal transformations as its basic symmetries.

3.2 Formal Solution Of The Eigenvalue Equation For OperatorD^+

One can formally solve the eigenvalue equation

$$D^+\Psi_z = \left[-t\frac{d}{dt} + t\frac{dF}{dt}\frac{1}{F}\right]\Psi_z = z\Psi_z. \tag{3.3}$$

for D^+ by factoring the eigenfunction to a product:

$$\Psi_z = f_z F. \tag{3.4}$$

The substitution into the eigenvalue equation gives

$$L_0 f_z = t\frac{d}{dt} f_z = -z f_z \tag{3.5}$$

allowing as its solution the functions

$$f_z(t) = t^z. \tag{3.6}$$

These functions are nothing but eigenfunctions of the scaling operator L_0 of the super-conformal algebra analogous to the eigen states of a translation operator. A priori all complex numbers z are candidates for the eigenvalues of D^+ and one must select the genuine eigenvalues by applying the requirement $D^\dagger = D^+$ in the space spanned by the genuine eigenfunctions.

It must be emphasized that Ψ_z is *not* an eigenfunction of D. Indeed, one has

$$D\Psi_z = -D^+\Psi_z + 2V\Psi_z = z\Psi_z + 2V\Psi_z. \tag{3.7}$$

This is in accordance with the analogy with the coherent states which are eigen states of annihilation operator but not those of creation operator.

4 Miscellaneous Ideas About Riemann Hypothesis

This section contains ideas about Riemann hypothesis which I regard as miscellaneous. I took them rather seriously for about more than decade ago but seeing them now makes me blush. I do not however have heart to throw away all these pieces of text away so that "miscellaneous" is a good attribute serving as a warning for the reader.

4.1 Universality Principle

The function, what I call $\hat{\zeta}$, is defined by the product formula for ζ and exists in the infinite-dimensional algebraic extension of rationals containing all roots of primes. $\hat{\zeta}$ is defined for all values of s for which the partition functions $1/(1-p^{-s})$ appearing in the product formula have value in the algebraic extension. Universality Principle states that $|\hat{\zeta}|^2$, defined as the product of the p-adic norms of $|\hat{\zeta}|^2$ by reversing the order of producting in the adelic formula, equals to $|\zeta|^2$ and, being an infinite dimensional vector in the algebraic extension of the rationals, vanishes only if it contains a rational factor which vanishes. This factor is present only provided an infinite number of partition functions appearing in the product formula of $\hat{\zeta}$ have rational valued norm squared: this locates the plausible candidates for the zeros on the lines $Re[s] = n/2$.

Universality Principle generalizes the original sharpened form of the Riemann hypothesis: the real parts of the phases p^{-iy} are rational. Universality Principle, even if proven, does not however yield a proof of the Riemann hypothesis. The failure of Riemann hypothesis becomes however extremely implausible and one could consider the possibility of regarding Riemann Hypothesis as an axiom.

4.2 How To Understand Riemann Hypothesis

The considerations of the preceding subsection lead to the requirement that the logarithmic waves $e^{iKlog(u)}$ exist in all number fields for $u = n$ (and thus for any rational value of u) implying number theoretical quantization of the scaling momenta K. Since the logarithmic waves appear also in Riemann Zeta as the basic building blocks, there is an interesting connection with Riemann hypothesis, which states that all non-trivial zeros of $\zeta(z) = \sum_n 1/n^z$ lie at the line $Re(z) = 1/2$.

I have applied two basic strategies in my attempts to understand Riemann hypothesis. Both approaches rely heavily on conformal invariance but being realized in a different manner. The universality of the scaling momentum spectrum implied by the number theoretical quantization allows to understand the relationship between these approaches.

4.2.1 Some approaches to RH

It is appropriate to list various approaches to RH that I have considered during years.

1. Coherent state approach to RH

In this approach (see the preprint in [L1] in Los Alamos archives and the article published in Acta Mathematica Universitatis Comeniae [H1]) one constructs a simple conformally invariant dynamical system for which the vanishing of Riemann Zeta at the critical line states that the coherent quantum states, which are eigen states of a generalized annihilation operator, are orthogonal to a vacuum state possessing a negative norm. This condition implies that the eigenvalues are given by the nontrivial zeros of ζ. Riemann hypothesis reduces to conformal invariance and the outcome is an analytic reductio ad absurdum argument proving Riemann hypothesis with the standards of rigor applied in theoretical physics.

2. The approach based on number theoretical universality

The basic idea is that Riemann Zeta is in some sense defined for all number fields. The basic question is what “some” could mean. Since Riemann Zeta decomposes into a product of harmonic oscillator partition functions $Z_p(z) = 1/(1-p^z)$ associated with primes p the natural guess is that $p^{1/2+iy}$ exists p-adically for the zeros of Zeta. The first guess was that for every prime p (and hence every integer n) and every zero of Zeta p^{iy} might define complex rational number (Pythagorean phase) or perhaps a complex algebraic number.

The transcendental considerations that one should try to generalize this idea: for every p and y appearing in the zero of Zeta the number p^{iy} belongs to a finite-dimensional extension of rationals involving also rational roots of e. This would imply that also the quantities n^{iy} make sense for all number fields and one can develop Zeta into a p-adic power series. Riemann Zeta would be defined for any number field in the set linearly spanned by the integer multiples of the zeros y of Zeta and it is easy to get convinced that this set is dense at the Y-axis. Zeta would therefore be defined at least in the set $X \times Y$ where X is some subset of real axis depending on the extension used.

If $log(p) = q_1 exp(q_2)/\pi$ holds true, then $y = q(y)\pi$ should hold true for the zeros of ζ. In this case one would have

$$p^{iy} = exp\left[iq(y)q_1(p)exp\left(q_2(p)\right)\right] \quad .$$

This quantity exists p-adically if the exponent has p-adic norm smaller than one. $q_1(p)$ is divisible by finite number of primes p_1 so that p^{iy} does not exist in a finite-dimensional extension of R_{p_1} unless $q(y)$ is proportional to a positive power of p_1. Also in this case the multiplication of y by the units defined by infinite primes (to be discussed later) would save the day and would be completely invisible operation in real context.

3. Logarithmic plane waves and Hilbert-Polya conjecture

Logarithmic plane waves allow also a fresh insight on how to physically understand Riemann hypothesis and the Hilbert-Polya conjecture stating that the imaginary parts of the zeros of Riemann Zeta correspond to the eigenvalues of some Hamiltonian in some Hilbert space.

1. At the critical line $Re(z) = 1/2$ (z=x+iy) the numbers $n^{-z} = n^{-1/2-iy}$ appearing in the definition of the Riemann Zeta allow an interpretation as logarithmic plane waves $\Psi_y(v) = e^{iylog(v)}v^{-1/2}$ with the scaling momentum $K = 1/2 - iy$ estimated at integer valued points $v = n$. Riemann hypothesis would follow from two facts. First, logarithmic plane waves form a complete basis equivalent with the ordinary plane wave basis from which sub-basis is selected by number theoretical quantization. Secondly, for all other powers v^k other than $v^{-1/2}$ in the denominator the norm diverges due to the contributions coming from either short ($k < -1/2$) or long distances ($k > -1/2$).

2. Obviously the logarithmic plane waves provide a concrete blood and flesh realization for the conjecture of Hilbert and Polya and the eigenvalues of the Hamiltonian correspond to the universal scaling momenta. Note that Hilbert-Polya realization is based on mutually orthogonal plane waves whereas the Approach 1 relies on coherent states orthogonal to the negative norm vacuum state. That eigenvalue spectra coincide follows from the universality of the number theoretical quantization conditions. The universality of the number theoretical quantization predicts that the zeros should appear in the scaling eigenvalue spectrum of any physical system obeying conformal invariance. Also the Hamiltonian generating by definition an infinitesimal time translation could act as an infinitesimal scaling.

3. The vanishing of the Riemann Zeta could code the conditions stating that the extensions involved are finite-dimensional: it would be interesting to understand this aspect more clearly.

4. The approach based on zero energy ontology

The approach based on zero energy ontology is the newest one and generalizes the thermodynamical approach by replacing thermodynamics with its square root. The amplitudes p^s define quantities proportional to time-like entanglement coefficients between positive and negative energy parts of a zero energy state having opposite energies given by $\pm log(p)$. The hypothesis that the sum over moduli squared for the coefficients diverges states that the zero energy state is not normalizable and has a physical interpretation as a critical state representing Bose-Einstein condensation. The additional condition that zero of zeta is in question is analogous to the condition $\int \Psi dV = 0$ and should be given a better physical justification. The interpretation as a zero energy counterpart of a coherent state seems to makes sense also now. Note that in ZEO coherent state property is in accordance with energy conservation. In the case of coherent states of Cooper pairs same applies to fermion number conservation. With this interpretation the condition would state orthogonality with respect to the coherent zero energy state characterized by $s = 0$.

4.2.2 Connection with the conjecture of Berry and Keating

The idea that the imaginary parts y for the zeros of Riemann zeta function correspond to eigenvalues of some Hermitian operator H is not new. Berry and Keating [A9] however proposed quite recently that the Hamilton in question is super-symmetric and given by

$$H = xp - \frac{i}{2} . \qquad (4.1)$$

Here the momentum operator p is defined as $p = -id/dx$ and x has non-negative real values.

H can be indeed expressed as a square $H = Q^2$ of a Hermitian super symmetry generator Q:

$$
\begin{aligned}
Q &= \sqrt{i}\,[ix\sigma_1 + p\sigma_2] + \sqrt{\frac{i}{2}}\sigma_3 \ , \\
\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \\
\sigma_2 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \\
\sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .
\end{aligned} \tag{4.2}
$$

By a direct calculation one finds that the following relationship holds true:

$$
Q^2 = \begin{pmatrix} xp + \frac{i}{2} & 0 \\ 0 & xp - \frac{i}{2} \end{pmatrix} \ .
$$

The eigen spinors of Q can be written as

$$
\psi = \begin{pmatrix} u \\ v \end{pmatrix} = x^{-iy} \begin{pmatrix} x^{1/2} \\ \sqrt{\frac{y}{i}}x^{-1/2} \end{pmatrix} \ .
$$

The eigenvalues of Q are $q = \sqrt{y}$. For $y \geq 0$ the eigenvalues are real so that Q is Hermitian when inner product is defined appropriately. Obviously y is eigenvalue of Hamiltonian.

Orthogonality requirement for the solutions of the Dirac equation requires that the inner product reduces to the inner product for plane waves $exp(iu)$, $u = log(x)$. This is achieved if inner product for spinors $\psi_i = (u_i, v_i)$ is defined as

$$
\langle \psi_1 | \psi_2 \rangle = \int_0^\infty \frac{dx}{x} \left[\overline{u}_1 v_2 + \overline{v}_1 u_2 \right] \ . \tag{4.3}
$$

In the basis formed by solutions of Dirac equation this inner product is indeed positive definite as one finds by a direct calculation.

The actual spectrum assumed to give the zeros of the Riemann Zeta function however remains open without additional hypothesis. An attractive hypothesis motivated by previous considerations is that the sharpened form of Riemann hypothesis stating that n^{iy} exists for any number field provided finite-dimensional extensions are allowed for the zeros of Riemann zeta function, holds true. This implies that x^{iy} satisfies the same condition for any rational value of x. $x^{\pm 1/2}$ in turn belongs to the infinite-dimensional algebraic extension Q_C^∞ of complex rationals, when x is rational. Therefore the solutions of Dirac equation, being of form $x^{iy}x^{\pm 1/2}$, exist for all number fields for rational values of argument x.

4.2.3 Connection with arithmetic quantum field theory and quantization of time

There is also a very interesting connection with arithmetic quantum field theory and sharpened form of Riemann hypothesis. The Hamiltonian for a bosonic/fermionic arithmetic quantum field theory is given by

$$
H = \sum_p log(p) a_p^\dagger a_p \ . \tag{4.4}
$$

where $a_p^\dagger$ and a_p satisfy standard bosonic/fermionic anti-commutation relations

$$
\{a_{p_1}^\dagger, a_{p_2}\}_\pm = \delta(p_1, p_2) \ . \tag{4.5}
$$

Here $\pm$ refers to anti-commutator/commutator. The sum of Hamiltonians defines super-symmetric arithmetic QFT. The states of the bosonic QFT are in one-one correspondence with non-negative integers and the decomposition of a non-negative integer to powers or prime corresponds to the decomposition of state to many boson states corresponding to various modes p. Analogous statement holds true for fermionic QFT.

The matrix element for the time development operator $U(t) \equiv exp(iHt)$ between states $|m\rangle$ and $|n\rangle$ can be written as

$$\langle m|U(t)|n\rangle \quad = \quad \delta(m,n)n^{it} \ . \tag{4.6}$$

Same form holds true both in bosonic and fermionic QFT: s. These matrix elements are defined for all number fields allowing finite-dimensional extensions if this holds true for n^{it} so that the allowed values of t corresponds to zeros of Riemann Zeta if one accepts Universality Principle. Similar statement holds in the case of fermionic QFT. One can say that the durations for the time evolutions are quantized in a well defined sense and allowed values of time coordinate correspond to the zeros of Riemann zeta function!

The result is very interesting from the point of view of quantum TGD since it would mean that $U(t)$ allows for the preferred values of the time parameter p-adicization ($p\ mod\ 4 = 3$) obtained by mapping the diagonal phases to their p-adic counterparts by phase preserving canonical identification. For phases this map means only the re-interpretation of the rational phase factor as a complexified p-adic number. For these quantized values of the time parameter time evolution operator of the arithmetic quantum field theory makes sense in all p-adic number fields besides complex numbers.

In the case of Berry's super-symmetric Hamiltonian the assumption that p^{iy} exists in all number fields with finite extensions allowed and the requirement that same holds true for the time evolution operator implies that allowed time durations for time evolution are given by $t = log(n)$. This means that there is nice duality between Berry's theory and arithmetic QFT. The allowed time durations (energies) in Berry's theory correspond to energies (allowed time durations) in arithmetic QFT.

4.3 Stronger Variants For The Sharpened Form Of The RiemannHypothesis

The previous form of the sharpened form of Riemann hypothesis was preceded by conjectures, which were much stronger. The strongest variant of the sharpening is that the phases p^{iy} are complex rational numbers for all primes and for all zeros ζ. A weaker form assumes that these phases belong to the square root allowing infinite-dimensional extension of rationals. Although these conjectures are probably unrealistic, they deserve a brief discussion.

4.3.1 Could the phases p^{iy} exist as complex rationals for the zeros of ζ?

The set $z = n/2 + iy$, $n > 0$ such that p^{-iy} is Pythagorean phase, is the set in which both real Riemann zeta function and the p-adic counterparts of Z_p exist for $p\ mod\ 4 = 3$. They exists also for $p\ mod\ 4 = 1$, if one *defines* $exp(ix) \equiv cos(x) + \sqrt{-1}sin(x)$: $\sqrt{-1}$ would be ordinary p-adic number for $p\ mod\ 4 = 1$. One could also allow phase factors in square root allowing algebraic extension of p-adics.

What is important that $x = 1/2$ is the smallest value of x for which the p-adic counterpart of $Z_B(p, x_p)$ exists. Already Riemann showed that the nontrivial zeros of Riemann Zeta function lie symmetrically around the line $x = 1/2$ in the interval $0 \leq x \leq 1$.

If one assumes that the zeros of Riemann zeta belong to the set at which the p-adic counterparts of Riemann zeta are defined, Riemann hypothesis follows in sharpened form.

1. Sharpened form of Riemann hypothesis does not necessarily exclude zeros with $x = 0$ or $x = 1$ as zeros of Riemann zeta unless they are explicitly excluded. It is however known that the lines $x = 0$ and $x = 1$ do not contains zeros of Riemann Zeta so that sharpened form implies also Riemann hypothesis.

2. The sharpening of the Riemann hypothesis following from p-adic considerations implies that the phases p^{iy} exist as rational complex phases for all values of $p\ mod\ 4 = 3$ when y corresponds to a zero of Riemann Zeta. Obviously the rational phases p^{iy} form a group with

respect to multiplication isomorphic with the group of integers in case that y does not vanish. The same is also true for the phases corresponding to integers continuing only powers of primes $p \ mod \ 4 = 3$ phase factor.

3. A stronger form of sharpened hypothesis is that all primes p and all integers are allowed. This would mean that each zero of the Riemann Zeta would generate naturally group isomorphic with the group of integers. Pythagorean phases form a group and should contain this group as a subgroup. It might be that very simple number theoretic considerations exclude this possibility. If not, one would have infinite number of conditions on each zero of Riemann function and much sharper form of Riemann hypothesis which could fix the zeros of Riemann zeta completely:

   The zeros of Riemann Zeta function lie on axis $x = 1/2$ and correspond to values of y such that the phase factor p^{iy} is rational complex number for all values of prime $p \ mod \ 4 = 3$ or perhaps even for all primes p.

   Of course, the proposed condition might be quite too strong. A milder condition is that $U_p(x_p)$ is rational for single value of p only: this would mean that the zeros of Riemann Zeta would correspond to Pythagorean angles labeled by primes. One can consider also the possibility that p^{iy} is rational for all y but for some primes only and that these preferred primes correspond to the p-adic primes characterizing the effective p-adic topologies realized in the physical world.

4. If this hypothesis is correct then each zero defines a subgroup of Pythagorean phases and also zeros have a natural group structure. Pythagorean phases contain an infinite number of subgroups generated by integer powers of phase. Each such subgroup has some number N of generators such that the subgroup is generated as products of these phases. From the fact that Pythagorean phases are in a one-one correspondence with rationals, it is obvious that there exists large number of subgroups of this kind. Every zero defines infinite number of Pythagorean phases and there are infinite number of zeros. The entire group generated by the phases is in one-one correspondence with the pairs (p, y).

5. If n^{iy} are rational numbers, there must exist imbedding map f: $(n, y) \to (r, s)$ from the set of phases n^{iy} to Pythagorean phases characterized by rationals $q = r/s$:

   $$(r, s) = (f_1(n, y), f_2(n, y)) \ .$$

   The multiplication of Pythagorean phases corresponds to certain map g

   $$\begin{aligned}(r_1, s_1) \circ (r_2, s_2) &= [g_1(r_1, s_1; r_2, s_2), \ g_2(r_1, s_1; r_2, s_2)] \\ &= (r_1 r_2 - s_1 s_2, r_1 s_2 + r_2 s_1) \equiv (r, s)\end{aligned}$$

   such that the values of r and s associated with the product can be calculated. Thus the product operation rise to functional equations giving constraints on the functional form of the map f.

   i) Multiplication of n^{iy_1} and n^{iy_2} gives rise to a condition

   $$f(n, y_1) \circ f(n, y_2) = f(n, y_1 + y_2) \ .$$

   ii) Multiplication of n_1^{iy} and n_2^{iy} gives rise to a condition

   $$f(n_1, y) \circ f(n_2, y) = f(n_1 n_2, y) \ .$$

This variant of the sharpened form of the Riemann hypothesis has turned out to be unnecessarily strong. Universality Principle requires only that the real parts of the factors $p^{-x}p^{-iy}$ are rational numbers: this means that allowed phases correspond to triangles whose two sides have integer-valued length squared whereas the third side has integer-valued length.

4.3.2 Sharpened form of Riemann hypothesis and infinite-dimensional algebraic extension of rationals

The proposed variant for the sharpened form of Riemann hypothesis states that the zeros of Riemann zeta are on the line $x = 1/2$ and that p^{iy}, where p is prime, are complex rational (Pythagorean) phases for zeros. Furthermore, Riemann hypothesis is equivalent with the corresponding statement for the fermionic partition function Z_F. If the sharpened form of Riemann hypothesis holds true, the value of $Z_F(z)$ in the set of zeros $z = 1/2 + iy$ of Z_F can be interpreted as a complex (vanishing) image of certain function $Z_F^{\infty}(1/2 + iy)$ having values in the infinite-dimensional algebraic extension of rationals defined by adding the square roots of all primes to the set of rational numbers.

1. The general element q of the infinite-dimensional extension Q_C^{∞} of complex rationals Q_C can be written as

$$
\begin{aligned}
q &= \sum_U q_U e_U \ , \\
e_U &= \prod_{i \in U} \sqrt{p}_i \ .
\end{aligned}
\tag{4.7}
$$

   Here q_U are complex rational numbers, U runs over the subsets of primes and e_U are the units of the algebraic extension analogous to the imaginary unit. One can map the elements of Q_C^{∞} to reals by interpreting the generating units $\sqrt{p}$ as real numbers. The real images $(e_U)_R$ of e_U are thus real numbers:

$$
e_U \rightarrow [e_U]_R = \prod_i \sqrt{p_i} \ .
$$

2. The value of $Z_F(z)$ at $z = 1/2 + iy$ can be written as

$$
Z_F(z = 1/2 + iy) = \sum_U \left[\frac{1}{e_U} \right]_R \times (e_U^2)^{-iy} \ .
\tag{4.8}
$$

   Here $(e_U)_R$ means that e_U are interpreted as real numbers.

3. If one restricts the set of values of $z = 1/2 + iy$ to such values of y that p^{iy} is complex rational for every value of p, then the value of $Z_F(1/2 + iy)$ can be also interpreted as the real image of the value of a function $Z_F(Q_{\infty}|z = 1/2 + iy)$ restricted to the set of zeros of Riemann zeta and having values at Q_C^{∞}:

$$
\begin{aligned}
Z_F(1/2 + iy) &= [Z_F(Q_{\infty}|1/2 + iy)]_R \ , \\
Z_F(Q_{\infty}|1/2 + iy) &\equiv \sum_U \frac{1}{e_U} \times (e_U^2)^{-iy} \ .
\end{aligned}
\tag{4.9}
$$

   Note that $Z_F(Q_{\infty}|z = 1/2 + iy)$ cannot vanish as element of Q_{∞}. One can also define the Q_C^{∞} valued counterparts of the partition functions $Z_F(p, 1/2 + iy)$

$$
\begin{aligned}
Z_F(Q_{\infty}|1/2 + iy) &= \prod_p Z_F(Q_{\infty}|p, z = 1/2 + iy) \ , \\
Z_F(Q_{\infty}|1/2 + iy) &\equiv 1 + p^{-1/2} p^{-iy} \ , \\
Z_F(p, 1/2 + iy) &= [Z_F(Q_{\infty}|p, 1/2 + iy)]_R \ .
\end{aligned}
\tag{4.10}
$$

   $Z_F(Q_{\infty}|1/2 + iy)$ and $Z_F(Q_{\infty}|p, 1/2 + iy)$ belong to Q_C^{∞} only provided p^{iy} is Pythagorean phase.

4. The requirement that p^{iy} is rational does not yet imply Riemann hypothesis. One can however strengthen this condition. The simplest condition is that the real image of $Z_F(Q_\infty|1/2+iy)$ is complex rational number for any value of Z_F. A stronger condition is that the complex images of the functions

$$\frac{Z_F^\infty}{\prod_{p\in U} Z_p^\infty}$$

are complex rational and U is finite set of primes. The complex counterparts of these functions are given by

$$\left[\frac{Z_F^\infty}{\prod_{p\in U} Z_p^\infty}\right]_R = \frac{Z_F}{\prod_{p\in U} Z_F(p,..)} \ . \tag{4.11}$$

Obviously these conditions can be true only provided that $Z_F(1/2+iy)$ vanishes identically for allowed values of y. This implies that sharpened form of Riemann hypothesis is true. "Physically" this means that the fermionic partition function restricted to any subset of integers not divisible by some finite set of primes, has real counterpart which is complex rational valued.

4.4 Are The Imaginary Parts Of The Zeros Of Zeta LinearlyIndependent Of Not?

Concerning the structure of the weight space of super-symplectic algebra the crucial question is whether the imaginary parts of the zeros of Zeta are linearly independent or not. If they are independent, the space of conformal weights is infinite-dimensional lattice. Otherwise points of this lattice must be identified. The model of the scalar propagator identified as a suitable partition function in the super-symplectic algebra for which the generators have zeros of Riemann Zeta as conformal weights demonstrates that the assumption of linear independence leads to physically unrealistic results and the the propagator does not exist mathematically for the entire super-symplectic algebra. Also the findings about the distribution of zeros of Zeta favor a hypothesis about the structure of zeros implying a linear dependence.

4.4.1 Imaginary parts of non-trivial zeros as additive counterparts of primes?

The natural looking (and probably wrong) working hypothesis is that the imaginary parts y_i of the nontrivial zeros $z_i = 1/2 + y_i$, $y_i > 0$, of Riemann Zeta are linearly independent. This would mean that y_i define play the role of primes but with respect to addition instead of multiplication. If there exists no relationship of form $y_i = n2\pi + y_j$, the exponents e^{iy_i} define a multiplicative representation of the additive group, and these factors satisfy the defining condition for primeness in the conventional sense. The inverses e^{-iy_i} are analogous to the inverses of ordinary primes, and the products of the phases are analogous to rational numbers.

There would exist an algebra homomorphism from $\{y_i\}$ to ordinary primes ordered in the obvious manner and defined as the map as $y_i \leftrightarrow p_i$. The beauty of this identification would be that the hierarchies of p-adic cutoffs identifiable in terms of the p-adic length scale hierarchy and y-cutoffs identifiable in terms p-adic phase resolution (the higher the p-adic phase resolution, the higher-dimensional extension of p-adic numbers is needed) would be closely related. The identification would allow to see Riemann Zeta as a function relating two kinds of primes to each other.

A rather general assumption is that the phases p^{iy_i} are expressible as products of roots of unity and Pythagorean phases:

$$\begin{aligned}
p^{iy} &= e^{i\phi_P(p,y)} \times e^{i\phi(p,y)} \ , \\
e^{i\phi_P(p,y)} &= \frac{r^2 - s^2 + i2rs}{r^2+s^2} \ , \quad r = r(p,y) \ , \quad s = s(p,y) \ , \\
e^{i\phi(p,y)} &= e^{i\frac{2\pi m}{n}} \ , \quad m = m(p,y) \ , \quad n = n(p,y) \ .
\end{aligned} \tag{4.12}$$

If the Pythagorean phases associated with two different zeros of zeta are different a linear independence over integers follows as a consequence.

Pythagorean phases form a multiplicative group having "prime" phases, which are in one-one correspondence with the squares of Gaussian primes, as its generators and Gaussian primes which are in many-to-one correspondence with primes $p_1\ mod\ 4 = 1$. If p^{iy} is a product of algebraic phase and Pythagorean phase for any prime p, one should be able to decompose any zero y into two parts $y = y_1(p) + y_P(p)$ such that one has

$$log(p)y_1(p) = \frac{m2\pi}{n} \ , \ log(p)y_P(p) = \Phi_P = arctan\left[\frac{2rs}{r^2+s^2}\right] \ . \tag{4.13}$$

Note that the decomposition is not unique without additional conditions. The integers appearing in the formula of course depend on p.

4.4.2 Does the space of zeros factorize to a direct sum of multiples Pythagorean prime phase angles and algebraic phase angles?

As already noticed, the linear independence of the y_i follows if the Pythagorean prime phases associated with different zeros are different. The reverse of this implication holds also true. Suppose that there are two zeros $log(p)y_{1i} = \Phi_{P_1} + q_{1i}2\pi$, $i = a, b$ and two zeros $log(p)y_{2i} = \Phi_{P_2} + q_{2i}2\pi$, $i = a, b$, where q_{ij} are rational numbers. Then the linear combinations $n_1y_{1a} + n_2y_{2a}$ and $n_1y_{1b} + n_2y_{2b}$ represent same zeros if one has $n_1/n_2 = (q_{2a} - q_{2b})/(q_{1b} - q_{1a})$.

One can of course consider the possibility that linear independence holds true only in the weaker sense that one cannot express any zero of zeta as a linear combination of other zeros. For instance, this guarantees that the super-symplectic algebra generated by generators labeled by the zeros has indeed these generates as a minimal set of generating elements.

For instance, one can imagine the possibility that for any prime p a given Pythagorean phase angle $log(p)y_{P_k}$ corresponds to a set of zeros by adding to $\Phi_{P_k} = log(p)y_{P_k}$ rational multiples $q_{k,i}2\pi$ of 2π, where $Q_p(k) = \{q_{k,i}|i = 1, 2, ..\}$ is a subset of rationals so that one obtains subset $\{\Phi_{P_k} + q_{k,i}2\pi | q_{k,i} \in Q_p(k)\}$. Note that the definition of y_P involves an integer multiple of 2π which must be chosen judiciously: for instance, if y_P is taken to be minimal possible (that is in the range $(0, \pi/2)$, one obviously ends up with a contradiction. The same is true if $q_{k,i} < 1$ is assumed. Needless to say, the existence of this kind of decomposition for every prime p is extremely strong number theoretic condition.

The facts that Pythagorean phases are linearly independent and not expressible as a rational multiple of 2π imply that no zero is expressible as a linear combination of other zeros whereas the linear independence fails in a more general sense as already found. An especially interesting situation results if the set $Q_p(k)$ for given p does not depend on the Pythagorean phase so that one can write $Q_p(k) = Q_p$. In this case the set of zeros of Zeta would be obtained as a union of translates of the set Q_p by a subset of Pythagorean phase angles and approximate translational invariance realized in a statistical sense would result. Note that the Pythagorean phases need not correspond to Pythagorean prime phases: what is needed is that a multiple of the same prime phase appears only once.

An attractive interpretation for the existence of this decomposition to Pythagorean and algebraic phases factors for every prime is in terms of the p-adic length scale evolution. The possibility to express the zeros of Zeta in an infinite number of manners labeled by primes could be seen as a number theoretic realization of the renormalization group symmetry of quantum field theories. Primes p define kind of length scale resolution and in each length scale resolution the decomposition of the phases makes sense. This assumption implies the following relationship between the phases associated with y:

$$\frac{\left[\Phi_{P(p_1)} + q(p_1)2\pi\right]}{log(p_1)} = \frac{\left[\Phi_{P(p_2)} + q(p_2)2\pi\right]}{log(p_2)} \ . \tag{4.14}$$

In accordance with earlier number theoretical speculations, assume that $log(p_2)/log(p_1) \equiv Q(p_2, p_1)$ is rational. This condition allows to deduce how the phases p_1^{iy} transform in $p_1 \rightarrow p_2$ transformation. Let $p_1^{iy} = U_{P,p_1,y}U_{q,p_1,y}$ be the representation of p_1^{iy} as a product of Pythagorean and algebraic phases. Using the previous equation, one can write

$$p_2^{iy} = U_{P,p_2,y}U_{q,p_2,y} = U_{P,p_1,y}^{Q(p_2,p_1)}U_{q,p_1,y}^{Q(p_2,p_1)} \quad . \tag{4.15}$$

This means that the phases are mapped to rational powers of phases. In the case of Pythagorean phases this means that Pythagorean phase becomes a product of some Pythagorean and an algebraic phase whereas algebraic phases are mapped to algebraic phases. The requirement that the set of phases p_2^{iy} is same as the set of phases p_1^{iy} implies that the rational power $U_{P,p_1,y}^{Q(p_2,p_1)}$ is proportional to some Pythagorean phase U_{P,p_1,y_1} times algebraic phase U_q such that the the product of $U_qU_{q,p_1,y}^{Q(p_2,p_1)}$ gives an allowed algebraic phase. The map $U_{P,p_1,y} \rightarrow U_{P,p_1,y_1}$ from Pythagorean phases to Pythagorean phases induced in this manner must be one-to one must be the map between algebraic phases. Thus it seems that in principle the hypothesis might make sense.

The basic question is why the phases q^{iy} should exist p-adically in some finite-dimensional extension of R_p for every p. Obviously some function coding for the zeros of Zeta should exist p-adically. The factors $G_q = 1/(1-q^{-iy-1/2})$ of the product representation of Zeta obviously exist if this assumption is made for every prime p but the product is not expected to converge p-adically.

Also the logarithmic derivative of Zeta codes for the zeros and can be written as

$$\frac{\zeta'}{\zeta} = -\sum_q log(q)\frac{q^{-1/2-iy}}{1-q^{-1/2-iy}} \quad . \tag{4.16}$$

As such this function does not exist p-adically but dividing by $log(p)$ one obtains

$$\frac{1}{log(p)}\frac{\zeta'}{\zeta} = -\sum_q Q(q,p)\frac{q^{-1/2-iy}}{1-q^{-1/2-iy}} \quad . \tag{4.17}$$

This function exists if the the p-adic norms rational numbers $Q(q,p)$ approach to zero for $q \rightarrow \infty$: $|Q(q,p)|_p \rightarrow 0$ for $q \rightarrow \infty$. The p-adic existence of the logarithmic derivative would thus give hopes of universal coding for the zeros of Zeta and also give strong constraints to the behavior of the factors $Q(q,p)$. The simplest guess would be $Q(q,p) \propto p^q$ for $q \rightarrow \infty$.

4.4.3 Correlation functions for the spectrum of zeros favors the factorization of the space of zeros

The idea that the imaginary parts of the zeros of Zeta are linearly independent is a very attractive but must be tested against what is known about the distribution of the zeros of Zeta.

There exists numerical evidence for the linear independence of y_i as well as for the hypothesis that the zeros correspond to a union of translates of a basic set Q_1 by subset of Pythagorean phase angles. Lu and Sridhar have studied the correlation among the zeros of ζ [A15]. They consider the correlation functions for the fluctuating part of the spectral function of zeros smoothed out from a sum of delta functions to a sum of Lorentzian peaks. The correlation function between two zeros with a constant distance $K_2 - K_1 + s$ with the first zero in the interval $[K_1, K_1 + \Delta]$ and second zero in the interval $[K_2, K_2 + \Delta]$ is studied. The choice $K_1 = K_2$ assigns a correlation function for single interval at K_1 as a function of distance s between the zeros.

1. The first interesting finding, made already by Berry and Keating, is that the peaks for the negative values of the correlation function correspond to the lowest zeros of Riemann Zeta (only those contained in the interval Δ can appear as minima of correlation function). This phenomenon observed already by Berry and Keating is known as resurgence. That the anti-correlation is maximal when the distance of two zeros corresponds to a low lying zero of zeta can be understood if linear combinations of the zeros of Zeta are the least probable candidates for zeros. Stating it differently, large zeros tend to avoid the points which represent linear combinations of the smaller zeros.

2. Direct numerical support the hypothesis that the correlation function is approximately translationally invariant, which means that it depends on $K_2 - K_1 + s$ only. Correlation function is

also independent of the width of the spectral window Δ. In the special $K_1 = K_2$ the finding means that correlation function does not depend at all on the position K_1 of the window and depends only on the variable s. Prophecy means that the correlation function between the interval $[K, K + \Delta]$ and its mirror image $[-K - \Delta, -K]$ is the correlation function for the interval $[2K + \Delta]$ and depends only on the variable $2K + s$ allowing to allows to deduce information about the distribution of zeros outside the range $[-K, K]$. This property obviously follows from the proposed hypothesis implying that the spectral function is a sum of translates of a basic distribution by a subset of Pythagorean prime phase angles.

This hypothesis is consistent with the properties of the the smoothed out spectral density for the zeros given by

$$\langle\rho(k)\rangle = \frac{1}{2\pi}log(\frac{k}{2\pi}) \ . \tag{4.18}$$

This implies that the smoothed out number of zeros y smaller than Y is given by

$$N(Y) = \frac{Y}{2\pi}(log(\frac{Y}{2\pi}) - 1) \ . \tag{4.19}$$

$N(Y)$ increases faster than linearly, which is consistent with the assumption that the distribution of zeros with positive imaginary part is sum over translates of a single spectral function ρ_{Q_0} for the rational multiples $q_i X_p$, $X_p = 2\pi/log(p)$, $q_i \in Q_p$, for every prime p.

If the smoothed out spectral function for $q_i \in Q_p$ is constant:

$$\rho_{Q_p} = \frac{1}{K_p 2\pi} \ , \ K_p > 0 \ , \tag{4.20}$$

the number $N_P(Y, p)$ of Pythagorean prime phases increases as

$$N_P(Y|p) = K_p(log(\frac{Y}{2\pi}) - 1) \ , \tag{4.21}$$

so that the smoothed out spectral function associated with $N_P(Y|p)$ is given by the function

$$\rho_P(k|p) = \frac{K_p}{k} \tag{4.22}$$

for sufficiently large values of k. Therefore the distances between subsequent zeros could quite well correspond to the same Pythagorean phase for a given p and thus should allow to deduce information about the spectral function ρ_{Q_0}. A convenient parameterization of K_p is as $K = K_{p,0}/4\pi^2$ since the points of Q_p are of form $q_i 2\pi = (n(q_i) + q_1(q_i))2\pi$, $q_1 < 1$, and $n(q_i)$ must in the average sense form an evenly spaced subset of reals.

5 Could Local Zeta Functions Take The Role Of Riemann Zeta In TGD Framework?

The recent view about TGD leads to some conjectures about Riemann Zeta.

1. Non-trivial zeros should be algebraic numbers.
2. The building blocks in the product decomposition of ζ should be algebraic numbers for non-trivial zeros of zeta.
3. The values of zeta for their combinations with positive imaginary part with positive integer coefficients should be algebraic numbers.

These conjectures are motivated by the findings that Riemann Zeta seems to be associated with critical systems and by the fact that non-trivial zeros of zeta are analogous to complex conformal weights or perhaps more naturally, to complex square roots of real conformal weights [K1]. The necessity to make such a strong conjectures, in particular conjecture c), is an unsatisfactory feature of the theory and one could ask how to modify this picture. Also a clear physical interpretation of Riemann zeta is lacking.

It was also found that there are good reasons for expecting that the zetas in question should have only a finite number zeros. In the same section the self-referentiality hypothesis for ζ was proposed on basis of physical arguments. In this section (written before the emergence of self-referentiality hypothesis) the situation will be discussed from different view point.

5.1 Local Zeta Functions And Weil Conjectures

Riemann Zeta is not the only zeta [A1, A8]. There is entire zoo of zeta functions and the natural question is whether some other zeta sharing the basic properties of Riemann zeta having zeros at critical line could be more appropriate in TGD framework.

The so called local zeta functions analogous to the factors $\zeta_p(s) = 1/(1-p^{-s})$ of Riemann Zeta can be used to code algebraic data about say numbers about solutions of algebraic equations reduced to finite fields. The local zeta functions appearing in Weil's conjectures [A7] associated with finite fields $G(p,k)$ and thus to single prime. The extensions $G(p,nk)$ of this finite field are considered. These local zeta functions code the number for the points of algebraic variety for given value of n. Weil's conjectures also state that if X is a mod p reduction of non-singular complex projective variety then the degree for the polynomial multiplying the product $\zeta(s)\times\zeta(s-1)$ equals to Betti number. Betti number is 2 times genus in 2-D case.

It has been proven that the zetas of Weil are associated with single prime p, they satisfy functional equation, their zeros are at critical lines, and rather remarkably, they are rational functions of p^{-s}. For instance, for elliptic curves zeros are at critical line [A7].

The general form for the local zeta is $\zeta(s) = exp(G(s))$, where $G = \sum g_n p^{-ns}$, $g_n = N_n/n$, codes for the numbers N_n of points of algebraic variety for n^{th} extension of finite field F with nk elements assuming that F has $k = p^r$ elements. This transformation resembles the relationship $Z = exp(F)$ between partition function and free energy $Z = exp(F)$ in thermodynamics.

The exponential form is motivated by the possibility to factorize the zeta function into a product of zeta functions. Note also that in the situation when N_n approaches constant N_∞, the division of N_n by n gives essentially $1/(1-N_\infty p^{-s})$ and one obtains the factor of Riemann Zeta at a shifted argument $s - log_p(N_\infty)$. The local zeta associated with Riemann Zeta corresponds to $N_n = 1$.

5.2 Local Zeta Functions And TGD

The local zetas are associated with single prime p, they satisfy functional equation, their zeros lie at the critical lines, and they are rational functions of p^{-s}. These features are highly desirable from the TGD point of view.

5.2.1 Why local zeta functions could be natural in TGD framework?

In TGD framework Kähler-Dirac equation assigns to a partonic 2-surface a p-adic prime p and inverse of the zeta defines local conformal weight. The intersection of the real and corresponding p-adic parton 2-surface is the set containing the points that one is interested in. Hence local zeta sharing the basic properties of Riemann zeta is highly desirable and natural. In particular, if the local zeta is a rational function then the inverse images of rational points of the geodesic sphere are algebraic numbers. Of course, one might consider a stronger constraint that the inverse image is rational. Note that one must still require that p^{-s} as well as s are algebraic numbers for the zeros of the local zeta (conditions 1) and 2) listed in the beginning) if one wants the number theoretical universality.

Ifthe Kähler-Dirac operator indeed assigns to a given partonic 2-surface a p-adic prime p, one can ask whether the inverse $\zeta_p^{-1}(z)$ of some kind of local zeta directly coding data about partonic 2-surface could define the generalized eigenvalues of the Kähler-Dirac operator and radial super-symplectic conformal weights so that the conjectures about Riemann Zeta would not be needed at all.

The eigenvalues of the mass squared assignable to the modes of the Kähler-Dirac operator, whose ground state part codes information about four-surface [K2] could in a holographic manner code for information about partonic 2-surface. This kind of algebraic geometric data are absolutely relevant for TGD since U-matrix and S-matrix must be formulated in terms of the data related to the intersection of real and partonic 2-surfaces (number theoretic braids) obeying same algebraic equations and consisting of algebraic points in the appropriate algebraic extension of p-adic numbers. Note that the hierarchy of algebraic extensions of p-adic number fields would give rise to a hierarchy of zetas so that the algebraic extension used would directly reflect itself in the eigenvalue spectrum of the Kähler-Dirac operator and super-symplectic conformal weights. This is highly desirable but not achieved if one uses Riemann Zeta.

One must of course leave open the possibility that for real-real transitions the inverse of the zeta defined as a product of the local zetas (very much analogous to Riemann Zeta defines the conformal weights. This kind of picture would conform with the idea about real physics as a kind of adele formed from p-adic physics.

5.2.2 Finite field hierarchy is not natural in TGD context

That local zeta functions are assigned with a hierarchy of finite field extensions do not look natural in TGD context. The reason is that these extensions are regarded as abstract extensions of $G(p,k)$ as opposed to a large number of algebraic extensions isomorphic with finite fields as abstract number fields and induced from the extensions of p-adic number fields. Sub-field property is clearly highly relevant in TGD framework just as the sub-manifold property is crucial for geometrizing also other interactions than gravitation in TGD framework.

The $O(p^n)$ hierarchy for the p-adic cutoffs would naturally replace the hierarchy of finite fields. This hierarchy is quite different from the hierarchy of finite fields since one expects that the number of solutions becomes constant at the limit of large n and also at the limit of large p so that powers in the function G coding for the numbers of solutions of algebraic equations as function of n should not increase but approach constant N_∞. The possibility to factorize $exp(G)$ to a product $exp(G_0)exp(G_\infty)$ would mean a reduction to a product of a rational function and factor(s) $\zeta_p(s) = 1/(1-p^{-s_1})$ associated with Riemann Zeta with argument s shifted to $s_1 = s - log_p(N_\infty)$.

5.2.3 What data local zetas could code?

The next question is what data the local zeta functions could code.

1. It is not at clear whether it is useful to code global data such as the numbers of points of partonic 2-surface modulo p^n. The notion of number theoretic braid occurring in the proposed approach to S-matrix suggests that the zeta at an algebraic point z of the geodesic sphere S^2 of CP_2 or of light-cone boundary should code purely local data such as the numbers N_n of points which project to z as function of p-adic cutoff p^n. In the generic case this number would be finite for non-vacuum extremals with 2-D S^2 projection. The n^{th} coefficient $g_n = N_n/n$ of the function G_p would code the number N_n of these points in the approximation $O(p^{n+1}) = 0$ for the algebraic equations defining the p-adic counterpart of the partonic 2-surface.

2. In a region of partonic 2-surface where the numbers N_n of these points remain constant, $\zeta(s)$ would have constant functional form and therefore the information in this discrete set of algebraic points would allow to deduce deduce information about the numbers N_n. Both the algebraic points and generalized eigenvalues would carry the algebraic information.

3. A rather fascinating self referentiality would result: the generalized eigen values of the Kähler-Dirac operator expressible in terms of inverse of zeta would code data for a sequence of approximations for the p-adic variant of the partonic 2-surface. This would be natural since second quantized induced spinor fields are correlates for logical thought in TGD inspired theory of consciousness. Even more, the data would be given at points $\zeta(s)$, s a rational value of a super-symplectic conformal weight or a value of generalized eigenvalue of Kähler-Dirac operator (which is essentially function $s = \zeta_p^{-1}(z)$ at geodesic sphere of CP_2 or of light-cone boundary).

5.3 Galois Groups, Jones Inclusions, And Infinite Primes

Langlands program [A3, A10] is an attempt to unify mathematics using the idea that all zeta functions and corresponding theta functions could emerge as automorphic functions giving rise to finite-dimensional representations for Galois groups (Galois group is defined as a group of automorphisms of the extension of field F leaving invariant the elements of F). The basic example corresponds to rationals and their extensions. Finite fields $G(p,k)$ and their extensions $G(p,nk)$ represents another example. The largest extension of rationals corresponds to algebraic numbers (algebraically closed set). Although this non-Abelian group is huge and does not exist in the usual sense of the word its finite-dimensional representations in groups $GL(n,Z)$ make sense.

For instance, Edward Witten is working with the idea that geometric variant of Langlands duality could correspond to the dualities discovered in string model framework and be understood in terms of topological version of four-dimensional $N=4$ super-symmetric YM theory [A11]. In particular, Witten assigns surface operators to the 2-D surfaces of 4-D space-time. This brings unavoidably in mind partonic 2-surfaces and TGD as $N=4$ super-conformal almost topological QFT.

This observation stimulates some ideas about the role of zeta functions in TGD if one takes the vision about physics as a generalized number theory seriously.

5.3.1 Galois groups, Jones inclusions, and quantum measurement theory

The Galois representations appearing in Langlands program could have a concrete physical/cognitive meaning.

1. The Galois groups associated with the extensions of rationals have a natural action on partonic 2-surfaces represented by algebraic equations. Their action would reduce to permutations of roots of the polynomial equations defining the points with a fixed projection to the above mentioned geodesic sphere S^2 of CP_2 or δM^4_+. This makes possible to define modes of induced spinor fields transforming under representations of Galois groups. Galois groups would also have a natural action on WCW -spinor fields. One can also speak about WCW spinor s invariant under Galois group.

2. Galois groups could be assigned to Jones inclusions having an interpretation in terms of a finite measurement resolution in the sense that the discrete group defining the inclusion leaves invariant the operators generating excitations which are not detectable.

3. The physical interpretation of the finite resolution represented by Galois group would be based on the analogy with particle physics. The field extension K/F implies that the primes (more precisely, prime ideals) of F decompose into products of primes (prime ideals) of K. Physically this corresponds to the decomposition of particle into more elementary constituents, say hadrons into quarks in the improved resolution implied by the extension $F \to K$. The interpretation in terms of cognitive resolution would be that the primes associated with the higher extensions of rationals are not cognizable: in other words, the observed states are singlets under corresponding Galois groups: one has algebraic/cognitive counterpart of color confinement.

4. For instance, the system labeled by an ordinary p-adic prime could decompose to a system which is a composite of Gaussian primes. Interestingly, the biologically highly interesting p-adic length scale range 10 nm-5 μm contains as many as four scaled up electron Compton lengths $L_e(k) = \sqrt{5}L(k)$ associated with Gaussian Mersennes ($M_k = (1+i)^k - 1$, $k = 151, 157, 163, 167$), which suggests that the emergence of living matter means an improved cognitive resolution.

5.3.2 Galois groups and infinite primes

In particular, the notion of infinite prime suggests a manner to realize the modular functions as representations of Galois groups. Infinite primes might also provide a new perspective to the concrete realization of Langlands program.

1. The discrete Galois groups associated with various extensions of rationals and involved with modular functions which are in one-one correspondence with zeta functions via Mellin transform defined as $\sum x_n n^{-s} \rightarrow \sum x_n z^n$ [A4]. Various Galois groups would have a natural action in the space of infinite primes having interpretation as Fock states and more general bound states of an arithmetic quantum field theory.

2. The number theoretic anatomy of space-time points due to the possibility to define infinite number of number theoretically non-equivalent real units using infinite rationals [L2] allows the imbedding space points themselves to code holographically various things. Galois groups would have a natural action in the space of real units and thus on the number theoretical anatomy of a point of imbedding space.

3. Since the repeated second quantization of the super-symmetric arithmetic quantum field theory defined by infinite primes gives rise to a huge space of quantum states, the conjecture that the number theoretic anatomy of imbedding space point allows to represent WCW (the world of classical worlds associated with the light-cone of a given point of H) and WCW spinor fields emerges naturally [L2].

4. Since Galois groups G are associated with inclusions of number fields to their extensions, this inclusion could correspond at quantum level to a generalized Jones inclusion $\mathcal{N} \subset \mathcal{M}$ such that G acts as automorphisms of $\mathcal{M}$ and leaves invariant the elements of $\mathcal{N}$. This might be possible if one allows the replacement of complex numbers as coefficient fields of hyper-finite factors of type II_1 with various algebraic extensions of rationals. Quantum measurement theory with a finite measurement resolution defined by Jones inclusion $\mathcal{N} \subset \mathcal{M}$ [L3] could thus have also a purely number theoretic meaning provided it is possible to define a non-trivial action of various Galois groups on WCW spinor fields via the imbedding of the configuration space spinors to the space of infinite integers and rationals (analogous to the imbedding of space-time surface to imbedding space).

This picture allows to develop rather fascinating ideas about mathematical structures and their relationship to physical world. For instance, the functional form of a map between two sets the points of the domain and target rather than only its value could be coded in a holographic manner by using the number theoretic anatomy of the points. Modular functions giving rise to generalized zeta functions would emerge in especially natural manner in this framework. WCW spinor fields would allow a physical realization of the holographic representations of various maps as quantum states.

5.4 About Hurwitz Zetas

The action of modular group SL(2, Z) on Riemann zeta [A5] is induced by its action on theta function [A6]. The action of the generator $\tau \rightarrow -1/\tau$ on theta function is essential in providing the functional equation for Riemann Zeta. Usually the action of the generator $\tau \rightarrow \tau + 1$ on Zeta is not considered explicitly. The surprise was that the action of the generator $\tau \rightarrow \tau + 1$ on Riemann Zeta does not give back Riemann zeta but a more general function known as Hurwitz zeta $\zeta(s, z)$ for $z = 1/2$. One finds that Hurwitz zetas for certain rational values of argument define in a well defined sense representations of fractional modular group to which quantum group can be assigned naturally. Could they allow to code the value of the quantum phase $q = exp(i2\pi/n)$ to the solution spectrum of the Kähler-Dirac operator D? As already shown the answer to this question is negative. Despite this Hurwitz zetas deserve a closer examination.

5.4.1 Definition

Hurwitz zeta is obtained by replacing integers m with $m + z$ in the defining sum formula for Riemann Zeta:

$$\zeta(s, z) = \sum_m (m + z)^{-s} \quad . \tag{5.1}$$

Riemann zeta results for $z = n$ apart from finite number of terms.

Hurwitz zeta obeys the following functional equation for rational $z = m/n$ of the second argument [A2]:

$$\zeta(1-s, \frac{m}{n}) = \frac{2\Gamma(s)}{2\pi n}^{s} \sum_{k=1}^{n} cos(\frac{\pi s}{2} - \frac{2\pi km}{n})\zeta(s, \frac{k}{n}) \quad . \tag{5.2}$$

The representation of Hurwitz zeta in terms of θ [A2] is given by the equation

$$\int_0^\infty [\theta(z, it) - 1]\, t^{s/2} \frac{dt}{t} = \pi^{(1-s)/2}\Gamma(\frac{1-s}{2})\, [\zeta(1-s, z) + \zeta(1-s, 1-z)] \quad . \tag{5.3}$$

By the periodicity of theta function this gives for $z = n$ Riemann zeta apart from finite number of terms.

5.4.2 The action of $\tau \to \tau + 1$ transforms $\zeta(s, 0)$ to $\zeta(s, 1/2)$

The action of the transformations $\tau \to \tau + 1$ on the integral representation of Riemann Zeta [A5] in terms of θ function [A6]

$$\theta(z; \tau) - 1 = 2 \sum_{n=1}^{\infty} [exp(i\pi\tau)]^{n^2} cos(2\pi nz) \tag{5.4}$$

is given by

$$\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) = \int_0^\infty [\theta(0; it) - 1]\, t^{s/2} \frac{dt}{t} \quad . \tag{5.5}$$

Using the first formula one finds that the shift $\tau = it \to \tau + 1$ in the argument θ induces the shift $\theta(0; \tau) \to \theta(1/2; \tau)$. Hence the result is Hurwitz zeta $\zeta(s, 1/2)$. For $\tau \to \tau + 2$ one obtains Riemann Zeta.

Thus $\zeta(s, 0)$ and $\zeta(s, 1/2)$ behave like a doublet under modular transformations. Under the subgroup of modular group obtained by replacing $\tau \to \tau + 1$ with $\tau \to \tau + 2$ Riemann Zeta forms a singlet. The functional equation for Hurwitz zeta relates $\zeta(1 - s, 1/2)$ to $\zeta(s, 1/2)$ and $\zeta(s, 1) = \zeta(s, 0)$ so that also now one obtains a doublet, which is not surprising since the functional equations directly reflects the modular transformation properties of theta functions. This doublet might be the proper object to study instead of singlet if one considers full modular invariance.

5.4.3 Hurwitz zetas form n-plets closed under the action of fractional modular group

The inspection of the functional equation for Hurwitz zeta given above demonstrates that $\zeta(s, m/n)$, $m = 0, 1, ..., n$, form in a well-defined sense an n-plet under fractional modular transformations obtained by using generators $\tau \to -1/\tau$ and $\tau \to \tau + 2/n$. The latter corresponds to the unimodular matrix $(a, b; c, d) = (1, 2/n; 0, 1)$. These matrices obviously form a group. Note that Riemann zeta is always one member of the multiplet containing n Hurwitz zetas.

These observations bring in mind fractionization of quantum numbers, quantum groups corresponding to the quantum phase $q = exp(i2\pi/n)$, and the inclusions for hyper-finite factors of type II_1 partially characterized by these quantum phases. Fractional modular group obtained using generator $\tau \to \tau + 2/n$ and Hurwitz zetas $\zeta(s, k/n)$ could very naturally relate to these and related structures.

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