Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry?

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Contents

Abstract

TGD leads to several proposals for the exact solution of field equations defining space-time surfaces as preferred extremals of twistor lift of Kähler action. So called M^8-H duality is one of these approaches. The beauty of M^8-H duality is that it could reduce classical TGD to algebraic geometry and would immediately provide deep insights to cognitive representation identified as sets of rational points of these surfaces.

In the sequel I shall consider the following topics.

1. I will discuss basic notions of algebraic geometry such as algebraic variety, surface, and curve, all rational point of variety central for TGD view about cognitive representation, elliptic curves and surfaces, and rational and potentially rational varieties. Also the notion of Zariski topology and Kodaira dimension are discussed briefly. I am not a mathematician and what hopefully saves me from horrible blunders is physical intuition developed during 4 decades of TGD.

2. It will be shown how M^8-H duality could reduce TGD at fundamental level to octonionic algebraic geometry. Space-time surfaces in M^8 would be algebraic surfaces identified as zero loci for imaginary part $IM(P)$ or real part $RE(P)$ of octonionic polynomial of complexified octonionic variable o_c decomposing as $o_c = q_c^1 + q_c^2 I^4$ and projected to a Minkowskian sub-space M^8 of complexified O. Single real valued polynomial of real variable with algebraic coefficients would determine space-time surface! As proposed already earlier, spacetime surfaces would form commutative and associative algebra with addition, product and functional composition.

   One can interpret the products of polynomials as correlates for free many-particle states with interactions described by added interaction polynomial, which can vanish at boundaries of CDs thanks to the vanishing in Minkowski signature of the complexified norm $q_c\overline{q_c}$ appearing in $RE(P)$ or $IM(P)$ caused by the quaternionic non-commutativity. This leads to the same picture as the view about preferred extremals reducing to minimal surfaces near boundaries of CD. Also zero zero energy ontology (ZEO) could emerge naturally from the failure of number field property for for quaternions at light-cone boundaries.

3. The fundamental challenge is to prove that the octonionic polynomials with real coefficients determine associative (co-associative) surfaces as the zero loci of their real part $RE(P)$ (imaginary parts $IM(P)$). $RE(P)$ and $IM(P)$ are defined in quaternionic sense. Contrary to the first naive working hypothesis, the identification $M^4 \subset O$ as as a co-associative region turns out to be the correct choice making light-cone boundary a counterpart of point-like singularity essential for the emergence of causal diamonds (CDs).

   The hierarchy of notions involved is well-ordering for 1-D structures, commutativity for complex numbers, and associativity for quaternions. This suggests a generalization of Cauchy-Riemann conditions for complex analytic functions to quaternions and octonions. Cauchy Riemann conditions are linear and constant value manifolds are 1-D and thus well-ordered. Quaternionic polynomials with real coefficients define maps for which the 2-D spaces corresponding to vanishing of real/imaginary parts of the polynomial are complex/co-complex or equivalently commutative/co-commutative. Commutativity is expressed by conditions bilinear in partial derivatives. Octonionic polynomials with real coefficients define maps for which 4-D surfaces for which real/imaginary part are quaternionic/co-quaternionic, or equivalently associative/co-associative. The conditions are now 3-linear.

   In fact, all algebras obtained by Cayley-Dickson construction adding imaginary units to octonionic algebra are power associative so that polynomials with real coefficients define an associative and commutative algebra. Hence octonion analyticity and M^8-H correspondence could generalize.

4. It turns out that in the generic case associative surfaces are 3-D and are obtained by requiring that one of the coordinates $RE(Y)^i$ or $IM(Y)^i$ in the decomposition $Y^i = RE(Y)^i + IM(Y)^i I_4$ of the gradient of $RE(P) = Y = 0$ with respect to the complex coordinates z_i^k, $k = 1,2$, of O vanishes that is critical as function of quaternionic components z_1^k or z_2^k associated with q_1 and q_2 in the decomposition $o = q_1 + q_2 I_4$, call this component X_i. In the generic case this gives 3-D surface.

   In this generic case M^8-H duality can map only the 3-surfaces at the boundaries of CD and light-like partonic orbits to H, and only determines the boundary conditions of the dynamics in H determined by the twistor lift of Kähler action. M^8-H duality would

allow to solve the gauge conditions for SSA (vanishing of infinite number of Noether charges) explicitly.

One can also have criticality. 4-dimensionality can be achieved by posing conditions on the coefficients of the octonionic polynomial P so that the criticality conditions do not reduce the dimension: X_i would have possibly degenerate zero at space-time variety. This can allow 4-D associativity with at most 3 critical components X_i. Space-time surface would be analogous to a polynomial with a multiple root. The criticality of X_i conforms with the general vision about quantum criticality of TGD Universe and provides polynomials with universal dynamics of criticality. A generalization of Thom's catastrophe theory emerges. Criticality should be equivalent to the universal dynamics determined by the twistor lift of Kähler action in H in regions, where Kähler action and volume term decouple and dynamics does not depend on coupling constants.

One obtains two types of space-time surfaces. Critical and associative (co-associative) surfaces can be mapped by M^8-H duality to preferred critical extremals for the twistor lift of Kähler action obeying universal dynamics with no dependence on coupling constants and due to the decoupling of Kähler action and volume term: these represent external particles. M^8-H duality does not apply to non-associative (non-co-associative) space-time surfaces except at 3-D boundary surfaces. These regions correspond to interaction regions in which Kähler action and volume term couple and coupling constants make themselves visible in the dynamics. M^8-H duality determines boundary conditions.

5. This picture generalizes to the level of complex/co-complex surfaces assigned with fermionic dynamics. Why in some cases 1-D light-like curves at partonic orbits seem to be enough to represent fermions? Why fermionic strings serve as correlates of entanglement for bound states? What selects string world sheets and partonic 2-surfaces from the slicing of space-time surfaces? I have proposed commutativity or co-commutatitivity of string worlds sheets/partonic 2-surfaces in quaternionic sense as number theoretic explanation (tangent space as a sub-space of quaternionic space is commutative/co-commutative at each point). Why not all string world sheets/partonic 2-surfaces in the slicing are not commutative/co-commutative? The answer to these questions is criticality again: in the generic case commutative varieties are 1-D curves. In critical case one has 2-D string worlds sheets and partonic 2-surfaces.

6. Also CDs and therefore also ZEO emerge from this approach naturally.

7. The super variant of octonionic geometry makes sense and the geometry of the variety correlates with the fermion and antifermion numbers assigned with it. The twistorial construction of scattering amplitudes in $\mathcal{N}$ SUSY might generalize to TGD in rather straightforward manner. Functional integral over WCW would reduce to summations over polynomials with coefficients in extension of rationals and criticality conditions on the coefficients could make the summation well-defined by bringing in finite measurement resolution.

1 Introduction

There are good reasons to hope that TGD is integrable theory in some sense. During years I have ended up with several proposals for the general solution of classical field equations (classical TGD is an exact part of quantum TGD).

The first approach is based on the geometry of the "world of classical worlds" (WCW) [K5, K3, K14].

1. The study of classical field equations led rather early to the realization that preferred extremals for the twistor lift of Kähler action with Minkowskian signature of induced metric define a slicing of space-time surfaces defined by 2-D string world sheets and partonic two-surfaces locally orthogonal to them. The interpretation is in terms of position dependent light-like momentum vector and polarization vector defining the local decompositions $M^2(x) \times E^2(x)$ of tangent space integrating to a foliation by partonic 2-surfaces and string world sheets. I christened this structure Hamilton-Jacobi structure. Its Euclidian counterpart is complex structure in Euclidian regions of space-time surface.

2. The formulation of quantum TGD in terms of spinor fields in WCW [K12] leads to the conclusion that WCW must have Kähler geometry [K5, K3] and has it only if it has maximal group of isometries identified as symplectic transformations of $\delta M^4_\pm \times CP_2$, where $\delta M^4_\pm$ denotes light cone boundary two which upper/lower boundary of causal diamond (CD) belongs. Symplectic Lie algebra extends naturally to supersymplectic algebra (SSA).

3. Space-time surfaces would be preferred extremals of twistor lift of Kähler action [K20] and the conditions realizing strong form of holography (SH) would state that sub-algebra of SSA isomorphic with it and its commutator with SSA give rise to vanishing Noether charges and these charges annihilate physical states or create zero norm states from them. One should solve these conditions.

4. The dynamics involves also fermions. Induced spinor fields are located inside space-time surface but for some yet not completely understood reason only the information about spinor at 2-D string world sheets is needed in the construction of scattering amplitudes. This dynamics would be 2-dimensional. The construction of twistor amplitudes even suggests that it is 1-dimensional in the sense that 1-D light-like curves at light-like partonic orbits defining boundaries of Minkowskian and Euclidian regions determines the scattering amplitudes. String world sheets are however needed only as correlates for entanglement between fermions at different partonic orbits.

   The 2-D character of fermionic dynamics conforms with the strong form of holography (SH) but how the string world sheets and partonic 2-surfaces are selected from Hamilton-Jacobi slicing? Electromagnetic neutrality could select string worlds sheets but one can actually always find a gauge in which the induced classical electroweak field at these surfaces is purely electromagnetic.

Second approach to preferred extremals is based on TGD version [K11, K19, K18, K20] of twistor Grassmann approach [B1, B3, B2].

1. The twistor lift of TGD leads to a proposal that space-time surfaces can be represented as sections in their 6-D twistor spaces identified as twistor bundles in the product $T(H) = T(M^4) \times T(CP_2)$ of 6-D twistor spaces of M^4 and CP_2. Twistor structure would be induced from $T(H)$. Kähler action can be lifted to the level of twistor spaces only for $M^4 \times CP_2$ since only for these spaces twistor space allows Kähler structure [A2]. Twistors were originally introduced by Penrose with the motivation that one could apply algebraic geometry in Minkowskian signature. The bundle property is extremely powerful and should be consistent with the algebraic geometrization at the level of M^8_c. The challenge is to formulate the twistor lift at the level of M^8.

2. The twistor lift of Kähler action contains also volume term. Field equations have two kinds of solutions. For the solutions of first kind the dynamics of volume term and Käction are coupled and the interpretation is in terms of interaction regions. Solutions of second kind are minimal surfaces and extremals of both Kähler action and volume term, whose dynamics decouple completely and all coupling constants disappear from the dynamics. These extremals are natural candidates for external particles. For these solutions at least the field equations reduce to the existence of Hamilton-Jacobi structure. The completely universal dynamics of these regions suggests interpretation in terms of maximal quantum criticality characterized by the extension of the usual conformal invariance to its quaternionic analog.

3. A connection with zero energy ontology (ZEO) emerges. Causal diamond (CD, intersection of future and past directed light-cones of M^4 with points replaced by CP_2) would naturally determine the interaction region to which external particles enter through its 2 future and past boundaries. But where does ZEO emerge?

The third approach is based on number theoretic vision [K9, K10, K8, K15].

1. M^8-H duality [K10, K15, K16] means that one can see space-times as 4-surfaces in either M^8 or $H = M^4 \times CP_2$. One could speak "number theoretical compactification" having however nothing to do with stringy version of compactification, which is dynamical. $M^8 - H$ duality

suggests that space-time surfaces in $H = M^4 \times CP_2$ are images of space-time surfaces in M^8 or actually of M^8 projections of complexified space-time surfaces in M^8_c identified as space of complexified octonions. These space-time surfaces could contain the integrated distributions of string world sheets and partonic 2-surfaces mentioned in the previous item. Space-time surfaces must have associative tangent or normal space for $M^8 - H$ correspondence to exist.

2. The fascinating possibility mentioned already earlier is that in M^8 these surfaces could correspond to zero loci for real or imaginary parts of real analytic octonionic polynomials $P(o) = RE(P) + IM(P)I_4$, I_4 an octonionic imaginary unit orthogonal to quaternionic ones. The condition $IM(P) = 0$ ($RE(P) = 0$) would give associative (co-associative) space-time surface. In the simplest case these functions would be polynomials so that one would have algebraic geometry for algebraically 4-D complex surfaces in 8-D complex space.

   Remark: The naive guess that space-time surfaces reduce to quaternionic curves in quaternionic plane fails due to the non-commutativity of quaternions meaning that one has $P(o) = P(q_1, q_2, \overline{q}_1, \overline{q}_2)$ rather than $P(o) = P(q_1, q_2)$.

3. The objection against this proposal is obvious. $M^8 - H$ correspondence cannot hold true since the dynamics defined by octonionic polynomials in M^8 contains no coupling constants whereas the dynamics of twistor lift of Kähler action depends on coupling constants in the generic space-time region. However, for space-time surfaces representing external particles entering inside CD at its boundaries this is however not the case! They could satisfy $M^8 - H$ correspondence!

   This suggests that inside CDs the space-time surfaces are not associative/co-associative in M^8 so that $M^8 - H$ correspondence cannot map them to H and the twistor lifted Kähler action and SH take care of the dynamics. External particles are associative and quantum critical and $M^8 - H$ correspondence makes sense. The quantum criticality and associativity at the boundaries of CD poses extremely powerful conditions and allows to satisfy infinite number of vanishing conditions for SSA charges.

4. This picture is consistent with the the explicit formulation of the associativity conditions $Re(P) = 0$ and $IM(P) = 0$ for varieties. The calculations shows that associativity can be realized either by posing a condition making them 3-dimensional except, when the situation is critical in the sense that the 4-D variety is analogous to a double root of polynomial: now however the polynomial corresponds to prime polynomial decomposing to product of polynomials in the extension of rationals such that the product contains higher powers of the factors. One has ramification at the level of polynomial primes so that the criticality condition does not bring anything new but need not make the situation associative. At most 3 conditions need to be applied to guarantee associativity and they might leave the space-time surface 4-D.

5. This octonionic view as also lower-dimensional quaternionic counterpart. In this case one considers 2-D commutative/co-commutative surfaces tentatively identifiable as string world sheets and partonic 2-surfaces. Why not all 2-surfaces appearing in the Hamilton-Jacobi slicing are not selected? The above mechanism would work also now. The commutativity conditions reduce in the generic case give 1-D curve as a solution. The interpretation would be as orbit of point like particle at 3-D partonic orbit appearing in the construction of twistorial amplitudes. In critical situation one would obtains string world sheet serving as a correlate for entanglement between point like particles at its ends: one would have quantum critical bound state.

I have considered also other attempts to define what quaternion structure could mean.

1. One could also consider the possibility that the tangent spaces of space-time surfaces in H are associative or co-associative [K15]. This is not necessary although it seems that this might be the case for the known extremals. If this holds true, one can construct further preferred extremals by functional composition by generalization of $M^8 - H$ correspondence to $H - H$ correspondence.

2. I have considered also the possibility of quaternion analyticity in the sense of generalization of Cauchy-Riemann equations, which tell that left- or right quaternionic differentiation makes sense [L4]. It however seems that this approach is not promising. The conditions are quite too restrictive and bring nothing essentially new. Octonion/quaternion analyticity in the above mentioned sense does not require the analogs of Cauchy-Riemann conditions.

The identification of space-time surfaces as zero loci of real or imaginary part of octonionic polynomial has several extremely nice features.

1. Octonionic polynomial is an algebraic continuation of a real valued polynomial on real line so that the situation is effectively 1-dimensional! Once the degree of polynomial is known, the value of polynomial at finite number of points are needed to determine it and cognitive representation could give this information! This would strengthen the view strong form of holography (SH) - this conforms with the fact that states in conformal field theory are determined by 1-D data.

2. One can add, sum, multiply, and functionally compose these polynomials provided they correspond to the same quaternionic moduli labelled by CP_2 points and share same time-line containing the origin of quaternionic and octonionic coordinates and real octonions (or actually their complexification by commuting imaginary unit). Classical space-time surfaces - classical worlds - would form an associative and commutative algebra. This algebra induces an analog of group algebra since these operations can be lifted to the level of functions defined in this algebra. These functions form a basic building brick of WCW spinor fields defining quantum states.

3. One can interpret the products of polynomials as correlates for free many-particle states with interactions described by added interaction polynomial, which can vanish at boundaries of CDs. This leads to the same picture as the view about preferred extremals reducing to minimal surfaces near boundaries of CD [L1]. Also zero zero energy ontology (ZEO) could be forced by the failure of number field property for quaternions at light-cone boundaries. It indeed turns out that light-cone boundary emerges quite generally as singular zero locus of polynomials $P(o)$ containing no linear part: this is essentially due to the non-commutativity of the octonionic units. Also the emergence of CDs can be understood. At this surface the region with $RE(P) = 0$ can transform to $IM(P) = 0$ region. In Euclidian signature this singularity corresponds to single point. A natural conjecture is that also the light-like orbits of partonic 2-surfaces correspond to this kind of singularities for non-trivial Hamilton-Jacobi structures.

4. The reduction to algebraic geometry would mean enormous boost to the vision about cognition with cognitive representations identified as generalized rational points common to reals rationals and various p-adic number fields defining the adele for given extension of rationals. Hamilton-Jacobi structure would result automatically from the decomposition of quaternions to real and imaginary parts which would be now complex numbers.

5. Also a connection with infinite primes is suggestive [K10]. The light-like partonic orbits, partonic 2-surfaces at their ends, and points at the corners of string world sheets might be interpreted in terms of singularities of varying rank and the analog of catastrophe theory emerges.

The great challenge is to prove rigorously that these approaches - or at least some of them - are indeed equivalent. Also it remains to be proven that the zero loci of real/imaginary parts of octonionic polynomials with real coefficients are associative or co-associative. I shall restrict the considerations of this article mostly to $M^8 - H$ duality. The strategy is simple: try to remember all previous objections against $M^8 - H$ duality and invent new ones since this is the best manner to make real progress.

In the sequel I shall consider the following topics.

1. I will discuss basic notions of algebraic geometry such as algebraic variety (see `http://tinyurl.com/hl6sjmz`), - surface (see `http://tinyurl.com/y8d5wsmj`), and - curve (see

http://tinyurl.com/nt6tkey), rational point of variety central for TGD view about cognitive representation, elliptic curves (see http://tinyurl.com/lovksny) and - surfaces (see http://tinyurl.com/yc33a6dg), and rational points (see http://tinyurl.com/ybbnnysu) and potentially rational varieties (see http://tinyurl.com/yablk4xt). Also the notion of Zariski topology (see http://tinyurl.com/h5pv4vk) and Kodaira dimension (see http://tinyurl.com/yadoj2ut) are discussed briefly. I am not a mathematician. What hopefully saves me from horrible blunders is physical intuition developed during 4 decades of TGD.

2. M^8-H duality [K16, K10, K15] would reduce classical TGD to the algebraic geometry and would immediately provide deep insights to cognitive representation identified as sets of rational points of these surfaces. Space-time surfaces in M^8 would be algebraic varieties identified as zero loci for imaginary part $IM(P)$ or real part $RE(P)$ of octonionic polynomial of complexified octonionic variable o decomposing as $o = q_c^1 + q_c^2 I_4$ and projected to a Minkowskian sub-space M^8 of o. Single real valued polynomial of real variable with algebraic coefficients would determine space-time surface! As proposed already earlier, spacetime surfaces in M^8 would form commutative and associative algebra with addition, product and functional composition.

   As already noticed, the associativity conditions do not allow 4-D solutions except for criticality so that M^8-H correspondence can hold true only in these space-time regions and one has these nice features at the level of M^8. In critical regions M^8-H correspondence is true and these features have H counterparts

   The basic problem is to understand the map mediating M^8-H duality mapping the point (m,e) of $M^8 = M_0^4 \times E^4$ to a point (m,s) of $M_0^4 \times CP_2$, where M_0^4 point is obtained as a projection to a suitably chosen $M_0^4 \subset M^8$ and CP_2 point parameterizes the tangent space as quaternionic sub-space containing preferred $M_0^2(x) \subset M^4(x)$. This map involves slightly non-local information and could allow to understand why the preferred extremals at the level of H are determined by partial differential equations rather than algebraic equations. Also the generalization to the level of twistor lift is briefly touched.

3. The fundamental challenge is to prove that the octonionic polynomials with real coefficients determine associative (co-associative) surfaces as the zero loci of their real part $RE(P)$ (imaginary parts $IM(P)$). $RE(P)$ and $IM(P)$ are defined in quaternionic sense. Contrary to the first naive working hypothesis, the identification $M^4 \subset O$ as as a co-associative region turns out to be the correct choice making light-cone boundary a counterpart of point-like singularity essential for the emergence of causal diamonds (CDs).

   This suggests a generalization of Cauchy-Riemann conditions for complex analytic functions to quaternions and octonions. Cauchy Riemann conditions are linear. Quaternionic polynomials with real coefficients define maps for which the 2-D spaces corresponding to vanishing of real/imaginary parts of the polynomial are complex/co-complex or equivalently commutative/co-commutative. Commutativity is expressed by conditions bilinear in partial derivatives. Octonionic polynomials with real coefficients define maps for which 4-D surfaces for which real/imaginary part are quaternionic/co-quaternionic, or equivalently associative/co-associative. The conditions are now 3-linear.

   In fact, all algebras obtained by Cayley-Dickson construction (see http://tinyurl.com/ybuyla2k) by adding imaginary unit repeatedly to octonionic algebra are power associative so that polynomials with real coefficients define an associative and commutative algebra. Hence octonion analyticity and a M^8-H correspondence could generalize (maybe even TGD!).

4. It turns out that in the generic case associative surfaces are 3-D and are obtained by requiring that one of the coordinates $RE(Y)^i$ or $IM(Y)^i$ in the decomposition $Y^i = RE(Y)^i + IM(Y)^i I_4$ of the gradient of $RE(P) = Y = 0$ with respect to the complex coordinates z_i^k, $k = 1,2$, of O vanishes that is critical as function of quaternionic components z_1^k or z_2^k associated with q_1 and q_2 in the decomposition $o = q_1 + q_2 I_4$, call this component X_i. In the generic case this gives 3-D surface.

   In this generic case M^8-H duality can map only the 3-surfaces at the boundaries of CD and light-like partonic orbits to H, and only determines the boundary conditions of the dynamics in H determined by the twistor lift of Kähler action. $M^8 - H$ duality would allow to solve the gauge conditions for SSA (vanishing of infinite number of Noether charges) explicitly.

One can also have criticality. 4-dimensionality can be achieved by posing conditions on the coefficients of the octonionic polynomial P so that the criticality conditions do not reduce the dimension: X_i would have possibly degenerate zero at space-time variety. This can allow 4-D associativity with at most 3 critical components X_i. Space-time surface would be analogous to a polynomial with a multiple root. The criticality of X_i conforms with the general vision about quantum criticality of TGD Universe and provides polynomials with universal dynamics of criticality. A generalization of Thom's catastrophe theory [A1] emerges. Criticality should be equivalent to the universal dynamics determined by the twistor lift of Kähler action in H in regions, where Kähler action and volume term decouple and dynamics does not depend on coupling constants.

One obtains two types of space-time surfaces. Critical and associative (co-associative) surfaces can be mapped by $M^8 - H$ duality to preferred critical extremals for the twistor lift of Kähler action obeying universal dynamics with no dependence on coupling constants and due to the decoupling of Kähler action and volume term: these represent external particles. $M^8 - H$ duality does not apply to non-associative (non-co-associative) space-time surfaces except at 3-D boundary surfaces. These regions correspond to interaction regions in which Kähler action and volume term couple and coupling constants make themselves visible in the dynamics. $M^8 - H$ duality determines boundary conditions.

An open question is whether one could allow 5-D co-associative surfaces.

5. This picture generalizes also to the level of complex/co-complex surfaces associated with fermionic dynamics. Why in some cases 1-D light-like curves at partonic orbits seem to be enough to represent fermions? Why fermionic strings serve as correlates of entanglement for bound states? What selects string world sheets and partonic 2-surfaces from the slicing of space-time surfaces? I have proposed commutativity or co-commutatitivity of string worlds sheets/partonic 2-surfaces in quaternionic sense as number theoretic explanation (tangent space as a sub-space of quaternionic space is commutative/co-commutative at each point). Why not all string world sheets/partonic 2-surfaces in the slicing are not commutative/co-commutative? The answer to these questions is criticality again: in the generic case commutative varieties are 1-D curves. In critical case one has 2-D string worlds sheets and partonic 2-surfaces.

6. Also CDs and therefore also ZEO emerge from this approach naturally.

7. The super variant of octonionic geometry makes sense and the geometry of the variety correlates with the fermion and antifermion numbers assigned with it. The twistorial construction of scattering amplitudes in $\mathcal{N}$ SUSY might generalize to TGD in rather straightforward manner. Functional integral over WCW would reduce to summations over polynomials with coefficients in extension of rationals and criticality conditions on the coefficients could make the summation well-defined by bringing in finite measurement resolution.

As I started writing this article I had in mind cognitive representations. My hope was that $M^8 - H$ duality could help to improve my understanding about them. It indeed did so and I have therefore included two sections strictly speaking do not represent the central topic of the article.

1. Cognitive representations are identified as sets of rational points for algebraic varieties with "active" points containing fermion. The representations are discussed at both M^8- and H level. General conjectures from algebraic geometry support the vision that these sets are concentrated at lower-dimensional algebraic varieties such as string world sheets and partonic 2-surfaces and their 3-D orbits identifiable also as singularities of these surfaces. For the earlier work related to adelic TGD and cognitive representations see [L6, L7, L3, L5].

   In TGD the reason would be simple: associativity and quantum criticality are satisfied in the generic case only at lower dimensional selected varieties: 3-surfaces at the ends of space-time surface and partonic orbits and also at string world sheets and fermion lines. For external particles these properties hold true in 4-D sense and cognitive representation could be 4-D-perhaps because rational points (in extension of rationals) form a dense set in these cases. This indeed conforms with the fact that we can solve free field theories!

2. Some aspects related to homology charge (Kähler magnetic charge) and genus-generation correspondence are discussed. Both topological quantum numbers are central in the proposed model of elementary particles and it is interesting to see whether the picture is internally consistent and how algebraic variety property affects the situation. Also possible problems related to $h_{eff}/h = n$ hierarchy [K4, K13] [L6] realized in terms of n-fold coverings of space-time surfaces are discussed from this perspective.

The easiest manner to kill $M^8 - H$ duality in the form it is represented here is to prove that 4-D zero loci for imaginary/real parts of octonionic polynomials with real coefficients can never be associative/co-associative being always 3-D. I hope that some professional mathematician would bother to check this.

In the sequel I will use some shorthand notations for key principles and key notions. Quantum Field Theory (QFT); Relativity Principle (RP); Equivalence Principle (EP); General Coordinate Invariance (GCI); Strong Form of GCI (SGCI); Quantum Criticality (QC); Strong Form of Holography (SH); World of Classical Worlds (WCW); Preferred Extremal (PE); Zero Energy Ontology (ZEO); Causal Diamond (CD); Number Theoretical Universality (NTU) are the most often occurring acronyms.

2 Some basic notions, ideas, results, and conjectures of algebraic geometry

In this section I will summarize very briefly the basic notions of algebraic geometry needed in the sequel.

2.1 Algebraic varieties, curves and surfaces

The basic notion of algebraic geometry is algebraic variety.

1. One considers affine space A^n with n coordinates $x^1, ..., x^n$ having values in a number field K usually assumed to be algebraically closed (note that affine space has no preferred origin like linear space). Algebraic variety is defined as a solution of one or more algebraic equations stating the vanishing of polynomials of n variables: $P^i(x^1, ..., x^n) = 0$, $i = 1, ..., r \leq n$. One can restrict the coefficients of polynomials to p-adic number field or or its extension to an extension of rationals. One talks about polynomials on $k \subset K$.

2. The basic condition is that the variety is not a union of disjoint varieties. This for instance happens, when the polynomial $P(x^1, .., x^n)$ defining co-dimension 1 manifold is product of polynomials $P = \prod_r P_r$. Algebraic variety need not be a manifold meaning that it can have singular points. For instance, for co-dimension 1 variety the Jacobian matrix $\partial P/\partial x^i$ of the polynomial can vanish at singularity.

3. One can define projective varieties (see `http://tinyurl.com/ybsqvy3r`) in projective space P^n having coordinatization in terms of $n+1$ homogenous coordinates $(x^1, ..., x^{n+1})$ in K with points differing by an overall scaling identified. Projective variety is defined as zero locus of homogenous polynomials of $n + 1$ coordinates so that solutions remain solutions under the overall scaling of all coordinates. By identifying the points related by scaling one obtains a surface in P^n. Grassmannian of linear space V^n (not affine space!) is a projective spaces defined as space of k-planes of V^n. These spaces are encountered in twistor Grassmannian approach to scattering amplitudes.

For polynomials of single variable one obtains just the roots of $P_n(x) = 0$ in an algebraic extension assignable to the polynomial. For several variables one can in principle proceed step by step by solving variable x^1 as algebraic function of others from $P_1(x^1, ..., x^n) = 0$, proceed to solve x^2 from $P_2(x^1(x^2, ...), x^2, ...) = 0$ as as algebraic function of the remaining variables, and so one. The algebraic functions involved get increasingly complex but in some exceptional situations the solution has parametric representation in terms of *rational* rather than algebraic functions of parameters t^k. For co-dimension $d_c > 1$ case the intersection of surfaces $P^i = 0$ need not be complete and the tangent spaces of the hyper-surfaces $P^i = 0$ need not intersect transversally in the generic case. Therefore $d_c > 1$ case is not gained so much attention as $d_c = 1$ case.

A more advanced treatment relies on ring theory by assigning to polynomials a ring as the ring of polynomials in the space involved divided by the ring of polynomials vanishing at zero loci of polynomials P^i.

1. The notion of ideal is central and determined as a subring invariant under the multiplication by elements of ring. Prime ideal generalizes the notion of prime and one can say that the notion of integer generalizes to that of ideal. One can also define the notion of fractional ideal.

2. Zariski topology (see `http://tinyurl.com/h5pv4vk`) replacing the topology based on real norm is second highly advanced notion. The closed sets in this topology are algebraic varieties of various dimensions. Since the complement of any algebraic variety is open set this topology and open also in the ordinary real topology, this topology is considerable rougher than the ordinary than the ordinary topology.

Some remarks from the point of view of TGD are in order.

1. In the scenario inspired by $M^8 - H$ duality one has co-dimension 4 surfaces in 8-D complex space. Octonionicity of polynomials however implies huge symmetries since the polynomial is determined by single real polynomial of real variable, whose values at finite number of points determined the polynomial.

2. In TGD the extension of rationals can be assumed to contain also powers for some root of e since in p-adic context this gives rise to a finite-dimensional extensions due to the fact that e^p is ordinary p-adic number. Also a restriction to a finite field are possible and restriction of rational coefficients to their modulo p counterparts reduces the polynomial to polynomial in finite field. This reduction is used as a technical tool. In the case of Diophantine equations (see `http://tinyurl.com/nt6tkey` and `http://tinyurl.com/y8hm4zce`) the coefficients are restricted to be integers.

3. In adelic TGD [L7] [L6] the number fields involved are reals and extensions of p-adic numbers. The coefficient field for the coefficients of polynomials would be naturally extension of rationals or extension of p-adics induced by it. The coefficients of polynomials serve as coordinates of adelic WCW. p-Adic numbers are not algebraically closed and one must assume an extension of p-adic numbers from that for the coefficients one to allow maximal number of roots.

   This suggests an evolutionary process [L9] extending the number field for the coefficients of polynomials. Arbitrary root of polynomial for given extension can be realized only if the original extension is extended further. But this allows polynomial coefficients in this new extension: WCW is now larger. Now one has however roots in even larger extension so that the unavoidable outcome is number theoretic evolution as increase of complexity.

4. What is so remarkable is that octonionic polynomials with rational coefficients could be determined by their values at finite set of points for a polynomial of real argument once the order of polynomial is fixed. Real coordinate corresponds to preferred time axis naturally. A cognitive representation consisting of finite number of rational points could fix the entire space-time surface! This would extend ordinary holography to its discrete variant!

5. Algebraic variety is rather simple object as compared to the solutions of partial differential equations encountered in physics - say those for minimal surfaces. Now one must fix boundary values or initial values at $n-1$-dimensional surface to fix the solution. For integrable theories the situation can change. In TGD SH suggests that the classical solutions are determined by data at 2-surfaces, which together with conformal invariance could reduce the data to one-dimensional data specified by a polynomial. $M^8 - H$ correspondence allows to consider this option seriously.

6. $M^8 - H$ duality suggests that space-time surfaces are co-dimension $d_c = 4$ algebraic curves in M^8. Could space-time surfaces define closed sets for the analog of Zariski topology? Could string world sheets and partonic 2-surfaces do the same inside space-time surfaces? An interesting question is whether this generalizes also to the level of imbedding space H and could perhaps define a topology rougher than real topology in better accord with the notion of finite measurement resolution.

2.2 About algebraic curves and surfaces

The realization $M^8 - H$ correspondence to be considered allows to understand space-time surfaces as 4-D complex algebraic surfaces X^4_c in the space o of complexified octonions projected to real sub-space of O^c with Minkowskian signature. Due to the non-commutativity of quaternions, the reduction of space-time surfaces to curves in quaternionic plane is not possible. Despite this it is instructive to start from the algebraic geometry of curves and surfaces.

2.2.1 Degree and genus of the algebraic curve

Algebraic curve is defined as zero locus of a polynomial $P(x^1, x^2, ..., x^n)$ with x^n in some - preferably algebraically closed - number field K and coefficients in some number field $k \subset K$. In adelic physics K corresponds to real or complex numbers and k to the extension of rationals defining adeles. In p-adic sectors k corresponds to tje extension of p-adic numbers induced by k. In general roots belong to an extension of k.

Degree, genus, and Euler characteristic are the basic characterizers of algebraic curve.

1. The degree d of algebraic curve corresponds to the highest power for the variables appearing in the polynomial. One can also define multi-degree in an obvious manner. A useful geometric interpretation for the degree is that line intersects curve (also complex) of degree d in at most d points as is clear from the fact that the equation of curve reduces the equation for curve to an equation for the roots of d:th order polynomial of single variable.

2. Also the genus g of the curve (see `http://tinyurl.com/ybm3wfue`) is important characteristic. One can distinguish between topological genus, geometric genus and arithmetic genus. For curves these notions are equivalent. The connection between genus and degree d of non-singular algebraic curve is very useful:

$$g = \frac{(d-1)(d-2)}{2} . \tag{2.1}$$

   Spherical topology for complex curves corresponds to $n = 1$ and $n = 2$.

   A more general formula reads as:

$$g = \frac{(d-1)(d-2)}{2} + \frac{n_s}{2} . \tag{2.2}$$

   Here n_s is the number of holes of the curve behaving like holes and increasing the genus.

3. Euler characteristic (for Euler characteristic see `http://tinyurl.com/pp52zd4`) is a homological invariant making sense in arbitrary dimension and also for manifolds. Homological definition based on simplicial homology relies on counting of simplexes of various dimension. The definition in terms of dimensions of homology groups H_n is given by

$$\chi = b_0 - b_1 + b_2 ... + (-1)^n b_n , \tag{2.3}$$

   where b_k is the dimension of k:th homology group (see `http://tinyurl.com/j48ojys`).

The following gives the engineering rules for obtaining Euler characteristic of the surface obtained from simpler building blocks. Note that algebraic variety property is not essential here.

1. Euler characteristic is homotopy invariant so that it does not change one adds homologically trivial space such as E^n as a Cartesian factor.

2. χ is additive under disjoint union. Inclusion-exclusion principle states that if M and N intersect, one has $\chi(M\cup N)=\chi(M)+\chi(N)-\chi(M\cap N)$.

3. Euler characteristic for the connected sum $A\#B$ of n-dimensional manifolds obtained by drilling balls B^n from summands, giving opposite orientation to the boundaries of the hole, and connecting with cylinder $D\times S^{n-1}$ is given by $\chi(A)+\chi(B)-\chi(S^{n-1})$. One has $\chi(S^2)=2$ and $\chi(D^2)=1$.

4. The Euler characteristic for product $M\times N$ is $\chi(M)\times\chi(N)$.

5. The Euler characteristic for N-fold covering space M_n is $N\times\chi(M)$ with a correction term coming from the singularities of the covering (ramified covering space).

6. For a fibration $M\rightarrow B$ with fiber S, which differs from fiber bundle in that the fibers are only homeomorphic, one has $\chi(M)=\chi(B)\times\chi(S)$.

Euler characteristic and the genus of 2-surface (or complex) curve are related by the equation

$$\chi \ = \ 2(1-g) \ . \tag{2.4}$$

having values $2,0,-2,.....$ If the 2-surface has n_s holes (punctures), one has

$$\chi=2(1-g)-n_s \ . \tag{2.5}$$

Punctures must be distinguished from singularities at which some sheets of covering meet at single point.

A formal generalization of the definition of genus for varieties in terms of Euler characteristic makes sense.

$$g \ = \ -\frac{\chi}{2}+1-\frac{n_s}{2} \ . \tag{2.6}$$

Disk has genus $1/2$ and drilling of n holes increases genus by $n/2$. Pair of holes gives same contribution to g and the cylinder connecting the holes. Note that for complex curves the definition of puncture is obvious. For real curves the puncture would mean missing point of the curve.

The latter definitions of genus can be identified in terms of Euler characteristic also for higher-dimensional varieties. For curves these notions are equivalent if there are no singularities. For algebraic curves g is same for the real and complex variants of the curve in RP_1 and CP_1 respectively.

2.2.2 Elliptic curves and elliptic surfaces

Elliptic curves (see `http://tinyurl.com/lovksny`) are cubic curves with no singularities (cusps or self-intersections) having representation of form $y^2-x^3-ax-b=0$. These singularities can occur only at special values of parameters ($(a=0,b=0)$. Since the degree equals to $d=3$, elliptic curve has genus $g=1$.

Elliptic curves allow a group of Abelian symmetries generated by a finite number of generators. The emergence of abelian group structure can be intuitively understood as follows.

1. Given line intersects the curve of degree 3 in at most 3 points. Let P and Q be two of these points. Then there can be also a third intersection point R and by the Z^2 symmetry changing the sign of y also the reflection of R - identify it as $-R$ - belongs to the curve. Define the sum of $P+Q$ to be $-R$.

The actual proof is slightly more complicated since the number of intersection points for the line with curve can be also 2 or 1. By writing explicit expressions for the coordinates x_R and y_R, one can also find that they are indeed rational if the points P and Q are rational. If the elliptic curve as single rational point it has infinite number of them.

2. The generators with finite order give rise to torsion. The rank of generators of infinite order is called rank and conjectured to be arbitrarily large (see `http://tinyurl.com/lovksny`) . Therefore elliptic curve is an Abelian group and one talks about Abelian variety. If elliptic curve contains a rational point it contains entire lattice of rational points obtained as shifts of this point.

Remark: Complex elliptic curves are 2-surfaces in complex projective plane CP_2 and therefore highly interesting from TGD point of view. $g = 1$ partonic 2-surfaces would in TGD framework correspond to second generation fermions [K2]. Abelian varieties define a generalization of elliptic curves to higher dimensions and simplest space-time surfaces allowing also large cognitive representations could correspond to such.

Elliptic surfaces (see `http://tinyurl.com/yc33a6dg`) are fibrations with an algebraic curve as base space and elliptic curve as fiber (fibration is more general notion than fiber space since the fibers are only homeomorphic). The singular fibers failing to be elliptic curves have been classified by Kodaira.

2.3 The notion of rational point and its generalization

The notion of algebraic integer (see `http://tinyurl.com/y8z389a7`) makes sense for any number field as a root of a monic polynomial (polynomial with integer coefficients with coefficient of highest power equal to unity). The field of fractions for given number field consists of ratios of algebraic integers. The same is true for the notion of prime. The more precise definition forces to replace integers and primes with ideals.

Rational varieties are expressible as maps defined by rational functions with rational coefficients in some extension of Q and contain infinite number of rational points. If the variety is not rational, one can ask whether it could allow a dense set of rational points with rational number replaced with the ratio of algebraic integers for some extension of Q. This leads to the idea of potentially rational point, and one can classify algebraic varieties according to whether they are potentially rational or not. The variety is potentially rational if it allows a parameteric representation using rational functions. Otherwise the parametric representation involves algebraic functions such as roots of rational functions.

The interpretation in terms of cognition would be that large enough extension makes the situation “cognitively easy” since cognitive representations involving fermions at the rational points and defining discretizations of the algebraic variety could be arbitrary large. The simpler the surface is cognitively, the large the number of rational points or potentially rational points is.

Complexity of algebraic varieties is measured by Kodaira dimension d_K (see `http://tinyurl.com/yadoj2ut`). The spectrum for this dimension varies in the range $(-\infty, 0, 1, 2, ...d)$, where d is the algebraic dimension of the variety. Maximal value equals to the ordinary topological dimension d and corresponds to maximal complexity: in this case the set of rational points is finite. Minimal Kodaira dimension is $d_K = -\infty$: in this case the set of rational points is infinite. Rational surfaces are maximally simple and this corresponds to the existence of parametric representations using only rational functions.

2.3.1 Rational points for algebraic curves

The sets of rational points for algebraic curves are well understood. Mordelli conjecture proved by Falting as a theorem (see `http://tinyurl.com/y9oq37ce`) states that a curve over Q with genus $g = (d-1)(d-2)/2 > 1$ (degree $d > 3$) has only finitely many rational points.

1. Sphere CP_1 in CP_2 has rational points as a dense set. Quite generally rational surfaces, which by definition allow parametric representation using polynomials with rational coefficients (encountered in context of Du Val singularities characterized by the extended Dynkin diagrams for finite subgroups of $SU(2)$) allow dense set of rational points [A3, A5]).

$g = 0$ does not yet guarantee that there is dense set of rational points. It is possible to have complex conics (quadratic surface) in CP_2 with no rational points. Note however that this depends on the choice of the coordinates: if origin belongs to the surface, there is at least one rational point

2. Elliptic curve $y^2 - x^3 - ax - b$ in CP_2 (see `http://tinyurl.com/lovksny`) has genus $g = 1$ and has a union of lattices of rational points and of finite cyclic groups of them since it has origin as a rational point. This lattice of points are generated by translations. Note that elliptic curve has no singularities that is self intersections or cusps (for $a = 0, b = 0$ origin is a singularity).

   $g = 1$ does not guarantee that there is infinite number of rational points. Fermat's last theorem and CP_2 as example. $x^d + y^d = z^d$ is projectively invariant statement and therefore defines a curve with genus $g = (d-1)(d-2)/2$ in CP_2 (one has $g = 0, 0, 2, 3, 6, 10, ...$). For $d > 2$, in particular $d = 3$, there are no rational points.

3. $g \geq 2$ curves do not allow a dense set of rational points nor even potentially cense set of rational points.

Remark: In TGD framework algebraic varieties could be zero loci of octonionic polynomials and have algebraic dimension 4 so that the classification for algebraic curves does not help. Octonion analyticity must bring in symmetries which simplify the situation.

2.3.2 Enriques-Kodaira classification

The tables of (see `http://tinyurl.com/ydelr4np`) give an overall view about the Enriques-Kodaira classification of algebraic curves, surfaces, and varieties in terms of Kodaira dimension (see `http://tinyurl.com/yadoj2ut`).

1. For instance, general curves ($g \geq 2$) have $d_K = 1$, elliptic curves ($g = 1$) have $d_K = 0$ and projective line ($g = 0$) has $d_K = -\infty$. $CP_1 \subset CP_2$ is a rational curve so that rational points are dense. Elliptic curves allow infinite number or rational points forming an Abelian group if they containing single rational point and are therefore cognitively easy.

2. Algebraic varieties (with real dimension $d_R = 4$ in complex case) with $d_K = 2$ are surfaces of general type, elliptic surfaces (see `http://tinyurl.com/yc33a6dg`) have $d_K = 1$, surfaces with attribute abelian, hyper-elliptic, K3, and Enriques, have $d_K = 0$.

   Remark: All real 2-surfaces are hyper-elliptic for $g \leq 2$, in other words allow Z_2 as global conformal symmetry. Genus-generation correspondence [K2] for fermions allows to assign to the 3 lowest generations of fermions hyper-elliptic partonic 2-surfaces with genus $g = 0, 1, 2$. These surfaces would have $d_K = 0$ and be rather simple as real surfaces in Kodaira classification. Could one regard them as M^4 projection of complex hyper-elliptic surfaces of real dimension $d_R = 4$? $d_K = -\infty$ holds true for rational surfaces and ruled surfaces, which allow straight line through any point are maximally simple. In complex case the line would be CP_1.

3. The Wikipedia article gives also a table about the classification of algebraic 3-folds. Real algebraic 3-surfaces might well occur in TGD framework. The twistor space for space-time surface might allow realization as complex 3-fold and since it has S^2 has fiber, it would naturally correspond to an uni-ruled surface with $d_K = -\infty$. The table shows that one can build higher dimensional algebraic varieties with $d_K < d$ from lower-dimensional ones as fiber-space like structures, which based or fiber having $d_K < d$. 3-D Abelian varieties and Calabi-Yau 3-folds are complex manifolds with $d_K = 0$, which cannot be engineered in this manner.

4. Space-time surfaces would be surfaces of algebraic dimension 4. Wikipedia tables do not give direct information about this case but one can make guesses on basis of the three tables. Octonionic polynomials are analytic continuations of real polynomials of real variable, which must mean a huge simplification, which also favor cognitive representability. The best that one might have infinite sets of rational points. The examples about extremals of Kähler action does not however favor this wish.

Bombieri-Lang conjecture (see http://tinyurl.com/y887yn5b) states that, for any variety X of general type over a number field k, the set of k-rational points of X fails to be Zariski dense (see http://tinyurl.com/jm9fh74) in X. This means that , the k-rational points are contained in a *finite* union of lower-dimensional sub-varieties of X. In dimension 1, this is exactly Faltings theorem, since a curve is of general type if and only if it has $g \geq 2$. The conjecture of Vojta (see http://tinyurl.com/y9sttuu4) states that varieties of general type cannot be potentially dense. As will be found, these conjectures might be highly relevant for TGD.

3 Does $M^8 - H$ duality allow to use the machinery of algebraic geometry?

The machinery of algebraic geometry is extremely powerful. In particular, the number theoretical universality of algebraic geometry implies that same equations make sense for all number fields: this is just what adelic physics [L7] [L6] demands. Therefore it would be extremely nice if one could somehow use this machinery also in TGD framework as it is used in string models. How this could be achieved? There are several guide lines.

1. Twistor lift of TGD [K11, K19, K18, K20] is now a rather well-established idea although a lot of work remains to be done with the details. Twistors were originally introduced in order to be able to use this machinery and involves complexification of Minkowski space M^4 to M^4_c as an auxiliary tool. Complexification in sufficiently general sense seems to be a necessary auxiliary tool but it cannot be a trick (like Wick rotation) but something fundamental and here complexification at the level of M^8 is suggestive. In the sequel I will used M^4 for M^4_c and M^8 for M^8_c unless it is necessary to emphasize that M^8_c is in question. The essential point is that the Euclidian metric is complexified and it reduces to a real metric in various sub-spaces defining besides Euclidian space also Minkowski spaces with varying signature when the complex coordinates are real or imaginary.

2. If $M^8 - H$ duality holds true, one can solve field equations in $M^8 = M^4 \times E^8$ by assuming that either the tangent space or normal space of the space-time surface X^4 is associative (quaternionic) at each point and contains preferred M^2 in its tangent space. M^2 could depend on x but $M^2(x)$:s should integrate to a 2-surface. This allows to map space-time surface M^8 to a surface in $M^4 \times CP_2$ since tangent spaces are parameterized by points of CP_2 and CP_2 takes the role of moduli space. The image of tangent space as point of CP_2 is same irrespective of whether one has quaternions or complexified quaternions (H_c).

   It came a surprise that associativity/co-associativity is possible only if the space-time surface is critical in the sense that some gradients of 8 complex components of the octonionic polynomial P vanish without posing them as additional conditions reducing thus the dimension of the space-time surface. This occurs when the coefficients of P satisfy additional conditions. One obtains associative/co-associative space-time regions and regions without either property and they correspond nicely to two solution types for the twistor lift of Kähler action.

3. Contrary to the original expectations, $M^4 \subset M^8_c$ must be identified as co-associative (co-quaternionic) subspace so that E^4 is the associative/quaternionic sub-space. This allows to have light-cone boundary as the counterpart of point-like singularity in ordinary algebraic geometry and also allows to understand the emergence of CDs and ZEO.

Remark: A useful convention to be used in the sequel. $RE(o)$ and $IM(o)$ denote the real and imaginary parts of the octionion in the decomposition $o = RE(o) + IM(o)I_4$ and $Re(o)$ and $Im(o)$ its real number valued and purely imaginary parts in the usual decomposition.

The problems related to the signature of M^4 have been a longstanding head-ache of M^8 duality.

1. The intuitive vision has been that the problems can be solved by replacing M^8 with its complexification M^8_c identifiable as complexified octonions o. This requires introduction of imaginary unit i commuting with the octonionic units $E^k \leftrightarrow (1, I_1, ..., I_7)$. The real octonionic components are thus replaced with ordinary complex numbers $z_i = x_i + iy_i$.

2. Importantly, complex conjugation $o \to \overline{o}$ changes only the sign of I_i but *not!* that of i so that the octonionic inner product $(o_1, o_2) = o_1\overline{o}_2 = o_1^k o_2^l \delta_{k,l}$ becomes complex valued. Norm is equal to $O\overline{O} = \sum_i z_i^2$. Both norm and inner product are in general complex valued and real valued only in sub-spaces in which octonionic coordinates are real or imaginary. Sub-spaces have all possible signatures of metric. These sub-spaces are not closed under multiplication and this has been an obstacle in the earlier attempts based on the notion of octonion analyticity. This argument applies also to quaternions and one obtains signatures $(1,1,1,1)$, $(1,1,1,-1)$, $(1,1,-1,-1)$, and $(1,-1,-1,-1)$. Why just the usual Minkowskian signature $(1,-1,-1,-1)$ is physical, should be understood.

   The convention consistent with that used in TGD corresponds to a negative length squared for space-like vectors and positive for time-like vectors. This gives $m = (o^0, io^1, ..., io^7)$ with real o^k. The projection $M_c^8 \to M^8$ defines the projection of $X_c^4 \subset M_c^8$ to $X^4 \subset M^8$ serving as the pre-image of $X^4 \subset M^8$ in $M^8 - H$ correspondence.

3. o is not field anymore as is clear from the fact that $1/o = \overline{o}/o\overline{o}$ is formally infinite in Minkowskian sub-spaces, when octonion defines a light-like vector. o (and H_c) remains however a ring so that sum and products are well-defined but division can lead to problems unless one stays inside 7+7-dimensional light-cone with $Re(o\overline{o}) > 0$ ($Re(q\overline{q}) > 0$).

   Although the number field structure is lost, one can still define polynomials needed to define algebraic varieties by requiring their simultaneous vanishing and rational functions make sense inside the light-cone. Also rational functions can be defined but poles are replaced with light-cones in Minkowskian section. Algebraic geometry would thus be forced by the complexification of octonions. This looks to me highly non-trivial! The extension of zeros and poles to light-cones making propagation possible could be a good reason for why Minkowskian signature is physical. Interestingly, the allowed octonionic momenta are light-like quaternions [K20].

4. An interesting question is whether ZEO and the emergence of CDs relates to the failure of field property. It seems now clear that CDs must be assigned even with elementary particles. I have asked whether they could form an analog for the covering of manifold by coordinate patches (in TGD inspired theory of consciousness CDs would be correlates for perceptive fields for conscious entities assignable to CDs [L9]). These observations encourage to ask whether the tips of CD should correspond to a pair formed by two poles/two zeros or by pole and zero assignable to positive and negative energy states.

   It turns out that the space-time surfaces in the interior of CD would naturally correspond to non-associative surfaces and only their 3-D boundaries would have associative 4-D tangent spaces allowing mapping to H by M^8-duality, which is enough by holography.

5. The relationship between light-like 3-surface bounding Minkowskian and Euclidian space-time regions and light-like boundaries of CDs is interesting. Could also the partonic orbits be understood a singularities of octonionic polynomials with $IM(P) = RE(P) = 0$?

3.1 What does one really mean with $M^8 - H$ duality?

The original proposal was that M^8 duality should map the associative tangent/normal planes of associative/co-associative space-time surface containing preferred M^2, call it M_0^2, to CP_2: the map read as $(m, e) \in M^4 \times E^4 \to (m, s) \in M^4 \times CP_2$. Eventually it became clear that the choice of M^2 can depend on position with $M^2(x)$ forming an integrable distribution to CP_2: this would define what I have called Hamilton-Jacobi structures [K16]. String like objects have minimal surface as M^4 projection for almost any general coordinate invariant action, and internal consistency requires that $M^2(x)$ integrate to a minimal surface. The details are however not understood well enough.

1. M^4 coordinate would correspond simply to projection to a fixed M_0^4 in the decomposition $M^8 = M_0^4 \times E_0^4$. One can however challenge this interpretation. How M_0^4 is chosen? Is it possible to choose it uniquely? Could also M^4 coordinates represent moduli analogous to CP_2 coordinates? What about ZEO?

There is an elegant general manner to formulate the choice of M^4_0 at the level of M^8. The complexified quaternionic sub-spaces of M^8_c (M^8) are parameterized by moduli space defining the quaternionic moduli. The moduli space in question is CP_2. The choice of M^4_0 corresponds to fixing of the quaternionic moduli by fixing a point of CP_2.

Warning: Note that one should be very careful in distinguishing between quaternionic as sub-spaces of M^8 and as the tangent space M^8 of given point of M^8.

2. One can ask whether there could be a connection with ZEO, where CDs play a key role. Indeed, the complexified Minkowski inner product means that the complexified octonions (quaternions) inside M^8_c (M^4_c) have inverse only inside 7-D (4-D) complexified light-cone and this would motivate the restriction of space-time surfaces inside future or past light-cone or both but not yet force CD.

   If one allows rational functions and even meromorphic functions of octonionic or quaternionic variable, one could consider the possibility of restricting the space-time surface defined as their zeros to a maximally sized region containing no poles.

3. Consider complexified quaternions H_c. Poles $(q\overline{q})^{-n}$, $n \geq 1$ would correspond M^4 light-cone boundaries since $q\overline{q} = 0$ at them. Also zeros $q\overline{q} = 0$, for $n \geq 1$ correspond to light-like boundaries. Could one have two poles with with time-like distance defining CD or a pair of pole and zero?

   There is also a possible connection with the notion of infinite primes [K8]. The notion of infinite prime leads to the proposal that rationals defined as ratios of infinite integers but having unit real norm (and also p-adic norms) could correspond pairs of positive and negative energy states with identical total quantum numbers and located at opposite boundaries of CD. Infinite rationals can be mapped to rational functions. Could positive energy states correspond to the numerators with zeros at second boundary of CD and negative energy states to denominators with zeros at opposite boundary of CD?

3.1.1 Is the choice of the pair (M^2_0, M^4_0) consistent with the properties of known extremals in H

It should be made clear that the notion of associativity/co-associativity (quaternionicity/co-quaternionicity) of the tangent/normal space need not make sense at the level of H. I shall however study this working hypothesis in the sequel.

The choice of the pair (M^2_0, M^4_0) means choosing preferred co-commutative (commutative) sub-space M^2_0 of M^8 defining a subspace of fixed co-quaternionic (quaternionic) sub-space $M^4_0 \subset M^8$.

Remark: M^4 should indeed be the co-associative/co-quaternionic subspace of M^8 if the argument about emergence of CDs is accepted and if $M^8 - H$ correspondence maps associative to associative and co-associative to co-associative.

M^4_0 in turn contains preferred M^2_0 defining co-commutative (hyper-complex) structure. Both M^2_0 and M^4_0 are needed in order to label the choice by CP_2 point (that is as a point of Grassmannian).

Is the projection to a fixed factor $M^4_0 \subset M^4_0 \times E^4$ as a choice of co-quaternionic moduli consistent with what we know about the extremals of twistor lift of Kähler action in H? How could one fix M^4_0 from the data about the extremal in H? One can make similar equations about the choice of M^2_0 as a fixed co-complex moduli characterized by a unit quaternion. Note that this choice is expected to relate closely to the twistor structure and Kähler structure.

It is best to check the proposal for the known extremals in H [K16]. Consider first CP_2 type extremals for which M^4 projection is a piece of light-like geodesic.

1. The CP_2 projection for the image of $X^4 \subset M^8$ differs from single point only if the tangent space isomorphic to M^4 and parameterized by CP_2 point varies. Consider CP_2 type extremals for the twistor lift of Kähler action [**?**]n H having light-like geodesic as M^4 projection as an example. The light-like geodesic defines a light-like vector in the tangent space of CP_2 type extremal. This light-like vector together with its dual spans fixed M^2, which however does not belong to the tangent space so that associative surface would not be in question.

What about co-associativity or associativity (the latter is favored by above argument)? This property should hold true for the pre-image of CP_2 type extremal in M^8 but I am not able to say anything about this. It is questionable to require this property at the level H but one can of course look what it would give.

What about associativity for CP_2 tangent space? The normal space of CP_2 type extremal is 3-D (!) since the only the light-like tangent vector of the geodesic and 2 vectors orthogonal to it are orthogonal to CP_2 tangent vectors. For Euclidian signature this would mean that tangent space is 5-D and cannot be associative but now the tangent space is 4-D. Can one still say that tangent space is associative. The co-associativity of the tangent space makes sense trivially. Can one conclude that CP_2 is co-associative.

The associativity for CP_2 tangent space might make sense since the tangent space is 4-D. The light-like vector k defines M_0^2. The associativity conditions involving two tangent space vectors of CP_2 and the light-like vector k contracted with the corresponding octonion components. The contributions from the components of k to the associator should cancel each other. Since one can change the relative sign of the components of k, this mechanism does not seem to work for all components. Hence associativity cannot hold true. Neither does M_0^2 belong to the normal space since k and its dual are not orthogonal.

Could one conclude that CP_2 type extremal is co-associative in accordance with the original belief thanks to the light-like projection to M^4? This does not conform with what the singularity considerations for the octonionic polynomials would suggest. Or is it simply not correct to try to apply associativity at the level of H. Or does $M^8 - H$ correspondence map associative tangent spaces to co-associative ones?

2. The normal space M^4 of CP_2 type extremal have all orientations characterized by its CP_2 projection. The normal space must contain the M_0^2 determined by the tangent of the light-like geodesic and this is indeed the case. Note that CP_2 type extremals cannot have entire CP_2 as CP_2 projection: they necessarily have hole at either end, which would be naturally be at the boundary of CD.

   CP_2 type extremals seem to be consistent with $M^8 - H$ correspondence. It however seems that one cannot fix the choice of M_0^4 uniquely in terms of the properties of the extremal. There is a moduli space for M_0^4:s defined by CP_2 and obviously codes for moduli for quaternion structures in octonionic space. The distributions of $M^2(x)$ (minimal surfaces) would code for quaternion structures (decomposition of octonionic coordinates to quaternionic coordinates in turn decomposing to pairs of complex coordinates).

Consider next the associativity condition for cosmic strings in $X^2 \times Y^2 \subset M^4 \times CP_2$. Now CP_2 projection is 2-D complex surfaces and M^4 projection is minimal surface. Situation is clearly associative. How unique the choice of M_0^4 is now?

1. Now $M^2(x)$ depends on position but $M^2(x)$:s define an integrable distribution defining string orbit X^2 as a minimal surface. M_0^4 must contain all surfaces $M^2(x)$, which would fix M_0^4 to a high degree for complex enough cosmic strings.

2. Each point of X^2 corresponds to the same partonic surface $Y^2 \subset CP_2$ labelling the tangent spaces for its pre-image in M^8. All the tangent surfaces $M^2(x) \times E^2(y)$ for $X^2 \times Y^2 \subset M^8$ share only $M^2(x) \subset M_0^4$. M_0^4 must contain all tangent spaces $M^2(x)$ and the inverse image of $Y^2 \subset CP_2$ must belong to the orthogonal complement E^4 of M_0^4. This is completely analogous with the condition $X^2 = X^2 \times Y^2 \subset M^4 \times CP_2$.

   Consider a decomposition $M^8 = M_0^4 \times E^4$, $M_0^4 = M_0^2 \times E_0^2$. If the inverse image of Y^2 at point x belongs to E^4, the M_0^4 projection belongs to M_0^4 also in M^8. If this does not pose any condition on the tangent spaces assignable to the points of Y^2 defining points of CP_2, there are no problems. What could happen that the tangent spaces assignable to Y^2 could force the projection of the inverse image of Y^2 to intersect M_0^4.

One should also understand massless extremals (MEs). How to choose M_0^4 in this case?

1. MEs are given as zeros of arbitrary functions of CP_2 coordinates and 2 M^4 coordinates u and v representing local light-like direction and polarization direction orthogonal to it. In the simplest situation these directions are constant and define $M^4_0 = M^2_0 \times E^2_0$ decomposition everywhere so that M^4_0 is uniquely defined. Same applies also when the directions are not constant. In the general case light-like direction would define the local tangent plane of string world sheet and local polarization plane. Since the dimension of M^4 projection is 4 there seems to be no problems involved.

2. Tangent plane of X^4 is parameterized by CP_2 coordinates depending on 2 coordinates u and v. The surface $X^4 \subset M^8$ must be graph for a map $M^4_0 \rightarrow E^4$ so that a 2-parameter deformation of M^4_0 as tangent plane is in question. The distribution of tangent planes of $X^4 \subset M^8$ is 2-D as is also the CP_2 projection in H.

To sum up, the original vision about the associativity properties of the known extremals at level of H survives. On the other hand, CDs emerge if M^4 corresponds to the co-associative part of O. Does this mean that $M^8 - H$ correspondence maps associative to co-associative by multiplying the quaternionic tangent space in M^8 by I_4 to get that in H and vice versa or that the notions of associative and co-associative do not make sense at the level of H? This does not affect the correspondence since the same CP_2 point parametrizes both associative tangent space and its complement.

3.1.2 Space-time surfaces as co-dimension 4 algebraic varieties defined by the vanishing of real or imaginary part of octonionic polynomial?

If the theory intended to be a theory of everything, the solution ansatz for the field equations defining space-time surfaces should be ambitious enough: nothing less than a general solution of field equations should be in question.

1. One cannot exclude the possibility that all analytic functions of complexified octonionic variable with real Taylor or even Laurent coefficients. These would would a commutative and associative algebra. Space-time surfaces would be identified as their zero loci. This option is however number theoretically attractive and can also leads to problems with adelic physics. Since Taylor series at rational point need not anymore give a rational value.

2. Polynomials of complexified octonion variable o with real coefficients define the simplest option but also rational functions formed as ratios of this kind of polynomials must be considered. Polynomials form a non-associative ring allowing sum, product, and functional decomposition as basic operations. If the coefficients o_n of polynomials are complex numbers $o_n = a_n + ib_n$, a_n, b_n real, where i refers to the commutative imaginary unit complexifying the octonions, the ring is associative. It is essential to allow only powers o^n (or $(o - o_0))^n$ with $o_0 = a_0 + ib_0$, a_0, b_0 real numbers). Physically this means that a preferred time axis is fixed. This time axis could connect the tips of CD in ZEO.

   One can write

$$P(o) = \sum_k p_k o^k \equiv RE(P)(q_1, q_2, \overline{q_1}, \overline{q}_2) + IM(P)(q_1, q_2, \overline{q_1}, \overline{q}_2) \times I_4 \quad , p_k \ \text{real} \quad , \tag{3.1}$$

   where the notations

$$o = q_1 + q_2 I_4 \quad , \quad q_i = z^1_i + z^2_i I_2 \quad , \quad \overline{q}_i = z^1_i - z^2_i I_2 \quad , z^j_i = x^j_i + i y^j_i \tag{3.2}$$

   Note that the conjugation does *not* change the sign of i. Due to the non-commutativity of octonions P^i as functions of quaternions are in general *not* analytic in the sense that they would be polynomials of q_i with real coefficients! They are however analytic functions of z_i. The real and imaginary parts of x_i^j correspond to Minkowskian and Euclidian signatures.

In adelic physics coefficients o_n of the octonionic polynomials define WCW coordinates and should be rational numbers or rationals in the extension of rationals defining the adele. The polynomials form an associative algebra since associativity holds for powers o^n multiplied by real number. Thus complex analyticity crucial in algebraic geometry would be a key element of adelic physics.

3. If the preferred extremals correspond to the associative algebra formed by these polynomials, one could construct a completely general solution of the field equations as zero loci of their real or imaginary parts and build up of new solutions using algebra operation sum, product, and functional decomposition. One could identify space-time regions as associative or co-associative algebraic varieties in terms of these polynomials and they would form an algebra.

The motivation for this dream comes from 2-D electrostatics, where conducting surfaces correspond to curves at which the real part u or imaginary part v of analytic function $w = f(z) = u+iv$ vanishes. In electrostatics curves form families with curves orthogonal to each other locally and the map $w = u+iv \rightarrow v-iu$ defines a duality in which curves of constant potential and the curves defining their normal vectors are mapped to each other.

1. The generalization to the recent situation would be vanishing of the imaginary part $IM(P)$ or real part $RE(P)$ of the octonionic polynomial, where real and imaginary parts are defined via $o = q_c^1 + q_c^2 I_4$. One can consider also the possibility that imaginary or real part has constant value c are restricted to be rational so that one can regard the constant value set also as zero set for a polynomial with constant shift. Note that the rationals could be also complexified by addition of i. One would have

$$RE(P)(z_i^k) \quad \text{or} \quad IM(P)(z_i^k) = c \quad , \quad c = c_0 \ \text{rational} \quad . \tag{3.3}$$

c_0 must be real. These two options should correspond to the situations in which tangent space or normal space is associative (associativity/co-associativity). Complexified space-time surfaces X_c^4 corresponding to different constant values c of imaginary or real part (with respect to i) would define foliations of M_c^8 by locally orthogonal 4-dimensional surfaces in M_c^8 such that normal space for surface X_c^4 would be tangent space for its co-surface.

CDs and ZEO emerges naturally if the $IM(o)$ corresponds to co-quaternionic part of octonion.

2. It must be noticed that one has moduli space for the quaternionic structures even when M_0^4 is fixed. The simplest choices of complexified quaternionic space $H_c = M_{c,0}^4$ containing preferred complex plane $M_{c,0}^2$ and its orthogonal complement are parameterized by CP_2. More complex choices are characterized by the choice of distribution of $M^2(x)$ integrable to (presumably minimal) 2-surface in M^4. Also the choice of the origin matters as found and one has preferred coordinates. Also the 8-D Lorentz boosts give rise to further quaternionic moduli. The physically interesting question concerns the interpretation of space-time surfaces with different moduli. For instance, under which conditions they can interact?

The proposal has several extremely nice features.

1. Single real valued polynomial of real coordinate extended to octonionic polynomial and fixed by real coefficients in extension of rationals would determine space-time surfaces.

2. The notion of analyticity needed in concrete equations is just the ordinary complex analyticity forced by the octonionic complexification: there is no need for the application to have left- or right quaternion analyticity since quaternionic derivatives are not needed. Algebraically one has the most obvious guess for the counterpart of real analyticity for polynomials generalized to octonionic framework and there is no need for the quaternionic generalization of Cauchy-Riemann equations [A7, A4] [A7, A4] (`http://tinyurl.com/yb8l34b5`) plagued by

the problems with the definition of differentiation in non-commutative and non-associative context. There would be no problems with non-associativity and non-commutativity thanks to commutativity of complex coordinates with octonionic units.

3. The vanishing of the real or imaginary part gives rise to 4 conditions for 8 complex coordinates z_1^k and z_2^k allowing to solve z_2^k as algebraic functions $z_2^k = f^k(z_1^l)$ or vice versa. The conditions would reduce to algebraic geometry in complex co-dimension $d_c = 4$ and all methods and concepts of algebraic geometry can be used! Algebraic geometry would become part of TGD as it is part of M-theory too.

3.2 Is the associativity of tangent-/normal spaces really achieved?

The non-trivial challenge is to prove that the tangent/normal spaces are indeed associative for the two options. The surfaces X_c^4 are indeed associative/co-associative if one considers the *internal* geometry since points are in M_c^4 or its orthogonal complement.

One should however prove that X_c^4 are also associative *as sub-manifolds* of O and therefore have quaternionic tangent space or normal space at each point parameterized by a point of CP_2 in the case that tangent space containing position dependent M_c^2, which integrate to what might be called a 2-D complexified string world sheet inside M_c^4.

1. The first thing to notice that associativity and quaternionicity need not be identical concepts. Any surface with complex dimension $d < 4$ in O is associative and any surface with dimension $d > 4$ co-associative. Quaternionic and co-quaternionic surfaces are 4-D by definition. One can of course ask whether one should consider a generalization of brane hierarchy of M-theory also in TGD context and allow associativity in its most general sense. In fact, the study of singularity of o^2 shows that 6 and 5-dimensional surfaces are allowed for which the only interpretation would be as co-associative spaces. This exceptional situation is due to the additional symmetries increasing the dimension of the zero locus.

2. One has clearly quaternionicity at the level of o obtained by putting $Y = 0$ and at the level of the tangent space for the resulting surface. The tangent space should be quaternionic. The Jacobian of the map defined by P is such that it takes fixed quaternionic subspace $H_c \rightarrow M_{0,c}^4$ of O to a quaternionic tangent space of X^4. The Jacobian applied to the vectors of H_c gives the octonionic tangent vectors and they should span a quaternionic sub-space.

3. The notion of quaternionic surface is rigorous. $M^8 - H$ correspondence could be actually interpreted in terms of the construction of quaternionic surface in M^8. One has 4-D integrable distribution of quaternionic planes in O with given quaternion structure labelled by points of CP_2 and has representation at the level of H as space-time surface and should be preferred extremals. These quaternion planes should integrate to a slicing by 4-surfaces and their duals. One obtains this slicing by fixing the values 4 of the suitably defined octonionic coordinates P^i, $i = 1,..,8$, to a real constants depending on the surface of the slicing. This gives a space-time surfaces for which tangent space-spaces or normal spaces are quaternionic.

   The first guess for these coordinates P^i be as real or imaginary parts of real polynomials $P(o)$. But how to prove and understand this?

Could the following argument be more than wishful thinking?

1. In complex case an analytic function $w(z) = u + iv$ of $z = x + iy$ mediates a map between complex planes Z and W. One can interpret the imaginary unit appearing in w locally as a tangent vector along $u = constant$ coordinate line.

2. One can interpret also octonionic polynomials with real coefficients as mediating a map from octonionic plane O to second octonionic plane, call if W. The decomposition $P = P^{1)} + P^{2)} I_4$ would have interpretation in terms of coordinates of W with coordinate lines representing quaternions and co-quaternions.

3. This would suggests that the quaternionic coordinate lines in W can be identified as coordinate curves in O - that space-time surfaces - which are quaternionic/co-quaternionic surfaces

for $P^1 = constant/P^2 = constant$ lines. One would have a representation of the same thing in two spaces, and if sameness includes also quaternionicity/co-quaternionicity as attributes, then also associativity and co-associativity should hold true.

The most reasonable approach is based on generality. Associativity/quaternionicity means a slicing of octonion space by orthogonal quaternionic and co-quaternionic 4-D surfaces defined by constant value surfaces of octonionic polynomial with real coefficients. This slicing should make sense also for quaternions: one should have a slicing by complex and co-complex (commutative/co-commutative) surfaces and in TGD string world sheets and partonic 2-surfaces assignable to Hamilton-Jacobi structure would define this kind of slicing. In the case of complex numbers one has a slicing in terms of constant value curves for real and imaginary parts of analytic function and Cauchy-Riemann equations should define the property and co-property. The first guess that the tangent space of the curve is real or imaginary is wrong.

3.2.1 Could associativity and commutativity conditions be seen as a generalization of Cauchy-Rieman conditions?

Quaternionicity in the octonionic case, complexity in quaternionic case, and what-ever-it-is in complex case should be seen as a 3-levelled hierarchy of geometric conditions satisfied by polynomial maps with real coefficients for polynomials in case of octonions and quaternions. Of course, also Taylor and even Laurent series might be considered. The “Whatever it is” cannot be nothing but Cauchy-Riemann conditions defining complex analyticity for complex maps.

The hierarchy looks obvious. In the case of Cauchy-Riemann conditions one has commutative and associative structure and Cauchy-Riemann conditions are linear in the partial derivatives. In the case of commutative sub-manifolds of quaternionic space the conditions are quadratic in the partial derivatives. In the case of associative sub-manifolds of octonionic space the conditions are trilinear in partial derivatives. One would have nothing but a generalization of Cauchy-Riemann equations to multilinear equations in dimensions $D = 2^k$, $k = 1, 2, 3$: k-linearity with $k = 1, 2, 3$!

One can continue the hierarchy of division algebras by assuming only algebra property by using Cayley-Dickson construction (see `http://tinyurl.com/ybuyla2k`) by adding repeatedly a non-commuting imaginary unit to the structure already obtained and thus doubling the dimension of the algebra each time. Polynomials with real coefficients should still define an associative and commutative algebra if the proposal is to make sense. All these algebras are indeed power associative: one has $x^m x^n = x^{m+n}$. For instance, sedenions define 16-D algebra. Could this hierarchy corresponds to a hierarchy of analyticities satisfying generalized Cauchy-Riemann conditions?

3.2.2 Complex curves in real plane cannot have real tangent space

Going from octonions to quaternions to complex numbers, could constant value curves of real and imaginary parts of ordinary analytic function in complex plane make sense? The curves $u = 0$ and $v = 0$ of functions $f(z) = u + iv$, $z = x + iy$ define a slicing of plane by orthogonal curves completely analogous to its octonionic and quaternionic variants. Can one say that the tangent vectors for these curves are real/imaginary? For $u = 0$ these curves have tangent $\partial_x u + i\partial_y u$, which is not real unless one has $f(z) = k(x + iy)$, k real.

Reality condition is clearly too strong. In fact, it is the well-ordering of the points of the 1-dimensional curve, which is the property in question and lost for complex numbers and regained at $u = 0$ and $v = 0$ curves. The reasonable interpretation is in terms of hierarchy of conditions multilinear in the gradients of coordinates proposed above and linear Cauchy-Riemann conditions is the only option in the case of complex plane. What is special in this curves that the tangent vectors define flows which by Cauchy-Riemann conditions are divergenceless and irrotational locally.

Pessimistic would conclude that since the conjecture fails except for linear polynomials in complex case, it fails also in the case of quaternions and octonions. For quaternionic polynomial q^2 the conditions are however satisfied and it turns out that the resulting conditions make sense also in the general case. Optimistic would argue that reality condition is not analogous to commutativity and associativity so that this example tells nothing. Less enthusiastic optimist might admit that the reality condition is a natural generalization to complex case but that the conjecture might be true only for a restricted set of polynomials - in complex case of for $f(z) = kz$, k real. In quaternionic and octonionic case but hopefully for a larger set of polynomials with real coefficients, maybe even all polynomials with real coefficients.

3.2.3 Associativity and commmutativity conditions as a generalization of Cauchy-Rieman conditions?

Quaternionicity in the octonionic case, complexity in quaternionic case, and what-ever-it-is in complex case should be seen as a 3-levelled hierarchy of geometric conditions satisfied by polynomial maps with real coefficients for polynomials in case of octonions and quaternions. Of course, also Taylor and even Laurent series might be considered. The "whatever-it-is" cannot be nothing but Cauchy-Riemann conditions defining complex analyticity for complex maps.

The hierarchy looks obvious. In the case of Cauchy-Riemann conditions one has commutative and associative structure and Cauchy-Riemann conditions are linear in the partial derivatives. In the case of commutative sub-manifolds of quaternionic space the conditions are quadratic in the partial derivatives. In the case of associative sub-manifolds of octonionic space the conditions are trilinear in partial derivatives. One would have nothing but a generalization of Cauchy-Riemann equations to multilinear equations in dimensions $D = 2^k$, $k = 1, 2, 3$: k-linearity with $k = 1, 2, 3$!

One can continue the hierarchy of number fields by assuming only algebra property by adding additional imaginary units as done in Cayley-Hamilton construction (see `http://tinyurl.com/ybuyla2k`) by adding repeatedly a non-commuting imaginary unit to the algebra already obtained and thus doubling the dimension of the algebra each time. Polynomials with real coefficients should still define an associative and commutative algebra if the proposal is to make sense. All these algebras are indeed power associative: one has $x^m x^n = x^{m+n}$. For instance, sedenions define 16-D algebra. Could this hierarchy corresponds to a hierarchy of analyticities satisfying generalized Cauchy-Riemann conditions? Could this hierarchy corresponds to a hierarchy of analyticities satisfying generalized Cauchy-Riemann conditions?

One would have also a nice physical interpretation: in the case of quaternions one would have "quaternionic conformal invariance" as conformal invariances inside string world sheets and partonic 2-surfaces in a nice agreement with basic vision about TGD. At the level of octonions would have "quaternionic conformal invariance" inside space-time surfaces and their duals. What selects the preferred commutative or co-commutative surfaces is of course an interesting problem. Is a gauge choice in question? Are these surfaces selected by some special property such as singular character? Or does one have wave function in the set of these surfaces for a given space-time surface?

3.2.4 Could quaternionic polynomials define complex and co-complex surfaces in H_c?

What about complex and co-complex (commutative/co-commutative) surfaces in the space of quaternions? One would have a slicing of the quaternionic space by pairs of complex and co-complex surfaces and would have natural identification as quaternion/Hamilton-Jacobi structure and relate to the decomposition of space-time to string world sheets and partonic 2-surfaces. Now the condition of associativity would be replaced with commutativity.

1. In the quaternionic case the tangent vectors of the 2-D complex sub-variety would be commuting. Can this be the case for the zero loci real polynomials $P(q)$ with $IM(P) = 0$ or $RE(P) = 0$? In this case the commutativity condition is that the tangent vectors have imaginary parts (as quaternions) proportional to each other but can have different real parts. The vanishing of cross product is the condition now and involves only two vectors whereas associativity condition involves 3 vectors and is more difficult.

2. The tangent vectors of a commutative 2-surface commute: $[t^1, t^2] = 0$. The commutator reduces to the vanishing of the cross product for the imaginary parts:

$$Im(t^1) \times Im(t^2) = 0 \quad . \tag{3.4}$$

3. Expressing z_1^i as a function of z_2^k and using (z_1^i, z_2^k) as coordinates in quaternionic space, the tangent vectors in quaternionic spaces can be written in terms of partial derivatives $\partial z_1^{1)}/\partial z_2^{k)}$ as

$$t_k^i = (\frac{\partial z_1^{i)}}{\partial z_2^{k)}}, \delta_k^i) \ , \tag{3.5}$$

Here the first part corresponds to $RE(t^i)$ as quaternion and second part to $IM(t^i)$ as quaternion.

The condition that the vectors are parallel implies

$$\frac{\partial z_1^{1)}}{\partial z_2^{k)}} = 0 \ . \tag{3.6}$$

At the commutative 2-surface X^2 $z_1^{1)}$ is constant and $z_1^{2)}$ is a function of $z_2^{1)}$ and $z_2^{2)}$. One would have a graph of a function $z_1^{2)} = f_2(z_2^{k)})$ at X^2 but not elsewhere. One could regard $z_1^{1)}$ as an extremum of a function $z_1^{1)} = f_1(z_2^{k)})$.

How to interpret this result?

1. In the generic case this condition eliminates 1 dimension so that that 2-D surface would reduce to a 1-D curve.

2. If one poses constraints on the coefficients of $P(q)$ analogous to the conditions forcing the potential function for say cusp catastrophe to have degenerate extrema at the boundaries of the catastrophe one can get 2-D solution. For these values of parameters the conditions would be equivalent with $RE(P) = 0$ or $IM(P) = 0$ conditions.

   The vanishing of the gradient of z_1^1 would indeed correspond in the case of cups catastrophe to the condition for the co-incidence of two roots of the behavior variable x as extremum of potential function $V(x, a, b)$ fixing the control variable a as function of b.

   This would pose constraints on the coefficients of P not all polynomials would be allowed. This kind of conditions would realize the idea of quantum criticality of TGD at the level of quaternion polynomials. This option is attractive if realizable also at the level of octonion polynomials. This turns out to be the case.

3. One would thus have two kinds of commutative/co-commutative surfaces. The generic 1-D surfaces and 2-D ones which are commutative/commutative and critical and assignable to string world sheets and partonic 2-surfaces. 1-D surfaces would correspond to fermion lines at the orbits of partonic 2-surfaces appearing in the twistor amplitudes in the interaction regions defined by CDS. 2-D surfaces would correspond to the orbits of fermionic strings connecting point-like fermions at their ends and serving as correlates for bound state entanglement for external fermions arriving into CD. This picture would allow also to understand why just some string world sheets and partonic 2-surfaces are selected.

The simplest manner to kill the proposal is to look for $P = q^2$ and $RE(P(q^2)) = 0$ surface. In this case this condition is indeed satisfied. One has

$$\begin{aligned}
&RE(P) = X^{1)} + X^{2)}I_1 \ , \\
&X^{1)} = (z_1^{1)})^2 - (z_1^{2)})^2 + (z_2^{1)})^2 - (z_2^{2)})^2 \ , \quad X^{2)} = 2z_1^{1)}z_1^{2)}I_1 \ , \\
&IM(P) = Y^{1)} + Y^{2)}I_1 \ , \\
&Y^{1)} = (z_2^{1)} + \overline{z_2^{1)}})z_1^{1)} \ , \qquad Y^{2)} = (z_2^{2)} + \overline{z_2^{2)}})z_1^{2)}
\end{aligned} \tag{3.7}$$

$X^{2)} = 0$ gives $z_1^{1)} z_1^{2)} = 0$ so that one has either $z_1^{1)} = 0$ or $z_1^{2)} = 0$. $X^{1)} = 0$ gives for $z_1^{1)} = 0$ $z_1^{2)} = \pm\sqrt{(z_2^{1)})^2 + (z_2^{2)})^2}$.

The partial derivative $\partial z_1^{1)}/\partial z_2^{k)}$ is from implicit function theorem - following from the vanishing of the differential $d(RE(P))$ along the surface - proportional $\partial X^{1)}/\partial z_2^{k)}$, but vanishes as required.

Clearly, the quaternionic variant of the proposal survives in the simplest case its simplest test. 2-D character of the surface would be due to the criticality of q^2 making it possible to satisfy the conditions without the reduction of dimension.

3.2.5 Explicit form of associativity/quaternionicity conditions

Consider now the explicit conditions for associativity in the octonionic case.

1. One should calculate the octonionic tangent (normal) vectors t^i for $X = 0$ in associative ($Y = 0$ in co-associative case) and show that there associators $Ass(t^i, t^j, t^k)$ vanish for all possible or all possible combinations i, j, k. In other words, one that one has

$$Ass(t^i, t^j, t^k) = 0 \quad , \quad i,j,k \in \{1,..,4\} \quad , \quad Ass(a,b,c) \equiv (ab)c - a(bc) \quad . \tag{3.8}$$

   One can cast the condition to simpler from by expressing t^i as octonionic vectors $t^i_k E^k$:

$$\begin{aligned} Ass(E^a, E^b, E^b) &=\equiv f^{abcd} E_d \quad , \quad a,b,c,d \in \{1,..,7\} \quad , \\ f^{abcd} &= \epsilon^{abe}\epsilon_e^{\ cd} - \epsilon^{aed}\epsilon^{bc}_{\ \ e} = 2\epsilon^{abe}\epsilon_e^{\ cd} \quad . \end{aligned} \tag{3.9}$$

   The permutation symbols for a given triplet i, j, k are structures constants for quaternionic inner product and completely antisymmetric (see `http://tinyurl.com/p42tqsq`).. $\epsilon_{ijk} = 1$ is true for the seven triplets $123, 145, 176, 246, 257, 347, 365$ defining quaternionic sub-spaces with 1-D intersections. The anti-associativity condition $(E_iE_j)E_k = -(E_iE_j)E_k$ holds true so that one has obtains the simpler expression for f^{ijks} having values ± 2.

   Using this representation $Ass(t^i, t^j, t^k)$ reduces to 7 conditions for each triplet:

$$t^i_r t^j_s t^k_t f^{rstu} = 0 \quad , \quad i,j,k \in \{1,..,4\} \quad , \quad r,s,t,u \in \{1,..,7\} \quad . \tag{3.10}$$

2. If the vanishing condition $X = 0$ or $Y = 0$ is crucial for associativity then every polynomial is its own case to be studied separately and a general principle behind associativity should be identified: the proposal is as a non-linear generalization of Cauchy-Riemann conditions. As the following little calculation shows, the vanishing condition indeed appears as one calculates partial derivatives $\partial z_1^{k)}/\partial z_2^{l)}$ in the expression for the tangent vectors of the surface deduced from the vanishing gradient of X or Y.

3. I have proposed the octonionic polynomial ansatz already earlier but failed to prove that it gives associative tangent or normal spaces. Besides the intuitive geometric argument I failed to notice that the complex 8-D tangent vectors in coordinates $z_1^{k)}$ or $z_2^{k)}$ for complexified space-time surface and coordinates $(z_1^{k)}, z_2^{k)})$ for o have components

$$\frac{\partial o^i}{\partial z^1_k} \leftrightarrow (\delta^i_k, \frac{\partial z_2^{i)}}{\partial z_1^{k)}})$$

or

$$(\frac{\partial o^i}{\partial z^2_k}) \leftrightarrow (\frac{\partial z_1^{i)}}{\partial z_2^{k)}}, \delta^i_k) \quad . \tag{3.11}$$

These vectors correspond to complexified octonions O_i given by

$$\delta^i_k E^k + \frac{\partial z_2^{i)}}{\partial z_1^{k)}} E^k E_4 \quad , \tag{3.12}$$

where the unit octonions are given by $(E_0, E_1, E_2, E_3) = (1, I_1, I_2, I_3)$ and $(E_5, E_5, E_7, E_8) = (1, I_1, I_2, I_3)E_4$. The vanishing of the associators stating that one has

4. One can calculate the partial derivatives $\frac{\partial z^k_i}{\partial z^k_j}$ explicitly without solving the equations or the complex valued quaternionic components of $RE(P) \equiv X = 0$ or $IM(P) \equiv Y = 0$ (note that X and Y have for complex components labelled by X^i and Y^i respectively.

$$\begin{aligned} &Y^i(z_1^{k)}, z_2^{l)}) = c \in R \quad , \quad i = 1, ..., 4 \quad , \quad \text{associativity} \quad , \\ &\text{or} \\ &X^i(z_1^{k)}, z_2^{l)}) = c \in R \quad , \quad i = 1, ..., 4 \quad , \quad \text{co-associativity} \quad . \end{aligned} \tag{3.13}$$

explicitly and check whether associativity holds true. The derivatives can be deduced from the constancy of Y or X.

5. For instance, if one has $z_2^{k)}$ as function of $z_1^{k)}$, one obtains in the associative case

$$\begin{aligned} &RE(Y)^i{}_k + IM(Y)^i{}_k \frac{\partial z_2^{r)}}{\partial z_1^{k)}} = 0 \\ &RE(Y)^i{}_k \equiv \frac{\partial Y^i}{\partial z_1^{k)}} \quad , \qquad IM(Y)^i{}_k \equiv \frac{\partial Y^i}{\partial z_2^{k)}} \quad . \end{aligned} \tag{3.14}$$

In co-associative case one must consider normal vectors expressible in terms of Y^i so that X is replaced with Y in these equations.

This allows to solve the partial derivatives needed in associator conditions

$$\frac{\partial z_2^{i)}}{\partial z_1^{k)}} = \left[Im(Y)^{-1}\right]^i{}_r Re(Y)^r{}_k \quad . \tag{3.15}$$

6. The vanishing conditions for the associators are however multilinear and one can multiply each factor by the matrix $IM(P)$ without affecting the condition so that $IM(P)^{-1}$ disappears and one obtains the conditions for vectors

$$\begin{aligned} &T^i_r T^j_s T^k_t f^{rstu} = 0 \ , \qquad\qquad i,j,k \in \{1,..,4\} \ , \quad r,s,t,u \in \{1,..,7\} \ , \\ &T^i = IM(Y)^i{}_k E^k - RE(Y)^i{}_k E^k E_4 \ . \end{aligned} \tag{3.16}$$

This form of conditions is computationally much more convenient.

How to solve these equations?

1. The antisymmetry of f^{rstu} with respect to the first two indices r,s leads one to ask whether one could have

$$T^i_r T^j_s T^k_t = 0 \tag{3.17}$$

for the 7 quaternionic triplets. This is guaranteed if one has either $RE(Y)^i{}_k = \partial Y^i/\partial z_1^k = 0$ (coquaternionic part of T^i) or $IM(Y)^i{}_k = \partial Y^i/\partial z_2^k = 0$ (co-quaternionic part of T^i) for *one* member in each triplet.

The study of the structure constants listed above shows that indices 1,2,3 are contained in all 7 triplets. Same holds for the indices appearing in any quaternionic triplet. Hence it is enough to require that three gradients $RE(Y)^i k = 0$ or $IM(Y)^i{}_k = 0$ $k \in \{1,2,3\}$ vanish. This condition is obviously too strong since already single vanishing condition reduces the dimension of space-time variety to 3 in the generic case and it becomes trivially associative.

Octonionic automorphism group G_2 gives additional basis with their own quaternion triplets and the general condition would be that 3 partial derivatives vanish for a triplet obtained from the basic triplet $\{1,2,3\}$ by G_2 transformation. It is not quite clear to me whether the G_2 transformation can depend on position on space-time surface.

2. As noticed, the vanishing of all triplets is an un-necessarily strong condition. Already the vanishing of single gradient $RE(Y)^i{}_k$ or $IM(Y)^i{}_k$ reduces the dimension of the surface from 4 to 3 in the generic case. If one accepts that the dimension of associative surface is lower than 4 then single criticality condition is enough to obtain 3-D surface.

In the generic case associativity holds true only at the 4-D tangent spaces of 3-surfaces at the ends of CD (at light-like partonic orbits it holds true trivially in 4-D) and that the twistor lift of Kähler action determines the space-time surfaces in their interior.

In this case one can map only the boundaries of space-time surface by $M^8 - H$ duality to H. The criticality at these 3-surfaces dictates the boundary conditions and provides a solution to infinite number of conditions stating the vanishing of SSA Noether charges at space-like boundaries. These space-time regions would correspond to the regions of space-time surfaces inside CDs identifiable as interaction regions, where Kähler action and volume term couple and dynamics depends on coupling constants.

The mappability of M^8 dynamics to H dynamics in all space-time regions does not look feasible: the dynamics of octonionic polynomials involves no coupling constants whereas twistor lift of Kähler action involves couplings parameters. The dynamics would be non-associative in the geometric sense in the interior of CDs. Notice that also conformal field theories involve slight breaking of associativity and that octonions break associativity only slightly ($a(bc) = -(ab)c$ for octonionic imaginary units). I have discussed the breaking of associativity from TGD viewpoint in [K17] .

3. Twistor lift of Kähler action allows also space-time regions, which are minimal surfaces [L1] and for which the coupling between Kähler action and volume term vanishes. Preferred extremal property reduces to the existence of Hamilton-Jacobi structure as image of the quaternionic structure at the level of M^8. The dynamics is universal as also critical dynamics and independent of coupling constants so that $M^8 - H$ duality makes sense for it. External particles arriving into CD via its boundaries would correspond to critical 4-surfaces: I have discussed their interpretation from the perspective of physics and biology in [L2].

4. One should be able to produce associativity without the reduction of dimension. One can indeed hope of obtaining 4-D associative surfaces by posing conditions on the coefficients of the polynomial P by requiring that one $RE(Y)^i_k$ or $IM(Y)^i_k$, $i = i_1$ -call it just X_1 - should vanish so that Y^i would be critical as function of z_1^k or z_2^k.

   At $X_1 = 0$ would have degenerate zero at the 4-surface. The decomposition of X_1 to a product of monomial factors with root in extension of rationals would have one or more factors appearing at least twice. The associative 4-surfaces would be ramified. Also the physically interesting p-adic primes are conjectured to be ramified in the sense that their decomposition to primes of extension of rationals contains powers of primes of extension. The ramification of the monomial factors is nothing but ramification for polynomials primes in field of rationals in terms of polynomial primes in its extension.

   This could lead to vanishing of say one triplet while keeping $D = 4$. This need not however give rise to associativity in which case also second $RE(Y)^i_i$ or $IM(Y)^i_k$, $i = i_2$, call it X_2, should vanish. The maximal number of X_i would be $n_{max} = 3$. The natural condition consistent with quantum criticality of TGD Universe would be that the variety is associative but maximally quantum critical and has therefore dimension $D = 3$ or $D = 4$. Stronger condition allows only $D = 4$.

   These $n \leq 3$ additional conditions make the space-time surface analogous to a catastrophe with $n \leq 3$ behavior variables in Thom's classification of 7 elementary catastrophes with less than 11 control variables [A1]. Thom's theory does not apply now since it has only one potential function $V(x)$ (now $n \leq 3$ corresponding to the critical coordinates Y^i!) as a function of behaviour variables and control variables). Also the number of non-vanishing coefficients in the polynomial having values in an extension of rationals and acting as control variables is unlimited. In quaternionic case the number of potential functions is indeed 1 but the number of control variables unlimited.

5. One should be able to understand the $D = 3$ associative objects - say light-like 3-surfaces or 3-surfaces at the boundaries of CD - as 3-surfaces along which 4-D associative (co-associative) and non-associative (non-co-associative) surfaces are glued together.

   Consider a product P of polynomials allowing 3-D surfaces as necessarily associative zero loci to which a small interaction polynomial vanishing at the boundaries of CD (proportional to o^n, $n > 1$) is added. Could P allow 4-D surface as a zero locus of real or imaginary part and containing the light-like 3-surfaces thanks to the presence of additional parameters coming from the interaction polynomial. Can one say that this small interaction polynomial would generate 4-D space-time in some sense? 4-D associative space-time regions would naturally emerge from the increasing algebraic complexity both via the increase of the degree of the polynomial and the increase of the dimension of the extension of rationals making it easier to satisfy the criticality conditions!

   There are two regions to be considered: the interior and exterior of CD. Could associativity/co-associativity be possible outside CD but not inside CD so that one would indeed have free external particles entering to the non-associative interaction region. Why associativity conditions would be more difficult to satisfy inside CD? Certainly the space-likeness of M^4 points with respect to the preferred origin of M^8 in this region should be crucial since Minkowski norm appears in the expressions of $RE(P)$ and $IM(P)$.

Do the calculations for the associative case generalize to the co-associative case?

1. Suppose that one has possibly associative surface having $RE(P) = 0$. One would have $IM(P) = 0$ for dual space-time surface defining locally normal space of $RE(P) = 0$ surface. This would transform the co-associativity conditions to associativity conditions and the preceding arguments should go through essentially as such.

   Associative and co-associative surfaces would meet at singularity $RE(P) = IM(P) = 0$, which need not be point in Minkowskian signature (see $P = o^2$ example in the Appendix) and can be even 4-D! This raises the possibility that the associative and co-associative surfaces defined by $RE(P) = 0$ and $IM(P) = 0$ meet along 3-D light-like orbits partonic surfaces or 3-D ends of space-time surfaces at the ends of CD.

2. If $D = 3$ for associative surfaces are allowed besides $D = 4$ as boundaries of 4-surfaces, one can ask why not allow $D = 5$ for co-associative surfaces. It seems that they do not have a reasonable interpretation as a surface at which co-associative and non-co-associative 4-D space-time regions would meet. Or could they in some sense be geometric "co-boundaries" of 4-surfaces like branes in M-theory serve as co-boundaries of strings? Could this mean that 4-D space-time-surface is boundary of 5-D co-associative surface defining a TGD variant of brane with strings world sheets replaced with 4-D space-time surfaces?

   What came as a surprise that $P = o^2$ allows 5-D and 6-D surfaces as zero loci of $RE(P)$ or $IM(P)$ as shown in Appendix. The vanishing of the entire o^2 gives 4-D interior or exterior of CD forced also by associativity/co-associativity and thus maximal quantum criticality. This is very probably due to the special properties of o^2 as polynomial: in the generic case the zero loci should be 4-D.

This discussion can be repeated for complex/co-complex surfaces inside space-time surfaces associated with fermionic dynamics.

1. Associativity condition does not force string world sheets and partonic 2-surfaces but they could naturally correspond to commutative or co-commutative varieties inside associative/co-associative varieties.

   In the generic case commutativity/co-commutativity allows only 1-D curves - naturally light-like fermionic world lines at the boundaries of partonic orbits and representing interacting point-like fermions inside CDs and used in the construction of twistor amplitudes [K19, K20]. There is coupling between Kähler part and volume parts of modified Dirac action inside CDs so that coupling constants are visible in the spinor dynamics and in dynamics of string world sheet.

2. At criticality one obtains 2-D commutative/co-commutative surfaces necessarily associated with external particles quantum critical in 4-D sense and allowing quaternionic structure. String world sheets would serve as correlates for bound state entanglement between fermions at their ends. Criticality condition would select string world sheets and partonic 2-surfacs from the slicing of space-time surface provided by quaternionic structure (having Hamilton-Jacobi structure as H-counterpart).

If associativity holds true and fixed M^2_c is contained in the tangent space of space-time surface, one can map the M^4 projection of the space-time surface to a surface in $M^4 \times CP_2$ so that the quaternionic tangent space at given point is mapped to CP_2 point. One obtains 4-D surface in $H = M^4 \times CP_2$.

1. The condition that fixed M^2_c belongs to the tangent space of X^4_c is true in the sense that the coordinates $z_2^{k)}$ do not depend on $z_1^{1)}$ and $z_1^{2)}$ defining the coordinates of M^2_c. It is not clear whether this condition can be satisfied in the general case: octonionic polynomials are expected to imply this dependence un-avoidably.

   The more general condition allows M^2_c to depend on position but assumes that M^2_c:s associated with different points integrate to a family 2-D surfaces defining a family of complexified string world sheets. In the similar manner the orthogonal complements E^2_c of M^2_c would integrate to a family of partonic 2-surfaces. At each point these two tangent spaces and their real projections would define a decomposition analogous to that define by light-like momentum vector and polarization vector orthogonal to it. This decomposition would define decomposition of quaternionic sub-spaces to complexified complex subspace and its co-complex normal space. The decomposition would correspond to Hamilton-Jacobi structure proposed to be central aspect of extremals [K16].

2. What is nice that this decomposition of M^4_c (M^4) would (and of course should!) follow automatically from the octonionic decomposition. This decomposition is lower-dimensional analog to that of the complexified octonionic space induced by level sets of real octonionic polymials but at lower level and extremely natural due to the inclusion hierarchy of classical number fields. Also M^2_c could have similar decomposition orthogonal complex curves by the

value sets of polynomials. The hierarchy of grids means the realization of the coordinate grid consisting of quaternionic, complex, and real curves for complexified coordinates o^k and their quaternionic and complex variants and is accompanied by corresponding real grids obtained by projecting to M^4 and mapping to CP_2.

Thus these decompositions would be obtained from the octonionic polynomial decomposing it to real quaternionic and imaginary quaternionic parts first to get a grid of space-time surfaces as constant value sets of either real or imaginary part, doing the same for the non-constant quaternionic part of the octonionic polynomial to get similar grid of complexified 2-surfaces, and repeating this for the complexified complex octonionic part.

Unfortunately, I do not have computer power to check the associativity directly by symbolic calculation. I hope that the reader could perform the numerical calculations in non-trivial cases to to this!

3.2.6 General view about solutions to $RE(P) = 0$ and $IM(P) = 0$ conditions

The first challenge is to understand at general level the nature of $RE(P) = 0$ and $IM(P) = 0$ conditions. Appendix shows explicitly for $P(o) = o^2$ that Minkowski signature gives rise to unexpected phenomena. In the following these phenomena are shown to be completely general but not quite what one obtains for $P(o) = o^2$ having double root at origin.

1. Consider first the octonionic polynomials $P(o)$ satisfying $P(0) = 0$ restricted to the light-like boundary δM^8_+ assignable to 8-D CD, where the octonionic norm of o vanishes.

   (a) $P(o)$ reduces along each light-ray of δM^8_+ to the same real valued polynomial $P(t)$ of a real variable t apart from a multiplicative unit $E = (1 + in)/2$ satisfying $E^2 = E$. Here n is purely octonion-imaginary unit vector defining the direction of the light-ray.

   $IM(P) = 0$ corresponds to quaterniocity. If the E^4 ($M^8 = M^4 \times E^4$) projection is vanishing, there is no additional condition. 4-D light-cones $M^4_\pm$ are obtained as solutions of $IM(P) = 0$. Note that $M^4_\pm$ can correspond to any quaternionic subspace.

   If the light-like ray has a non-vanishing projection to E^4, one must have $P(t) = 0$. The solutions form a collection of 6-spheres labelled by the roots t_n of $P(t) = 0$. 6-spheres are not associative.

   (b) $RE(PE) = 0$ corresponding to co-quaternionicity leads to $P(t) = 0$ always and gives a collection of 6-spheres.

2. Suppose now that $P(t)$ is shifted to $P_1(t) = P(t) - c$, c a real number. Also now $M^4_\pm$ is obtained as solutions to $IM(P) = 0$. For $RE(P) = 0$ one obtains two conditions $P(t) = 0$ and $P(t - c) = 0$. The common roots define a subset of 6-spheres which for special values of c is not empty.

The above discussion was limited to δM^8_+ and light-likeness of its points played a central role. What about the interior of 8-D CD?

1. The natural expectation is that in the interior of CD one obtains a 4-D variety X^4. For $IM(P) = 0$ the outcome would be union of X^4 with M^4_+ and the set of 6-spheres for $IM(P) = 0$. 4-D variety would intersect M^4_+ in a discrete set of points and the 6-spheres along 2-D varieties X^2. The higher the degree of P, the larger the number of 6-spheres and these 2-varieties.

2. For $RE(P) = 0$ X^4 would intersect the union of 6-spheres along 2-D varieties. What comes in mind that these 2-varieties correspond in H to partonic 2-surfaces defining light-like 3-surfaces at which the induced metric is degenerate.

3. One can consider also the situation in the complement of 8-D CD which corresponds to the complement of 4-D CD. One expects that $RE(P) = 0$ condition is replaced with $IM(P) = 0$ condition in the complement and $RE(P) = IM(P) = 0$ holds true at the boundary of 4-D CD.

6-spheres and 4-D empty light-cones are special solutions of the conditions and clearly analogs of branes. Should one make the (reluctant-to-me) conclusion that they might be relevant for TGD at the level of M^8.

1. Could M^4_+ (or CDs as 4-D objects) and 6-spheres integrate the space-time varieties inside different 4-D CDs to single connected structure with space-time varieties glued to the 6-spheres along 2-surfaces X^2 perhaps identifiable as pre-images of partonic 2-surfaces and maybe string world sheets? Could the interactions between space-time varieties X^4_i assignable with different CDs be describable by regarding 6-spheres as bridges between X^4_i having only a discrete set of common points. Could one say that X^2_i interact via the 6-sphere somehow. Note however that 6-spheres are not dynamical.

2. One can also have Poincare transforms of 8-D CDs. Could the description of their interactions involve 4-D intersections of corresponding 6-spheres?

3. 6-spheres in $IM(P) = 0$ case do not have image under $M^8 - H$ correspondence. This does not seem to be possible for $RE(P) = 0$ either: it is not possible to map the 2-D normal space to a unique CP_2 point since there is 2-D continuum of quaternionic sub-spaces containing it.

3.3 $M^8 - H$ duality: objections and challenges

In the following I try to recall all objections against the reduction of classical physics to octonionic algebraic geometry and against the notion of $M^8 - H$ duality and also invent some new counter arguments and challenges.

3.3.1 Can on really assume distribution of $M^2(x)$?

Hamilton-Jacobi structure means that $M^2(x)$ depends on position and $M^2(x)$ should define an integrable distribution integrating to a 2-D surface. For cosmic string extremals this surface would be minimal surface so that the term "string world sheet" is appropriate. There are objections.

1. It seems that the coefficients of octonionic polynomials cannot contain information about string world sheet, and the only possible choice seems to be that string world sheets and partonic 2-surfaces parallel to it assigned with integrable distribution of orthogonal complements $E^2(x)$ should be coded by quaternionic moduli. It should be possible to define quaternionic coordinates q_i decomposing to pairs of complex coordinates to each choice of $M^2(x) \times E^2(x)$ decomposition of given M^4_0. Octonionic coordinates would be given by $o = q_1 + q_2 I_4$ where q_i are associated with the same quaternionic moduli. The choice of Hamilton-Jacobi structure would not be ad hoc procedure anymore but part of the definition of solutions of field equations at the level of M^8.

2. It would be very nice if the quaternionic structure could be induced from a fixed structure defined for M^8_c once the choice of curvilinear M^4 coordinates is made. Since Hamiltoni-Jacobi structure [K16] involves a choice of generalized Kähler form for M^4 and since quaternionic structure means that there is full S^2 of Kähler structures determined by quaternionic imaginary units (ordinary Kähler form for sub-space $E^8 \subset M^8_c$) the natural proposal is that Hamilton-Jacobi structures is determined by a particular local choice of the Kähler form for M^4 involving fixing of quaternionic imaginary unit fixing $M^2(x) \subset M^4_0$ identifiable as point of S^2. This might relate closely also to the fixing of twistor structure, which indeed involves also self-dual Kähler form and a similar choice.

3. One can argue that it is not completely clear whether massless extremals (MEs) [K16] allow a general Hamilton-Jacobi structure. It is certainly true that if the light-like direction and orthogonal polarization direction are constant, MEs exist. It is clear that if the form of field equations is preserved and thus reduces to contractions of various tensors with second fundamental form one obtains only contractions of light-like vector with itself or polarization vector and these contractions vanish. For spatially varying directions one could argue that light-like direction codes for a direction of light-like momentum and that problems with local conservation laws expressed by field equations might emerge.

3.3.2 Can one assign to the tangent plane of $X^4 \subset M^8$ a unique CP_2 point when M^2 depends on position

One should show that the choice $s(x) \in CP_2$ for a given distribution of $M^2(x) \subset M^4(x)$ is unique in order to realize the $M^8 - H$ correspondence as a map $M^8 \to H$. It would be even better if one had an analytic formula for $s(x)$ using tangent space-data for $X^4 \subset H$.

1. If $M^2(x) = M_0^2$ holds true but the tangent space $M^4(x)$ depends on position, the assignment of CP_2 point $s(x)$ to the tangent space of $X^4 \subset M^8$ is trivial. When $M^4(x)$ is not constant, the situation is not so easy.

2. The space $M^2(x) \subset M^4(x)$ satisfies also the constraint $M^2(x) \subset M_0^4$ since quaternionic moduli are fixed. To avoid confusion notice that $M^4(x)$ denotes tangent space of X^4 and is different from M_0^4 fixing the quaternionic moduli.

3. $M^2(x)$ determines the local complex subspace and its completion to quaternionic tangent space $M^4(x)$ determines a point $s(x)$ of CP_2. The idea is that M_0^2 defines a standard reference and that one should be able to map $M^2(x)$ to M_0^2 by G_2 automorphism mapping also the $s(x)$ to a unique point $s_0(x) \in CP_2$ defining the CP_2 point assignable to the point of $X^4 \subset M^8$.

4. One can assign to the point x quaternionic unit vector $n(x)$ determining $M^2(x)$ as the direction of the preferred imaginary unit. The G_2 transformation must rotate $n(x)$ to n_0 defining M_0^2 and acts on s. G_2 transformation is not unique since $u_1 g u_2$ has the same effect for $u_i \subset U(2)$ leaving invariant the point of CP_2 for initial and final situation. Hence the equivalence classes of transformations should correspond to a point of 6-dimensional double coset space $U(2)\backslash G_2/U(2)$. Intuitively it seems obvious that the $s_0(x)$ is unique but proof is required.

3.3.3 What about the inverse of $M^8 - H$ duality?

$M^8 - H$ duality should have inverse in the critical regions of $X^4 \subset M^8$, where associativity conditions are satisfied. How could one construct the inverse of $M^8 - H$ duality in these regions? One should map space-time points $(m,s) \in M^4 \times CP_2$ to points $(m,e) = (m, f(m,s)) \in M^8$. $M_0^4 \supset M_0^2$ parameterized by CP_2 point can be chosen arbitrarily and one can require that it corresponds to some space-time point $(m_0, s_0) \in H$. CP_2 point $s(x)$ characterizes the quaternionic tangent space containing $M^2(x)$ and can choose M_0^2 to be $M^2(x_0)$ for conveniently chosen x_0. Coordinates x can be used also for $X^4 \subset M^8$.

One obtains set of points $(m,e) = (m(x), f(m(x), s(x)) \in M^8$ and a distribution of 4-D spaces of labelled by $s(x)$. This requires that the 4-D tangent space spanned by the gradients of $m(x)$ and $f(m(x), s(x))$ and characterized by $s_1 \subset CP_2$ for given $M^2(x)$ by using the above procedure mapping the situation to that for M_0^2 is same as the tangent space determined by $s(x)$: $s(x) = s_1(x)$. Also the associativity conditions should hold true. One should have a formula for s_1 as function of tangent vectors of space-time surface in M^8. The ansatz based on algebraic geometry in M_c^8 should be equivalent with this ansatz. The problem is that the ansatz leads to algebraic functions which cannot be found explicitly. It might be that in practice the correspondence is easy only in the direction $M^8 \to H$.

3.3.4 What one can say about twistor lift of $M^8 - H$ duality?

One can argue that the twistor spaces CP_1 associated with M^4 and E^4 are in no way visible in the dynamics of octonion polynomials and in $M^8 - H$ duality. Hence one could argue that they are not needed for any reasonable purpose. I cannot decide whether this is indeed the case. There I will consider the existence of twistor lift of the M^8 and also the twistor lift $M^8 - H$ duality in the space-time regions, where the tangent spaces satisfy the conditions for the existence of the duality as a map $(m,e) \in M^8 \to (m,s) \in M^4 \times CP_2$ must be considered. This involves some non-trivial delicacies.

1. The twistor bundles of M^4_c and E^4_c would be simply $M^4_c \times CP_1$ and $E^4_c \times CP_1$. $CP_1 = S^2$ parameterizes direction vectors in 3-D Euclidian space having interpretation as unit quaternions so that this interpretation might make sense. The definition of twistor structure means a selection of a preferred quaternion unit and its representation as Kähler form so that these twistor bundles would have thus Kähler structure. Twistor lift replaces complex quaternionic surfaces with their twistor spaces with induced twistor structure.

2. In M^8 the radii of the spheres CP_1 associated with M^4 and E^4 would be most naturally identical whereas in $M^4 \times CP_2$ they can be different since CP_2 is moduli space. Is the value of the CP_2 radius visible at all in the classical dynamics in the critical associative/co-associative space-time regions, where one has minimal surfaces. Criticality would suggest that besides coupling constants also parameters with dimension of length should disappear from the field equations. At least for the known extremals such as massless extremals, CP_2 type extremals, and cosmic strings CP_2 radius plays no role in the equations. CP_2 radius comes however into play only in interaction regions defined by CDs since $M^8 - H$ duality works only at the 3-D ends of space-time surface and at the partonic orbits. Therefore the different radii for the CP_1 associated with CP_2 and E^4 cause no obvious problems.

Consider now the idea about twistor space as real part of octonionic twistor space regarded as quaternion-complex space.

1. One can regard $CP_1 = S^2$ as the space of unit quaternions and it is natural to replace it with the 6-sphere S^6 of octonionic imaginary units at the level of complexified octonions. The sphere of complexified (by i) unit octonions is non-compact space since the norm is complex valued and this generalization looks neither attractive nor necessary since the projection to real numbers would eliminate the complex part.

   The equations determining the twistor bundle of space-time surface can be indeed formulated as vanishing of the quaternionic imaginary part of S^6 coordinates, and one obtains a reduction to quaternionic sphere S^2 at space-time level.

   If S^2 is identified as sub-manifold $S^2 \subset S^6$, it can be chosen in very many manners (this is of course not necessary). The choices are parameterized by $SO(7)/SO(3) \times SO(4)$ having dimension $D = 12$. This choice has no physical content visible at the level of H. Note that the Kähler structure determining Hamilton-Jaboci structure is fixed by the choice of preferred direction ($M^2(x)$). If all these moduli are allowed, it seems that one has something resembling multiverse, the description at the level of M^8 is deeper one and one must ask whether the space-time surfaces with different twistorial, octonionic, and quaternionic moduli can interact.

2. The resulting octonionic analog of twistor space should be mapped by $M^8 - H$ corresponds to twistor space of space-time surface $T(M^4) \times T(CP_2)$. The radii of twistor spheres of $T(M^4)$ and $T(CP_2)$ are different and this should be also understood. It would seem that the radius of $T(M^4)$ at $H = M^4 \times CP_2$ side should correspond to that of $T(M^4)$ at M^8 side and thus to that of S^6 as its geodesic sphere: Planck length is the natural proposal inspired by the physical interpretation of the twistor lift. The radius of $T(CP_2)$ twistor sphere should correspond to that of CP_2 and is about 2^{12} Planck lengths.

   Therefore the scale of CP_2 would emerge as a scale of moduli space and does not seem to be present at the level of M^8 as a separate scale. M^8 level would correspond to what might be called Planckian realm analogous to that associated with strings before dynamical compactification which is now replaced with number theoretic compactification. The key question is what determines the ratio of the radii of CP_2 scale to Planck for which favored value is 2^{12} [K18]. Could quantum criticality determine this ratio?

4 Challenges of octonionic algebraic geometry

Space-time surfaces in $H = M^4 \times CP_2$ identified as preferred extremals of twistor lift of Kähler action leads to rather detailed view about space-time surfaces as counterparts of particles. Does this picture follow from $X^4 \subset M^8$ picture and does this description bring in something genuinely new?

4.1 Could free many-particle states as zero loci for real or imaginary parts for products of octonionic polynomials

In algebraic geometry zeros for the products of polynomials give rise to disjoint varieties, which are disjoint unions of surfaces assignable to the individual surfaces and possibly having lower-dimensional intersections. For instance, for complex curves these intersections consist of points. For complex surfaces they are complex curves.

In the case of octonionic polynomial $P = RE(P) + IM(P)I_4$ (Re and Im are defined in quaternionic sense) one considers zeros of quaternionic polynomial $RE(P)$ or $IM(P)$.

1. Product polynomial $P = P_1P_2$ decomposes to

$$P = RE(P_1)RE(P_2) - IM(P_1)IM(P_2) + (RE(P_1)IM(P_1) + IM(P_1)RE(P_2)I_4 \ .$$

   One can require vanishing of $RE(P)$ or $IM(P)$.

   (a) $IM(P)$ vanishes for

$$(RE(P_1) = 0, RE(P_2) = 0)$$

   or

$$I(m(P_1) = 0, IM(P_2) = 0) \ .$$

   (b) $RE(P)$ vanishes for

$$(RE(P_1) = 0, IM(P_2) = 0)$$

   or

$$IM(P_1) = 0, RE(P_2) = 0) \ .$$

   One could reduce the condition $RE(P) = 0$ to $IM(P) = 0$ by replacing $P = P_1 + P_2I_4$ with $P_2 - P_1I_4$. If this condition is satisfied for the factors, it is satisfied also for the product. The set of surfaces is a commutative and associative algebra for the condition $IM(P) = 0$. Note that the quaternionic moduli must be same for the members of product. If one has quantum superposition of quaternionic moduli, the many-particle state involves a superposition of products with same moduli.

   As found, the condition $IM(P) = 0$ can transform to $RE(P) = 0$ at singularities having $RE(P) = 0, IM(P) = 0$.

2. The commutativity of the product means that the products are analogous to many-boson states. P^n would define an algebraic analog of Bose-Einstein condensate. Does this surface correspond to a state consisting of n identical particles or is this artefact of representation? As a limiting case of product of different polynomials it might have interpretation as genuine n-boson states.

3. The product of two polynomials defines a union of disjoint surfaces having discrete intersection in Euclidian signature. In Minkowskian signature the vanishing of $q\overline{q}$ (conjugation does not affect the sign of i and changes only the sign of I_k!) can give rise to 3-D light-cone. The non-commutativity of quaternions indeed can give rise to combinations of type $q\overline{q}$ in $RE(P)$ and $IM(P)$.

What about interactions?

1. Could one introduce interaction by simply adding a polynomial P_{int} to the product? This polynomial should be small outside interaction region. CD would would define naturally interaction regions and the interaction terms should vanish at the boundaries of CD. This might be possible in Minkowskian signature, where $f(q^2)$ multiplying the interaction term might vanish at the boundary of CD: in Euclidian sector $q\overline{q} = 0$ would imply $q = 0$ but in Minkowskian sector it would give light-cone as solution. One should arrange $IM(P_{int})$ to be proportional to $q\overline{q}$ vanishing at the boundary of CD. Minkowskian signature could be crucial for the possibility to "turning interactions on".

2. If the imaginary part of the interaction term is proportional $f_1(q^2)f_2((q-T)^2)$ (T is real and corresponds to the temporal distance between the tips of CD) with $f_i(0) = 0$, one could obtain asymptotic states reducing to disjoint unions of zero loci of P^i at the boundaries of CD. If the order of of the perturbation terms is higher than the total order of polynomials P^i, one would obtain new roots and particle emission. Non-perturbative situation would correspond to a dramatic modification of the space-time surface as a zero locus of $IM(P)$. This picture would be M^8 counterpart for the reduction of preferred extremals to minimal surfaces analogous to geodesic lines near the boundaries of CD: preferred extremals reduce to extremals of both Kähler action and volume term in these regions [L1].

The singularities of scattering amplitudes at algebraic varieties of Grassmann manifolds are central in the twistor Grassmann program [B1, B3, B2]. Since twistor lift of TGD seems to be the correct manner to formulate classical TGD in H, one can wonder about the connection between space-time surfaces in M^8_c and scattering amplitudes. Witten's formulation of twistor amplitudes in terms of algebraic curves in CP_3 suggests a formulation of scattering amplitudes in terms of the 4-D algebraic varieties in M^8_c as of course, also TGD itself [K19, K20]! Could the huge multi-local Yangian symmetries of twistor Grassmann amplitudes reduce to octonion analyticity.

4.2 Questions related to ZEO and CDs

Octonionic polynomials provide a promising approach to the understanding of ZEO and CDs. Light-like boundary of CD as also light-cone emerge naturally as zeros of octonionic polynomials. This does not yet give CDs and ZEO: one should have intersection of future and past directed light-cones. The intuitive picture is that one has a hierarchy of CDs and that also the space-time surfaces inside different CDs an interact.

4.2.1 Some general observations about CDs

It is good to list some basic features of CDS, which appear as both 4-D and 8-D variants.

1. There are both 4-D and 8-D CDs defined as intersections of future and past directed light-cones with tips at say origin 0 at real point T at quaternionic or octonionic time axis. CDs can be contained inside each other. CDs form a fractal hierarchy with CDs within CDs: one can add smaller CDs with given CD in all possible manners and repeat the process for the sub-CDs. One can also allow overlapping CDs and one can ask whether CDs define the analog of covering of O so that one would have something analogous to a manifold.

2. The boundaries of two CDs (both 4-D and 8-D) can intersect along light-like ray. For 4-D CD the image of this ray in H is light-like ray in M^4 at boundary of CD. For 8-D CD the image is in general curved line and the question is whether the light-like curves representing fermion orbits at the orbits of partonic 2-surfaces could be images of these lines.

3. The 3-surfaces at the boundaries of the two 4-D CDs are expected to have a discrete intersection since 4 + 4 conditions must be satisfied (say $RE(P_i^k)) = 0$ for $i = 1, 2$, $k = 1, 4$. Along line octonionic coordinate reduces effectively to real coordinate since one has $E^2 = E$ for $E = (1 + in)/2$, n octonionic unit. The origins of CDs are shifted by a light-like vector kE so that the light-like coordinates differ by a shift: $t_2 = t_1 - k$. Therefore one has common zero for real polynomials $RE(P_1^k(t))$ and $RE(P_2^k(t-k))$.

   Are these intersection points somehow special physically? Could they correspond to the ends of fermionic lines? Could it happen that the intersection is 1-D in some special cases? The example of o^2 suggest that this might be the case. Does 1-D intersection of 3-surfaces at boundaries of 8-D CDs make possible interaction between space-time surfaces assignable to separate CDs as suggested by the proposed TGD based twistorial construction of scattering amplitudes?

4. Both tips of CD define naturally an origin of quaternionic coordinates for $D = 4$ and the origin of octonionic coordinates for $D = 8$. Real analyticity requires that the octonionic polynomials have real coefficients. This forces the origin of octonionic coordinates to be along the real line (time axis) connecting the tips of CD. Only the translations in this specified direction are symmetries preserving the commutativity and associativity of the polynomial algebra.

5. One expects that also Lorentz boosts of 4-D CDs are relevant. Lorentz boosts leave second boundary of CD invariant and Lorentz transforms the other one. Same applies to 8-D CDs. Lorentz boosts define non-equivalent octonionic and quaternionic structures and it seems that one assume moduli spaces of them.

One can of course ask whether the still somewhat ad hoc notion of CD general enough. Should one generalize it to the analog of the polygonal diagram with light-like geodesic lines as its edges appearing in the twistor Grassmannian approach to scattering diagrams? Octonionic approach gives naturally the light-like boundaries assignable to CDs but leaves open the question whether more complex structures with light-like boundaries are possible. How do the space-time surfaces associated with different quaternionic structures of M^8 and with different positions of tips of CD interact?

4.2.2 The emergence of causal diamonds (CDs)

CDs are a key notion of zero energy ontology (ZEO). Could the emergence of CDs be understood in terms of singularities of octonion polynomials located at the light-like boundaries of CDs? In Minkowskian case the complex norm $q\overline{q}_i$ is present in P. Could this allow to blow up the singular point to a 3-D boundary of light-cone and allow to understand the emergence of causal diamonds (CDs) crucial in ZEO.

The study of the special properties for zero loci of general polynomial $P(o)$ at light-rays of O indeed demonstrated that both 8-D land 4-D light-cones and their complements emerge naturally, and that the M^4 projections of these light-cones and even of their boundaries are 4-D future - or past directed light-cones. What one should understand is how CDs as their intersections, and therefore ZEO, emerge.

1. One manner to obtain CDs naturally is that the polynomials are sums $P(t) = \sum_k P_k(o)$ of products of form $P_k(o) = P_{1,k}(o)P_{2,k}(o - T)$, where T is real octonion defining the time coordinate. Single product of this kind gives two disjoint 4-varieties inside future and past directed light-cones $M^4_+(0)$ and $M^4_-(T)$ for either $RE(P) = 0$ (or $IM(P) = 0$) condition. The complements of these cones correspond to $IM(P) = 0$ (or $RE(P) = 0$) condition.

2. If one has nontrivial sum over the products, one obtains a connected 4-variety due the interaction terms. One has also as special solutions $M^4_\pm$ and the 6-spheres associated with the zeros $P(t)$ or equivalently $P_1(t_1) \equiv P(t)$, $t_1 = T - t$ vanishing at the upper tip of CD. The causal diamond $M^4_+(0) \cap M^4_-(T)$ belongs to the intersection.

   Remark: Also the union $M^4_-(0) \cup M^4_+(T)$ past and future directed light-cones belongs to the intersection but the latter is not considered in the proposed physical interpretation.

3. The time values defined by the roots t_n of $P(t)$ define a sequence of 6-spheres intersecting 4-D CD along 3-balls at times t_n. These time slices of CD must be physically somehow special. Space-time variety intersects 6-spheres along 2-varieties X^2_n at times t_n. The varieties X^2_n are perhaps identifiable as 2-D interaction vertices, pre-images of corresponding vertices in H at which the light-like orbits of partonic 2-surfaces arriving from the opposite boundaries of CD meet.

   The expectation is that in H one as generalized Feynman diagram with interaction vertices at times t_n. The higher the evolutionary level in algebraic sense is, the higher the degree of the polynomial $P(t)$, the number of t_n, and more complex the algebraic numbers t_n. $P(t)$ would be coded by the values of interaction times t_n. If their number is measurable, it would provide important information about the extension of rationals defining the evolutionary level. One can also hope of measuring t_n with some accuracy! Octonionic dynamics would solve the roots of a polynomial! This would give a direct connection with adelic physics [L7] [L8].

Remark: Could corresponding construction for higher algebras obtained by Cayley-Dickson construction solve the "roots" of polynomials with larger number of variables? Or could Cartesian product of octonionic spaces perhaps needed to describe interactions of CDs with arbitrary positions of tips lead to this?

4. Above I have considered only the interiors of light-cones. Also their complements are possible. The natural possibility is that varieties with $RE(P) = 0$ and $IM(P) = 0$ are glued at the boundary of CD, where $RE(P) = IM(P) = 0$ is satisfied. The complement should contain the external (free) particles, and the natural expectation is that in this region the associativity/co-associativity conditions can be satisfied.

5. The 4-varieties representing external particles would be glued at boundaries of CD to the interacting non-associative solution in the complement of CD. The interaction terms should be non-vanishing only inside CD so that in the exterior one would have just product $P(o) = P_{1,k_0}(o)P_{2,k_0}(o-T)$ giving rise to a disjoint union of associative varieties representing external particles. In the interior one could have interaction terms proportional to say $t^2(T-t)^2$ vanishing at the boundaries of CD in accordance with the idea that the interactions are switched one slowly. These terms would spoil the associativity.

Remark: One can also consider sums of the products $\prod_k P_k(o-T_k)$ of n polynomials and this gives a sequence CDs intersecting at their tips. It seems that something else is required to make the picture physical.

4.2.3 How could the space-time varieties associated with different CDs interact?

The interaction of space-time surfaces inside given CD is well-defined. Sitation is not so clear for different CDs for which the choice of the origin of octonionic coordinates is in general different and polynomial bases for different CDs do not commute nor associate.

The intuitive expectation is that 4-D/8-D CDs can be located everywhere in M^4/M^8. The polynomials with different origins neither commute nor are associative. Their sum is a polynomial whose coefficients are not real. How could one avoid losing the extremely beautiful associative and commutative algebra of polynomials?

1. Should one assume that the physics observable by single conscious observer corresponds to single CD defining the perceptive field of this observer [L9].

2. Or should one give up associativity and allow products (but not sums since one should give up the assumption that the coefficients of polynomials are real) of polynomials associated with different CDs as an analog for the formation of free many-particle states.

Consider first what happens for the single particle solutions defines as solutions of either $RE(P_i) = 0$ or $IM(P_i) = 0$.

1. The polynomials associated with different 8-D CDs do not commute nor associate. Should one allow their products so that one would still *effectively* have a Cartesian product of commutative and associative algebras? This would realize non-commutative and non-associative physics emerging in conformal field theories also at the level of space-time geometry.

2. If the CDs differ by a *real* (time) translation $o_2 = o_1 + T$ one still obtains $IM(P_1) = 0$ and $IM(P_2) = 0$ as solutions to $IM(P_1P_2) = 0$. This applies also to states with more particles. The identification would be in terms of external particles. For $RE(P_1P_2) = 0$ this is not the case. If the interior of CD corresponds to $RE(P_1P_2) = 0$, the dynamics in the interior is not only non-trivial but also non-commutative and non-associative. Non-trivial interaction would

be obtained even without interaction terms in the polynomial vanishing at the boundaries of CD!

Could one consider allowing only CDs with tips at the same real axis but having all sizes scales? This hierarchy of CD would characterize a particular hierarchy of conscious observers - selves having sub-selves (sub-CDs) [L9]. The allowance of only these CD would be analogous to a fixing of quantization axes.

3. What happens if one allows CDs differing by arbitrary octonion translation? Consider external particles. For P_1 and P_2 RE and IM are defined for different decompositions $o_i = RE(o_i) + n_i Im(o_i)$, where n_i, $i = 1,2$ is a unit octonion.

   What decomposition should one use for P_1P_2? The decomposition for P_1 or P_2 or some other decomposition? One can express $P_2(o_2)$ using o_1 as coordinate but the coefficients multiplying powers of o_1 from *right* would not be real numbers anymore implying $IM(P_2)_1 \neq IM(P_2)_2$. $IM(P_2)_1 = 0$ makes sense but the presence of particle 1 would have affected particle 2 or vice versa.

   Could one argue that the coordinate systems satisfying the condition that some external particles described by P_i have real coefficients and perhaps serving in the role of observers are preferred? Or could one imagine that o_{12} is a kind of center of mass coordinate? In this case the 4-varieties associated with both particles would be affected. What is clear that the choice of the octonionic coordinate origin would affect the space-time varieties of external particles even if they could remain associative/critical. Are there preferred coordinates in which criticality is preserved? For instance, can one achiever criticality for P_2 on coordinates of o_1 if P_1 is critical. Could one see this as a kind of number theoretic observer effect at the level of space-time geometry?

   Remark: $P_i(o)$ would reduce to a real polynomial at light-like rays with origin for o_i irrespective of the octonionic coordinate used so that the spheres S^6_i with origin at the origin of o_i as solutions of $P_i(o) = 0$ would not be lost.

If one does not give up associativity and commutativity, how can one describe the interactions between space-time surfaces inside different CDs at the level of M^8?

1. The most straightforward manner would be to introduce Cartesian powers of O and CD:s inside these powers to describe the interaction between CDs with different origin. This would be analogous to what one does in condensed matter physics. What seems clear is that $M^8 - H$ correspondence should map all the factors of $(M^8)^n$ to the same $M^4 \times CP_2$ by a kind of diagonal projection.

2. The simplest interaction would be associated with the common stable intersection points of the space-time surfaces associated with different CDs. In the generic case the intersection would consist of discrete points. A stronger condition would be that these points belong to the extension of rationals defining adeles. The interaction points could be ramified points at which the the action of a subgroup Galois group would leave the sheets of the Galois covering invariant so that some number of them would touch other. I have discussed this proposal in [L5]. These points could be seen as analogs of interaction points in QFT description in terms of n-point functions and the sum over polynomials would give rise to the analog over integral over different n-point configurations.

3. Could the intersection of space-time surfaces with zero loci for $RE(P_i)$ and $IM(P_i)$ define the loci of interaction? As already found, the 6-D spheres S^6 with radii t_n given by the zeros of $P(t)$ are universal and have interpretation as $t = t_n$ snapshots of 7-D spherical light front.

   The 2-D intersections X^2 of 4-D space-time variety X^4 with S^6 would define natural candidates for the intersections and might allow interpretation as pre-images of partonic 2-surfaces. X^2 would be the contact of X^4 with S^6 associated with second 8-D CD. Together with SH this gives hopes about an elegant description of interactions in terms of connected space-time varieties.

4. For instance, the following picture can be considered. Consider two space-time varieties X^4_i, $i=1,2$ associated with CDs with different origins and connected by a connected sum contact, which at the level of H corresponds to a wormhole contact connecting space-time sheets with different octonionic coordinates. The partonic 2-varieties $X^2_i = X^4_i \cap S^6_i$ are labelled by time values $t = t_{i,n_i}$.

   Assume that there is tube-like 3-surface $X^3_{1,2}$ connecting X^2_1 and X^2_2. The union $X^2_1 \cap X^2_2$ of partonic 2-surfaces must be homologically trivial in order to define a boundary of 3-surface $X^3_{1,2}$. The surfaces X^2_i must therefore have opposite homology charges. $X^3_{1,2}$ would be pre-image of a wormhole contact connecting different space-time sheets to which the CDs are assigned.

   The 6-spheres S^6_i intersect along 4-D surface $X^4_{1,2} = S^6_1 \cap S^6_2$ in M^8. One should have $X^3_{1,2} \subset X^4_{1,2}$ and $X^3_{1,2}$ should be non-critical but associative and therefore 3-D. This surface should allow a realization as a zero locus of $RE(P_{1,2}(u))$ or $IM(P_{1,2}(u))$ and belong to $X^4_{1,2}$. One would not have manifold-topology. Rather, one could speak of two 4-D branes X^4_i (3-branes) connected by a 3-D brane $X^3_{1,2}$ (2-brane). Two 2 parallel 4-planes joint by a 1-D curve is the lower-dimensional analogy. The interaction would be instantaneous inside X^4_i. I must admit that this looks to me somewhat troublesome.

5. Partonic 2-surfaces define wormhole throats and appear in pairs if they carry monopole charges. Could one think that the above mentioned 2-surfaces are intersections of X^1_i with S^k_{i+1} for the pair of space-time sheets assignable to different CDs? Could the image in H of the structure formed by $\{X^2_1, X^2_2, S^6_1, S^6_2\}$ under $M^8 - H$ correspondences be wormhole contact.

4.3 About singularities of octonionic algebraic varieties

In Minkowskian signature the notion of singularity for octonionic polynomials involves new aspects as the study of o^2 singular at origin shows (see Appendix). The region in which $RE(o^2) = 0, IM(o^2) = 0$ holds true is 4-D rather than a discrete set of points as one would naively expect.

1. At singularity the local dimension of the algebraic variety is reduced. For instance, double cone of 3-space has origin as singular point where it becomes 0-dimensional. A more general example is local pinch in which cylinder becomes infinitely thin at some point. This kind of pinching could occur for fibrations as the fiber contracts to a lower-dimensional space along a sub-variety of the base space.

   A very simple analogy for this kind of singularity is the singularity of $P(x,y) = y^2 - x = 0$ at origin: now the sheets $y = \pm\sqrt{x}$ co-incide at origin. The algebraic functions $y \mp \sqrt{x}$ defining the factorization of $P(x,y)$ co-incide at origin. Quite generally, two or more factors in the factorization of polynomial using algebraic functions co-incide at the singularity. This is completely analogous to the degeneracy or roots of polynomials of single variable.

   The signature of the singularity of algebraic variety determined by the conditions $P^i(z^j) = 0$ is the reduction of the maximal rank r for the matrix formed by the partial derivatives $P^i_j \equiv \partial IM(P)^i/\partial z^j$ ("RE" could replace "IM"). Rank corresponds to the largest dimension of the minor of P^i_j with non-vanishing determinant. Determinant vanishes when two rows of the minor are proportional to each other meaning that two tangent vectors become linearly dependent. When the rank is reduced by Δr, one has $r = r_{max} - \Delta r$ and the local dimension is locally reduced by Δr. One has hierarchy of singularities within singularities.

   The conditions that all independent minors of the P^i_j have reduced rank gives additional constraints and define a sub-variety of the algebraic variety. Note that the dimension of the singularity corresponds to $d_s = \Delta r$ in the sense that the dimension of tangent space at singularity is effectively d_s.

2. In the recent case there are 4 polynomials and 4 complex variables so that $IM(P)^i_j$ is 4×4-matrix. Its rank r can have values in $r = 1, 2, 3, 2, 4$. One can use Thom's catastrophe theory as a guideline. Catastrophe decomposes to pieces of various dimensions characterized by the reduction of the rank of the matrix defined by the second derivatives $V_{ij} = \partial_i\partial_j V$

of the potential function defining the catastrophe. For instance, for cusp catastrophe with $V(x,a,b) = x^4 + ax^2 + bx$ one has V-shaped region in (a,b) plane with maximal reduction of rank to $r = 0$ ($\partial_x^2 V = 0$) at the tip $(a,b) = 0$ at reduction to $r = 1$ at the sides of V, where two roots of $\partial_x V = 4x^3 + 2ax + b = 0$ co-incide requiring that the discriminant of this equation vanishes.

3. In the recent case $IM(P)$ takes the role of complex quaternion valued potential function and the 4 coordinates $z_1^{k)}$ that of behavior variable x for cusp and $z_2^{k)}$ that of control parameters (a,b). The reduction of the rank of $n \times n$ matrix by Δr means that there are r linearly independent rows in the matrix. These give Δr additional conditions besides $IM(P) = 0$ so that the sub-variety along which the singularity takes places as dimension r. One can say that the r-dimensional tangent spaces integrate to the singular variety of dimension r.

   The analogy with branes would be realized as a hierarchical structure of singularities of the spacetime surfaces. This hierarchy of singularities would realize space-time correlates for quantum criticality, which is basic principle of quantum TGD. For instance, the reduction by 3-units would correspond to strings - say at the ends of CD and along the partonic orbits (fermion lines), and maximal reduction might correspond to discrete points - say the ends of fermion lines at partonic 2-surfaces. Also isolated intersection points can be regarded as singularities and are stably present but it does not make sense to add fermions to these points so that cognitive representations are not possible.

4. Note that also the associativity - and commutativity conditions already discuss involved the gradients of $IM(P)^i$ and $RE(P)^i$, which would suggests that these regions can be interpreted as singularities for which the dimension is not lowered by on unit since the vanishing conditions hold true identically by criticality.

There are two cases to be considered. The usual Euclidian case in which pinch reducing the dimension and the Minkowskian case in which metric dimension is reduced locally.

Consider first the Euclidian case.

1. In Euclidian case it is difficult to tell whether all values of Δr are possible since octonion analyticity poses strong conditions on the singularities. The pinch could correspond to the singularity of the covering associated with the space-time surface defined by Galois group for the covering associated with $h_{eff}/h = n$ identifiable as the order of Galois group [L5]. Therefore there would be very close connection between the extensions of rationals defining the Galois group and the extension of polynomial ring of 8 complex variables z_i^k, $i = 1,2$, $k = 1,..,4$ by algebraic functions. At the pinch, which would be algebraic point, the Galois group would have subgroup leaving the coordinates of the point invariant and some sheets of the covering defining roots would co-incide.

2. A very simple analogy for this kind of singularity is the singularity of $P(x,y) = y^2 - x = 0$ at origin: now the sheets $y = \pm\sqrt{x}$ co-incide at origin. The algebraic functions $y \mp \sqrt{x}$ defining the factorization of $P(x,y)$ co-incide at origin. Quite generally, two or more factors in the factorization of polynomial using algebraic functions co-incide at the singularity. This is completely analogous to the degeneracy or roots of polynomials of single variable.

3. Quaternion structure predicts the slicing of M^4 by string world sheets inducing that of space-time surfaces. One must ask whether singular space-time sheets emerge already for the slicing of M^4 by string world sheets. String world sheets could be considered as candidates for $\Delta r = 2$ singularities of this kind. The physical intuition strongly suggests that there indeed physically preferred string world sheets and identification as $\Delta r = 2$ singularities of Euclidian type is attractive. Partonic 2-surfaces are also candidates in this respect. Could some sheets of the $h_{eff}/h = n$ covering co-incide at string world sheets?

Consider next the Minkowskian case. At the level of H the rank of the induced metric is reduced. This reduction need not be same as that for the matrix P_j^i and it is of course not obvious that the partonic orbit allows description as a singularity of algebraic variety.

1. Could the matrix P^i_j take a role analogous to the dual of induced metric and one might hope that the change of the sign for P^i_j for a fixed polynomial at singular surface could be analogous to the change of the sign of $\sqrt{g_4}$ so that the idea about algebraization of this singularity at level of M^8 might make sense. The information about metric could come from the fact that $IM(P)$ depends on complex valued quaternion norm reducing to Minkowskian metric in Minkowskian sub-space.

2. The condition for the reduction of rank from its maximal value of $r = 4$ to $r = 3$ occurs if one has $det(P) = 0$, which defines co-dimension 1 surface as a sub-variety of space-time surface. The interpretation as co-incidence of two roots should make sense if $IM(P) = 0$. Root pairs would now correspond now to the points at different sides of the singular 3-surface.

   Minkowskian singularity cannot be identified as the 3-D space-like boundary of many-sheeted space-time surface located at the boundary of CD (induced metric is space-like).

   Could this sub-variety be identified as partonic orbit, the common boundary of the Euclidian and Minkowskian regions? This would require that associative region transforms to co-associative one here. $IM(P) = 0$ condition can transform to $RE(P) = 0$ condition if one has $P = 0$ at this surface. Minkowskian variant of point singularity (P^i_j vanishes) would explode it to a light-like partonic orbit.

   What does this imply about the rank of singularity? The condition $IM(P) = RE(P) = 0$ does not reduce the rank if P is linear polynomial and one could consider a hierarchy of reductions of rank. Since $q\overline{q}$ vanishes in Minkowskian sub-space at light-cone boundary rather than at point $q = 0$ only, there are reasons to expect that it appears in P and reduces the rank by $\Delta r = 4$ (see Appendix for the discussion of o^2 case). The rank of the induced 4-metric is however reduced only by $\Delta r = 1$ at partonic orbit. If the complexified complex norm $z\overline{z}$, $z = z_1 + z_2 I_2$ can take the role of $q\overline{q}$, one has $\Delta r = 2$.

3. The reduction of rank to $r = 2$ would give rise to 2-surfaces, which are at the boundaries of 3-D singularities. If partonic orbits correspond to $\Delta r = 1$ singularities one could identify them as partonic 2-surfaces at the ends partonic orbits.

   Could the singularity at partonic 2-surface correspond to the reduction of the rank of the induced metric by 2 units? This is impossible in strict sense since there is only one light-like direction in signature $(1, -1, -1, -1)$. Partonic 2-surface singularity would however correspond to a corner for both Euclidian and Minkowskian regions at which the metrically 2-D but topologically 3-D partonic orbit meets the the space-like 3-surface along the light-like boundary of CD. Also the radial direction for space-like 3-surface could become light-like at partonic 2-surface if the CP_2 coordinates have vanishing gradient with respect to the light-like radial coordinate r_M at the partonic 2-surface. In this sense the rank could be reduced by 2 units. The situation is analogous to that for fold singularity $y^2 - x = 0$.

   String world sheets cannot be subsets of $r = 3$ singularities, which suggests different interpretation for partonic 2-surfaces and string world sheets.

What could this different interpretation be?

1. Perhaps the most convincing interpretation of string world sheets/partonic 2-surfaces has been already discussed (this interpretation would generalize to associative space-time surfaces). They could be commutative/co-commutative (here permutation might be allowed!) sub-manifolds of associative regions of the space-time surface allowing quaternionic tangent spaces so that the notions of commutative and co-commutative make sense. The criticality conditions are satisfied without the reduction of dimension from $d = 2$ to $d = 1$. In non-associative regions string world sheets would reduce to 1-D curves. This would happen at the boundaries of partonic orbits and 3-surfaces at the ends of space-time surface and only the ends of strings at partonic orbits carrying fermion number would be needed to determine twistorial scattering amplitudes [K19, K20].

2. I have also considered an interpretation in terms of singularities of space-time surfaces represented as a sections of their own twistor bundle. Self-intersections of the space-time surface would correspond to 2-D surfaces in this case [L5] and perhaps identifiable as string world

sheets. The interpretation mentioned above would be in terms of Euclidian singularities. If this is true, the question is only about whether these two interpretations are consistent with each other.

If I were forced to draw conclusion on basis of these notices, it would be that only $r = 4$ Minkowskian singularities could be interesting and at them $RE(P) = 0$ regions could be transformed to $IM(P) = 0$ regions. Furthermore, the reduction of rank for the induced metric cannot be equal to the reduction of the rank for P^i_j.

4.4 The decomposition of space-time surface to Euclidian and Minkowskian regions in octonionic description

The unavoidable outcome of H picture is the decomposition of space-time surface to regions with Minkowskian or Euclidian signature of the induced metric. These regions are bounded by 3-D regions at which the signature of the induced metric is $(0, -1, -1, -1)$ due to the vanishing of the determinant of the induced metric. The boundary is naturally the light-like orbit of partonic 2-surface although one can consider also the possibility that these regions have boundaries intersecting along light-like curves defining boundaries of string world sheets. A more detailed view inspired by the study of extremals is following.

1. Let us assume that the above picture about decomposition of space-time surfaces in H to two kinds regions takes place. The regions where the dynamicis universal minimal surface dynamics have associative pre-image in M^8. The regions where Kähler action and volume term couple the associative pre-image in M^8 exists only at the 3-D boundary regions and M^8 dynamics determines the boundary conditions for H dynamics, which by holography is enough.

2. In the space-time regions having associative pre-image in M^8 one has a fibration of X^4 with with partonic surface as a local base and string world sheet as local fiber. In the interior of space-time region there are no singularities but at the boundary 2-D string world sheets becomes metrically 1-D as 1-D string boundary reduces metrically to 0-D structure analogous to a point. This reduction of dimension would be metric, but not topological.

   The singularity for plane curve $P(x, y) = y^2 - x^3 = 0$ at origin illustrates the difference between Minkowskian and Euclidian singularity. One has $(\partial_x P, \partial_y P) = (-3x^2, 2y)$ vanishing at origin so that $\Delta r = 1$ singularity is in question and the dimension of singular manifold is indeed $r = 0$. From $y = \pm x^{3/2}$, $x \geq 0$. The induced metric $g_{xx} = 1 + (dy/dx)^2 = 1 + (9/4)x$, $x \geq 0$ is however non-singular at origin.

3. If the Euclidian region with pre-image corresponds to a deformation of wormhole contact, the identification as image of a co-associative space-time region in M^8 is natural so that normal space is associative and contains also the preferred $M^2(x)$. In Minkowskian regions the identification as image of associative space-time region in M^8 is natural.

What can one say about the relationship of the M^8 counterparts of neighboring Minkowskian and Euclidian regions?

1. Do these regions intersect along light-like 3-surfaces, 1-D light-like curve (orbit of fermion) or is the intersection disrete set of points possibly assignable to the partonic 2-surface at the boundaries of CD? The M^4 projections of the inverse image of the light-like partonic orbit should co-incide but E^4 projections need not do so. They could be however mappable to the same partonic two surface in $M^8 - H$ correspondence or the images could have at least have light-like curve as common.

2. Is seems impossible for the space-time surfaces determined as zeros of octonionic polynomials to have boundaries. Rather, it seems that the boundary must be between Minkowskian and Euclidian regions of the space-time surface determined by the same octonionic polynomial. At the boundary also associate region would transform to co-associative region suggesting that $IM(P) = RE(P) = 0$ holds allowing to change the condition from $IM(P) = 0$ to $RE(P) = 0$.

Consider now in more detail whether this view can be realized.

1. In $H = M^4 \times CP_2$ the boundary between the Minkowskian and Euclidian space-time regions - light-like partonic 3-surface - is a singularity possible only in Minkowskian signature. Space-time surface X^4 at the boundary is effectively 3-D since one has $\sqrt{g_4} = 0$ meaning that tangent space is effectively 3-D. The 3-D boundary itself is metrically 2-D and this gives rise to the extended conformal invariance defining crucial distinction between TGD and super string models.

2. The singularities of $P(o)$ for o identified as linear coordinate of M^8_c were already considered. The singularities correspond to the boundaries of light-cone and the emergence of CDs can be understood. Could also the light-like orbits of partonic 2-surfaces be understood in the same manner? Does the pre-image of this singularity in M^8 emerge as a singularity of an algebraic variety determined by the vanishing of $IM(P)$ for the octonionic polynomial?

   What is common is that the rank of the induced metric by one unit also now. Now one has however also $det(g_4) = 0$. The singularities correspond to curved light-like 3-surfaces inside space-time surfaces rather than light-like surfaces in M^8: induced metric matters rather than M^4 metric.

3. Could also these regions correspond to singularities of octonionic polynomials at which $P(o) = 0$ is satisfied and associative region transforms to a co-associative region? For $M^2(x) = M^2_0$ this is impossible. Partonic 2-surfaces are planes E^2 now. One should have closed partonic 2-surfaces.

   Could the allowance of quaternionic structures with slicing of X^4 by string world sheets and partonic 2-surfaces help? If one has slicing of string world sheets by dual light-like curves corresponding to light-like coordinates u and v, this slicing gives also rise to a slicing of light-like 3-surfaces and dual light-like coordinate. The pair (u, v) in fact defines the analog of z and $\overline{z}$ in hypercomplex case. Could the singularity of $P(o)$ using the quaternionic coordinates defined by (u, v) and coordinates of partonic 2-surface allow to identify light-like partonic orbits with $det(g_4) = 0$ as a generalization of light-cone boundaries in M^4?

   The decomposition $M^4_0 = M^2_x \times E^2(x)$ associated with quaternionic structure is independent of E^4. In the other hand, tangent space of space-time surface at point decomposes $M^2(x) \times E^2_T(x)$, where $E^2_T(x)$ is in general different from $E^2(x)$. Is this enough to obtain partonic 2-surfaces as singularities with $RE(P) = IM(P) = 0$?

The question whether the boundaries between Minkowskian and Euclidian can correspond to singular regions at which $P(o)$ vanishes and the surface $RE(P) = 0$ transforms to $IM(P) = 0$ surface remains open. What remains poorly understood is the role of the induced metric. My hope is that with a further work the picture could be made more detailed.

4.5 About rational points of space-time surface

What one can say about rational points of space-time surface?

1. An important special case corresponds to a generalization of so called rational surfaces for which a parametric representation in terms of 4 complex coordinates t^k exists such that o^k_1 are *rational* functions of t^k. The singularities for 2-complex dimensional surfaces in C^3 or equivalently CP_3 are classified by Du Val [A3, A5] (see `http://tinyurl.com/ydz93hle`).

2. In [L5] [L3] I considered possible singularities of the twistor bundle. These would correspond typically 2-D self-intersections of the imbedding of space-time surfaces as 4-D base space of 6-D twistor bundle with sphere as a fiber. They could relate to string world sheets and partonic 2-surfaces and - as already found - are different from singularities at the level of M^8_c. The singularities of string world sheets and partonic 2-surfaces as hyper-complex and co-complex surfaces consist of points and could relate to the singularities at octonionic level.

As already mentioned, Bombieri-Lang conjecture (see `http://tinyurl.com/y887yn5b`) states that, for any variety X of general type over a number field k, the set of k-rational points of X is not Zariski dense (see `http://tinyurl.com/jm9fh74`) in X. Even more, the k-rational points are contained in a *finite* union of lower-dimensional sub-varieties of X.

This conjecture is highly interesting from TGD point of view if one believes in M^8-H duality. Space-time surfaces $X^4 \subset M^8_c$ can be seen as $M^8 = M^4 \times E^4$ projections of zero loci for real or imaginary parts of octonionic polynomials in o. In complex sense they reduce to $M^4 \times E^4$ projections of algebraic co-dimension 4 surfaces in C^8. If Bombieri-Lang conjectures makes sense in this context, it would state that for a space-time surface $X^4 \subset M^8$ of general type the rational points are contained in a *finite* union of lower-dimensional sub-varieties. Also the conjecture of Vojta (see `http://tinyurl.com/y9sttuu4`) stating that varieties of general type cannot be potentially dense is known to be true for curves and support this general vision.

Could the finite union of sub-varieties correspond to string world sheets, partonic 2-surfaces, and their light-like orbits define singularities? But why just singular sub-varieties would be cognitively simple and have small Kodaira dimension d_K allowing large number of rational points? In the case of partonic orbits one might understand this as a reduction of metric dimension. The orbit is effectively 2-dimensional partonic surface metrically and for the genera $g = 0, 1$ rational points are dense. For string world sheets with handle number smaller than 2 the situation is same.

The proposed realizations of associativity and commutativity provide additional support for this picture. Criticality guaranteeing associativity/commutativity would select preferred space-time surfaces as also string world sheets and partonic 2-surfaces.

Concluding, the general wisdom of algebraic geometry conforms with SH and with the vision about the localization of cognitive representations at 2-surfaces. There are of many possible options for detailed interpretation and certainly the above sketch cannot be correct at the level of details.

4.6 Connection with infinite primes

The idea about space-time surfaces as zero loci of polynomials emerged for the first time as I tried to understand the physical interpretation of infinite primes [K8], which were motivated by TGD inspired theory of consciousness. Infinite primes form an infinite hierarchy. At the lowest level the basic entity is the product $X = \prod_p p$ of all finite primes. The physical interpretation could be as an analog of fermionic sea with fermion states labelled by finite primes p.

1. The simplest infinite primes are of form $P = X \pm 1$ as is easy to see. One can construct more complex infinite primes as infinite integers of form $nX/r + mr$. Here r is square free integer, n is integer having no common factors with r, and m can have only factors possessed also by r.

   The interpretation is that r defines fermionic state obtained by kicking from Dirac sea the fermions labelled by the prime factors of r. The integers n and m define bosonic excitations in which k:th power of p corresponds to k bosons in state labelled by p. One can also construct more complex infinite primes as polynomials of X and having no rational factors. In fact, X becomes coordinate variable in the correspondence with polynomials.

2. This process can be repeated at the next level. Now one introduces product $Y = \prod_P P$ of all primes at the previous level and repeats the same construction. These infinite correspond to polynomials of Y with coefficients given by rational functions of X. Primality means irreducibility in the field of rational functions so that solving Y in terms of X would give algebraic function.

3. At the lowest level are ordinary primes. At the next level the infinite primes are indeed infinite in real sense but have p-adic norms equal to unity. They can be mapped to polynomials $P(x_1)$ with rational coefficients and the simplest polynomials are monomials with rational root. Higher polynomials are irreducible polynomials with algebraic roots. At the third level of hierarchy one has polynomials $P(x_2|x_1)$ of two variables. They are polynomials of x_1 with coefficients with are rational functions of x_1. This hierarchy can be continued.

   One can define also infinite integers as products of infinite primes at various levels of hierarchy and even infinite rationals.

4. This hierarchy can be interpreted in terms of a repeated quantization of an arithmetic super-symmetric quantum field theory with elementary particles labelled by primes at given level

of hierarchy. Physical picture suggests that the hierarchy of second quantizations is realized also in Nature and corresponds to the hierarchy of space-time sheets.

5. One could consider a mapping $P(x_n|x_{n-1}|..|x_1)$ by a diagonal projection $x_i = x$ to polynomials of single variable x. One could replace x with complexified octonic coordinate o_c. Could this correspondence give rise to octonionic polynomials and could the connection with second quantization give classical space-time correlates of real quantum states assignable to infinite primes and integers? Even quantum states defining counterparts of infinite rationals could be considered. One could require that the real norm of these infinite rationals equals to one. They would define infinite number of real units with arbitrarily complex number theoretical anatomy. The extension of real numbers by these units would mean huge extension of the notion of real number and one could say that each real point corresponds to platonic defined by these units closed under multiplication.

   In ZEO zero energy states formed by pairs of positive and negative energy could correspond to these states physically. The condition that the ratio is unit would have also a physical interpretation in terms of particle content.

6. As already noticed, the notions of analyticity, quaternionicity, and octonionicity could be seen as a manifestation of polynomials in algebras defined by adding repeatedly a new non-commuting imaginary unit to already existing algebra. The dimension of the algebra is doubled in each step so that dimension comes as a power of 2. The algebra of polynomials with real coefficients is commutative and associative. This encourages the crazy idea that the spaces are indeed realized and the generalization of $M^8 - H$ duality holds true at each level. At level k the counterpart for CP_2 (for $k = 3$) would be as moduli space for sub-spaces of dimension 2^{k-1} for which tangent space reduces to the algebra at level $k - 1$. For $k = 2$ CP_1 is the moduli space and could correspond to twistor sphere. Essentially Grassmannian $Gl(2^k, 2^{k-1})$ would be in question. This brings in mind twistor Grassmann approach involving hierarchy of Grassmannians too, which however allows all dimensions. What is interesting that the spinor bundle for space of even dimension d has fiber with dimension $2^{d/2}$.

   The number of arguments for the hierarchy of polynomials assignable to the hierarchy of infinite primes increases by one at each step. Hence these two hierarchies are different.

The vanishing of the octonionic polynomials indeed allow a decomposition to products of prime polynomials with roots which in general are algebraic numbers and an exciting possibility is that the prime polynomials have interpretation as counterparts of elementary particles in very general sense.

4.7 What about fermions and scattering amplitudes at the level of O?

Could the octonionic level provide an elegant description of fermions in terms of super variant of octonionic algebraic geometry? Could one even construct scattering amplitudes at the level of M^8 using the variant of the twistor approach discussed in [K19, K20]?

The idea about super-geometry is of course very different from the idea that fermionic statistics is realized in terms of the spinor structure of "world of classical worlds" (WCW) but M^8-H duality could however map these ideas and also number theoretic and geometric vision to each other. The angel of geometry and the devil of algebra could be dual to each other.

4.7.1 Octonionic superspace

Consider now what super version of the octonionic algebraic geometry might look like.

1. What makes octonions so nice is the octonionic triality. One has three 8-D representations: vector representation 8_V, spinor representation 8_s and its conjugate $\overline{8}_s$. The tensor products of two representations gives the third representation in the triplet. This is the completely unique feature of dimension 8 and makes octonionic physics so fascinating an option. The octonionic triality is central also in super-string models but in a different manner since of starts from 10-D situation and ends up with effectively 8-D situation for physical states.

2. One can define super octonion as $o_s = o + \theta_1 + \theta_2$. Here o is bosonic octonionic coordinate. $\theta_i = \theta_i^k E_k$, where E_k are octonionic units, is Grassmann valued octonion in 8_s satisfying the usual anti-commutations and θ_2 transforms as $\overline{8}_s$. (I have already earlier considered as natural candidates for spinors in octonionic M^8).

   The first interpretation is that θ_1 and θ_2 correspond to objects with opposite fermion numbers. If this is not the case, one could perhaps define the conjugate of super-coordinate as $\overline{o}_s = \overline{o} + \overline{\theta}_1 + \overline{\theta}_2$. This looks however ugly.

3. What could be the physical interpretation? One should obtain particles and antiparticles naturally as also separately conserved baryon and lepton numbers (I have also considered the identification of hadrons in terms of anyonic bound states of leptons with fractional charges).

   Quarks and leptons have different coupling to the induced Kähler form at the level of H. It seems impossible to understand this at the level of M^8, where the dynamics is purely algebraic and contains no gauge couplings.

   The difference between quarks and leptons is that they allow color partial waves with triality $t = \pm 1$ and triality $t = 0$. Color partial waves correspond to wave functions in the moduli space CP_2 for $M_0^4 \supset M_0^2$. Could the distinction between quarks and leptons emerge at the level of this moduli space rather than at the fundamental octonionic level? There would be no need for gauge couplings to distinguish between quarks and leptons at the level of M^8. All couplings would follow from the criticality conditions guaranteeing 4-D associativity for external particles (on mass shell states would be critical).

   If so, one would have only the super octonions $o_s = \theta_1 + \theta_2 = \overline{\theta}_1$ and θ_1 and $\theta_2 = \overline{\theta}_1$ would correspond to fermions and antifermions with no differentiation to quarks or leptons. Fermion number conservation would be coded by the Grassmann algebra.

   One can imagine also other options but they have their problems. Therefore this option will be considered in the sequel.

4.7.2 Super version of octonionic algebraic geometry

Instead of super-fields one would have a super variant of octonionic algebraic geometry.

1. Super polynomials make still sense and reduce to a sum of octonionic polynomials $P_{kl}\theta_1^k\theta_2^l$, where the integers k and l would be tentatively identified as fermion numbers. The coefficients in $P_{kl} = P_{kl,n}o^n$ would be given by $P_{kl,n} = P_{n+k+l}B(n+k+l, k+l)$, where $B(r,s) = r!/(r-s)!s!$ is binomial coefficient. The space-time surfaces associated with P_{kl} would be different and they need not be simultaneously critical, which could give rise to a breaking of supersymmetry.

   One would clearly have an upper bound for k and l for given CD. Therefore these many-fermion states must correspond to fundamental particles rather than many-fermion Fock states. One would obtain bosons with non-vanishing fermion numbers if the proposed identification is correct. Octonionic algebraic geometry for single CD would describe only fundamental particles or states with bounded fermion numbers. Fundamental particles would be indeed fundamental also geometrically.

2. I have already earlier considered the question whether the partonic 2-surfaces can carry also many-fermion states or not [K19, K20], and adopted the working hypothesis that fermion numbers is not larger than 1 for given wormhole throat, possibly for purely dynamical reasons. This picture however looks too limited. The many fermion states might not however propagate as ordinary particles (the proposal has been that their propagator pole corresponds to higher power of p^2).

3. The result looks somewhat disappointing at first. It would seem that the states with high fermion numbers must be described in terms of Cartesian products just like in condensed matter physics with interactions described by the proposed braney mechanism in which intersection of space-time surfaces with S^6 giving analogs of partonic 2-surfaces are involved.

4. One can also now define space-time varieties as zero loci via the conditions $RE(P_s)(o_s) = 0$ or $IM(P_s)(o_s) = 0$. One obtains a collection of 4-surfaces as zero loci of P_{kl}. One would have a correlation with between fermion content and algebraic geometry of the space-time surface unlike in the ordinary super-space approach, where the notion of the geometry remains rather formal and there is no natural coupling between fermionic content and classical geometry. At the level of H this comes from quantum classical correspondence (QCC) stating that the classical Noether charges are equal to eigenvalues of fermionic Noether charges.

4.7.3 Questions about quantum numbers

This raises several questions.

1. Could octonionic super geometry code for the quantum numbers of the particle states? It seems that super-octonionic polynomials multiplied by octonionic multi-spinors inside single CD can code only for the electroweak quantum numbers of fundamental particles besides their fermion and anti-fermion numbers.

   As already suggested, color corresponds to partial waves in CP_2 serving as moduli space for $M_0^4 \supset M_0^2$ and quarks and leptons have different trialities. Also four-momentum and angular momentum are naturally assigned with the translational degrees for the tip of CD assignable with the fundamental particle.

   Remark: There is a funny accident that deserves to be noticed. Octonionic spinor decomposes to $1 \oplus 1 \oplus 3 \oplus \overline{3}$ under $SU(3) \subset G_2$. Could it be that $1 \oplus 1$ corresponding to real unit and preferred imaginary unit assignable to M_0^2 correspond to color wave functions in CP_2 transforming like leptons and $3 + \overline{3}$ corresponds to wave functions transforming like quarks and antiquarks? Unfortunately, one cannot understand electroweak quantum numbers in this framework. There would be uncertainty principle allowing to measure either of these quantum numbers but not both.

2. What about twistors in this framework? $M^4 \times CP_1$ as twistor space with CP_1 coding for the choice of $M_0^2 \subset M_0^4$ allows projection to the usual twistor space CP_3. Twistor wave functions describing spin elegantly would correspond to wave functions in the twistor space and one expects that the notion of super-twistor is well-defined also now. The 6-D twistor space $SU(3)/U(2) \times U(1)$ of CP_2 would code besides the choice of $M_0^4 \supset M_0^2$ also quantization axis for color hypercharge and isospin.

3. What about the sphere S^6 serving as the moduli space for the choices of M_+^8? Should one have wave functions in S^6 or can one restrict the consideration to single M_+^8? As found, one obtains S^6 also as the zero locus of $Im(P) = 0$ for some radii identifiable as values t_n of time coordinates given as roots of $P(t)$. This would be crucial for the braney description of interactions between space-time surfaces associated with different CDs.

4.7.4 Could scattering amplitudes be computed at the level of M^8?

It would be extremely nice if the scattering amplitudes could be computed at the octonionic level by using a generalization of twistor approach in ZEO finding a nice justification at the level of M^8. Something rather similar to $\mathcal{N} = 4$ twistor Grassmann approach suggests itself.

1. In ZEO picture one would consider the situation in which the passive boundary of CD and members of state pairs at it appearing in zero energy state remain fixed during the sequence of state function reductions inducing stepwise drift of the active boundary of CD and change of states at it by unitary U-matrix at each step following by a localization in the moduli space for the positions of the active boundary.

2. At the active boundary one would obtain quantum superposition of states corresponding to different octonionic geometries for the outgoing particles. Instead of functional integral one would have sum over discrete points of WCW. WCW coordinates would be the coefficients of polynomial P in the extension of rationals. This would give undefined result without additional constraints since rationals are a dense set of reals.

Criticality however serves as a constraint on the coefficients of the polynomials and is expected to realize finite measurement resolution, and hopefully give a well defined finite result in the summation. Criticality for the outgoing states would realize purely number theoretically the cutoff due to finite measurement resolution and would be absolutely essential for the finiteness and well-definedness of the theory.

5 Cognitive representations and algebraic geometry

The general vision about cognition is realized in terms of adelic physics as physics of sensory experience and cognition [L7, L6]. Rational points and their generalization as ratios of algebraic integers for geometric objects would define cognitive representations as points common for real and various p-adic variants of the space-time surface. The finite-dimensionality for induced p-adic extensions allows also extensions of rationals involving root of e and its powers. This picture applies both at space-time level, imbedding space level, and at the level of space-time surfaces but basically reduces to imbedding space level. Hence counting of the (generalized) rational points for geometric objects would be determination of the cognitive representability.

5.1 Cognitive representations as sets of generalized rational points

The set of rational points depends on the coordinates chosen and one can argue that one must allow different cognitive representations and classify them according to their effectiveness.

How uniquely the M^8_c coordinates can be chosen?

1. Polynomial property allows only linear transformations of the complex octonionic coordinates with coefficients which belong to the extension of rationals used. This poses extremely strong restrictions on the allowed representations once the quaternionic moduli defining a foliation of M^4_0 is chosen. One has therefore moduli space of quaternionic structures. One must also fix the time axis in M^4 assignable to real octonions.

2. One can also define several inequivalent octonionic structures and associate a moduli space to these. The moduli space for octonionic structures would correspond to the space of $M^4_0 \subset M^8$s as quaternionic planes containing fixed M^2_0. One can allow even allow Lorentz transforms mixing real and imaginary octonionic coordinates. It seems that these moduli are not relevant at the level of H.

What could the precise definition of rationality?

1. The coordinates of point are rational in the sense defined by the extension of rationals used. Suppose that one considers parametric representations of surfaces as maps from space-time surface to imbedding space. Suppose that one uses as space-time coordinates subset of preferred coordinates for imbedding space. These coordinate changes cannot be global and one space-time surface decomposes to regions in which different coordinates apply.

2. The coordinate transformations between over-lapping regions are birational in the sense that both the map and its inverse are in terms of rational functions. This makes the notion of rationality global.

3. When cognitively easy rational parametric representations are possible? For algebraic curves with $g \geq 2$ in CP_2 represented as zeros of polynomials this cannot the case since the number of rational points is finite for instance for $g \geq 2$ surfaces. There is simple explanation for this. Solving second complex coordinate in terms of the other one gives it as an algebraic function for $g \geq 2$: this must be the reason for the loss of dense set of rational points. For elliptic surfaces $y^2 - x^3 - ax - b = 0$ y^2 is however polynomial of x and one can find rational parametric representation by taking y^2 as coordinate [L5]. For $g = 0$ one has linear equations and one obtains dense set of rational points. For conic sections one can also have dense set of rational points but not always. Generalizing from this it would seem that the failure to have rational parametric representation is the basic reason for the loss of dense set of rational points.

This picture does not work for general surfaces but generalizes for algebraic varieties defined by several polynomial equations. The co-dimension $d_c = 1$ case is however unique and the most studied one since for several polynomial equations one encounters technical difficulties when the intersection of the surfaces defined by the d_c polynomials need not be complete for $d_c > 1$. In the recent situation one has $d_c = 4$ but octonion analyticity could be powerful enough symmetry to solve the problem of non-complete intersections by eliminating them or providing a physical interpretation for them.

5.2 Cognitive representations assuming $M^8 - H$ duality

Many questions should be answered.

1. Can one generalize the results applying to algebraic varieties? Could the general vision about rational and potentially dense set of rational points generalize?. At M^8 side the description of space-time surfaces as algebraic varieties indeed conforms with this picture. Could one understand SH from the fact that real analyticity octonionic polynomials are determined by ordinary polynomial real coordinate completely? In information theoretic sense sense SH reduces to 1-D holography and the polynomial property makes the situation effectively discrete since finite number of points of real axis allows to determine the octonionic polynomial completely! It is a pity that one cannot measure octonionic polynomial directly!

2. Also the notion of Zariski dimension should make sense in TGD at M^8 side. Preferred extremals define the notion of closed set for given CD at M^8 side? It would indeed seem that one define Zariski topology at the level of M^8_c. Zariski topology would require 4-surfaces, string world sheets, or partonic 2-surfaces and even 1-D curves. This picture conforms with the recent view about TGD and and resembles the M-theory picture, where one has branes. SH suggests that the analog of Zariski dimension of space-time surface reduces to that for strings world sheets and partonic 2-surfaces and that even these are analogous to 1-D curves by complex analyticity. Integrability of TGD and preferred extremal property would indeed suggest simplicity.

   $M^8 - H$ hypothesis suggests that these conjectures make sense also at H side. String world sheets, partonic 2-surface, space-like 3-surfaces at the ends of space-time surface at boundaries of CD, and light-like 3-surfaces correspond to closed sets also at the level of WCW in the topology most natural for WCW.

3. Also the problems related to Minkowskian signature could be solved. String world sheets are problematic because of the Minkowskian signature. They however have the topology of disk plus handles suggesting immediately a vision about cognitive representations in terms of rational points. One can can complexify string world sheets and it seems possible to apply the results of algebraic geometry holding true in Euclidian signature. This would be analogous to the Wick rotation used in QFTs and also in twistor Grassmann approach.

4. What about algebraic geometrization of the twistor lift? How complex are twistor spaces of M^4, CP_2 and space-time surface? How can one generalize twistor lift to the level of M^8. S^2 bundle structure and the fact that S^2 allows a dense set of rational suggests that the complexity of twistor space is that of space-time surface itself so that the situation actually reduces to the level of space-time surfaces.

Suppose one accepts $M^8 - H$ duality requiring that the tangent space of space-time surface at given point x contains $M^2(x)$ such that $M^2(x)$ define an integrable distribution giving rise to string world sheets and their orthogonal complements give rise to partonic 2-surfaces. This would give rise to a foliation of the space-time surface by string world sheets and partonic 2-surface conjecture on basis of the properties of extremals of Kähler action. As found these foliations could correspond to quaternion structures that is allowed choices of quaterionic coordinates.

Should one define cognitive representations at the level of M^8 or at the level of $M^4 \times CP_2$? Or both? For M^8 option the condition that space-time point belongs to an extension of rationals applies at the level of M^8 coordinates. For $M^4 \times CP_2$ option cognitive representations are at the level of M^4 and CP_2 parameterizing the points of M^4 and their tangent spaces. The formal study of partial differential equations alone does not help much in counting the number of rational points. One can define cognitive representation in very many manners, and some cognitive representation could be preferred only because they are more efficient than others. Hence both cognitive representations seems to be acceptable.

Some cognitive representations are more efficient than others. General coordinate invariance (GCI) at the level of cognition is broken. The precise determination of cognitive efficiency is a challenge in itself. For instance, the use of coordinates for which coordinate lines are orbits of subgroups of the symmetry group should be highly efficient. Only coordinate transformations mediated by bi-rational maps can take polynomial representations to polynomial representations. It might well be that only a rational (in generalized sense) sub-group G_2 of octonionic automorphisms is allowed. For rational surfaces allowing parametric representation in terms of polynomial functions the rational points form a dense set.

The cognitive resolution for a dense set of rational points is unrealistically high since cognitive representation would contain infinite number of points. Hence one must tighten the notion of cognitive representation. The rational points must contain a fermion. Fermions are indeed are identified as correlates for Boolean cognition [K1]. This would suggests a view in which cognitive representations are realized at the light-like orbits of partonic 2-surfaces at which Minkowskian associative and Euclidian co-associative space-time surfaces meet. The general wisdom is that rational points are localized to lower-dimensional sub-varieties (Bombieri-Lang conjecture): this conforms with the view that fermion lines reside at the orbits of partonic 2-surfaces.

5.3 Are the known extremals in H easily cognitively representable?

Suppose that one takes TGD inspired adelic view about cognition seriously. If cognitive representations correspond to rational points for an extension of rationals, then the surfaces allowing large number of this kind of points are easily representable cognitively by adding fermions to these points. One could even speculate that mathematical cognition invents those geometric objects, which are easily cognitively representable and thus have a large number of rational points.

5.3.1 Could the known extremals of twistor lift be cognitively easy?

Also TGD is outcome of mathematical cognition. Could the known extremals of the twistor lift of Kähler action be cognitively easy? This is suggested by the fact that even such a pariah class theoretician as I am have managed to discover then! Positive answer could be seen as support for the proposed description of cognition!

1. If one believes in $M^8 - H$ duality and the proposed identification of associative and co-associative space-time surfaces in terms of algebraic surfaces in octonionic space M^8_c, the generalization of the results of algebraic geometry should give overall view about the cognitive representations at the level of M^8. In particular, surfaces allowing rational parametric representation (polynomials would have rational coefficients) would allow dense set or rational points since the images of rational points are rational. Rationals are understood here as ratios of algebraic integers in extension of rationals.

2. Also for H the existence of parameter representation using preferred H-coordinates and rational functions with rational coefficients implies that rational points are dense. If $M^8 - H$ correspondence maps the parametric representations in terms of rational functions to similar representations, dense set of rational points is preserved in the correspondence. There is however no obvious reason why $M^8 - H$ duality should have this nice property.

   One can even play with the idea that the surfaces, which are cognitively difficult at the M^8 side, might be cognitively easy at H-side or vice versa. Of course, if the explicit representation as algebraic functions makes sense at M^8 side, this side looks cognitively ridiculously easy as compared to H side. The preferred extremal property and SH can however change the situation.

3. At M^8 side and for a given point of M^4 there are several points of E^4 (or vice versa) if the degree of the polynomial is larger than $n = 1$ so that for the image of the surface H there

are several CP_2 points for a given point of M^4 (or vice versa) depending on the choice of coordinates. This is what the notion of the many-sheeted space-time predicts.

4. The equations for the surface at H side are obtained by a composite map assigning first to the coordinates of $X^4 \subset M^8$ point of $M^4 \times E^4$, and then assigning to the points of $X^4 \subset M^8$ CP_2 coordinates of the tangent space of the point. At this step the slightly non-local tangent space information is fed in and the surfaces in $M^4 \times CP_2$ cannot be given by zeros of polynomials. The indeed satisfy instead of algebraic equations partial differential equations given by the Kähler action for the twistor lift TGD. Algebraic equations instead of partial differential equations suggests that the M^8 representation is much simpler than H-representation. On the other hand, reduction to algebraic equations at M^8 side could have interpretation in terms of the conjectured complete integrability of TGD [K16, K11].

5.3.2 Testing the idea about self-reference

In any case, it is possible to test the idea about self-reference by looking whether the known extremals in H are cognitively easy and even have a dense set of rational points in natural coordinates. Here I will consider the situation at the level of $M^4 \times CP_2$. It was already found that the known extremals can have inverse images in M^8.

1. Canonically imbedded M^4 with linear coordinates and constant CP_2 coordinates rational is the simple example about preferred extremal and it seems that TGD based cosmology at microscopic relies on these extremals. In this case it is obvious that one has a dense set of rational points at both sides. Could this somehow relate to the fact that physics as physics M^4 was discovered before general relativity?

   Canonically imbedded M^4 corresponds to a first order octonionic polynomial for which imaginary part is put to constant so that tangent space is same everywhere and corresponds to a constant CP_2 coordinate.

2. CP_2 type extremals have 4-D CP_2 projection and light-like geodesic line of M^4 as M^4 projection. One can choose the time parameter as a function of CP_2 coordinates in infinitely many manners. Clearly the rational points are dense in any CP_2 coordinates.

3. Massless extremals (MEs) are given as zeros of arbitrary functions of CP_2 coordinates and 2 M^4 coordinates representing local light-like direction and polarization direction orthogonal to it. In the simplest situation these directions are constant. In the general case light-like direction would define tangent space of string world sheet giving rise also to a distribution of ortogonal polarization planes. This is consistent with the general properties of the M^8 representation and corresponds to the decomposition of quaternionic tangent plane to complex plane and its complement. One can ask whether one should allow only polynomials with rational coefficients as octonionic polynomials.

4. String like objects $X^2 \times Y^2$ with $X^2 \subset M^4$ a minimal surface and Y^2 complex or Lagrangian surface of CP_2 are also basic extremals and their deformations in M^4 directions are expected to give rise to magnetic flux tubes.

   If Y^2 is complex surface with genus $g = 0$ rational points are dense. Also for $g = 1$ one obtains a dense set of rational points in some extension of rationals. For elliptic curves one has lattice of rational points. What happens for Lagrangian surfaces Y^2? In this case one does not have complex curves but real co-dimension 2 surfaces. There is no obvious objection why these surfaces would not be possible.

5. What about string world sheets? If the string world is static $M^2 \subset M^4$ one has a dense set of rational points. One however expects something more complex. If the string world sheet is rational map M^2 to its orthogonal complement E^2 one has rational surface. For rotating strings this does not make sense except for certain period of time. If the choice of the quaternion structure corresponds to a choice of minimal surface in M^4 as integrable distribution for $M^2(x)$, the coordinates associated with the Hamilton-Jacobi structure could make the situation simple.

If one restricts the consideration the intersections of partonic 2-surfaces and string world sheets at two boundaries of CD the situation simplifies and the question is only about the rationality of the M^4 coordinates at rational points of $Y^2 \subset CP_2$. This would simplify the situation enormously and might even allow to use existing knowledge.

6. The slicing of of space-time surfaces by string world sheets and partonic 2-surfaces required by Hamilton-Jacobi structure could be seen as a fibering analogous to that possessed by elliptic surfaces. This suggest that M^8 counterparts of spacetime surfaces are not of general type in Kodaira classification and that the number of rational points can be large. If the existence of Hamilton-Jacobi structure does not allow handles, this factor would be cognitively simple. This would however suggests that fermion number is not localized at the ends of strings only - as assumed in the construction of scattering amplitudes inspired by twistor Grassmann approach [K19] - but also to the interior of the light-like curves inside string world sheets.

5.4 Twistor lift and cognitive representations

What about twistor lift of TGD replacing space-time surfaces with their twistor spaces. Consider first M^8 side.

1. At M^8 side S^2 seems to introduce nothing new. One might expect that the situation does not change at H-side since space-time surfaces are obtained essentially by dimensional reduction and the possible problem relates to the choice of base space as section of is twistor bundle and the imbedding of space-time as base space could have singularities at the boundary of Euclidian and Minkowskian space-time regions as discussed in [L5].

   At the side of M^8 the proposed induction of twistor structure is just a projection of the twistor sphere S^6 to its geodesic sphere and one has 4-D moduli space for geodesic spheres $S^2 \subset S^6$. If one interprets the choice of $S^2 \subset S^6$ as as a section in the moduli space, the moduli of S^2 can depend on the point of space-time surface. Note that there are is also a position dependent choice of preferred point of S^2 representing Kähler form, and this choice is good candidate for giving rise to Hamilton-Jacobi structures with position dependent M^2.

2. The notion of Kodaira dimension is defined also for co-dimension 4 algebraic varieties in M^8_c. The cognitively easiest spacetime surfaces would allow rational parametric representation with complex coordinates serving as parameters. If this is not possible, one has algebraic functions, which makes the situation much more complex so that the number of rational points would be small.

3. For some complex enough extensions of rationals the set of rational points can be dense. $g \geq 2$ genera are basic example and one expects also in more general case that polynomials involving powers larger than $n = 4$ make the situation problematic. The condition that real or imaginary part of real analytic octonionic polynomial is in question is a strong symmetry expected to faciliate cognitive representability.

4. The general intuitive wisdom from algebraic geometry is that the rational points are dense only in lower-dimensional sub-varieties (Bombieri-Lang and Vojta conjectures mentioned in the first section). The general vision inspired by SH and the proposal for the construction of twistor amplitudes indeed is that the algebraic points (rational in generalized sense) defining cognitive representations are associated with the intersections of string world sheets and partonic 2-surfaces to which fermions are assigned. This would suggest that partonic 2-surfaces and string world sheets contain the cognitive representation, which under additional conditions can contain very many points.

5. An interesting question concerns the M^8 counterparts of partonic 2-surfaces as space-time regions with Minkowskian and Euclidian signature. The partonic orbits representing the boundaries between these regions should be mapped to each other by $M^8 - H$ duality. This conforms with the fact that induced metric must have degenerate signature $(0, -1, -1, -1)$ at partonic orbits. Can one assume that the topologies of partonic 2-surfaces at two sides are identical? Consider partonic 2-surface of genus g in $M^4 \times CP_2$ - say at the boundary of CD. It should be inverse image of a 2-surface in $M^4 \times E^4$ such that the tangent space of this

surface labelled by CP_2 coordinates is mapped to a 2-surface in $M^4 \times CP_2$. If the inverse of $M^8 - H$ correspondence is continuous one expects that g is preserved.

Consider next the H-side. There is a conjecture that for Cartesian product the Kodaira dimension is sum $d_K = \sum_i d_{K,i}$ of the Kodaira dimensions for factors. Suppose that CP_1 fiber as surface in the 12-D twistor bundle $T(M^4) \times T(CP_2)$ has Kodaira dimension $d_K(CP_1) = -\infty$ (it is expected to be rational surface) then the fact that the bundle decomposes to Cartesian product locally and rational points are pairs of rational points in the factors, is indeed consistent with the proposal. S^2 would give dense set of rational points in S^2 and the bundle would have infinite number of rational points.

In TGD context, it is however space-time surface which matters. Space-time surface as section of the bundle would not however have a dense set of points in the general case and the relevant Kodaira dimension be $d_K = d_K(X^4)$. One can of course ask whether the space-time surface as an algebraic section (not many of them) of the twistor bundle could chosen to be cognitively simple.

6 A possible connection with family replication phenomenon?

In TGD framework the genus g of the partonic 2-surfaces is proposed to label fermion families [K2, K6, K7]. One can characterize by genus g the topology of light-like partonic orbits and identify the three fermion generators as 2-surfaces with genus $g = 0, 1, 2$ with the special property that they are always hyper-elliptic. Quantum mechanically also topological mixing giving rise to CKM mixing is possible. The view is that given connected 3-surface can contain several light-like 3-surface with different genera. For instance, hadrons would be such surfaces.

There are however questions to be answered.

1. The homology and genus for 2-surfaces of CP_2 correlate with each other [A6]: is this consistent with the proposed topologicization of color hypercharge implying color confinement?

2. $h_{eff}/h = n$ hypothesis means that dark variant of particle particle characterized by genus g is n-fold covering of this surface. In the general case the genus of covering is different. Is this consistent with the genus-generation correspondence?

3. The degree of complex curve correlates with the genus of the curve. Is generation-genus correspondence consistent with the assumption that partonic 2-surfaces have algebraic curve as CP_2 projection (this need not be the case)?

6.1 How the homology charge and genus correlate?

Complex surfaces in CP_2 are highly interesting from TGD point of view.

1. The model for elementary particles assumes that the partonic 2-surfaces carrying fermion number are homologically non-trivial, in other words they carry Kähler magnetic monopole flux having values $q = \pm 1$ and $q = \pm 2$. The idea is that color hyper charge $Y = \{\pm 2/3, \pm 1/3\}$ is proportional to n for quarks and color confinement topologizes to the vanishing of total homology charge [K7].

2. The explanation of the family replication phenomenon [K2] in terms of genus-generation correspondence states that the three quarks and lepton generations correspond to the three lowest genera $g = 0, 1, 2$ for partonic 2-surfaces. Only these genera are always hyper-elliptic allowing thus a global Z_2 conformal symmetry. The physical vision is that for higher genera the handles behave like free particles. Is this proposal consistent with the proposal for the topologization of color confinement?

There is a result [A6] (page 124) stating that if the homology charge q is divisible by 2 then one must have $g \geq q^2/4 - 1$. If q is divisible by h, which is odd power of prime, one has $g \geq (q^2/4 - 1) - (q^2/4h^2)$. For $q = 2$ the theorem allows $g \geq 0$ so that all genera with color hyper charge $Y = \pm 2/3$ are realized.

The theorem says however nothing about $q = 0, 1$. These charges can be assigned to the two different geodesic spheres of CP_2 with $g = 0$ remaining invariant under SO(3) and U(2) subgroups of SU(3) respectively. Is $g > 0$ possible for $q = 1$ as the universality of topological color confinement would require? For $q = 3$ one would have $g \geq 1$. For $q = 4$ $h = 2$ divides q and one has $g \geq 2$. It would seem $g \geq 5$. The conditions become more restrictive for higher q, which suggests that for $q = 0, 1$ one has $g \geq 0$ so that the topologization of color hypercharge would make sense.

6.2 Euler characteristic and genus for the covering of partonic 2-surface

Hierarchy of Planck constants $h_{eff}/h = n$ means a hierarchy of space-time surfaces identifiable as n-fold coverings. The proposal is that the number of sheets in absence of singularities is the maximal possible one and equal to the order of Galois group for the extension of rationals but this result is not really proven. Second naive guess is that it corresponds to the dimension of extension.

The Euler characteristic of n-fold covering in absence of singular points is $\chi_n = n\chi$. If there are singular (ramified) points these give a correction term given by Riemann-Hurwitz formula (see `http://tinyurl.com/y7n2acub`.)

In absence of singularities one has from $\chi = -2(g - 1)$ and $\chi_n = n\chi$

$$g_n = n(g - 1) + 1 \ . \tag{6.1}$$

For $n = 1$ this indeed gives $g_1 = g$ independent of g. One can also combine this with the formula $g = (d - 1)(d - 2)/2$ holding for non-singular algebraic curves of degree d.

Singularities are unavoidable at algebraic points of cognitive representations at which some subgroup of Galois group leaves the point invariant (say rational point in ordinary sense). One can consider the possibility that fermions are located at the singular points at which several sheets of covering touch each other. This would give a correction factor to the formula. If the projection map from the covering to based is of form $\Pi(z) = z^n$ at the singular point P, one says that singularity has ramimifaction index $e_P = n$ and the algebraic genus would increase to

$$g_n = n(g - 1) + 1 + \frac{1}{2}\sum_P (e_P - 1) \ . \tag{6.2}$$

Indeed, singularities mean that sheets touch each other at singular points and this increases connectivity.

Under what conditions the genus of dark partonic surface with $n > 1$ can be same as that of the ordinary partonic surface representing visible matter? For the genera $g = 0$ and $g = 1$ this is possible so that these genera would be in an exceptional role also from the point of view of dark matter.

1. For $g = 1$ one has $g_n = g = 1$ independent of n in absence of singular point. Torus topology (assignable to muon and (c,s) quarks) is exceptional. In presence of singularities the genus would increase by the $\sum_P (e_P - 1)/2$ independent of the value of n. The lattice of points for elliptic surfaces would suggest existence of infinite number of singular points if the abelian group operations preserve the singular character of the points so that the genus would become infinite.

2. For $g = 0$ one would have $g_n = -n + 1$ in absence of singularities. Only $n = 1$ - ordinary matter - is possible without singularities. Dark matter is however possible if singularities are allowed. For sphere one would obtain $g_n = -n + 1 + \sum_P (e_P - 1)/2 \geq 0$. The condition $n \leq \sum_P (e_P - 1)/2 + 1$ must therefore hold true for $g \geq 0$.

   The condition $g_n = -n + 1 + \sum_P (e_P - 1)/2 = g = 0$ gives $\sum_P (e_P - 1) = 2(n - 1)$. For spherical topology it is possible to have dense set of rational points so that it is possible create cognitive representations with arbitrary number of points which can be also singular. One might argue that this kind of situation corresponds to a non-perturbative phase.

3. For $g = 2$ one would have $g_n = n + 1 + \sum_P (e_P - 1)/2$ and genus would grow with n even in absence of singularities and would be very large for large values of h_{eff}. $g_n = 2$ is obtained with $n = 1$ (ordinary matter) and no singular points not even allowed for $n = 1$. $g_n = g = 2$ is not possible for $n > 1$.

Note that dark $g \geq 2$ fermions cannot correspond to lower generation fermions with singular points of covering. More generally, one could say that $g \geq 2$ fermions can exists only with standard value of Planck constant unless they are singular coverings of $g < 2$ fermions.

What is clear that the model of dark matter predicts breaking of universality. This breaking is not seen in the standard model couplings but makes it visible in amore delicate manner and might allow to understand why the masses of fermions increase with generation index.

6.3 All genera are not representable as non-singular algebraic curves

Suppose for a moment that partonic 2-surfaces correspond to rational maps of algebraic curves in CP_2 to M^4 that is deformations of these curves in M^4 direction. This assumption is of course questionable but deserves to be sttudied.

The formula (for algebraic curve see `http://tinyurl.com/nt6tkey`)

$$g = \frac{(d-1)(d-2)}{2} + \frac{\sum \delta_s}{2} \ ,$$

where $\delta_s > 0$ characterizes the singularity, does not allow all genera for algebraic curves for $\sum \delta_s = 0$: one has $g = 0, 0, 1, 3, 6, 10, ..$ for $d = 1, 2, ...$

For instance, $g = 2$, which would correspond in TGD to third quark or lepton generation is not possible without singularities for $d = 3$ curve having $g = 1$ without singularities!

This raises questions. Could the third fermion generation actually correspond to $g = 3$? Or does it correspond to $g = 2$ 2-surface of CP_2, which is more general surface than algebraic curve meaning that it is not representable as complex surface? Or could third generation fermions correspond to $g = 0$ or $g = 1$ curves with singular point of covering by Galois group so that several sheets touch each other?

To sum up, if the results for algebraic varieties generalize to TGD framework, they suggest notable differences between different fermion families. Universality of standard model interactions says that the only differences between fermion families are due to the differ masses. It is not clear whether the different masses could be due to the differences at number theoretical level and dark matter sectors.

1. All genera can appear as as ordinary matter ($d = 1$). Dark variants of $g = 1$ states have $g_d = 1$ automatically in absence of singular points. Dark variants of $g = 0$ states must have singular point in order to give $g_n = 0$. Dark variants of $g = 2$ states with $g_d = 2$ are obtained from $g = 1$ states with singularities. The special role of the two lowest is analogous to their special role for algebraic curves.

2. If one assumes that partonic 2-surfacs correspond to algebraic curves, one obtains again that $g = 2$ surfaces must correspond to singular $g = 0$ and $g = 1$ which could be dark in TGD sense.

7 Summary and future prospects

In the following I give a brief summary about what has been done. I concentrate on $M^8 - H$ duality since the most significant results are achieved here.

It is fair to say that the new view answers the following a long list of open questions.

1. When $M^8 - H$ correspondence is true (to be honest, this question emerged during this work!)? What are the explicit formulas expressing associativity of the tangent space or normal space of the 4-surface?

   The key element is the formulation in terms of complexified M^8 identified in terms of octonions and restriction to M^8. One loses the number field property but for polynomials ring property is enough. The level surfaces for real and imaginary parts of octonionic polynomials with real coefficients define 4-D surfaces in the generic case.

   Associativity condition is an additional condition reducing the dimension of the space-time surface unless some components of $RE(P)$ or $IM(P)$ are critical meaning that also their gradients vanish. This conforms with the quantum criticality of TGD and provides a concrete first principle realization for it.

2. How this picture corresponds to twistor lift? The twistor lift of Kähler action (dimensionally reduced Kähler action in twistor space of space-time surface) one obtains two kinds of space-time regions. The regions, which are minimal surfaces and obey dynamics having no dependence on coupling constants, correspond naturally to the critical regions in M^8 and H.

   There are also regions in which one does not have extremal property for both Kähler action and volume term and the dynamics depends on coupling constant at the level of H. These regions are associative only at their 3-D ends at boundaries of CD and at partonic orbits, and the associativity conditions at these 3-surfaces force the initial values to satisfy the conditions guaranteeing preferred extremal property. The non-associative space-time regions are assigned with the interiors of CDs. . The particle orbit like space-time surfaces entering to CD are critical and correspond to external particles.

3. The surprise was that $M^4 \subset M^8$ is naturally co-associative. If associativity holds true also at the level of H, $M^4 \subset H$ must be associative. This is possible if $M^8 - H$ duality maps tangent space in M^8 to normal space in H and vice versa.

4. The connection to the realization of the preferred extremal property in terms of gauge conditions of subalgebra of SSA is highly suggestive. Octonionic polynomials critical at the boundaries of space-time surfaces would determine by $M^8 - H$ correspondence the solution to the gauge conditions and thus initial values and by holography the space-time surfaces in H.

5. A beautiful connection between algebraic geometry and particle physics emerges. Free many-particle states as disjoint critical 4-surfaces can be described by products of corresponding polynomials satisfying criticality conditions. These particles enter into CD , and the non-associative and non-critical portions of the space-time surface inside CD describe the interactions. One can define the notion of interaction polynomial as a term added to the product of polynomials. It can vanish at the boundary of CD and forces the 4-surface to be connected inside CD. It also spoils associativity: interactions are switched on. For bound states the coefficients of interaction polynomial are such that one obtains a bound state as associative space-time surface.

6. This picture generalizes to the level of quaternions. One can speak about 2-surfaces of space-time surface with commutative or co-commutative tangent space. Also these 2-surfaces would be critical. In the generic case commutativity/co-commutativity allows only 1-D curves.

   At partonic orbits defining boundaries between Minkowskian and Euclidian space-time regions inside CD the string world sheets degenerate to the 1-D orbits of point like particles at their boundaries. This conforms with the twistorial description of scattering amplitudes in terms of point like fermions.

   For critical space-time surfaces representing incoming states string world sheets are possible as commutative/co-commutative surfaces (as also partonic 2-surfaces) and serve as correlates for (long range) entaglement) assignable also to macroscopically quantum coherent system ($h_{eff}/h = n$ hierarchy implied by adelic physics).

7. The octonionic polynomials with real coefficients form a commutative and associative algebra allowing besides algebraic operations function composition. Space-time surfaces therefore form an algebra and WCW has algebra structure. This could be true for the entire hierarchy of Cayley-Dickson algebras, and one would have a highly non-trivial generalization of the conformal invariance and Cauchy-Riemann conditions to their n-linear counterparts at the n:th level of hierarchy with $n = 1, 2, 3, ..$ for complex numbers, quaternions, octonions,... One can even wonder whether TGD generalizes to this entire hierarchy!

All big pieces of quantum TGD are now tightly interlinked.

1. The notion of causal diamond (CD) and therefore also ZEO can be now regarded as a consequence of the number theoretic vision and $M^8 - H$ correspondence, which is also understood physically.

2. The hierarchy of algebraic extensions of rationals defining evolutionary hierarchy corresponds to the hierarchy of octonionic polynomials.
3. Associative varieties for which the dynamics is critical are mapped to minimal surfaces with universal dynamics without any dependence on coupling constants as predicted by twistor lift of TGD. The 3-D associative boundaries of non-associative 4-varieties are mapped to initial values of space-time surfaces inside CDs for which there is coupling between Kähler action and volume term.
4. Free many particle states as algebraic 4-varieties correspond to product polynomials in the complement of CD and are associative. Inside CD the addition of interaction terms vanishing at its boundaries spoils associativity and makes these varieties connected.
5. The basic building bricks of topological scattering diagrams identified as space-time surfaces having as vertices partonic 2-surfaces emerge from the special features of the octonionic algebraic geometry predicting sequence of 3-balls as intersections of hyperplanes $t = t_n$ with CD. One can say that octonionic dynamics solves roots of the polynomial $P(t)$, whose octonionic extension defines space-time surfaces as zero loci . Furthermore, the generic prediction is the existence of 6-spheres inside octonionic CDs having 2-D partonic 2-variety as intersection with space-time surface inside CD and interpreted as a vertex of generalized scattering diagram.
6. The super variant of the octonionic algebraic geometry makes sense, and one obtains a beautiful correlation between the fermion content of the state and corresponding space-time variety. This suggests that twistorial construction indeed generalizes. Criticality for the external particles giving rise to additional constraints on the coefficients of polynomials could make possible to have well-define summation over corresponding varieties.

What mathematical challenges one must meet?

1. One should prove more rigorously that criticality is possible without the reduction of dimension of the space-time surface.
2. One must demonstrate that SSA conditions can be true for the images of the associative regions (with 3-D or 4-D). This would obviously pose strong conditions on the values of coupling constants at the level of H.

What questions should be answered?

1. Do associative space-time regions have minimal surface extremals as images in H and indeed obeying universal critical dynamics? As found, the study of the known extremals supports this view.
2. Could one construct the scattering amplitudes at the level of M^8? Here the possible problems are caused by the exponents of action (Kähler action and volume term) at H side. Twistorial construction [K20] however leads to a proposal that the exponents actually cancel. This happens if the scattering amplitude can be thought as an analog of Gaussian path integral around single extremum of action and conforms with the integrability of the theory. In fact, nothing prevents from defining zero energy states in this manner! If this holds true then it might be possible to construct scattering amplitudes at the level of M^8.
3. What about coupling constants? Coupling constants make themselves visible at H side both via the vanishing conditions for Noether charges in sub-algebra of SSA and via the values of the non-vanishing Noether charges. $M^8 - H$ correspondence determining the 3-D boundaries of interaction regions within CDs suggests that these couplings must emerge from the level M^8 via the criticality conditions posing conditions on the coefficients of the octonionic polynomials coding for interactions.

   Could all coupling constant emerge from the criticality conditions at the level of M^8? The ratio of R^2/l_P^2 of CP_2 scale and Planck length appears at H level. Also this parameter should emerge from $M^8 - H$ correspondence and thus from criticality at M^8 level. Physics would reduce to a generalization of the catastrophe theory of Rene Thom!

4. Real analyticity requires that the octonionic polynomials have real coefficients. This forces the origin of octonionic coordinates to be at real line (time axis) in the octonionic sense. All CDs cannot be located along this line. How do the varieties associated with octonionic polynomials with different origins interact? The polynomials with different origins neither commute nor are associative. How could one avoid losing the extremely beautiful associative and commutative algebra? It seems that one cannot form their products and sums and must form the Cartesian product of M^8:s with different origins and formulate the interaction in this framework.

   How the space-time surfaces associated with different quaternionic structures of M^8 and with different positions of tips of CD interact? The spheres S^6 appearing as roots of $P(t)$ the boundary of 8-D light-cone have 2-D varieties as intersections with space-time variety and this leads to a proposal for how the interaction could take place in "braney" fashion.

   Is the interaction well-defined only at the level of H inside CD to which these 4-D varieties arrive through the boundary of this CD? All CDs, whose tips are along light-like ray of CD boundary, share this ray. There is a common M_0^2 shared by these CDs. Could M_0^2 make possible the interaction. The CDs able to interact with given CD would have tips at the 3-D boundary of this CD and share common M_0^2. These M_0^2:s are labelled by the points of twistor sphere so that twistoriality seems to enter into the game in non-trivial manner also at the level of M^8!

5. What is the connection with Yangian symmetry, whose generalization in TGD framework is highly suggestive?

8 Appendix: o^2 as a simple test case

Octonionic polynomial o^2 serves as a simple testing case. o^2 is not irreducible so that its properties might not be generic and it might be better to study polynomial of form $P(o) = o + po^2$ instead.

Before continuing, some conventions are needed.

1. The convention is that in $M^8 = M^1 \times E^7$ E^7 corresponds to purely imaginary complexified octonions in both octonionic sense and in the sense that they are proportional to i. M^1 corresponds to octonions real in both senses. This corresponds to the signature $(1, -1, -1, -1, ...)$ for M^8 metric obtained as restriction of complexified metric. For $M^4 = M^1 \times E^3$ analogous conventions hold true.

2. Conjugation $o = o_0 + o_k I_k \rightarrow \overline{o} \equiv o_0 - I_k o_k$ does not change the sign of i. Quaternions can be decomposed to real and imaginary parts and some notation is needed. The notation $q = Re(q) + Im(q)$ seems to be the least clumsy one: here $Im(q)$ is 3-vector.

The explicit expression in terms of quaternionic decomposition $o = q_1 + q_2 I_4$ reads as

$$P(o) = o^2 = q_1^2 - q_2\overline{q}_2 + (q_1 q_2 + q_2\overline{q}_1)I_4 \quad . \tag{8.1}$$

o corresponds to complexified octonion and there are two options concerning the interpretation of M^4 and E^4. M^4 could correspond to quaternionic or co-quaternionic sub-space. I have assumed the first interpretation hitherto but actually the identification is not obvious. This two cases are different and must be treated both.

With these notations quaternionic inner product reads as

$$\begin{aligned} q_1 q_2 &= Re(q_1 q_2) + Im(q_1 q_2) \ , \\ Re(q_1 q_2) &= Re(q_1)Re(q_2) - Im(q_1) \cdot Im(q_2) \ , \\ Im(q_1 q_2) &= Re(q_1)Im(q_2) + Re(q_2)Im(q_1) + Im(q_1) \times Im(q_2) \ . \end{aligned} \tag{8.2}$$

Here $a \cdot b$ denotes the inner product of 3-vectors and $a \times b$ their cross product.

Note that one has real and imaginary parts of octonions as two quaternions and real and imaginary parts of quaternions. To avoid confusion, I will use RE and IM to denote the decomposition of octonions to quaterions and Re and Im for the decomposition of quaternions to real and imaginary parts.

One can express the $RE(o^2)$ as

$$\begin{array}{l} RE(o^2) \equiv X \equiv q_1^2 - q_2\overline{q}_2 \ , \\ Re(X) = Re(q_1)^2 - Im(q_1)\cdot Im(q_2) - (Re(q_2)^2 + Im(q_2)\cdot Im(q_2)) \ , \\ Im(X) = Im(q_1^2) = 2Re(q_1)Im(q_1) \ . \end{array} \tag{8.3}$$

For $IM(o^2)$ one has

$$\begin{array}{l} IM(o^2) \equiv Y = q_1q_2 + q_2\overline{q}_1 \\ Re(Y) = 2Re(q_1)Re(q_2) \ , \\ Im(Y) = Re(q_1)Im(q_2) - Re(q_2)Im(q_1) + Im(q_1)\times Im(q_2) \ . \end{array} \tag{8.4}$$

The essential point is that only $RE(o^2)$ contains the complexified Euclidian norm $q_2\overline{q_2}$ which becomes Minkowskian of Euclidian norm depending on whether one identifies M^4 as associative or co-associative surface in o_c^8.

8.1 Option I: M^4 is quaternionic

Consider first the condition $RE(o^2) = 0$. The condition decomposes to two conditions stating the vanishing of quaternionic real and imaginary parts:

$$\begin{array}{l} Re(X) = Re(q_1)^2 - Im(q_1)\cdot Im(q_2) - (Re(q_2)^2 + Im(q_2)\cdot Im(q_2)) \equiv N_{M^4}(q_1) - N_{E^4}(q_2) = 0 \ , \\ Im(X) = Im(q_1^2) = 2Re(q_1)Im(q_1) = 0 \ . \end{array} \tag{8.5}$$

$Im(X) = 0$ is satisfied for $Re(q_1) = 0$ or $Im(q_1) = 0$ so that one has two options. This gives 1-D line in time direction of 3-D hyperplane as a solution for M^4 factor.

$Re(X) = 0$ states $N_{M^4}(q_1) = N_{E^4}(q_2)$. q_2 coordinate itself is free. $N_{E^4}(q_2)$ is negative so that q_1 must be space-like with respect to the N_{M^4} so that only the solution $Re(q_1) = 0$ is possible. Therefore one has $Re(q_1) = 0$ and $N_{M^4}(q_1) = N_{E^4}(q_2)$.

One can assign to each E^4 point a section of hyperboloid with $t = 0$ hyper-plane giving a sphere and the surface is 6-dimensional sphere bundle like variety! This is completely unexpected result and presumably is due to the additional accidental symmetries due to the octonionicity. Also the fact that o^2 is not irreducible polynomial is a probably reason since for o the surface is 4-D. The addition of linear term is expected to remove the degeneracy.

Consider next the case $IM(o^2) = 0$. The conditions read now as

$$\begin{array}{l} Re(Y) = 2Re(q_1)Re(q_2) = 0 \ , \\ Im(Y) = Re(q_1)Im(q_2) - Re(q_2)Im(q_1) + Im(q_1)\times Im(q_2) = 0 \ . \end{array} \tag{8.6}$$

Since cross product is orthogonal to the the factors $Im(Y) = 0$ condition requires that $Im(q_1)$ and $Im(q_2)$ are parallel vectors: $Im(q_1) = \lambda Im(q_2)$ and one has the condition $Re(q_1) = \lambda Re(q_2)$ implying $q_1 = \Lambda q_2$. Therefore to each point of E^4 is associated a line of M^4. The surface is 5-dimensional.

It is interesting to look what the situation is if both conditions are true so that one would have a singularity. In this case $Re(q_1) = 0$ and $Re(q_1) = \lambda Re(q_2)$ imply $\lambda = 0$ so that $q_1 = 0$ is obtained and the solution reduces to 4-D E^4, which would be co-associative.

8.2 Option II: M^4 is co-quaternionic

This case is obtained by the inspection of the previous calculation by looking what changes the identification of M^4 as co-quaternionic factor means. Now q_1 is Euclidian and q_2 Minkowskian coordinate and $q_2\overline{q}_2$ gives Minkowskian rather than Euclidian norm.

Consider first $RE(o^2) = 0$ case.

$$\begin{aligned} Re(X) &= Re(q_1)^2 - Im(q_1)\cdot Im(q_2) - (Re(q_2)^2 + Im(q_2)\cdot Im(q_2)) \equiv N_{M^4}(q_1) - N_{M^4}(q_2) = 0 \ , \\ Im(X) &= Im(q_1^2) = 2Re(q_1)Im(q_1) = 0 \ . \end{aligned} \tag{8.7}$$

$N_{M^4}(q_1) - N_{M^4}(q_2) = 0$ condition holds true now besides the condition $Re(q_1) = 0$ or $Im(q_1) = 0$ so that one has also now two options.

1. For $Re(q_1) = 0$ $N_{M^4}(q_1)$ is non-positive and this must be the case for $N_{M^4}(q_2))$ so that the *exterior* of the light-cone is selected. In this case the points of M^4 with fixed N_{M^4} give rise to a 2-D intersection with $Re(q_1) = 0$ hyper-plane that is sphere so that one has 6-D surface, kind of sphere bundle.

2. For $Im(q_1) = 0$ Minkowski norm is positive and so must be corresponding norm in E^4 so that in E^4 surface has future ligt-cone as projection. This surface is 4-D. The emergence of future light-cone might provide justification for the emergence of CDs and zero energy ontology.

For $IM(o^2)$ the discussion is same as in quaternionic case since norm does not appear in the equations.

At singularity both $RE(o^2)$ and $IM(o^2) = 0$ vanish. The condition $q_1 = \Lambda q_2$ reduces to $\Lambda = 0$ so that $q_1 = 0$ is only allowed. This leaves only light-cone boundary under consideration.

The appearance of surfaces with dimension higher than 4 raises the question whether something is wrong. One could of course argue that associativity allows also lower than 4-D surfaces as associative surfaces and higher than 4-D surfaces as co-associative surfaces. At H-level one can say that one has 4-D surfaces. A good guess is that this behavior disappears when the linear term is absent and origin ceases to be a singularity.

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