# p-Adic Numbers and Generalization of Number Concept

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# Contents

**Abstract**

In this chapter the general TGD inspired mathematical ideas related to p-adic numbers are discussed. The extensions of the p-adic numbers including extensions containing transcendentals, the correspondences between p-adic and real numbers, p-adic differential and integral calculus, and p-adic symmetries and Fourier analysis belong the topics of the chapter.

The basic hypothesis is that p-adic space-time regions correspond to cognitive representations for the real physics appearing already at the elementary particle level. The interpretation of the p-adic physics as a physics of cognition is justified by the inherent p-adic non-determinism of the p-adic differential equations making possible the extreme flexibility of imagination.

p-Adic canonical identification and the identification of reals and p-adics by common rationals are the two basic identification maps between p-adics and reals and can be interpreted as two basic types of cognitive maps. The concept of p-adic fractality is defined and p-adic fractality is the basic property of the cognitive maps mapping real world to the p-adic internal world. Canonical identification is not general coordinate invariant and at the fundamental level it is applied only to map p-adic probabilities and predictions of p-adic thermodynamics to real numbers. The correspondence via common rationals is general coordinate invariant correspondence when general coordinate transformations are restricted to rational or extended rational maps: this has interpretation in terms of fundamental length scale unit provided by $CP_2$ length.

A natural outcome is the generalization of the notion of number. Different number fields form a book like structure with number fields and their extensions representing the pages of the book glued together along common rationals representing the rim of the book. This generalization forces also the generalization of the manifold concept: both imbedding space and WCW are obtained as union of copies corresponding to various number fields glued together along common points, in particular rational ones. Space-time surfaces decompose naturally to real and p-adic space-time sheets. In this framework the fusion of real and various p-adic physics reduces more or less to to an algebraic continuation of rational number based physics to various number fields and their extensions.

The definition of p-adic manifold is not discussed although it has turned out to be highly non-trivial. The feasible definition of p-adic sub-manifold emerged two decades after the emergence of the notion of of p-adic space-time sheet. The definition relies on the idea that p-adic space-time surfaces serve as p-adic charts - cognitive maps - for real space-time surfaces and vice versa and that both real and p-adic space-time sheets are preferred extremals of Kähler action and defined only modulo finite measurement/cognitive resolution.

p-Adic differential calculus obeys the same rules as real one and an interesting outcome are p-adic fractals involving canonical identification. Perhaps the most crucial ingredient concerning the practical formulation of the p-adic physics is the concept of the p-adic valued definite integral. Quite generally, all general coordinate invariant definitions are based on algebraic continuation by common rationals. Integral functions can be defined using just the rules of ordinary calculus and the ordering of the integration limits is provided by the correspondence via common rationals. Residy calculus generalizes to p-adic context and also free Gaussian functional integral generalizes to p-adic context and is expected to play key role in quantum TGD at WCW level.

The special features of p-adic Lie-groups are briefly discussed: the most important of them being an infinite fractal hierarchy of nested groups. Various versions of the p-adic Fourier analysis are proposed: ordinary Fourier analysis generalizes naturally only if finite-dimensional extensions of p-adic numbers are allowed and this has interpretation in terms of p-adic length scale cutoff. Also p-adic Fourier analysis provides a possible definition of the definite integral in the p-adic context by using algebraic continuation.

# 1 Introduction

In this chapter basic facts about p-adic numbers and the question about their relation to real numbers are discussed. Also the basic technicalities related to the notion of p-adic physics are discussed.

## 1.1 Problems

It is far from obvious what the p-adic counterpart of real physics could mean and how one could fuse together real and p-adic physics. Therefore it is good to list the basic problems and proposals for their solution.

The first problem concerns the correspondence between real and p-adic numbers.

1. The success of p-adic mass calculations involves the notions of p-adic probability, thermodynamics, and the mapping of p-adic probababilities to the real ones by a continuous correspondence $x = \sum x_n p^n \rightarrow Id(x) = \sum x_n p^{-n}$ that I have christened canonical identification. The problem is that $I$ n does not respect symmetries defined by isometries and also general coordinate invariance is possible only if one can identify preferred imbedding space coordinates. The reason is that $I$ does not commute with the basic arithmetic operations. $I$ allows several variants and it is possible to have correspondence which respects symmetries in arbitrary accuracy in preferred coordinates. Thus $I$ can play a role at space-time level only if one defines symmetries modulo measurement resolution. $I$ would make sense only in the interval defining the measurement resolution for a given coordinate variable and the p-adic effective topology would make sense just because the finite measurement resolution does not allow to well-order the points.

2. The identification of real and p-adic numbers via rationals common to all number fields - or more generally along algebraic extension of rationals- respects symmetries and algebra but is not continuous. At the imbedding space level preferred coordinates are required also now. The maximal symmetries of the imbedding space allow identification of this kind of coordinates. They are not unique. For instance, $M^4$ linear coordinates look very natural but for $CP_2$ trigonometric functions of angle like coordinates look more suitablel and Fourier analysis suggests strongly the introduction of algebraic extensions involving roots of unity. Partly the non-uniqueness has an interpretation as an imbedding space correlate for the selection of the quantization axes. The symmetric space property of WCW gives hopes that general coordinate invariance in quantal sense can be realized. The existence of p-adic harmonic analysis suggests a discretization of the p-adic variant of imbedding space and WCW based on roots of unity.

3. One can consider a compromise between the two correspondences. Discretization via common algebraic points can be completed to a p-adic continuum by assigning to each real discretization interval (say angle increment $2\pi/N$) p-adic numbers with norm smaller than one.

Second problem relates to integration and Fourier analysis. Both these procedures are fundamental for physics -be it classical or quantum. The p-adic variant of definite integral does not exist in the sense required by the action principles of physics although classical partial differential equations assigned to a particular variational principle make perfect sense. Fourier analysis is also possible only if one allows algebraic extension of p-adic numbers allowing a sufficient number of roots of unity correlating with the measurement resolution of angle. The finite number of them has interpretation in terms of finite angle resolution. Fourier analysis provides also an algebraic realization of definite integral when one integrates over the entire manifold as one indeed does in the case of WCW. If the space in question allows maximal symmetries as WCW and imbedding space do, there are excellent hopes of having p-adic variants of both integration and harmonic analysis and the above described procedure allows a precise completion of the discretized variant of real manifold to its continuous p-adic variant.

The third problem relates to the definitions of the p-adic variants of Riemannian, symplectic, and Kähler geometries. It is possible to generalize formally the notion of Riemann metric although non-local quantities like areas and total curvatures do not make sense if defined in terms of integrals. If all relevant quantities assignable to the geometry (family of Hamiltonians defining isometries, Killing vector fields, components of metric and Kähler form, Kähler function, etc...) are expressible in terms of rational functions involving only rational numbers as coefficients of polynomials, they allow an algebraic continuation to the p-adic context and the p-adic variant of the geometry makes sense.

The fourth problem relates to the question what one means with p-adic quantum mechanics. In TGD framework p-adic quantum theory utilizes p-adic Hilbert space. The motivation is that the notions of p-adic probability and unitarity are well defined. From the beginnning it was clear that the straightforward generalization of Schrödinger equation is not very interesting physically and gradually the conviction has developed that the most realistic approach must rely on the attempt to find the p-adic variant of the TGD inspired quantum physics in all its complexity. The recent approach starts from a rather concrete view about generalized Feynman diagrams defining the points of WCW and leads to a rather detailed view about what the p-adic variants of QM could be and how they could be fused with real QM to a larger structure. Even more, just the requirement that this p-adicization exists, gives very powerful constraints on the real variant of the quantum TGD.

The fifth problem relates to the notion of information in p-adic context. p-Adic thermodynamics leads naturally to the question what p-adic entropy might mean and this in turn leads to the realization that for rational or even algebraic probabilities p-adic variant of Shannon entropy can be negative and has minimum for a unique prime. One can say that the entanglement in the intersection of real and p-adic worlds is negentropic. This leads to rather fascinating vision about how negentropic entanglement makes it possible for living systems to overcome the second law of thermodynamics. The formulation of quantum theory in the intersection of real and living worlds becomes the basic challenge.

The proposed solutions to the technical problems could be rephrased in terms of the notion of algebraic universality. Various p-adic physics are obtained as algebraic continuation of real physics through the common algebraic points of real and p-adic worlds and by performing completion in the sense that the interval corresponding to finite measurement resolution are replaced with their p-adic counterpart via canonical identification. This allows to have exact symmetries as their discrete variants and also a continuous correspondence if desired. Particular p-adicization is characterized by a choice fo preferred imbedding space coordinates, which has interpretation in terms of a particular cognitive representation. Hence one is forced to refine the view about general coordinate invariance. Different coordinate choices correspond to different cognitive representations having delicate effects on physics if it is assumed to include also the effects of cognition.

## 1.2 Program

These ideas lead to a reasonably well defined p-adicization program. Try to define precisely the concepts of the p-adic space-time and configuration space (WCW), formulate the finite-p p-adic versions of quantum TGD. Try to fuse together real and various p-adic quantum TGDs are to form a full theory of physics and cognition.

The construction of the p-adic TGD necessitates the generalization of the basic tools of standard physics such as differential and integral calculus, the concept of Hilbert space, Riemannian geometry, group theory, action principles, and the notions of probability and unitarity to the p-adic context. Also new physical thinking and philosophy is needed. The notions of zero energy ontology, hierarchy of Planck constants and the generalization of the notion of imbedding space required by it are essential but not discussed in detail in this chapter.

## 1.3 Topics Of The Chapter

The topics of the chapter are the following:

1. p-Adic numbers, their extensions (also those involving transcendentals) are described. The existence of a square root of an ordinary p-adic number is necessary in many applications of the p-adic numbers (p-adic group theory, p-adic unitarity, Riemannian geometry) and its existence implies a unique algebraic extension, which is 4-dimensional for $p > 2$ and 8-dimensional for $p = 2$. Contrary to the first expectations, all possible algebraic extensions are possible and one cannot interpret the algebraic dimension of the algebraic extension as a physical dimension.

2. The concepts of the p-adic differentiability and analyticity are discussed. The notion of p-adic fractal is introduced the properties of the fractals defined by p-adically differentiable functions are discussed.

3. Various approaches to the problem of defining p-adic valued definite integral are discussed. The only reasonable generalizations rely on algebraic continuation and correspondence via common rationals. p-Adic field equations do not necessitate p-adic definite integral but algebraic continuation allows to assign to a given real space-time sheets a p-adic space-time sheets if the definition of space-time sheet involves algebraic relations between imbedding space coordinates. There are also hopes that one can algebraically continue the value of Kähler action to p-adic context if finite-dimensional extensions are allowed.

4. Symmetries are discussed from p-adic point of view starting from the identification via common rationals. Also possible p-adic generalizations of Fourier analysis are considered. Besides a number theoretical approach, group theoretical approach providing a direct generalization of the ordinary Fourier analysis based on the utilization of exponent functions existing in algebraic extensions containing some root of $e$ and its powers up to $e^{p-1}$ is discussed. Also the generalization of Fourier analysis based on the Pythagorean phases is considered.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [?].

# 2 Summary Of The Basic Physical Ideas

In the following various manners to end up with p-adic physics and with the idea about p-adic topology as an effective topology of space-time surface are described.

## 2.1 P-Adic Mass Calculations Briefly

p-Adic mass calculations based on p-adic thermodynamics with energy replaced with the generator $L_0 = zd/dz$ of infinitesimal scaling are described in the first part of [K16].

1. p-Adic thermodynamics is justified by the randomness of the motion of partonic 2-surfaces restricted only by the light-likeness of the orbit.

2. It is essential that the conformal symmetries associated with the light-like coordinates of parton and light-cone boundary are not gauge symmetries but dynamical symmetries. The point is that there are two kinds of conformal symmetries: the super-symplectic conformal symmetries assignable to the light-like boundaries of $CD \times CP_2$ and super Kac-Moody symmetries assignable to light-like 3-surfaces defining fundamental dynamical objects. In so called coset construction the differences of super-conformal generators of these algebras annihilate the physical states. This leads to a generalization of equivalence principle since one can assign four-momentum to the generators of both algebras identifiable as inertial *resp.* gravitational four-momentum. A second important consequence is that the generators of either algebra do not act like gauge transformations so that it makes sense to construct p-adic thermodynamics for them.

3. In p-adic thermodynamics scaling generator $L_0$ having conformal weights as its eigen values replaces energy and Boltzmann weight $exp(H/T)$ is replaced by $p^{L_0/T_p}$. The quantization $T_p = 1/n$ of conformal temperature and thus quantization of mass squared scale is implied by number theoretical existence of Boltzmann weights. p-Adic length scale hypothesis states that primes $p \simeq 2^k$, $k$ integer. A stronger hypothesis is that $k$ is prime (in particular Mersenne prime or Gaussian Mersenne) makes the model very predictive and fine tuning is not possible.

The basic mystery number of elementary particle physics defined by the ratio of Planck mass and proton mass follows thus from number theory once $CP_2$ radius is fixed to about $10^4$ Planck lengths. Mass scale becomes additional discrete variable of particle physics so that there is not more need to force top quark and neutrinos with mass scales differing by 12 orders of magnitude to the same multiplet of gauge group. Electron, muon, and $\tau$ correspond to Mersenne prime $k = 127$ (the largest non-super-astrophysical Mersenne), and Mersenne primes $k = 113, 107$. Intermediate gauge bosons and photon correspond to Mersenne $M_{89}$, and graviton to $M_{127}$.

Mersenne primes are very special also number theoretically because bit as the unit of information unit corresponds to $log(2)$ and can be said to exists for $M_n$-adic topology. The reason is that $log(1+p)$ existing always p-adically corresponds for $M_n = 2^n - 1$ to $log(2^n) \equiv nlog(2)$ so that one has $log(2 \equiv log(1+M_n)/n$. Since the powers of 2 modulo $p$ give all integers $n \in \{1, p-1\}$ by Fermat's theorem, one can say that the logarithms of all integers modulo $M_n$ exist in this sense and therefore the logarithms of all p-adic integers not divisible by $p$ exist. For other primes one must introduce a transcendental extension containing $log(a)$ where are is so called primitive root. One could criticize the identification since $log(1+M_n)$ corresponding in the real sense to $n$ bits corresponds in p-adic sense to to a very small information content since the p-adic norm of the p-adic bit is $1/M_n$.

The value of $k$ for quark can depend on hadronic environment [K9] and this would produce precise mass formulas for low energy hadrons. This kind of dependence conforms also with the indications that neutrino mass scale depends on environment [C1]. Amazingly, the biologically most relevant length scale range between 10 nm and 4 $\mu$m contains four Gaussian Mersennes $(1+i)^n - 1$, $n = 151, 157, 163, 167$ and scaled copies of standard model physics in cell length scale could be an essential aspect of macroscopic quantum coherence prevailing in cell length scale.

p-Adic mass thermodynamics is not quite enough: also Higgs boson is needed and wormhole contact carrying fermion and anti-fermion quantum numbers at the light-like wormhole throats is excellent candidate for Higgs [K7]. The coupling of Higgs to fermions can be small and induce only a small shift of fermion mass: this could explain why Higgs has not been observed. Also the Higgs contribution to mass squared can be understood thermodynamically if identified as absolute value for the thermal expectation value of the eigenvalues of the Kähler-Dirac operator having interpretation as complex square root of conformal weight.

The original belief was that only Higgs corresponds to wormhole contact. The assumption that fermion fields are free in the conformal field theory applying at parton level forces to identify all gauge bosons as wormhole contacts connecting positive and negative energy space-time sheets [K7]. Fermions correspond to topologically condensed $CP_2$ type extremals with single light-like wormhole throat. Gravitons are identified as string like structures involving pair of fermions or gauge bosons connected by a flux tube. Partonic 2-surfaces are characterized by genus which explains family replication phenomenon and an explanation for why their number is three emerges [K1]. Gauge bosons are labeled by pairs $(g_1, g_2)$ of handle numbers and can be arranged to octet and singlet representations of the resulting dynamical SU(3) symmetry. Ordinary gauge bosons are SU(3) singlets and the heaviness of octet bosons explains why higher boson families are effectively absent. The different character of bosons could also explain why the p-adic temperature for bosons is $T_p = 1/n < 1$ so that Higgs contribution to the mass dominates.

## 2.2 P-Adic Length Scale Hypothesis, Zero Energy Ontology, And Hierarchy Of Planck Constants

Zero energy ontology and the hierarchy of Planck constants realized in terms of the generalization of the imbedding space lead to a deeper understanding of the origin of the p-adic length scale hypothesis.

### 2.2.1 Zero energy ontology

In zero energy ontology one replaces positive energy states with zero energy states with positive and negative energy parts of the state at the light-like boundaries of CD. All conserved quantum numbers of the positive and negative energy states are of opposite sign so that these states can be created from vacuum. "Any physical state is creatable from vacuum" becomes thus a basic principle of quantum TGD and together with the notion of quantum jump resolves several philosophical problems (What was the initial state of universe?, What are the values of conserved quantities for Universe?, Is theory building completely useless if only single solution of field equations is realized?). At the level of elementary particle physics positive and negative energy parts of zero energy state are interpreted as initial and final states of a particle reaction so that quantum states become physical events.

### 2.2.2 Does the finiteness of measurement resolution dictate the laws of physics?

The hypothesis that the mere finiteness of measurement resolution could determine the laws of quantum physics [K2] completely belongs to the category of not at all obvious first principles. The basic observation is that the Clifford algebra spanned by the gamma matrices of the "world of classical worlds" represents a von Neumann algebra [A11] known as hyperfinite factor of type $II_1$ (HFF) [K2, K14, K5]. HFF [A6, A10] is an algebraic fractal having infinite hierarchy of included subalgebras isomorphic to the algebra itself [A1]. The structure of HFF is closely related to several notions of modern theoretical physics such as integrable statistical physical systems [A15], anyons [D1] , quantum groups and conformal field theories [A16], and knots and topological quantum field theories [A13, A7].

Zero energy ontology is second key element. In zero energy ontology these inclusions allow an interpretation in terms of a finite measurement resolution: in the standard positive energy ontology this interpretation is not possible. Inclusion hierarchy defines in a natural manner the notion of coupling constant evolution and p-adic length scale hypothesis follows as a prediction. In this framework the extremely heavy machinery of renormalized quantum field theory involving the elimination of infinities is replaced by a precisely defined mathematical framework. More concretely, the included algebra creates states which are equivalent in the measurement resolution used. Zero energy state can be modified in a time scale shorter than the time scale of the zero energy state itself.

One can imagine two kinds of measurement resolutions. The element of the included algebra can leave the quantum numbers of the positive and negative energy parts of the state invariant, which means that the action of subalgebra leaves M-matrix invariant. The action of the included algebra can also modify the quantum numbers of the positive and negative energy parts of the state such that the zero energy property is respected. In this case the Hermitian operators subalgebra must commute with $M$-matrix.

The temporal distance between the tips of CD corresponds to the secondary p-adic time scale $T_{p,2} = \sqrt{p}T_p$ by a simple argument based on the observation that light-like randomness of light-like 3-surface is analogous to Brownian motion. This gives the relationship $T_p = L_p^2/Rc$, where $R$ is $CP_2$ size. The action of the included algebra corresponds to an addition of zero energy parts to either positive or negative energy part of the state and is like addition of quantum fluctuation below the time scale of the measurement resolution. The natural hierarchy of time scales is obtained as $T_n = 2^{-n}T$ since these insertions must belong to either upper or lower half of the causal diamond. This implies that preferred p-adic primes are near powers of 2. For electron the time scale in question is.1 seconds defining the fundamental biorhythm of 10 Hz.

M-matrix representing a generalization of S-matrix and expressible as a product of a positive square root of the density matrix and unitary S-matrix would define the dynamics of quantum theory [K2]. The notion of thermodynamical state would cease to be a theoretical fiction and in a well-defined sense quantum theory could be regarded as a square root of thermodynamics. Connes tensor product [A6] provides a mathematical description of the finite measurement resolution but does not fix the $M$-matrix as was the original hope. The remaining challenge is the calculation of M-matrix and the progress induced by zero energy ontology during last years has led to rather concrete proposal for the construction of $M$-matrix.

### 2.2.3 How do p-adic coupling constant evolution and p-adic length scale hypothesis emerge?

In zero energy ontology zero energy states have as imbedding space correlates causal diamonds for which the distance between the tips of the intersecting future and past directed light-cones comes as integer multiples of a fundamental time scale: $T_n = n \times T_0$. p-Adic length scale hypothesis allows to consider a stronger hypothesis $T_n = 2^n T_0$ and its generalization a slightly more general hypothesis $T_n = p^n T_0$, $p$ prime. It however seems that these scales are dynamically favored but that also other scales are possible.

Could the coupling constant evolution in powers of 2 implying time scale hierarchy $T_n = 2^n T_0$ (or $T_p = pT_0$) induce p-adic coupling constant evolution and explain why p-adic length scales correspond to $L_p \propto \sqrt{p}R$, $p \simeq 2^k$, $R$ $CP_2$ length scale? This looks attractive but there is a problem. p-Adic length scales come as powers of $\sqrt{2}$ rather than 2 and the strongly favored values of $k$ are primes and thus odd so that $n = k/2$ would be half odd integer. This problem can be solved.

1. The observation that the distance traveled by a Brownian particle during time $t$ satisfies $r^2 = Dt$ suggests a solution to the problem. p-Adic thermodynamics applies because the partonic 3-surfaces $X^2$ are as 2-D dynamical systems random apart from light-likeness of their orbit. For $CP_2$ type vacuum extremals the situation reduces to that for a one-dimensional random light-like curve in $M^4$. The orbits of Brownian particle would now correspond to light-like geodesics $\gamma_3$ at $X^3$. The projection of $\gamma_3$ to a time=constant section $X^2 \subset X^3$ would define the 2-D path $\gamma_2$ of the Brownian particle. The $M^4$ distance $r$ between the end points of $\gamma_2$ would be given $r^2 = Dt$. The favored values of $t$ would correspond to $T_n = 2^n T_0$ (the full light-like geodesic). p-Adic length scales would result as $L^2(k) = DT(k) = D2^k T_0$ for $D = R^2/T_0$. Since only $CP_2$ scale is available as a fundamental scale, one would have $T_0 = R$ and $D = R$ and $L^2(k) = T(k)R$.

2. p-Adic primes near powers of 2 would be in preferred position. p-Adic time scale would not relate to the p-adic length scale via $T_p = L_p/c$ as assumed implicitly earlier but via $T_p = L_p^2/R_0 = \sqrt{p}L_p$, which corresponds to secondary p-adic length scale. For instance, in the case of electron with $p = M_{127}$ one would have $T_{127} = .1$ second which defines a fundamental biological rhythm. Neutrinos with mass around.1 eV would correspond to $L(169) \simeq 5$ $\mu$m (size of a small cell) and $T(169) \simeq 1. \times 10^4$ years. A deep connection between elementary particle physics and biology becomes highly suggestive.

3. In the proposed picture the p-adic prime $p \simeq 2^k$ would characterize the thermodynamics of the random motion of light-like geodesics of $X^3$ so that p-adic prime $p$ would indeed be an inherent property of $X^3$. For $T_p = pT_0$ the above argument is not enough for p-adic length scale hypothesis and p-adic length scale hypothesis might be seen as an outcome of a process analogous to natural selection. Resonance like effect favoring octaves of a fundamental frequency might be in question. In this case, $p$ would a property of CD and all light-like 3-surfaces inside it and also that corresponding sector of WCW .

### 2.2.4 Mersenne primes and Gaussian Mersennes

The generalization of the imbedding space required by the postulated hierarchy of Planck constants [K5] means a book like structure for which the pages are products of singular coverings or factor spaces of CD (causal diamond defined as intersection of future and past directed light-cones) and of $CP_2$ [K5]. This predicts that Planck constants are rationals and that a given value of Planck constant corresponds to an infinite number of different pages of the Big Book, which might be seen as a drawback. If only singular covering spaces are allowed the values of Planck constant are products of integers and given value of Planck constant corresponds to a finite number of pages given by the number of decompositions of the integer to two different integers. The definition of the book like structure assigns to a given CD preferred quantization axes and so that quantum measurement has direct correlate at the level of moduli space of CDs.

TGD inspired quantum biology and number theoretical considerations suggest preferred values for $r = \hbar/\hbar_0$. For the most general option the values of $\hbar$ are products and ratios of two integers $n_a$ and $n_b$. Ruler and compass integers defined by the products of distinct Fermat primes and power of two are number theoretically favored values for these integers because the phases $exp(i2\pi/n_i)$, $i \in \{a, b\}$, in this case are number theoretically very simple and should have emerged first in the number theoretical evolution via algebraic extensions of p-adics and of rationals. p-Adic length scale hypothesis favors powers of two as values of $r$.

One can however ask whether a more precise characterization of preferred Mersennes could exist and whether there could exists a stronger correlation between hierarchies of p-adic length scales and Planck constants. Mersenne primes $M_k = 2^k - 1$, $k \in \{89, 107, 127\}$, and Gaussian Mersennes $M_{G,k} = (1 + i)k - 1$, $k \in \{113, 151, 157, 163, 167, 239, 241..\}$ are expected to be physically highly interesting and up to $k = 127$ indeed correspond to elementary particles. The number theoretical miracle is that all the four p-adic length scales with $k \in \{151, 157, 163, 167\}$ are in the biologically highly interesting range 10 nm-2.5 $\mu$m). The question has been whether these define scaled up copies of electro-weak and QCD type physics with ordinary value of $\hbar$. The proposal that this is the case and that these physics are in a well-defined sense induced by the dark scaled up variants of corresponding lower level physics leads to a prediction for the preferred values of $r = 2^{k_d}$, $k_d = k_i - k_j$.

What induction means is that dark variant of exotic nuclear physics induces exotic physics with ordinary value of Planck constant in the new scale in a resonant manner: dark gauge bosons transform to their ordinary variants with the same Compton length. This transformation is natural since in length scales below the Compton length the gauge bosons behave as massless and free particles. As a consequence, lighter variants of weak bosons emerge and QCD confinement scale becomes longer.

This proposal will be referred to as Mersenne hypothesis. It leads to strong predictions about EEG [K3] since it predicts a spectrum of preferred Josephson frequencies for a given value of membrane potential and also assigns to a given value of $\hbar$ a fixed size scale having interpretation as the size scale of the body part or magnetic body. Also a vision about evolution of life emerges. Mersenne hypothesis is especially interesting as far as new physics in condensed matter length scales is considered: this includes exotic scaled up variants of the ordinary nuclear physics and their dark variants. Even dark nucleons are possible and this gives justification for the model of dark nucleons predicting the counterparts of DNA, RNA, tRNa, and amino-acids as well as realization of vertebrate genetic code [K13].

These exotic nuclear physics with ordinary value of Planck constant could correspond to ground states that are almost vacuum extremals corresponding to homologically trivial geodesic sphere of $CP_2$ near criticality to a phase transition changing Planck constant. Ordinary nuclear physics would correspond to homologically non-trivial geodesic sphere and far from vacuum extremal property. For vacuum extremals of this kind classical $Z^0$ field proportional to electromagnetic field is present and this modifies dramatically the view about cell membrane as Josephson junction. The model for cell membrane as almost vacuum extremal indeed led to a quantitative breakthrough in TGD inspired model of EEG and is therefore something to be taken seriously. The safest option concerning empirical facts is that the copies of electro-weak and color physics with ordinary value of Planck constant are possible only for almost vacuum extremals - that is at criticality against phase transition changing Planck constant.

## 2.3 P-Adic Physics And The Notion Of Finite Measurement Resolution

Canonical identification mapping p-adic numbers to reals in a continuous manner plays a key role in some applications of TGD and together with the discretization necessary to define the p-adic variants of integration and harmonic analysis suggests that p-adic topology identified as an effective topology could provide an elegant manner to characterize finite measurement resolution.

1. Finite measurement resolution can be characterized as an interval of minimum length. Below this length scale one cannot distinguish points from each other. A natural definition for this inability could be as an inability to well-order the points. The real topology is too strong in the modelling in kind of situation since it brings in large amount of processing of pseudo information whereas p-adic topology which lacks the notion of well-ordering could be more appropriate as effective topology and together with a pinary cutoff could allow to get rid of the irrelevant information.

2. This suggest that canonical identification applies only inside the intervals defining finite measurement resolution in a given discretization of the space considered by say small cubes. The canonical identification is unique only modulo diffeomorphism applied on both real and p-adic side but this is not a problem since this would only reflect the absence of the well-ordering lost by finite measurement resolution. Also the fact that the map makes sense only at positive real axis would be natural if one accepts this identification.

This interpretation would suggest that there is an infinite hierarchy of measurement resolutions characterized by the value of the p-adic prime. This would mean quite interesting refinement of the notion of finite measurement resolution. At the level of quantum theory it could be interpreted as a maximization of p-adic entanglement negentropy as a function of the p-adic prime. Perhaps one might say that there is a unique p-adic effective topology allowing to maximize the information content of the theory relying on finite measurement resolution.

## 2.4 P-Adic Numbers And The Analogy Of TGD With Spin-Glass

The vacuum degeneracy of the Kähler action leads to a precise spin glass analogy at the level of the WCW geometry and the generalization of the energy landscape concept to TGD context leads to the hypothesis about how p-adicity could be realized at the level of WCW . Also the concept of p-adic space-time surface emerges rather naturally.

### 2.4.1 Spin glass briefly

The basic characteristic of the spin glass phase [B1] is that the direction of the magnetization varies spatially, being constant inside a given spatial region, but does not depend on time. In the real context this usually leads to large surface energies on the surfaces at which the magnetization direction changes. Regions with different direction of magnetization clearly correspond non-vacuum regions separated by almost vacuum regions. Amusingly, if 3-space is effectively p-adic and if magnetization direction is p-adic pseudo constant, no surface energies are generated so that p-adics might be useful even in the context of the ordinary spin glasses.

Spin glass phase allows a great number of different ground states minimizing the free energy. For the ordinary spin glass, the partition function is the average over a probability distribution of the coupling constants for the partition function with Hamiltonian depending on the coupling constants. Free energy as a function of the coupling constants defines "energy landscape" and the set of free energy minima can be endowed with an ultra-metric distance function using a standard construction [A14].

### 2.4.2 Vacuum degeneracy of Kähler action

The Kähler action defining WCW geometry allows enormous vacuum degeneracy: any four-surface for which the induced Kähler form vanishes, is an extremal of the Kähler action. Induced Kähler form vanishes if the $CP_2$ projection of the space-time surface is Lagrange manifold of $CP_2$: these manifolds are at most two-dimensional and any canonical transformation of $CP_2$ creates a new Lagrange manifold. An explicit representation for Lagrange manifolds is obtained using some canonical coordinates $P_i, Q_i$ for $CP_2$: by assuming

$$P_i = \partial_i f(Q_1, Q_2) \ , \ i = 1, 2 \ ,$$

where $f$ arbitrary function of its arguments. One obtains a 2-dimensional sub-manifold of $CP_2$ for which the induced Kähler form proportional to $dP_i \wedge dQ^i$ vanishes. The roles of $P_i$ and $Q_i$ can obviously be interchanged. A familiar example of Lagrange manifolds are $p_i = constant$ surfaces of the ordinary $(p_i, q_i)$ phase space.

Since vacuum degeneracy is removed only by the classical gravitational interaction there are good reasons to expect large ground state degeneracy, when the system corresponds to a small deformation of a vacuum extremal. This degeneracy is very much analogous to the ground state degeneracy of spin glass but is 4-dimensional.

### 2.4.3 Vacuum degeneracy of the Kähler action and physical spin glass analogy

Quite generally, the dynamical reason for the physical spin glass degeneracy is the fact that Kähler action has a huge vacuum degeneracy. Any 4-surface with $CP_2$ projection, which is a Lagrangian sub-manifold (generically two-dimensional), is vacuum extremal. This implies that space-time decomposes into non-vacuum regions characterized by non-vanishing Kähler magnetic and electric fields such that the (presumably thin) regions between the non-vacuum regions are vacuum extremals. Therefore no surface energies are generated. Also the fact that various charges and momentum and energy can flow to larger space-time sheets via wormholes is an important factor making possible strong field gradients without introducing large surfaces energies. From a given preferred extremal of Kähler action one obtains a new one by adding arbitrary space-time surfaces which is vacuum extremal and deforming them.

The symplectic invariance of the Kähler action for vacuum extremals allows a further understanding of the vacuum degeneracy. The presence of the classical gravitational interaction spoils the canonical group $Can(CP_2)$ as gauge symmetries of the action and transforms it to the isometry group of $CH$. As a consequence, the $U(1)$ gauge degeneracy is transformed to a spin glass type degeneracy and several, perhaps even infinite number of maxima of Kähler function become possible. Given sheet has naturally as its boundary the 3-surfaces for which two maxima of the Kähler function coalesce or are created from single maximum by a cusp catastrophe [A8]. In catastrophe regions there are several sheets and the value of the maximum Kähler function determines which give a measure for the importance of various sheets. The quantum jumps selecting one of these sheets can be regarded as phase transitions.

In TGD framework classical non-determinism forces to generalize the notion of the 3-surface by replacing it with a sequence of space like 3-surfaces having time like separations such that the sequence characterizes uniquely one branch of multi-furcation. This characterization works when non-determinism has discrete nature. For $CP_2$ type extremals which are bosonic vacua, basic objects are essentially four-dimensional since $M^4_+$ projection of $CP_2$ type extremal is random light like curve. This effective four-dimensionality of the basic objects makes it possible to topologize Feynman diagrammatics of quantum field theories by replacing the lines of Feynman diagrams with $CP_2$ type extremals.

In TGD framework spin glass analogy holds true also in the time direction, which reflects the fact that the vacuum extremals are non-deterministic. For instance, by gluing vacuum extremals with a finite space-time extension (also in time direction!) to a non-vacuum extremal and deforming slightly, one obtains good candidates for the degenerate preferred extremals. This non-determinism is expected to make the preferred extremals of the Kähler action highly degenerate. The construction of S-matrix at the high energy limit suggests that since a localization selecting one degenerate maximum occurs, one must accept as a fact that each choice of the parameters corresponds to a particular S-matrix and one must average over these choices to get scattering rates. This averaging for scattering rates corresponds to the averaging over the thermodynamical partition functions for spin glass. A more general is that one allows final state wave functions to depend on the zero modes which affect S-matrix elements: in the limit that wave functions are completely localized, one ends up with the simpler scenario.

### 2.4.4 p-Adic non-determinism and spin glass analogy

One must carefully distinguish between cognitive and physical spin-glass analogy. Cognitive spin-glass analogy is due to the p-adic non-determinism. p-Adic pseudo constants induce a non-determinism which essentially means that p-adic extrema depend on the p-adic pseudo constants which depend on a finite number of positive pinary digits of their arguments only. Thus p-adic extremals are glued from pieces for which the values of the integration constants are genuine constants. Obviously, an optimal cognitive representation is achieved if pseudo constants reduce to ordinary constants.

More precisely, any function

$$\begin{aligned} f(x) &= f(x_N) \ , \\ x_N &= \sum_{k\leq N} x_k p^k \ , \end{aligned} \tag{2.1}$$

which does not depend on the pinary digits $x_n$, $n > N$ has a vanishing p-adic derivative and is thus a pseudo constant. These functions are piecewise constant below some length scale, which in principle can be arbitrary small but finite. The result means that the constants appearing in the solutions the p-adic field equations are constants functions only below some length scale. For instance, for linear differential equations integration constants are arbitrary pseudo constants. In particular, the p-adic counterparts of the preferred extremals are highly degenerate because of the presence of the pseudo constants. This in turn means a characteristic randomness of the spin glass also in the time direction since the surfaces at which the pseudo constants change their values do not give rise to infinite surface energy densities as they would do in the real context.

The basic character of cognition would be spin glass like nature making possible "engineering" at the level of thoughts (planning) whereas classical non-determinism of the Kähler action would make possible "engineering" at the level of the real world.

## 2.5 Life As Islands Of Rational/Algebraic Numbers In The Seas Of Real And P-Adic Continua?

NMP and negentropic entanglement demanding entanglement probabilities which are equal to inverse of integer, is the starting point. Rational and even algebraic entanglement coefficients make sense in the intersection of real and p-adic words, which suggests that in some sense life and conscious intelligence reside in the intersection of the real and p-adic worlds.

What could be this intersection of realities and p-adicities?

1. The facts that fermionic oscillator operators are correlates for Boolean cognition and that induced spinor fields are restricted to string world sheets and partonic 2-surfaces suggests that the intersection consists of these 2-surfaces.

2. Strong form of holography allows a rather elegant adelization of TGD by a construction of space-time surfaces by algebraic continuations of these 2-surfaces defined by parameters in algebraic extension of rationals inducing that for various p-adic number fields to real or p-adic number fields. Scattering amplitudes could be defined also by a similar algebraic contination. By conformal invariance the conformal moduli characterizing the 2-surfaces would defined the parameters.

This suggests a rather concrete view about the fundamental quantum correlates of life and intelligence.

1. For the minimal option life would be effectively 2-dimensional phenomenon and essentially a boundary phenomenon as also number theoretical criticality suggests. There are good reasons to expect that only the data from the intersection of real and p-adic string world sheets partonic two-surfaces appears in $U$-matrix so that the data localizable to strings connecting partonic 2-surfaces would dictate the scattering amplitudes.

A good guess is that algebraic entanglement is essential for quantum computation, which therefore might correspond to a conscious process. Hence cognition could be seen as a quantum computation like process, a more appropriate term being quantum problem solving [K4]. Living-dead dichotomy could correspond to rational-irrational or to algebraic-transcendental dichotomy: this at least when life is interpreted as intelligent life. Life would in a well defined sense correspond to islands of rationality/algebraicity in the seas of real and p-adic continua. Life as a critical phenomenon in the number theoretical sense would be one aspect of quantum criticality of TGD Universe besides the criticality of the space-time dynamics and the criticality with respect to phase transitions changing the value of Planck constant and other more familiar criticalities. How closely these criticalities relate remains an open question [K10].

The view about the crucial role of rational and algebraic numbers as far as intelligent life is considered, could have been guessed on very general grounds from the analogy with the orbits of a dynamical system. Rational numbers allow a predictable periodic decimal/pinary expansion and are analogous to one-dimensional periodic orbits. Algebraic numbers are related to rationals by a finite number of algebraic operations and are intermediate between periodic and chaotic orbits allowing an interpretation as an element in an algebraic extension of any p-adic number field. The projections of the orbit to various coordinate directions of the algebraic extension represent now periodic orbits. The decimal/pinary expansions of transcendentals are un-predictable being analogous to chaotic orbits. The special role of rational and algebraic numbers was realized already by Pythagoras, and the fact that the ratios for the frequencies of the musical scale are rationals supports the special nature of rational and algebraic numbers. The special nature of the Golden Mean, which involves $\sqrt{5}$, conforms the view that algebraic numbers rather than only rationals are essential for life.

Later progress in understanding of quantum TGD allows to refine and simplify this view dramatically. The idea about p-adic-to-real transition for space-time sheets as a correlate for the transformation of intention to action has turned out to be un-necessary and also hard to realize mathematically. In adelic vision real and p-adic numbers are aspects of existence in all length scales and mean that cognition is present at all levels rather than emerging. Intentions have interpretation in terms of state function reductions in ZEO and there is no need to identify p-adic space-time sheets as their correlates.

## 2.6 P-Adic Physics As Physics Of Cognition

The vision about p-adic physics as physics of cognition has gradually established itself as one of the key idea of TGD inspired theory of consciousness. There are several motivations for this idea.

The strongest motivation is the vision about living matter as something residing in the intersection of real and p-adic worlds. One of the earliest motivations was p-adic non-determinism identified tentatively as a space-time correlate for the non-determinism of imagination. p-Adic non-determinism follows from the fact that functions with vanishing derivatives are piecewise constant functions in the p-adic context. More precisely, p-adic pseudo constants depend on the pinary cutoff of their arguments and replace integration constants in p-adic differential equations. In the case of field equations this means roughly that the initial data are replaced with initial data given for a discrete set of time values chosen in such a manner that unique solution of field equations results. Solution can be fixed also in a discrete subset of rational points of the imbedding space. Presumably the uniqueness requirement implies some unique pinary cutoff. Thus the space-time surfaces representing solutions of p-adic field equations are analogous to space-time surfaces consisting of pieces of solutions of the real field equations. p-Adic reality is much like the dream reality consisting of rational fragments glued together in illogical manner or pieces of child's drawing of body containing body parts in more or less chaotic order.

The obvious interpretation for the solutions of the p-adic field equations is as a geometric correlate of imagination. Plans, intentions, expectations, dreams, and cognition in general are expected to have p-adic cognitive space-time sheets as their geometric correlates. A deep principle seems to be involved: incompleteness is characteristic feature of p-adic physics but the flexibility made possible by this incompleteness is absolutely essential for imagination and cognitive consciousness in general.

If one accepts the idea that real and p-adic space-time regions are correlates for matter and cognitive mind, one encounters the question how matter and mind interact. The original candidate for this interaction was as a phase transition leading to a transformation of the real space-time regions to p-adic ones and vice versa. These transformations would take place in quantum jumps. p-Adic-to-real phase transition would have interpretation as a transformation of thought into a sensory experience (dream or hallucination) or to an action. The reverse phase transition might relate to the transformation of the sensory experience to cognition. Sensory experiences could be also transformed to cognition by initial values realized as common rational points of a real space-time sheet representing sensory input and a p-adic space-time sheet representing the cognitive output. In this case the cognitive mental image is unique only in case that p-adic pseudo constants are ordinary constants.

It turned out that this interpretation leads to grave mathematical difficulties: one should construct U-matrix and M-matrix for transitions between different number fields, and this makes sense only if all the parameters involved are rational or algebraic. A more realistic view is that the interaction between real and p-adic number fields is that p-adic space-time surfaces define cognitive representations of real space-time surfaces (preferred extremals). One could also say that real space-time surface represents sensory aspects of conscious experience and p-adic space-time surfaces its cognitive aspects. Both real and p-adics rather than real or p-adics. The notion of p-adic manifold [K17] tries to catch this idea mathematically.

Strong form of holography implied by strong form of General Coordinate Invariance leads to the suggestion that partonic 2-surfaces and string world sheets at which the induced spinor fields are localized in order to have a well-defined em charge (this is only one of the reasons) and having having discrete set as intersection points with partonic 2-surfaces define what might called "space-time genes". Space-time surfaces would be obtained as preferred extremals satisfying certain boundary conditions at string world sheets. Space-time surfaces are defined only modulo transformations of super-symplectic algebra defining its sub-algebra and acting as conformal gauge transformations so that one can talk about conformal gauge equivalences classes of space-time surfaces.

The map assigning to real space-time surface cognitive representation would be replaced by a correspondence assigning to the string world sheets preferred extremals of Kähler action in various number fields: string world sheets would be indeed like genes. Mathematically this formulation is much more elegant that that based on p-adic manifold since discretization seems to be un-necessary at space-time level and applies only to the parameters characterizing string world sheet.

String world sheets and partonic 2-surfaces would be in the intersection of realities and p-adicities in the sense that the parameters characterizing them would be algebraic numbers associated with the algebraic extension of p-adic numbers in question. It is not clear whether the preferred extremal is possible for all p-adic primes but this would fit nicely with the vision that elementary particles are characterized by p-adic primes. It could be also that the classical non-determinism of Kähler action responsible for the conformal gauge symmetry corresponds to p-adic non-determinism for some particular prime so that the cognitive map is especially good for this prime.

The idea about p-adic pseudo constants as correlates of imagination is however too nice to be thrown away without trying to find an alternative interpretation consistent with strong form of holography. Could the following argument allow to save p-adic view about imagination in a mathematically respectable manner?

1. The construction of preferred extremals from data at 2-surfaces is like boundary value problem. Integration constants are replaced with pseudo-constants depending on finite number pinary digits of variables depending on coordinates normal to string world sheets and partonic 2-surfaces.

2. Preferred extremal property in real context implies strong correlations between string world sheets and partonic 2-surfaces by boundary conditions a them. One cannot choose these 2- surfaces completely independently. Pseudo-constant could allow a large number of p-adic configurations involving string world sheets and partonic 2-surfaces not allowed in real context and realizing imagination.

3. Could imagination be realized as a larger size of the p-adic sectors of WCW? Could the realizable intentional actions belong to the intersection of real and p-adic WCWs? Could the modes of WCW spinor fields for which 2-surfaces are extendable to space-time surfaces only in some p-adic sectors make sense? The real space-time surface for them be somehow degenerate, for instance, consisting of string world sheets only.

   Could imagination be search for those collections of string world sheets and partonic 2-surfaces, which allow extension to (realization as) real preferred extremals? p-Adic physics would be there as an independent aspect of existence and this is just the original idea. Imagination could be realized in state function reduction, which always selects only those 2-surfaces which allow continuation to real space-time surfaces. The distinction between only imaginable and also realizable would be the extendability by using strong form of holography.

I have the feeling that this view allows respectable mathematical realization of imagination in terms of adelic quantum physics. It is remarkable that strong form of holography derivable from - you can guess, strong form of General Coordinate Invariance (the Big E again!), plays an absolutely central role in it.

## 2.7 P-Adic Numbers

### 2.7.1 *Basic properties of p-adic numbers*

p-Adic numbers ($p$ is prime: 2,3,5,... ) can be regarded as a completion of the rational numbers using a norm, which is different from the ordinary norm of real numbers [A4] . p-Adic numbers are representable as power expansion of the prime number $p$ of form:

$$x = \sum_{k\geq k_0} x(k)p^k ,\ x(k) = 0, ...., p-1 \ . \tag{2.2}$$

The norm of a p-adic number is given by

$$|x| = p^{-k_0(x)} \ . \tag{2.3}$$

Here $k_0(x)$ is the lowest power in the expansion of the p-adic number. The norm differs drastically from the norm of the ordinary real numbers since it depends on the lowest pinary digit of the p-adic number only. Arbitrarily high powers in the expansion are possible since the norm of the p-adic number is finite also for numbers, which are infinite with respect to the ordinary norm. A convenient representation for p-adic numbers is in the form

$$x = p^{k_0}\varepsilon(x) \ , \tag{2.4}$$

where $\varepsilon(x) = k + ....$ with $0 < k < p$, is p-adic number with unit norm and analogous to the phase factor $exp(i\phi)$ of a complex number.

The distance function $d(x,y) = |x-y|_p$ defined by the p-adic norm possesses a very general property called ultra-metricity:

$$d(x,z) \leq max\{d(x,y), d(y,z)\} \ . \tag{2.5}$$

The properties of the distance function make it possible to decompose $R_p$ into a union of disjoint sets using the criterion that $x$ and $y$ belong to same class if the distance between $x$ and $y$ satisfies the condition

$$d(x,y) \leq D \ . \tag{2.6}$$

This division of the metric space into classes has following properties:

1. Distances between the members of two different classes $X$ and $Y$ do not depend on the choice of points $x$ and $y$ inside classes. One can therefore speak about distance function between classes.
2. Distances of points $x$ and $y$ inside single class are smaller than distances between different classes.
3. Classes form a hierarchical tree.

Notice that the concept of the ultra-metricity emerged in physics from the models for spin glasses and is believed to have also applications in biology [B2] . The emergence of p-adic topology as the topology of the effective space-time would make ultra-metricity property basic feature of physics.

### 2.7.2 *Extensions of p-adic numbers*

Algebraic democracy suggests that all possible real algebraic extensions of the p-adic numbers are possible. This conclusion is also suggested by various physical requirements, say the fact that the eigenvalues of a Hamiltonian representable as a rational or p-adic $N \times N$-matrix, being roots of N:th order polynomial equation, in general belong to an algebraic extension of rationals or p-adics. The dimension of the algebraic extension cannot be interpreted as physical dimension. Algebraic extensions are characteristic for cognitive physics and provide a new manner to code information. A possible interpretation for the algebraic dimension is as a dimension for a cognitive representation of space and might explain how it is possible to mathematically imagine spaces with all possible dimensions although physical space-time dimension is four. The idea of algebraic hologram and other ideas related to the physical interpretation of the algebraic extensions of p-adic numbers are discussed in [K12] .

It seems however that algebraic democracy must be extended to include also transcendentals in the sense that finite-dimensional extensions involving also transcendental numbers are possible: for instance, Neper number $e$ defines a $p$-dimensional extension. It has become clear that these extensions fundamental for understanding how p-adic physics as physics of cognition is able to mimic real physics. The evolution of mathematical cognition can be seen as a process in which p-adic space-time sheets involving increasing value of p-adic prime $p$ and increasing dimension of algebraic extension appear in quantum jumps.

1. *Recipe for constructing algebraic extensions*

Real numbers allow only complex numbers as an algebraic extension. For p-adic numbers algebraic extensions of arbitrary dimension are possible
[A4] . The simplest manner to construct (n+1)-dimensional extensions is to consider irreducible polynomials $P_n(t)$ in $R_p$ assumed to have rational coefficients: irreducibility means that the polynomial does not possess roots in $R_p$ so that one cannot decompose it into a product of lower order $R_p$ valued polynomials. This condition is equivalent with the condition with irreducibility in the finite field $G(p,1)$, that is modulo $p$ in $R_p$.

Denoting one of the roots of $P_n(t)$ by $\theta$ and defining $\theta^0 = 1$ the general form of the extension is given by

$$Z = \sum_{k=0,..,n-1} x_k\theta^k \ . \tag{2.7}$$

Since $\theta$ is root of the polynomial in $R_p$ it follows that $\theta^n$ is expressible as a sum of lower powers of $\theta$ so that these numbers indeed form an n-dimensional linear space with respect to the p-adic topology.

Especially simple odd-dimensional extensions are cyclic extensions obtained by considering the roots of the polynomial

$$\begin{aligned} P_n(t) &= t^n + \epsilon d \ , \\ \epsilon &= \pm 1 \ . \end{aligned} \tag{2.8}$$

For $n = 2m+1$ and $(n = 2m, \epsilon = +1)$ the irreducibility of $P_n(t)$ is guaranteed if $d$ does not possess $n$:th root in $R_p$. For $(n = 2m, \epsilon = -1)$ one must assume that $d^{1/2}$ does not exist p-adically. In this case $\theta$ is one of the roots of the equation

$$t^n = \pm d \ , \tag{2.9}$$

where $d$ is a p-adic integer with a finite number of pinary digits. It is possible although not necessary to identify the roots as complex numbers. There exists $n$ complex roots of $d$ and $\theta$ can be chosen to be one of the real or complex roots satisfying the condition $\theta^n = \pm d$. The roots can be written in the general form

$$\begin{aligned} \theta &= d^{1/n} exp(i\phi(m)), \ m = 0,1,....,n-1 \ , \\ \phi(m) &= \frac{m2\pi}{n} \text{ or } \frac{m\pi}{n} \ . \end{aligned} \tag{2.10}$$

Here $d^{1/n}$ denotes the real root of the equation $\theta^n = d$. Each of the phase factors $\phi(m)$ gives rise to an algebraically equivalent extension: only the representation is different. Physically these extensions need not be equivalent since the identification of the algebraically extended p-adic numbers with the complex numbers plays a fundamental role in the applications. The cases $\theta^n = \pm d$ are physically and mathematically quite different.

*2. p-Adic valued norm for numbers in algebraic extension*

The p-adic valued norm of an algebraically extended p-adic number $x$ can be defined as some power of the ordinary p-adic norm of the determinant of the linear map $x :^e R_p^n \rightarrow^e R_p^n$ defined by the multiplication with $x$: $y \rightarrow xy$

$$N(x) = det(x)^{\alpha} \ , \ \alpha > 0 \ . \tag{2.11}$$

Real valued norm can be defined as the p-adic norm of $N(x)$. The requirement that the norm is homogenous function of degree one in the components of the algebraically extended 2-adic number (like also the standard norm of $R^n$ ) implies the condition $\alpha = 1/n$, where $n$ is the dimension of the algebraic extension.

The canonical correspondence between the points of $R_+$ and $R_p$ generalizes in obvious manner: the point $\sum_k x_k\theta^k$ of algebraic extension is identified as the point $(x_R^0, x_R^1, ..., x_R^k, .., )$ of $R^n$ using the pinary expansions of the components of p-adic number. The p-adic linear structure of the algebraic extension induces a linear structure in $R_+^n$ and p-adic multiplication induces a multiplication for the vectors of $R_+^n$.

*3. Algebraic extension allowing square root of ordinary p-adic numbers*

The existence of a square root of an ordinary p-adic number is a common theme in various applications of the p-adic numbers and for long time I erratically believed that only this extension is involved with p-adic physics. Despite this square root allowing extension is of central importance and deserves a more detailed discussion.

1. The p-adic generalization of the representation theory of the ordinary groups and Super Kac Moody and Super Virasoro algebras exists provided an extension of the p-adic numbers allowing square roots of the "real" p-adic numbers is used. The reason is that the matrix elements of the raising and lowering operators in Lie-algebras as well as of oscillator operators typically involve square roots. The existence of square root might play a key role in various p-adic considerations.

2. The existence of a square root of a real p-adic number is also a necessary ingredient in the definition of the p-adic unitarity and probability concepts since the solution of the requirement that $p_{mn} = S_{mn}\bar{S}_{mn}$ is ordinary p-adic number leads to expressions involving square roots.

3. p-Adic length scales hypothesis states that the p-adic length scale is proportional to the square root of p-adic prime.

4. Simple metric geometry of polygons involves square roots basically via the theorem of Pythagoras. p-Adic Riemannian geometry necessitates the existence of square root since the definition of the infinitesimal length $ds$ involves square root. Note however that p-adic Riemannian geometry can be formulated as a mere differential geometry without any reference to global concepts like lengths, areas, or volumes.

The original belief that square root allowing extensions of p-adic numbers are exceptional seems to be wrong in light of TGD as a generalized number theory vision. All algebraic extensions of p-adic numbers a possible and the interpretation of algebraic dimension of the extension as a physical dimension is not the correct thing to do. Rather, the possibility of arbitrarily high algebraic dimension reflects the ability of mathematical cognition to imagine higher-dimensional spaces. Square root allowing extension of the p-adic numbers is the simplest one imaginable, and it is fascinating that it indeed is the dimension of space-time surface for $p > 2$ and dimension of imbedding space for $p = 2$. Thus the square root allowing extensions deserve to be discussed.

The results can be summarized as follows.

1. In $p > 2$ case the general form of extension is

$$Z = (x + \theta y) + \sqrt{p}(u + \theta v) \ , \tag{2.12}$$

where the condition $\theta^2 = x$ for some p-adic number $x$ not allowing square root as a p-adic number. For $p \ mod \ 4 = 3$ $\theta$ can be taken to be imaginary unit. This extension is natural for p-adication of space-time surface so that space-time can be regarded as a number field locally. Imbedding space can be regarded as a cartesian product of two 4-dimensional extensions locally.

2. In $p = 2$ case 8-dimensional extension is needed to define square roots. The extension is defined by adding $\theta_1 = \sqrt{-1} \equiv i$, $\theta_2 = \sqrt{2}$, $\theta_3 = \sqrt{3}$ and the products of these so that the extension can be written in the form

$$Z = x_0 + \sum_k x_k\theta_k + \sum_{k<l} x_{kl}\theta_{kl} + x_{123}\theta_1\theta_2\theta_3 \quad . \tag{2.13}$$

Clearly, $p = 2$ case is exceptional as far as the construction of the conformal field theory limit is considered since the structure of the representations of Virasoro algebra and groups in general changes drastically in $p = 2$ case. The result suggest that in $p = 2$ limit space-time surface and $H$ are in same relation as real numbers and complex numbers: space-time surfaces defined as the absolute minima of 2-adiced Kähler action are perhaps identifiable as surfaces for which the imaginary part of 2-adically analytic function in $H$ vanishes.

The physically interesting feature of p-adic group representations is that if one doesn't use $\sqrt{p}$ in the extension the number of allowed spins for representations of $SU(2)$ is finite: only spins $j < p$ are allowed. In $p = 3$ case just the spins $j \leq 2$ are possible. If 4-dimensional extension is used for $p = 2$ rather than 8-dimensional then one gets the same restriction for allowed spins.

*4. Is e an exceptional transcendental?*

One can consider also the possibility of transcental extensions of p-adic numbers and an open problem is whether the infinite-dimensional extensions involving powers of $\pi$ and logarithms of primes make sense and whether they should be allowed. For instance, it is not clear whether the allowance of powers of $\pi$ is consistent with the extensions based on roots of unity. This question is not academic since Feynman amplitudes in real context involve powers of $\pi$ and algebraic universality forces the consider that also they p-adic variants might involve powers of $\pi$.

Neper number obviously defines the simplest transcendental extension since only the powers $e^k$, $k = 1, ..., p-1$ of $e$ are needed to define p-adic counterpart of $e^x$ for $x = n$ so that the extension is finite-dimensional. In the case of trigonometric functions deriving from $e^{ix}$, also $e^i$ and its $p-1$ powers must belong to the extension.

An interesting question is whether $e$ is a number theoretically exceptional transcendental or whether it could be easy to find also other transcendentals defining finite-dimensional extensions of p-adic numbers.

1. Consider functions $f(x)$, which are analytic functions with rational Taylor coefficients, when expanded around origin for $x > 0$. The values of $f(n)$, $n = 1, ..., p-1$ should belong to an extension, which should be finite-dimensional.

2. The expansion of these functions to Taylor series generalizes to the p-adic context if also the higher derivatives of $f$ at $x = n$ belong to the extension. This is achieved if the higher derivatives are expressible in terms of the lower derivatives using rational coefficients and rational functions or functions, which are defined at integer points (such as exponential and logarithm) by construction. A differential equation of some finite order involving only rational functions with rational coefficients must therefore be satisfied ($e^x$ satisfying the differential equation $df/dx = f$ is the optimal case in this sense). The higher derivatives could also reduce to rational functions at some step ($log(x)$ satisfying the differential equation $df/dx = 1/x$).

3. The differential equation allows to develop $f(x)$ in power series, say in origin

   $$f(x) = \sum f_n \frac{x^n}{n!}$$

   such that $f_{n+m}$ is expressible as a rational function of the $m$ lower derivatives and is therefore a rational number.

   The series converges when the p-adic norm of $x$ satisfies $|x|_p \leq p^k$ for some $k$. For definiteness one can assume $k = 1$. For $x = 1, ..., p-1$ the series does not converge in this case, and one can introduce and extension containing the values $f(k)$ and hope that a finite-dimensional extension results.

Finite-dimensionality requires that the values are related to each other algebraically although they need not be algebraic numbers. This means symmetry. In the case of exponent function this relationship is exceptionally simple. The algebraic relationship reflects the fact that exponential map represents translation and exponent function is an eigen function of a translation operator. The necessary presence of symmetry might mean that the situation reduces always to either exponential action. Also the phase factors $exp(iq\pi)$ could be interpreted in terms of exponential symmetry. Hence the reason for the exceptional role of exponent function reduces to group theory.

Also other extensions than those defined by roots of $e$ are possible. Any polynomial has $n$ roots and for transcendental coefficients the roots define a finite-dimensional extension of rationals. It would seem that one could allow the coefficients of the polynomial to be functions in an extension of rationals by powers of a root of $e$ and algebraic numbers so that one would obtain infinite hierarchy of transcendental extensions.

#### 2.7.3 *p-Adic Numbers and finite fields*

Finite fields (Galois fields) consists of finite number of elements and allow sum, multiplication and division. A convenient representation for the elements of a finite field is as the roots of the polynomial equation $t^{p^m} - t = 0 \ mod \ p$ , where $p$ is prime, $m$ an arbitrary integer and $t$ is element of a field of characteristic $p$ ($pt = 0$ for each $t$). The number of elements in a finite field is $p^m$, that is power of prime number and the multiplicative group of a finite field is group of order $p^m - 1$. $G(p, 1)$ is just cyclic group $Z_p$ with respect to addition and $G(p, m)$ is in rough sense $m$:th Cartesian power of $G(p, 1)$ .

The elements of the finite field $G(p, 1)$ can be identified as the p-adic numbers $0, ..., p - 1$ with p-adic arithmetics replaced with modulo $p$ arithmetics. The finite fields $G(p, m)$ can be obtained from m-dimensional algebraic extensions of the p-adic numbers by replacing the p-adic arithmetics with the modulo p arithmetics. In TGD context only the finite fields $G(p > 2, 2)$ , $p \ mod \ 4 = 3$ and $G(p = 2, 4)$ appear naturally. For $p > 2$, $p \ mod \ 4 = 3$ one has: $x+iy+\sqrt{p}(u+iv) \rightarrow x_0+iy_0 \in G(p, 2)$.

An interesting observation is that the unitary representations of the p-adic scalings $x \rightarrow p^k x$ $k \in Z$ lead naturally to finite field structures. These representations reduce to representations of a finite cyclic group $Z_m$ if $x \rightarrow p^m x$ acts trivially on representation functions for some value of $m$, $m = 1, 2, ...$ Representation functions, or equivalently the scaling momenta $k = 0, 1, ..., m - 1$ labelling them, have a structure of cyclic group. If $m \neq p$ is prime the scaling momenta form finite field $G(m, 1) = Z_m$ with respect to the summation and multiplication modulo $m$. Also the p-adic counterparts of the ordinary plane waves carrying p-adic momenta $k = 0, 1..., p - 1$ can be given the structure of Finite Field $G(p, 1)$: one can also define complexified plane waves as square roots of the real p-adic plane waves to obtain Finite Field $G(p, 2)$.

# 3 What Is The Correspondence Between P-Adic And Real Numbers?

There must be some kind of correspondence between reals and p-adic numbers. This correspondence can depend on context. In p-adic mass calculations one must map p-adic mass squared values to real numbers in a continuous manner and canonical identification is a natural guess. Presumably also p-adic probabilities should be mapped to their real counterparts. One can wonder whether p-adic valued S-matrix has any physical meaning or whether one should assume that the elements of S-matrix are algebraic numbers allowing interpretation as real or p-adic numbers: this would pose extremely strong constraints on S-matrix. If one wants to introduce p-adic physics at space-time level one must be able to relate p-adic and real space-time regions to each other and the identification along common rational points of real and various p-adic variants of the imbedding space suggests itself here.

## 3.1 Generalization Of The Number Concept

The recent view about the unification of real and p-adic physics is based on the generalization of number concept obtained by fusing together real and p-adic number fields along common rationals (see **Fig.** `http://tgdtheory.fi/appfigures/book.jpg` or **Fig. ??** in the appendix of this book).

### 3.1.1 Rational numbers as numbers common to all number fields

The unification of real physics of material work and p-adic physics of cognition and intentionality leads to the generalization of the notion of number field. Reals and various p-adic number fields are glued along their common rationals (and common algebraic numbers appearing in the extension of p-adic numbers too) to form a fractal book like structure. Allowing all possible finite-dimensional algebraic and perhaps even transcendental extensions of p-adic numbers adds additional pages to this "Big Book".

This leads to a generalization of the notion of manifold as a collection of a real manifold and its p-adic variants glued together along common points. The outcome of experimentation is that this generalization makes sense under very high symmetries and that it is safest to lean strongly on the physical picture provided by quantum TGD.

1. The most natural guess is that the coordinates of common points are rational or in some algebraic extension of rational numbers. General coordinate invariance and preservation of symmetries require preferred coordinates existing when the manifold has maximal number of isometries. This approach is especially natural in the case of linear spaces- in particular Minkowski space $M^4$. The natural coordinates are in this case linear Minkowski coordinates. The choice of coordinates is not completely unique and has interpretation as a geometric correlate for the choice of quantization axes for a given CD.

2. As will be found, the need to have a well-defined integration based on Fourier analysis (or its generalization to harmonic analysis in symmetric spaces) poses very strong constraints and allows p-adicization only if the space has maximal symmetries. Fourier analysis requires the introduction of an algebraic extension of p-adic numbers containing sufficiently many roots of unity.

   (a) This approach is especially natural in the case of compact symmetric spaces such as $CP_2$.

   (b) Also symmetric spaces such the 3-D proper time $a = constant$ hyperboloid of $M^4$- call it $H(a)$ -allowing Lorentz group as isometries allows a p-adic variant utilizing the hyperbolic counterparts for the roots of unity. $M^4 \times H(a = 2^n a_0)$ appears as a part of the moduli space of CDs.

   (c) For light-cone boundaries associated with CDs $SO(3)$ invariant radial coordinate $r_M$ defining the radius of sphere $S^2$ defines the hyperbolic coordinate and angle coordinates of $S^2$ would correspond to phase angles and $M^4_\pm$ projections for the common points of real and p-adic variants of partonic 2-surfaces would be this kind of points. Same applies to $CP_2$ projections. In the "intersection of real and p-adic worlds" real and p-adic partonic 2-surfaces would obey same algebraic equations and would be obtained by an algebraic continuation from the corresponding equations making sense in the discrete variant of $M^4_\pm \times CP_2$. This connection with discrete sub-manifolds geometries means very powerful constraints on the partonic 2-surfaces in the intersection.

3. The common algebraic points of real and p-adic variant of the manifold form a discrete space but one could identify the p-adic counterpart of the real discretization intervals $(0, 2\pi/N)$ for angle like variables as p-adic numbers of norm smaller than 1 using canonical identification or some variant of it. Same applies to the the hyperbolic counterpart of this interval. The non-uniqueness of this map could be interpreted in terms of a finite measurement resolution. In particular, the condition that WCW allows Kähler geometry requires a decomposition to a union of symmetric spaces so that there are good hopes that p-adic counterpart is analogous to that assigned to $CP_2$.

The idea about astrophysical size of the p-adic cognitive space-time sheets providing representation of body and brain is consistent with TGD inspired theory of consciousness, which forces to take very seriously the idea that even human consciousness involves astrophysical length scales.

### 3.1.2 Generalizing complex analysis by replacing complex numbers by generalized numbers

One general idea which results as an outcome of the generalized notion of number is the idea of a universal function continuable from a function mapping rationals to rationals or to a finite extension of rationals to a function in any number field. This algebraic continuation is analogous to the analytical continuation of a real analytic function to the complex plane. Rational functions for which polynomials have rational coefficients are obviously functions satisfying this constraint. Algebraic functions for which polynomials have rational coefficients satisfy this requirement if appropriate finite-dimensional algebraic extensions of p-adic numbers are allowed.

For instance, one can ask whether residue calculus might be generalized so that the value of an integral along the real axis could be calculated by continuing it instead of the complex plane to any number field via its values in the subset of rational numbers forming the back of the book like structure (in very metaphorical sense) having number fields as its pages. If the poles of the continued function in the finitely extended number field allow interpretation as real numbers it might be possible to generalize the residue formula. One can also imagine of extending residue calculus to any algebraic extension. An interesting situation arises when the poles correspond to extended p-adic rationals common to different pages of the "Big Book". Could this mean that the integral could be calculated at any page having the pole common. In particular, could a p-adic residue integral be calculated in the ordinary complex plane by utilizing the fact that in this case numerical approach makes sense. Contrary to the first expectations the algebraically continued residue calculus does not seem to have obvious applications in quantum TGD.

## 3.2 Canonical Identification

Canonical There exists a natural continuous map $Id : R_p \to R_+$ from p-adic numbers to non-negative real numbers given by the "pinary" expansion of the real number for $x \in R$ and $y \in R_p$ this correspondence reads

$$\begin{aligned} y &= \sum_{k>N} y_k p^k \to x = \sum_{k<N} y_k p^{-k} \ , \\ y_k &\in \{0,1,..,p-1\} \ . \end{aligned} \tag{3.1}$$

This map is continuous as one easily finds out. There is however a little difficulty associated with the definition of the inverse map since the pinary expansion like also decimal expansion is not unique (1 = 0.999...) for the real numbers $x$, which allow pinary expansion with finite number of pinary digits

$$\begin{aligned} x &= \sum_{k=N_0}^{N} x_k p^{-k} \ , \\ x &= \sum_{k=N_0}^{N-1} x_k p^{-k} + (x_N-1)p^{-N} + (p-1)p^{-N-1}\sum_{k=0,..} p^{-k} \ . \end{aligned} \tag{3.2}$$

The p-adic images associated with these expansions are different

$$\begin{aligned} y_1 &= \sum_{k=N_0}^{N} x_k p^{k} \ , \\ y_2 &= \sum_{k=N_0}^{N-1} x_k p^{k} + (x_N-1)p^{N} + (p-1)p^{N+1}\sum_{k=0,..} p^{k} \\ &= y_1 + (x_N-1)p^N - p^{N+1} \ , \end{aligned} \tag{3.3}$$

so that the inverse map is either two-valued for p-adic numbers having expansion with finite number of pinary digits or single valued and discontinuous and non-surjective if one makes pinary expansion unique by choosing the one with finite number of pinary digits. The finite number of pinary digits expansion is a natural choice since in the numerical work one always must use a pinary cutoff on the real axis.

### 3.2.1 Canonical identification is a continuous map of non-negative reals to p-adics

The topology induced by the inverse of the canonical identification map in the set of positive real numbers differs from the ordinary topology. The difference is easily understood by interpreting the p-adic norm as a norm in the set of the real numbers. The norm is constant in each interval $[p^k, p^{k+1})$ (see **Fig. 1** ) and is equal to the usual real norm at the points $x = p^k$: the usual linear norm is replaced with a piecewise constant norm. This means that p-adic topology is coarser than the usual real topology and the higher the value of $p$ is, the coarser the resulting topology is above a given length scale. This hierarchical ordering of the p-adic topologies will be a central feature as far as the proposed applications of the p-adic numbers are considered.

Ordinary continuity implies p-adic continuity since the norm induced from the p-adic topology is rougher than the ordinary norm. This allows two alternative interpretations. Either p-adic image of a physical systems provides a good representation of the system above some pinary cutoff or the physical system can be genuinely p-adic below certain length scale $L_p$ and become in good approximation real, when a length scale resolution $L_p$ is used in its description. The first interpretation is correct if canonical identification is interpreted as a cognitive map. p-Adic continuity implies ordinary continuity from right as is clear already from the properties of the p-adic norm (the graph of the norm is indeed continuous from right, see **Fig. 1** ). This feature is one clear signature of the p-adic topology.

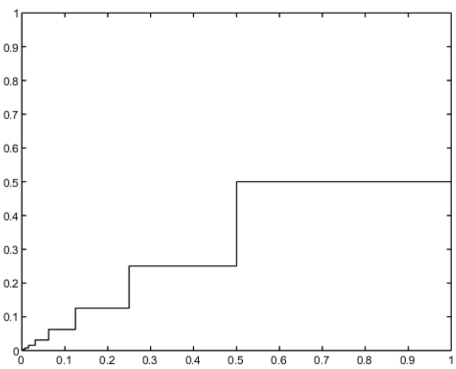


**Figure 1:** The real norm induced by canonical identification from 2-adic norm.

If one considers seriously the application of canonical identification to basic quantum TGD one cannot avoid the question about the p-adic counterparts of the negative real numbers. There is no satisfactory manner to circumvent the fact that canonical images of p-adic numbers are naturally non-negative. This is not a problem if canonical identification applies only to the coordinate interval $(0, 2\pi/N)$ or its hyperbolic variant defining the finite measurement resolution. That p-adicization program works only for highly symmetric spaces is not a problem from the point of view of TGD.

### 3.2.2 The interpretation of canonical identification in terms of finite measurement resolution

The question what the canonical identification really means could be a key to the understanding of the special aspects of this map. The notion of finite measurement resolution is a good candidate for the needed principle.

1. Finite measurement resolution can be characterized as an interval of minimum length. Below this length scale one cannot distinguish points from each other. A natural definition for this inability could be as an inability to well-order the points. The real topology is too strong in the modelling in kind of situation since it brings in large amount of processing of pseudo information whereas p-adic topology which lacks the notion of well-ordering could be more appropriate as effective topology and together with pinary cutoff could allow to get rid of the irrelevant information.

2. This suggest that canonical identification applies only inside the intervals defining finite measurement resolution in a given discretization of the space considered by say small cubes. The canonical identification is unique only modulo diffeomorphism applied on both real and p-adic side but this is not a problem since this would only reflect the absence of the well-ordering lost by finite measurement resolution. Also the fact that the map makes sense only at positive real axis would be natural if one accepts this identification.

### 3.2.3 The notion of p-adic linearity

The linear structure of the p-adic numbers induces a corresponding structure in the set of the non-negative real numbers and p-adic linearity in general differs from the ordinary concept of linearity. For example, p-adic sum is equal to real sum only provided the summands have no common pinary digits. Furthermore, the condition $x +_p y < max\{x, y\}$ holds in general for the p-adic sum of the real numbers. p-Adic multiplication is equivalent with the ordinary multiplication only provided that either of the members of the product is power of $p$. Moreover one has $x \times_p y < x \times y$ in general. An interesting possibility is that p-adic linearity might replace the ordinary linearity in some strongly nonlinear systems so these systems would look simple in the p-adic topology.

### 3.2.4 Does canonical identification define a generalized norm?

Canonical correspondence is quite essential in TGD applications. Canonical identification makes it possible to define a p-adic valued definite integral. Canonical identification is in a key role in the successful predictions of the elementary particle masses. Canonical identification makes also possible to understand the connection between p-adic and real probabilities. These and many other successful applications suggests that canonical identification is involved with some deeper mathematical structure. The following inequalities hold true:

$$\begin{aligned} (x+y)_R &\leq x_R + y_R \ , \\ |x|_p|y|_R \leq (xy)_R &\leq x_R y_R \ , \end{aligned} \tag{3.4}$$

where $|x|_p$ denotes p-adic norm. These inequalities can be generalized to the case of $(R_p)^n$ (a linear vector space over the p-adic numbers).

$$\begin{aligned} (x+y)_R &\leq x_R + y_R \ , \\ |\lambda|_p|y|_R \leq (\lambda y)_R &\leq \lambda_R y_R \ , \end{aligned} \tag{3.5}$$

where the norm of the vector $x \in T_p^n$ is defined in some manner. The case of Euclidian space suggests the definition

$$(x_R)^2 = (\sum_n x_n^2)_R \ . \tag{3.6}$$

These inequalities resemble those satisfied by the vector norm. The only difference is the failure of linearity in the sense that the norm of a scaled vector is not obtained by scaling the norm of the original vector. Ordinary situation prevails only if the scaling corresponds to a power of $p$.

These observations suggests that the concept of a normed space or Banach space might have a generalization and physically the generalization might apply to the description of some nonlinear systems. The nonlinearity would be concentrated in the nonlinear behavior of the norm under scaling.

## 3.3 The Interpretation Of Canonical Identification

During the development of p-adic TGD two seemingly mutually inconsistent competing identifications of reals and p-adics have caused a lot of painful tension. Canonical identification provides one possible identification map respecting continuity whereas the identification of rationals as points common to p-adics and reals respects algebra of rationals. The resolution of the tension came from the realization that canonical identification naturally maps the predictions of p-adic probability theory and thermodynamics to real numbers. Canonical identification also maps p-adic cognitive representations to symbolic ones in the real world world or vice versa. The identification by common rationals is in turn the correspondence implied by the generalized notion of number and natural in the construction of quantum TGD proper.

### 3.3.1 Canonical identification maps the predictions of the p-adic probability calculus and statistical physics to real numbers

p-Adic mass calculations based on p-adic thermodynamics were the first and rather successful application of the p-adic physics (see the four chapters in [K16]. The essential element of the approach was the replacement of the Boltzmann weight $e^{-E/T}$ with its p-adic generalization $p^{L_0/T_p}$, where $L_0$ is the Virasoro generator corresponding to scaling and representing essentially mass squared operator instead of energy. $T_p$ is inverse integer valued p-adic temperature. The predicted mass squared averages were mapped to real numbers by canonical identification.

One could also construct a real variant of this approach by considering instead of the ordinary Boltzmann weights the weights $p^{-L_0/T_p}$. The quantization of temperature to $T_p = log(p)/n$ would be a completely ad hoc assumption. In the case of real thermodynamics all particles are predicted to be light whereas in case of p-adic thermodynamics particle is light only if the ratio for the degeneracy of the lowest massive state to the degeneracy of the ground state is integer. Immense number of particles disappear from the spectrum of light particles by this criterion. For light particles the predictions are same as of p-adic thermodynamics in the lowest non-trivial order but in the next order deviations are possible.

Also p-adic probabilities and the p-adic entropy can be mapped to real numbers by canonical identification. The general idea is that a faithful enough cognitive representation of the real physics can by the number theoretical constraints involved make predictions, which would be extremely difficult to deduce from real physics.

### 3.3.2 The variant of canonical identification commuting with division of integers

The basic problems of canonical identification is that it does not respect unitarity. For this reason it is not well suited for relating p-adic and real scattering amplitudes. The problem of the correspondence via direct rationals is that it does not respect continuity.

A compromise between algebra and topology is achieved by using a modification of canonical identification $I_{R_p \rightarrow R}$ defined as $I_1(r/s) = I(r)/I(s)$. If the conditions $r \ll p$ and $s \ll p$ hold true, the map respects algebraic operations and also unitarity and various symmetries. It seems that this option must be used to relate p-adic transition amplitudes to real ones and vice versa [K8]. In particular, real and p-adic coupling constants are related by this map. Also some problems related to p-adic mass calculations find a nice resolution when $I_1$ is used.

This variant of canonical identification is not equivalent with the original one using the infinite expansion of $q$ in powers of $p$ since canonical identification does not commute with product and division. The variant is however unique in the recent context when $r$ and $s$ in $q = r/s$ have no common factors. For integers $n < p$ it reduces to direct correspondence.

Generalized numbers would be regarded in this picture as a generalized manifold obtained by gluing different number fields together along rationals. Instead of a direct identification of real and p-adic rationals, the p-adic rationals in $R_p$ are mapped to real rationals (or vice versa) using a variant of the canonical identification $I_{R \rightarrow R_p}$ in which the expansion of rational number $q = r/s = \sum r_n p^n / \sum s_n p^n$ is replaced with the rational number $q_1 = r_1/s_1 = \sum r_n p^{-n} / \sum s_n p^{-n}$ interpreted as a p-adic number:

$$q = \frac{r}{s} = \frac{\sum_n r_n p^n}{\sum_m s_n p^n} \rightarrow q_1 = \frac{\sum_n r_n p^{-n}}{\sum_m s_n p^{-n}} \quad . \tag{3.7}$$

$R_{p1}$ and $R_{p_2}$ are glued together along common rationals by an the composite map $I_{R\rightarrow R_{p_2}} I_{R_{p_1}\rightarrow R}$.

This variant of canonical identification seems to be excellent candidate for mapping the predictions of p-adic mass calculations to real numbers and also for relating p-adic and real scattering amplitudes to each other [K8].

### 3.3.3 p-Adic fractality, canonical identification, and symmetries

The original motivation for the canonical identification and its variants- in particular the variant mapping real rationals with the defining integers below a pinary cutoff to p-adic rationals- was that it defines a continuous map from p-adics to reals and produces beautiful p-adic fractals as a map from reals to p-adics by canonical identification followed by a p-adically smooth map in turn followed by the inverse of the canonical identification.

The first drawback was that the map does not commute with symmetries. Second drawback was that the standard canonical identification from reals to p-adics with finite pinary cutoff is two-valued for finite integers. The canonical real images of these transcendentals are also transcendentals. These are however countable whereas p-adic algebraics and transcendentals having by definition a non-periodic pinary expansion are uncountable. Therefore the map from reals to p-adics is single valued for almost all p-adic numbers.

On the other hand, p-adic rationals form a dense set of p-adic numbers and define "almost all" for the purposes of numerics! Which argument is heavier? The direct identification of reals and p-adics via common rationals commutes with symmetries in an approximation defined by the pinary cutoff an is used in the canonical identification with pinary cutoff mapping rationals to rationals.

Symmetries are of extreme importance in physics. Is it possible to imagine the action of say Poincare transformations commuting with the canonical identification in the sets of p-adic and real transcendentals? This might be the case.

1. Wick rotation (see `http://tinyurl.com/qh8jvoj`) is routinely used in quantum field theory to define Minkowskian momentum integrals. One Wick rotates Minkowski space to Euclidian space, performs the integrals, and returns to Minkowskian regime by using the inverse of Wick rotation. The generalization to the p-adic context is highly suggestive. One could map the real Minkowski space to its p-adic counterpart, perform Poincare transformation there, and return back to the real Minkowski space using the inverse of the rational canonical identification.

2. For p-adic transcendentals one would a formal automorph of Poincare group as $IPI^{-1}$ and these Poincare group would be the fractal counterpart of the ordinary Poincare group. Mathematician would regard $I$ as the analog of intertwining operator, which is linear map between Hilbert spaces. This variant of Poincare symmetry would be exact in the transcendental realm since canonical identification is continuous. For rationals this symmetry would fail.

3. For rationals which are constructed as ratios of small enough integers, the rational Poincare symmetry with group elements involving rationals constructed from small enough integers would be an exact symmetry. For both options the use of preferred coordinates, most naturally linear Minkowski coordinates would be essential since canonical identification does not commute with general coordinate transformations.

4. Which of these Poincare symmetries corresponds to the physical Poincare symmetry? The above argument does not make it easy to answer the question. One can however circumvent it. Maybe one could distinguish between rational and transcendental regime in the sense that Poincare group and other symmetries would be realized in different manner in these regimes?

Note that the analog of Wick rotation could be used also to define p-adic integrals by mapping the p-adic integration region to real one by some variant of canonical identification continuously, performing the integral in the real context, and mapping the outcome of the integral to p-adic number by canonical identification. Again preferred coordinates are essential and in TGD framework such coordinates are provided by symmetries. This would allow a numerical treatment of the p-adic integral but the map of the resulting rational to p-adic number would be two valued. The difference between the images would be determined by the numerical accuracy when p-adic expansions are used. This method would be a numerical analog of the analytic definition of p-adic integrals by analytic continuation from the intersection of real and p-adic worlds defined by rational values of parameters appearing in the expressions of integrals.

# 4 P-Adic Differential And Integral Calculus

p-Adic differential calculus differs from its real counterpart in that piecewise constant functions depending on a finite number of pinary digits have vanishing derivative. This property implies p-adic nondeterminism, which has natural interpretation as making possible imagination if one identifies p-adic regions of space-time as cognitive regions of space-time.

One of the stumbling blocks in the attempts to construct p-adic physics have been the difficulties involved with the definition of the p-adic version of a definite integral. There are several alternative options as how to define p-adic definite integral and it is quite possible that there is simply not a single correct version since p-adic physics itself is a cognitive model.

1. The first definition of the p-adic integration is based on three ideas. The ordering for the limits of integration is defined using canonical correspondence. $x < y$ holds true if $x_R < y_R$ holds true. The integral functions can be defined for Taylor series expansion by defining indefinite integral as the inverse of the differentiation. If p-adic pseudo constants are present in the integrand one must divide the integration range into pieces such that p-adic integration constant changes its value in the points where new piece begins.

2. Second definition is based on p-adic Fourier analysis based on the use of p-adic plane waves constructed in terms of Pythagorean phases. This definition is especially attractive in the definition of p-adic QFT limit and is discussed in detail later in the section "p-Adic Fourier analysis". In this case the integral is defined in the set of rationals and the ordering of the limits of integral is therefore not a problem.

3. For p-adic functions which are direct canonical images of real functions, p-adic integral can be defined also as a limit of Riemann sum and this in principle makes the numerical evaluation of p-adic integrals possible. As found in the chapter "Mathematical Ideas", Riemann sum representation leads to an educated guess for an *exact formula for the definite integral* holding true for functions which are p-adic counterparts of real-continuous functions and for p-adically analytic functions. The formula provides a calculational recipe of p-adic integrals, which converges extremely rapidly in powers of $p$. Ultrametricity guarantees the absence of divergences in arbitrary dimensions provided that integrand is a bounded function. It however seems that this definition of integral cannot hold true for the p-adically differentiable function whose real images are not continuous.

## 4.1 P-Adic Differential Calculus

The rules of the p-adic differential calculus are formally identical to those of the ordinary differential calculus and generalize in a trivial manner for the algebraic extensions.

The class of the functions having vanishing p-adic derivatives is larger than in the real case: any function depending on a finite number of positive pinary digits of p-adic number and of arbitrary number of negative pinary digits has a vanishing p-adic derivative. This becomes obvious, when one notices that the p-adic derivative must be calculated by comparing the values of the function at nearby points having the same p-adic norm (here is the crucial difference with respect to real case!). Hence, when the increment of the p-adic coordinate becomes sufficiently small, p-adic constant doesn't detect the variation of $x$ since it depends on finite number of positive p-adic pinary digits only. p-Adic constants correspond to real functions, which are constant below some length scale $\Delta x = 2^{-n}$. As a consequence p-adic differential equations are non-deterministic: integration constants are arbitrary functions depending on a finite number of the positive p-adic pinary digits. This feature is central as far applications are considered and leads to the interpretation of p-adic physics as physics of cognition which involves imagination in essential manner. The classical non-determinism of the Kähler action, which is the key feature of quantum TGD, corresponds in a natural manner to the non-determinism of volition in macroscopic length scales.

p-analytic maps $g : R_p \rightarrow R_p$ satisfy the usual criterion of differentiability and are representable as power series

$$g(x) = \sum_k g_k x^k \ . \tag{4.1}$$

Also negative powers are in principle allowed.

## 4.2 P-Adic Fractals

p-Adically analytic functions induce maps $R_+ \rightarrow R_+$ via the canonical identification map. The simplest manner to get some grasp on their properties is to plot graphs of some simple functions (see **Fig. 2** for the graph of p-adic $x^2$ and **Fig. 3** for the graph of p-adic $1/x$). These functions have quite characteristic features resulting from the special properties of the p-adic topology. These features should be universal characteristics of cognitive representations and should allow to deduce the value of the p-adic prime $p$ associated with a given cognitive system.

1. p-Analytic functions are continuous and differentiable from right: this peculiar asymmetry is a completely general signature of the p-adicity. As far as time dependence is considered, the interpretation of this property as a mathematical counterpart of the irreversibility looks attractive. This suggests that the transition from the reversible microscopic dynamics to irreversible macroscopic dynamics could correspond to the transition from the ordinary topology to an effective p-adic topology.

2. There are large discontinuities associated with the points $x = p^n$. This implies characteristic threshold phenomena. Consider a system whose output $f(n)$ is a function of input, which is integer $n$. For $n < p$ nothing peculiar happens but for $n = p$ the real counterpart of the output becomes very small for large values of $p$. In the bio-systems threshold phenomena are typical and p-adicity might be the key to their understanding. The discontinuities associated with the powers of $p = 2$ are indeed encountered in many physical situations. Auditory experience has the property that a given frequency $\omega_0$ and its multiples $2^k\omega_0$, octaves, are experienced as the same frequency, this suggests that the auditory response function for a given frequency $\omega_0$ is a 2-adicallly analytic function. Titius-Bode law states that the mutual distances of planets come in powers of 2, when suitable unit of distance is used. In turbulent systems period doubling spectrum has peaks at frequencies $\omega = 2^k\omega_0$.

3. A second signature of the p-adicity is "p-plicity" appearing in the graph of simple p-analytic functions. As an example, consider the graph of the p-adic $x^2$ demonstrating clearly the decomposition into $p$ steps at each interval $[p^k, p^{k+1})$.

4. The graphs of the p-analytic functions are in general ordered fractals as the examples demonstrate. For example, power functions $x^n$ are self-similar (the values of the function at some any interval $(p^k, p^{k+1})$ determines the function completely) and in general p-adic $x^n$ with non-negative (negative) $n$ is smaller (larger) than real $x^n$ expect at points $x = p^n$ as the graphs of p-adic $x^2$ and $1/x$ show (see **Fig. 2** and **3** ) These properties are easily understood from the properties of the p-adic multiplication. Therefore the first guess for the behavior of a p-adically analytic function is obtained by replacing $x$ and the coefficients $g_k$ with their p-adic norms: at points $x = p^n$ this approximation is exact if the coefficients of the power series are powers of $p$. This step function approximation is rather reasonable for simple functions such as $x^n$ as the figures demonstrate. Since p-adically analytic function can be approximated with $f(x) \sim f(x_0) + b(x - x_0)^n$ or as $a(x - x_0)^n$ (allowing non-analyticity at $x_0$) around any point the fractal associated with p-adically analytic function has universal geometrical form in sufficiently small length scales.

p-Adic analyticity is well defined for the algebraic extensions of $R_p$, too. The figures **??** and **??** visualize the behavior of the real and imaginary parts of the 2-adic $z^2$ function as a function of the real $x$ and $y$ coordinates in the parallelpiped $I^2$, $I = [1 + 2^{-7}, 2 - 2^{-7}]$. An interesting possibility is that the order parameters describing various phases of some physical systems are p-adically differentiable functions. The p-analyticity would therefore provide a means for coding the information about ordered fractal structures.

The order parameter could be one coordinate component of a p-adically analytic map $R^n \rightarrow R^n$, $n = 3, 4$. This is analogous to the possibility to regard the solution of the Laplace equation in two dimensions as a real or imaginary part of an analytic function. A given region $V$ of the order parameter space corresponds to a given phase and the volume of the ordinary space occupied by this phase corresponds to the inverse image $g^{-1}(V)$ of $V$. Very beautiful images are obtained if the order parameter is the real or imaginary part of a p-analytic function $f(z)$. A good example is p-adic $z^2$ function in the parallelpiped $[a, b] \times [a, b]$, $a = 1 + 2^{-9}$, $b = 2 - 2^9$ of $C$-plane. The value range of the order parameter can be divided into, say, 16 intervals of the same length so that each interval corresponds to a unique color. The resulting fractals possess features, which probably generalize to higher-dimensional extensions.

1. The inverse image is an ordered fractal and possesses lattice/cell like structure, with the sizes of cells appearing in powers of $p$. Cells are however not identical in analogy with the differentiation of the biological cells.

2. p-Analyticity implies the existence of a local vector valued order parameter given by the p-analytic derivative of $g(z)$: the geometric structure of the phase portrait indeed exhibits the local orientation clearly.

A second representation of the fractals is obtained by dividing the value range of $z$ into a finite number of intervals and associating different color to each interval. In a given resolution this representation makes obvious the presence of 0, 1- and 2-dimensional structures not obvious from the graph representation used in the figures of this book.

These observations suggests that p-analyticity might provide a means to code the information about ordered fractal structures in the spatial behavior of order parameters (such as enzyme concentrations in bio-systems). An elegant manner to achieve this is to use purely real algebraic extension for 3-space coordinates and for the order parameter: the image of the order parameter $\Phi = \phi_1 + \phi_2\theta + \phi_3\theta^2$ under the canonical identification is real and positive number automatically and might be regarded as concentration type quantity.

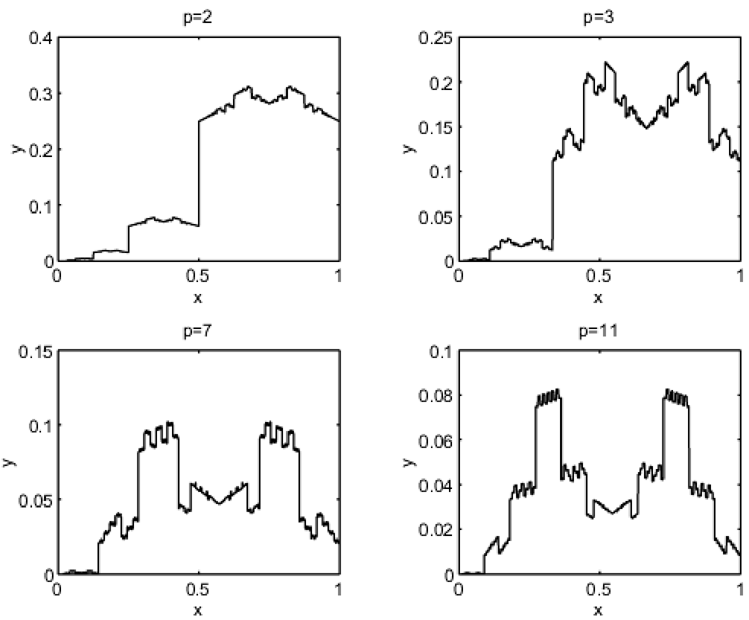


**Figure 2:** p-Adic $x^2$ function for some values of $p$

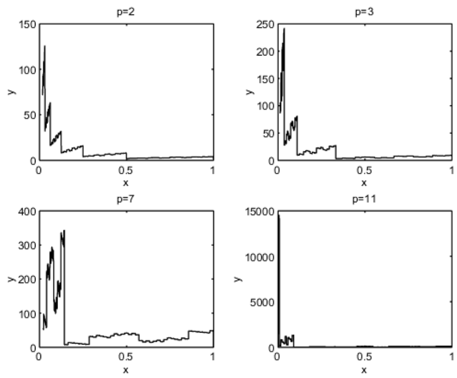


**Figure 3:** p-Adic $1/x$ function for some values of $p$

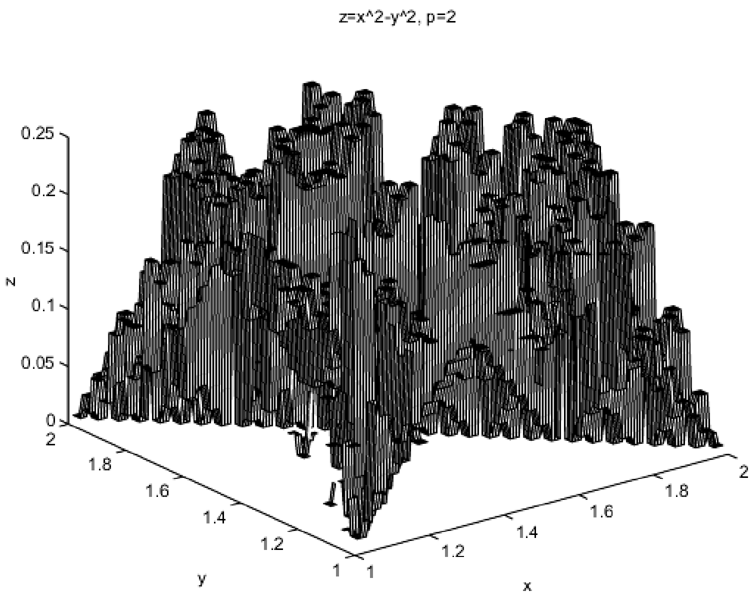


**Figure 4:** The graph of the real part of 2-adically analytic $z^2 =$ function.

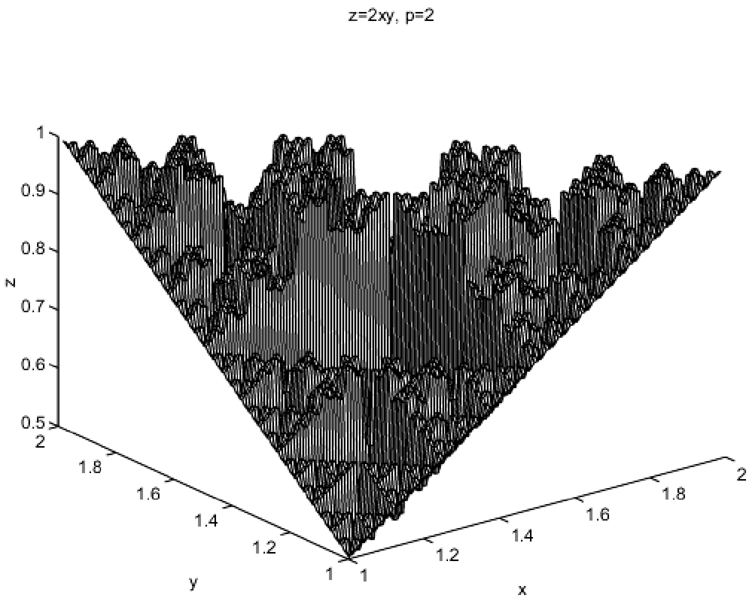


**Figure 5:** The graph of 2-adically analytic $Im(z^2) = 2xy$ function.

## 4.3 P-Adic Integral Calculus

The basic problems of the integration with p-adic values of integral are caused by the facts that p-adic numbers are not well-ordered and by the properties of p-adic norm. The general idea that p-adic physics can mimic real physics only at the algebraic level, leads to the idea that p-adic integration could be algebraized whereas numerical approaches analogous to Riemann sum are not possible. In the following three examples are discussed.

1. Definite integral can be defined using integral function and by defining integration limits via canonical identification: the drawback is the loss of general coordinate invariance. A more elegant general coordinate invariant approach is based on the identification of rationals as common to both reals and p-adics. This works for rational valued integration limits.

2. residue calculus allows to realize integrals of analytic functions over closed curves of complex plane. The generalization of the residue calculus makes possible to realize conformal invariance at elementary particle horizons which are metrically 2-dimensional and allow conformal invariance and has also p-adic counterpart.

3. The perturbative series using Gaussian integration is the only to perform in practice infinite-dimensional functional integrals and being purely algebraic procedure, allows a straightforward p-adic generalization. This is the only option for p-adicizing configuration space integral.

### 4.3.1 Definition of the definite integral using integral function concept and canonical identification or identification by common rationals

The concept of the p-adic definite integral can be defined for functions $R_p \rightarrow C$ [A5] using translationally invariant Haar measure for $R_p$. In present context one is however interested in defining a p-adic valued definite integral for functions $f : R_p \rightarrow R_p$: target and source spaces could of course be also some some algebraic extensions of the p-adic numbers.

What makes the definition nontrivial is that the ordinary definition as the limit of a Riemann sum doesn't seem to work: it seems that Riemann sum approaches to zero in the p-adic topology since, by ultra-metricity, the p-adic norm of a sum is never larger than the maximum p-adic norm for the summands. The second difficulty is related to the absence of a well-ordering for the p-adic numbers. The problems might be avoided by defining the integration essentially as the inverse of the differentiation and using the canonical correspondence to define ordering for the p-adic numbers. More generally, the concepts of the form, cohomology and homology are crucially based on the concept of the boundary. The concept of boundary reduces to the concept of an ordered interval and canonical identification makes it indeed possible to define this concept.

The definition of the p-adic integral functions defining integration as inverse of the differentiation is straightforward and one obtains just the generalization of the standard calculus. For instance, one has $\int z^n = \frac{z^{n+1}}{(n+1)} + C$ and integral of the Taylor series is obtained by generalizing this. One must however notice that the concept of integration constant generalizes: any function $R_p \rightarrow R_p$ depending on a finite number of pinary digits only, has a vanishing derivative.

Consider next the definite integral. The absence of the well ordering implies that the concept of the integration range $(a, b)$ is not well defined as a purely p-adic concept. As already mentioned there are two solutions of the problem.

1. The identification of rational numbers as common to both reals and p-adics allows to order the integration limits when the end points of the integral are rational numbers. This is perhaps the most elegant solution of the problem since it is consistent with the restricted general coordinate invariance allowing rational function based coordinate changes. This approach works for rational functions with rational coefficients and more general functions if algebraic extension or extension containing transcendentals like $e$ and logarithms of primes are allowed. The extension containing $e$, $\pi$, and $log(p)$ is finite-dimensional if $e/\pi$ and $\pi/log(p)$ are rational numbers for all primes $p$. Essentially algebraic continuation of real integral to p-adic context is in question.

2. An alternative resolution of the problem is based on the canonical identification. Consider p-adic numbers $a$ and $b$. It is natural to define $a$ to be smaller than $b$ if the canonical images of $a$ and $b$ satisfy $a_R < b_R$. One must notice that $a_R = b_R$ does not imply $a = b$, since the inverse of the canonical identification map is two-valued for the real numbers having a finite number of pinary digits. For two p-adic numbers $a, b$ with $a < b$, one can define the integration range $(a, b)$ as the set of the p-adic numbers $x$ satisfying $a \leq x \leq b$ or equivalently $a_R \leq x_R \leq b_R$. For a given value of $x_R$ with a finite number of pinary digits, one has two values of $x$ and $x$ can be made unique by requiring it to have a finite number of pinary digits.

One can define definite integral $\int_a^b f(x)dx$ formally as

$$\int_a^b f(x)dx = F(b) - F(a) \ , \tag{4.2}$$

where $F(x)$ is integral function obtained by allowing only ordinary integration constants and $b_R > a_R$ holds true. One encounters however a problem, when $a_R = b_R$ and $a$ and $b$ are different. Problem is avoided if the integration limits are assumed to correspond to p-adic numbers with a finite number of pinary digits.

One could perhaps relate the possibility of the p-adic integration constants depending on finite number of pinary digits to the possibility to decompose integration range $[a_R, b_R]$ as $a = x_0 < x_1 < ....x_n = b$ and to select in each subrange $[x_k, x_{k+1}]$ the inverse images of $x_k \leq x \leq x_{k+1}$, with $x$ having finite number of pinary digits in two different manners. These different choices correspond to different integration paths and the value of the integral for different paths could correspond to the different choices of the p-adic integration constant in integral function. The difference between a given integration path and "standard" path is simply the sum of differences $F(x_k) - F(y_k)$, $(x_k)_R = (y_k)_R$.

This definition has several nice features.

1. The definition generalizes in an obvious manner to the higher dimensional case. The standard connection between integral function and definite integral holds true and in the higher-dimensional case the integral of a total divergence reduces to integral over the boundaries of the integration volume. This property guarantees that p-adic action principle leads to same field equations as its real counterpart. It this in fact this property, which drops other alternatives from the consideration.

2. The basic results of the real integral calculus generalize as such to the p-adic case. For instance, integral is a linear operation and additive as a set function.

The ugly feature is the loss of the general coordinate invariance due to the fact that canonical identification does not commute with coordinate changes (except scalings by powers of $p$) and it seems that one cannot use canonical identification at the fundamental level to define definite integrals.

### 4.3.2 Definite integrals in p-adic complex plane using residue calculus

residue calculus allows to calculate the integrals $\oint_C f(z)dz$ around complex curves as sums over poles of the function inside the curve:

$$\oint f(z)dz = i2\pi \sum_k Res(f(z_k)) \ , \tag{4.3}$$

where $Res(f(z_k))$ at pole $z = z_k$ is defined as $Res(f(z_k)) = \lim_{z \rightarrow z_k}(z - z_k)f(z)$. This definition applies in case of 2-dimensional $\sqrt{-1}$-containing algebraic extension of p-adic numbers ($p \ mod \ 4 = 3$) but its seems that this is not relevant for quantum TGD.

Quaternion conformal invariance corresponds to the conformal invariance associated with topologically 3-dimensional elementary particle horizons surrounding wormhole contacts which have

Euclidian signature of induced metric. The induced metric is degenerate at the elementary particle horizon so that these surfaces are metrically two-dimensional. This implies a generalization of conformal invariance analogous to that at light cone cone boundary. In particular, a subfield of quaternions isomorphic with complex numbers is selected. One expects that residue calculus generalizes.

Elementary particle horizons are defined by a purely algebraic condition stating that the determinant of the induced metric vanishes, and thus the notion makes sense for p-adic space-time sheets too. Also residue calculus should make sense for all algebraic extensions of p-adic numbers and the algebra of quaternion conformal invariance would generalize to the p-adic context too. Note however that the notion of p-adic quaternions does not make sense: the reason is that p-adic Euclidian length squared for a non-vanishing p-adic quaternion can vanish so that the inverse of quaternion is not well defined always. In the set of rational numbers this failure does not however occur and this might be enough for p-adicization to work.

### 4.3.3 Definite integrals using Gaussian perturbation theory

In quantum field theories functional integrals are defined by Gaussian perturbation theory. For real infinite-dimensional Gaussians the procedure has a rigorous mathematical basis deriving from measure theory. For the imaginary infinite-dimensional Gaussians defining the Feynman path integrals of quantum field theory the rigorous mathematical justification is lacking.

In TGD framework the integral over WCW of three surface can be reduced to a real Gaussian perturbation theory around the maxima of Kähler function. The integration is over quantum fluctuating degrees of freedom defining infinite-dimensional symmetric space for given values of zero modes. According to the more detailed arguments about how to construct p-adic counterpart of real WCW physics described in the chapter "Construction of Quantum Theory", the following conjectures are trued.

1. The symmetric space property implies that there is only one maximum of Kähler function for given values of zero modes.

2. The generalization of Duistermaat-Heecke theorem holding true in finite-dimensional case suggests that by symmetric space property the integral of the exponent of Kähler gives just the exponent of Kähler function at the maximum and Gaussian determinant and metric determinant cancel each other.

3. The fact that free Gaussian field theory corresponds to a flat symmetric space inspires the hypothesis that S-matrix elements involving WCW spinor fields in the representations of the isometry group reduce to those given by free field theory with propagator defined by the inverse of WCW covariant Kähler metric evaluated in the tangent space basis defined by the isometry currents at the maximum of Kähler function. This implies that there is no perturbation series which would spoil any hopes about proving the rationality. The reduction to a free field theory does not make quantum TGD non-interacting since interactions are described as topologically (as decays and fusions of 3-surfaces) rather than algbraically as non-linearities of local action.

4. If the exponent function is a rational function with rational coefficients in the sense that for the points of WCW having finite number of rational valued coordinates (also zero modes), then the exponent $e^{K_{max}}$ is a rational number for rational values of zero modes. From the rationality of the exponent of the Kähler function follows the rational valuedness of the matrix elements of the metric. The undeniably very optimistic conclusion is that for rational values of the zero modes the S-matrix elements would be rational valued or have values if finite extension of rationals, so that they could be continued to the p-adic sectors of WCW . The S-matrix would have the same form in all number fields.

5. One could also interpret the outcome as an algebraic continuation of the rational quantum physics to real and p-adic physics. WCW -integrals can be thought of as being performed in the rational WCW . Of course, one can define also ordinary integrals over $R^n$ numerically using Riemann sums by considering the division of the integration region to very small n-cubes for which the sides have rational-number valued lengths and such that the value of the function is taken at rational valued point inside each cube.

The finite-dimensional real one-dimensional Gaussian $exp(-ax^2/2)$ provides a natural testing ground for this rather speculative picture. The integral of the Gaussian is $(2\pi)^{1/2}/\sqrt{a}$: in n-dimensional case where $a$ is replaced by a quadratic form defined by a matrix $A$ one obtains $(2\pi)^{n/2}/\sqrt{det(A)}$ in n-dimensional case. The integral of a function $exp(-ax^2 + kx^n)x^k$ reduces to a perturbation series as sum of graphs containing single vertex containing $k$ lines and arbitrary number of vertices containing $n$ lines and endowed with a factor $k$, and assigning with the lines the propagator factor $1/a$. For n-dimensional case the propagator factor would be inverse of the matrix $A$.

The result makes sense in the p-adic context if $a$ and $k$ are rational numbers. In the n-dimensional case matrix $A$ and the coefficients defining the polynomial defining the interaction term must be rational numbers. The only problematic factor is the power of $2\pi$, which seems to require algebraic extension containing $\pi$. Of course, one could define the normalization of the functional integral by dividing it by $(2\pi)^{n/2}$ to get rid of this fact. In the definition of S-matrix elements this normalization factor always disappears so that this problem has no physical significance.

In the case of free scalar quantum field theory n-point functions the perturbation theory are simply products of 2-point functions defined by the inverse of the infinite-dimensional Gaussian matrix. For plane wave basis for scalar field labelled by 4-momentum $k$ the inverse of the Gaussian matrix reduces to the propagator ($i/(k^2 + i\epsilon)$ for scalar field), which is rational function of the square of 4-momentum vector. In case of interacting quantum field the infinite summation over graphs spoils the hopes of obtaining end result which could be proven to be rational valued for rational values of incoming and outgoing four-momenta. The loop integrals are source of divergence problems and also number-theoretically problematic.

# 5 P-Adic Symmetries And Fourier Analysis

## 5.1 P-Adic Symmetries And Generalization Of The Notion Of Group

The most basic questions physicist can ask about the p-adic numbers are related to symmetries. It seems obvious that the concept of a Lie-group generalizes: nothing prevents from replacing the real or complex representation spaces associated with the definitions of the classical Lie-groups with the linear space associated with some algebraic extension of the p-adic numbers: the defining algebraic conditions, such as unitarity or orthogonality properties, make sense for the algebraically extended p-adic numbers, too.

For orthogonal groups one must replace the ordinary real inner product with the inner product $\sum_k X_k^2$ with a Cartesian power of a purely real extension of p-adic numbers. In the unitary case one must consider the complexification of a Cartesian power of a purely real extension with the inner product $\sum \bar{Z}_k Z_k$. Here $p\ mod\ 4 = 3$ is required. It should be emphasized however that the p-adic inner product differs from the ordinary one so that the action of, say, p-adic counterpart of a rotation group in $R_p^3$ induces in $R^3$ an action, which need not have much to do with ordinary rotations so that the generalization is physically highly nontrivial. Extensions of p-adic numbers also mean extreme richness of structure.

The exponentiation $t \rightarrow exp(tJ)$ of the Lie-algebra element $J$ is a central element of Lie group theory and allows to coordinatize that elements of Lie group by mapping tangent space points the points representing group elements. Without algebraic extensions involving $e$ or its roots one can exponentiate only the group parameters $t$ satisfying $|t|_p < 1$. Thus the values of the exponentiation parameter which are too small/large in real/p-adic sense are not possible and one can say that the standard p-adic Lie algebra is a ball with radius $|t|_p = 1/p$.

The study of ordinary one-dimensional translations gives an idea about what it is involved. For finite values of the p-adic integer $t$ the exponentiated group element corresponds in the case of translation group to a power of $e$ so that the points reached by exponentiation cannot correspond to rational points. Since logarithm function exist as an inverse of p-adic exponent and since rationals correspond to infinite but periodic pinary expansions, rational points having the same p-adic norm can be reached by p-adic exponentials using $t$ which is infinite as ordinary integer. This result is expected to generalize to the case of groups represented using rational-valued matrices.

One can define a hierarchy of p-adic Lie-groups by allowing extensions allowing $e$ and even its roots such that the algebras have p-adic radii $p^k$. Hence the fact that the powers $e, ..., e^{p-1}$ define a finite-dimensional extensions of p-adic numbers seems to have a deep group theoretical meaning. One can define a hierarchy of increasingly refined extensions by taking the generator of extension to be $e^{1/n}$. For instance, in the case of translation group this makes possible p-adic variant of Fourier analysis by using discrete plane wave basis.

One can generalize also the notion of group by using the generalized notion of number. This means that one starts from the restriction of the group in question to a group acting in say rational and complex rational linear space and requires that real and p-adic groups have rational group transformations as common. By performing various completions one obtains a generalized group having the characteristic book like structure. In this kind of situation the relationship between various groups is clear and also the role of extensions of p-adic numbers can be understood. The notion of Lie-algebra generalizes also to form a book like structure. Coefficients of the pages of the Lie-algebra belong to various number fields and rational valued coefficients correspond to a part partially (because of the restriction $|t|_p < p^k$) common to all Lie-algebras.

### 5.1.1 $SO(2)$ as example

A simple example is provided by the generalization of the rotation group $SO(2)$. The rows of a rotation matrix are in general $n$ orthonormalized vectors with the property that the components of these vectors have p-adic norm not larger than one. In case of $SO(2)$ this means the matrix elements $a_{11} = a_{22} = a, a_{12} = -a_{21} = b$ satisfy the conditions

$$\begin{aligned} a^2 + b^2 &= 1 \ , \\ |a|_p &\leq 1 \ , \\ |b|_p &\leq 1 \ . \end{aligned} \tag{5.1}$$

One can formally solve $a$ as $a = \sqrt{1 - b^2}$ but the solution doesn't exists always. There are various possibilities to define the orthogonal group.

1. One possibility is to allow only those values of $a$ for which square root exists as p-adic number. In case of orthogonal group this requires that both $b = sin(\Phi)$ and $a = cos(\Phi)$ exist as p-adic numbers. If one requires further that $a$ and $b$ make sense also as ordinary rational numbers, they define a Pythagorean triangle (orthogonal triangle with integer sides) and the group becomes discrete and cannot be regarded as a Lie-group. Pythagorean triangles emerge for rational counterpart of any Lie-group.

2. Other possibility is to allow an extension of the p-adic numbers allowing a square root of any ordinary p-adic number. The minimal extensions has dimension 4 (8) for $p > 2$ ($p = 2$). Therefore space-time dimension and imbedding space dimension emerge naturally as minimal dimensions for spaces, where p-adic $SO(2)$ acts "stably". The requirement that $a$ and $b$ are real is necessary unless one wants the complexification of $so(2)$ and gives constraints on the values of the group parameters and again Lie-group property is expected to be lost.

3. The Lie-group property is guaranteed if the allowed group elements are expressible as exponents of a Lie-algebra generator $Q$. $g(t) = exp(iQt)$. This exponents exists only provided the p-adic norm of $t$ is smaller than one. If one uses square root allowing extension, one can require that $t$ satisfies $|t| \leq p^{-n/2}$, $n > 0$ and one obtains a decreasing hierarchy of groups $G_1, G_2, ...$ For the physically interesting values of $p$ (typically of order $p = 2^{127} - 1$ ) the real counterparts of the transformations of these groups are extremely near to the unit element of the group. These conclusions hold true for any group. An especially interesting example physically is the group of "small" Lorentz transformations with $t = O(\sqrt{p})$. If the rest energy of the particle is of order $O(\sqrt{p})$: $E_0 = m = m_0\sqrt{p}$ (as it turns out) then the Lorentz boost with velocity $\beta = \beta_0\sqrt{p}$ gives particle with energy $E = m/\sqrt{1 - \beta_0^2 p} = m(1 + \frac{\beta_0^2 p}{2} + ..)$ so that $O(p^{1/2})$ term in energy is Lorentz invariant. This suggests that non-relativistic regime corresponds to small Lorentz transformations whereas in genuinely relativistic regime one must include also the discrete group of "large" Lorentz transformations with rational transformations matrices.

4. One can extend the group to contain products $G_1G_2$, such that $G_1$ is a rational matrix belonging to the restriction of the Lie-group to rational matrices not obtainable from a unit matrix p-adically by exponentiation, and $G_2$ is a group element obtainable from unit element by exponentiation. For instance, rational $CP_2$ is obtained from the group of rational $3 \times 3$ unitary matrices as by dividing it by the $U(2)$ subgroup of rational unitary matrices.

Even the construction of the representations of the translation group raises nontrivial issues since the construction of p-adic Fourier analysis is by no means a nontrivial task. One can however define the concept of p-adic plane wave group theoretically and p-adic plane waves are orthogonal with respect to the inner product defined by the proposed p-adic integral.

The representations of 3-dimensional rotation group $SO(3)$ can be constructed as homogenous functions of Cartesian coordinates of $E^3$ and in this case the phase factors $exp(im\phi)$ typically appearing in the expressions of spherical harmonics do not pose any problems. The construction of p-adic spherical harmonics is possible if one assumes that allowed spherical angles $(\theta, \phi)$ correspond to Pythagorean triangles.

A similar situation is encountered also in the case of $CP_2$ spherical harmonics in in fact, quite generally. This number theoretic quantization of angles could be perhaps interpreted as a kind of cognitive quantum effect consistent with the fact that only rationals can be visualized concretely and relate directly to the sensory experience. More generally, the possibility to realize only rationals numerically might reflect the facts that only rationals are common to reals and p-adics and that cognition is basically p-adic.

### 5.1.2 Fractal structure of the p-adic Poincare group

p-Adic Poincare group, just as any other p-adic Lie group, contains entire fractal hierarchy of subgroups with the same Lie-algebra. For instance, translations $m^k \rightarrow m^k + p^N a^k$, where $a^k$ has p-adic norm not larger than one form subgroup for all values of $N$. The larger the value of $N$ is, the smaller this subgroup is. Quite generally this implies orbits within orbits and representations within representations like structure so that p-adic symmetry concept contains hologram like aspect. This property of the p-adic symmetries conforms nicely with the interpretation of p-adic symmetries as cognitive representations of real symmetries since the symmetries can be realized in a p-adically finite spatiotemporal volume of the cognitive space-time sheet. Even more, this volume can be p-adically arbitrarily small. If one identifies both p-adics and reals as a completion of rationals, the corresponding real volumes are however strictly speaking infinite in absence of a pinary cutoff.

The hierarchy of subgroups implies that $M^4_+$ decomposes in a natural manner to 4-cubes with side $L_0 = N_p(L)L_p$, where $N_p(L) = p^{-N}$ denotes the p-adic norm of $L$ such that these 4-cubes are invariant under the group of sufficiently small Poincare transformations. In real context these cubes define a hierarchy of exteriors of cubes with decreasing sizes. One can have full p-adic Poincare invariance in p-adically arbitrarily small volume. Only those Poincare transformations, which leave the minimal p-adic cube invariant are symmetries. Also this picture suggest that the p-adic space-time sheets providing cognitive representations about finite space-time regions by canonical identification can have very large size.

The construction of the p-adic Fourier analysis is a nontrivial problem. The usual exponent functions $f_P(x) = exp(iPx)$, providing a representation of the p-adic translations do not make sense as a Fourier basis: $f_P$ is not a periodic function; $f_P$ does not converge if the norm of $Px$ is not smaller than one and the natural orthogonalization of the different momentum eigenstates does not seem to be possible using the proposed definition of the definite integral.

This state of affairs suggests that p-adic Fourier analysis involves number theory. It turns out that one can construct what might be called number theoretical plane waves and that p-adic momentum space has a natural fractal structure in this case. The basic idea is to reduce p-adic Fourier analysis to a Fourier analysis in a finite field $G(p, 1)$ plus fractality in the sense that all $p^m$-scaled versions of the $G(p, 1)$ plane waves are used. This means that p-adic plane waves in a given interval $[n, n + 1)p^m$ are piecewise constant plane waves in a finite field $G(p, 1)$. Number theoretical p-adic plane waves are pseudo constants so that the construction does not work for p-adically differentiable functions. The pseudo-constancy however turns out to be a highly desirable feature in the construction of the p-adic QFT limit of TGD based on the mapping of the real $H$-quantum fields to p-adic quantum fields using the canonical identification.

The unsatisfactory feature of this approach is that number theoretic p-adic plane waves do not behave in the desired manner under translations. It would be nice to have a p-adic generalization of the plane wave concept allowing a generalization of the standard Fourier analysis and a direct connection with the theory of the representations of the translation group. A natural idea is to to define exponential function as a solution of a p-adic differential equation representing the action of a translation generator and to introduce multiplicative pseudo constant making possible to define exponential function for all values of its argument. One can develop an argument suggesting that the plane waves obtained in this manner are indeed orthogonal.

Infinitesimal form of translational symmetry might be argued to be too strong requirement since p-adically infinitesimal translations typically correspond to real translations which are arbitrarily large: this is not consistent with the idea that cognitive representations with a finite spatial resolution are in question. This motivates a third approach to the p-adic Fourier analysis. The basic requirement is that discrete subgroup of translations commutes with the map of the real plane waves to their p-adic counterparts. This means that the products of the real phase factors are mapped to the products of the corresponding p-adic phase factors. This is possible if the phase factor is a rational complex number so that the phase angle corresponds to a Pythagorean triangle. The p-adic images of the real plane waves are defined for the momenta $k = nk_G$, $k_G = \phi_G/\Delta x$, where $\phi_G \in [0, 2\pi]$ is a Pythagorean phase angle and where the points $x_n = n\Delta x$ define a discretization of $x$-space, $\Delta x$ being a rational number. These plane waves form a complete and orthogonalized set.

## 5.2 P-Adic Fourier Analysis: Number Theoretical Approach

Contrary to the original expectations, number theoretical Fourier analysis is probably not basic mathematical tools of p-adic QFT since it fails to provide irreducible representation for the translational symmetries. Despite this it deserves documentation.

### 5.2.1 Fourier analysis in a finite field $G(p, 1)$

The p-adic numbers of unit norm modulo $p$ reduce to a finite field $G(p, 1)$ consisting of the integers $0, 1, ..., p-1$ with arithmetic operations defined by those of the ordinary integers taken modulo $p$. Since the elements $1, ..., p-1$ form a multiplicative group there must exists an element $a$ of $G(p, 1)$ (actually several) such that $a^{p-1} = 1$ holds true in $G(p, 1)$. This kind of element is called primitive root. If $n$ is a factor of $p-1$: $(p-1) = nm$, then also $a^m = 1$ holds true. This reflects the fact that $Z_{p-1}$ decomposes into a product $Z_{m_1}^{n_1} Z_{m_2}^{n_2} ... Z_{m_s}^{n_s}$ of commuting factors $Z_{m_i}$, such that $m_i^{n_i}$ divides $p-1$.

A Fourier basis in $G(p, 1)$ can be defined using $p$ functions $f_k(n)$, $k = 0, .., p-1$. For $k = 0, 1, ..., p-2$ these functions are defined as

$$f_k(n) = a^{nk} \ , \ n = 0 \ , ..., p-1 \ , \tag{5.2}$$

and satisfy the periodicity property

$$f_k(0) = f_k(p-1) \ .$$

The problem is to identify the lacking $p$: th function. Since $f_k(n)$ transforms irreducibly under translations $n \to n+m$ it is natural to require that also the $p$: th function transforms in a similar manner and satisfies the periodicity property. This is achieved by defining

$$f_{p-1}(n) = (-1)^n \ . \tag{5.3}$$

The counterpart of the complex conjugation for $f_k$ for $k \neq p-1$ is defined as $f_k \to f_{p-1-k}$. $f_{p-1}$ is invariant under the conjugation. The inner product is defined as

$$\langle f_k, f_l \rangle = \sum_{n=0}^{p-2} f_{p-1-k}(n) f_l(n) = \delta(k, l)(p-1) \ . \tag{5.4}$$

The dual basis $\hat{f}_k$ clearly differs only by the normalization factor $1/(p-1)$ from the basis $f_{p-k}$. The counterpart of Fourier expansion for any real function in $G(p,1)$ can be obviously constructed using this function basis and Fourier components are obtained as the inner products of the dual Fourier basis with the function in question.

A natural interpretation for the integer $k$ is as a p-adic momentum since in the translations $n \rightarrow n+m$ the plane wave with $k \neq p-1$ changes by a phase factor $a^{km}$. For $k = p-1$ it transforms by $(-1)^m$ so that also now an eigen state of finite field translations is in question.

### 5.2.2 p-Adic Fourier analysis based on p-adic plane waves

The basic idea is to reduce p-adic Fourier analysis to the Fourier analysis in $G(p,1)$ by using fractality.

1. Let the function $f(x)$ be such that the maximum p-adic norm of $f(x)$ is $p^{-m}$. One can uniquely decompose $f(x)$ to a sum of functions $f_n(x)$ such that $|f_n(x)|_p = p^n$ or vanishes in the entire range of definition for $f$:

$$\begin{aligned} f(x) &= \sum_{n\geq m} f_n(x) \ , \\ f_n(x) &= g_n(x)p^n \ , \\ |g_n(x)| &= 1 \ \text{for} \ g(x)\neq 0 \ . \end{aligned} \tag{5.3}$$

   The higher the value of $n$, the smaller the contribution of $f_n$. The expansion converges extremely rapidly for the physically interesting large values of $p$.

2. Assume that $f(x)$ is such that for each value of $n$ one can find some resolution $p^{m(n)}$ below which $g_n(x)$ is constant in the sense that for all intervals $[r, r+1)p^{m(n)}$ (defined in terms of the canonical identification) the function $f_n(x)$ is constant. For p-adically differentiable functions this cannot be the case since they would be pseudo constants if this were true. In the physical situation $CP_2$ size provides a natural p-adic cutoff so that only a finite number of $f_n$: s are needed and the resolution in question corresponds to $CP_2$ length scale. Hence ordinary plane waves (possibly with a natural UV cutoff) should have an expansion in terms of the p-adic plane waves.

3. The assumption implies that in each interval $(r, r+1)p^{m(n)-1}$, $g_n$ can be regarded as a function in $G(p,1)$ identified as the set $x = (r+sp)p^{m(n)-1}$, $s = 0, 1, ..., p-1$. Hence one can Fourier expand $f_n(x)$ using $G(p,1)$ plane waves $f^{ks}$. In this manner one obtains a rapidly converging expansion using p-adic plane waves.

### 5.2.3 Periodicity properties of the number theoretic p-adic plane waves

The periodicity properties of the p-adic plane waves make it possible to associate a definite wavelength with a given p-adic plane wave. For the p-adic momenta $k$ not dividing $p-1$, the wavelength corresponds to the entire range $(n, n+1)p^m$ and its real counterpart is

$$\lambda = p^{-m-1/2} l \ ,$$

where $l \sim 10^4\sqrt{\hbar G}$ is the fundamental p-adic length scale. If $k$ divides $p-1 = \prod_i m_i^{n_i}$, the period is $m_i$ and the real wavelength is

$$\lambda(m_i) = m_i p^{-m-1-1/2} l \ .$$

One might wonder whether this selection of preferred wavelengths has some physical consequences. The first thing to notice is that p-adic plane waves do *not* replace ordinary plane waves in the construction of the p-adic QFT limit of TGD. Rather, ordinary plane waves are expanded using the p-adic plane waves so that the selection of the preferred wavelengths, if it occurs at all, must be a dynamical process. The average value of the prime divisors, and hence the number of different wavelengths for a given value of $p$, counted with the degeneracy of the divisor is given by [A12]

$$\Omega(n) = ln(ln(n)) + 1.0346 \ ,$$

and is surprisingly small, or order 6 for numbers of order $M_{127}$! If one can apply probabilistic arguments or [A12] to the numbers of form $p-1$, too then one must conclude that very few wavelengths are possible for general prime $p$! This in turn means that to each $p$ there are associated only very few characteristic length scales, which are predictable. Furthermore, all the $p^k$-multiples of these scales are also possible if p-adic fractality holds true in macroscopic length scales.

Mersenne primes $M_n$ can be considered as an illustrative example of the phenomenon. From [A9] one finds that $M_{127}-1$ has 11 distinct prime factors and 3 and 7 occurs three and 2 times respectively. The number of distinct length scales is $3\cdot 2^{11}-1 \sim 2^{12}$. $M_{107}-1$ and $M_{89}-1$ have 7 and 11 singly occurring factors so that the numbers of length scales are $2^7-1 = 127 = M_7$ and $2^{11}-1$. Note that for hadrons ($M_{107}$) the number of possible wavelengths is especially small: does this have something to do with the collective behavior of color confined quarks and gluons? An interesting possibility is that this length scale generation mechanism works even macroscopically (for p-adic length scale hypothesis at macroscopic length scales see the third part of the book). One cannot exclude the possibility that long wavelength photons, gravitons and neutrinos might therefore provide a completely new mechanism for generating periodic structures with preferred sizes of period.

## 5.3 P-Adic Fourier Analysis: Group Theoretical Approach

The problem with the straightforward generalization of the Fourier analysis is that the standard Taylor expansion of the plane wave $exp(ikx)$ converges only provided $x$ has p-adic norm smaller than one and that the p-adic exponential function does not have the periodicity properties of the ordinary exponential function guaranteeing orthogonality of the functions of the Fourier basis. Besides this one must assume $p\ mod\ 4 = 3$ to guarantee that $\sqrt{-1}$ does not exist as ordinary p-adic number.

### 5.3.1 The approach based on algebraic extensions allowing trigonometry

In an attempt to construct Fourier analysis the safest approach is to start from the ordinary Fourier analysis at circle or that for a particle in a one-dimensional box. The function basis uses as the basic building blocks the functions $e^{in\phi}$ in the case of circle and functions $e^{in\pi x/L}$ in the case of a particle in a box of side $L$.

The view about rationals as common to both reals and p-adics, and the possibility of finite-dimensional extensions of p-adics generated by the roots $e^{i2\pi/p^k}$ suggest how to realize this idea.

1. Consider first the case of the circle. Fix some value of $N$ and select a set of points $\phi_n = in2\pi/p^k$ at which the phases are defined meaning $p^{k+1}$-dimensional algebraic extension. That powers of $p$ appear is consistent with p-adic fractality. If so spin 1/2 *resp.* spin 1 particles would be inherently 2-adic *resp.* 3-adic. The plane wave basis corresponds $exp(ik\phi_n)$, $k = 0, ..., N-1$. In the case of particle in the one-dimensional box such that $L$ corresponds to a rational number, the box is decomposed into $N$ intervals of length $L/N$.

2. One can assign to the phases a well defined angular momentum as integer $n = 0, ..., N-1$ whereas the momentum spectrum for a particle in a box are given by $n\pi/L$. It is possible to continue the phase factor to the neighborhood of each point by requiring that the differential equation

$$\frac{d}{dx}exp(ikx) = ikexp(ikx)$$

defining the exponential function is satisfied.

3. The inner product of the plane waves $f_{k_1}$) and $f_{k_2}$ can be defined as the sum

$$\langle k_1 \rangle \equiv \sum_n \overline{f}_{k_1}(x_n) f_{k_2}(x_n) \ , \tag{5.4}$$

and orthogonality and completeness differ by no means from those of ordinary Fourier analysis.

### 5.3.2 p-Adic Fourier analysis, Pythagorean phases, and Gaussian primes

An alternative approach is based on Pythagorean phases and discretization in x-space, which might be a natural thing to do if p-adic field theory is taken as a cognitive model rather than "real" physics. This is also natural because rational Minkowski space is in the algebraic approach the fundamental object and reals and p-adics emerge as its completions.

Rational phase factors are common to the complexified p-adics ($p \ mod \ 4 = 3$) and reals and this suggests that one should define p-adic plane waves so that their values are in the set of the Pythagorean phases. Pythagorean phases are in one-one correspondence with the phases of the squares of Gaussian integers $N_G$ and thus generated as products of squares of Gaussian primes $\pi_G$, which are complex integers with modulus squared equal to prime $p \ mod \ 4 = 1$. Thus the set of phases $\phi(\pi_G)$ for the phases for $\pi_G^2$ form an algebraically infinite-dimensional linear space in the sense that the phases representable as superpositions

$$2\phi_G = \sum_{\pi_G} n_{\pi_G} 2\phi(\pi_G)$$

of these phases with integer coefficients belong to the set.

Consider now the definition of the plane wave basis based on Pythagorean phases and the identification of the p-adics and reals via common rationals.

1. Let $x_0 = q = m/n$ denote a value of $x$-coordinate and let $k$ denote some value of momentum. If $exp(ikx_0)$ is a Pythagorean phase then also the multiples $nk$ correspond to Pythagorean phases. $k$ itself cannot be a rational number so that $k$ is not defined as an ordinary p-adic number: this could be seen as a defect of the approach since one cannot speak of a well-defined momentum. Neither can $k$ be a rational multiple of $\pi$ so that Pythagorean phases have nothing to do with the phases defined by algebraic extensions containing the phase $exp(i\pi/n)$ already discussed.

   For a given value of $x_0 = q$ the momenta $k$ for which $exp(ikq)$ is a Pythagorean phase are in one-one correspondence with Pythagorean phases. Moreover, Pythagorean phases result in the lattice defined by the multiples of the $x_0$. Thus a natural definition of the p-adic plane waves emerges predicting a maximal momentum spectrum with one-one correspondence with Pythagorean phases, and selecting a preferred lattice of points at the real axis. This definition is also in accordance with the idea that p-adic plane waves are related with a cognitive representation for real physics.

2. Pythagorean phases are in one-one correspondence with the phase factors associated with the squares of the Gaussian integers and generating phases correspond to the phases $\phi(\pi_G)$ associated with the squares of Gaussian primes $\pi_G$. The moduli squared for the Gaussian primes correspond to squares of rational primes $p \ mod \ 4 = 1$. Thus set of allowed momenta $k_G$ for given spatial resolution $m/n$ is the set

$$\{k_G(q)\} = \{\tfrac{2\phi_G/}{q} + \tfrac{2\pi n}{q} | n \in Z\} \ ,$$

$$\{\phi_G\} = \{\textstyle\sum_{\pi_G} n_{\pi_G}\phi(\pi_G)\} \ .$$

   When the spatial resolution $x_0 = q$ is replaced with $q_1 = r/s$, the spectrum is scaled by a rational factor $q/q_1$. The set of momenta is a dense subset of the real axis. There is no observable difference between the real momenta differing by a multiple of $2\pi/q$ and one must drop them from consideration. This conclusion is forced also by the fact that p-adically the momenta $k = nk_0$ do not exist, it is only the phase factors which exist.

3. It is easy to see that the p-adic plane waves with different momenta are orthogonal to each other as complex rational numbers:

$$\sum_n exp\left[in(k_G(1) - k_G(2))\right] = 0 \ \ .$$

4. Also completeness relations are satisfied in the sense that the condition

$$\sum_{k_G} exp\left[i(n_1 - n_2)k_G\right] = 0$$

is satisfied for $n_1 \neq n_2$. This is due to the fact that all integer multiples of $k_G$ define Pythagorean phases. This means that the Fourier series of a function with respect to Pythagorean phases makes sense and one can expand p-adic-valued functions of space-time coordinates as Fourier series using Pythagorean phases. In particle expansion of the imbedding space coordinates as functions of p-adic space-time coordinates might be carried out in this manner.

5. One can criticize this approach for the fact that there is no unique continuation of the phase factors from the set of the rationals $x_n = nx_0$ to p-adic numbers neighborhoods of these points. Although eigen states of finite translations are in question one cannot regard the states as eigen states of infinitesimal translations since the momenta are not well defined as p-adic numbers. One could of course arbitrarily assign momentum eigenstate $e^{in\pi(x-x_k)}$ the point $x_k$ to the eigenstate characterized by the dimensionless momentum $n$ but the momentum spectrum associated with different Pythagorean phases would be same.

## 5.4 How To Define Integration And P-Adic Fourier Analysis, Integral Calculus, And P-Adic Counterparts Of Geometric Objects?

p-Adic differential calculus exists and obeys essentially the same rules as ordinary differential calculus. The only difference from real context is the existence of p-adic pseudo-constants: any function which depends on finite number of pinary digits has vanishing p-adic derivative. This implies non-determinism of p-adic differential equations. One can defined p-adic integral functions using the fact that indefinite integral is the inverse of differentiation. The basis problem with the definite integrals is that p-adic numbers are not well-ordered so that the crucial ordering of the points of real axis in definite integral is not unique. Also p-adic Fourier analysis is problematic since direct counterparts of ep(ix) and trigonometric functions are not periodic. Also exp(-x) fails to converse exponentially since it has p-adic norm equal to 1. Note also that these functions exists only when the p-adic norm of $x$ is smaller than 1.

The following considerations support the view that the p-adic variant of a geometric objects, integration and p-adic Fourier analysis exists but only when one considers highly symmetric geometric objects such as symmetric spaces. This is welcome news from the point of view of physics. At the level of space-time surfaces this is problematic. The field equations associated with Kähler action and Kähler-Dirac equation make sense. Kähler action defined as integral over p-adic space-time surface fails to exist. If however the Kähler function identified as Kähler for a preferred extremal of Kähler action is rational or algebraic function of preferred complex coordinates of WCW with rational coefficients, its p-adic continuation is expected to exist.

### 5.4.1 Circle with rotational symmetries and its hyperbolic counterparts

Consider first circle with emphasis on symmetries and Fourier analysis.

1. In this case angle coordinate $\phi$ is the natural coordinate. It however does not make sense as such p-adically and one must consider either trigonometric functions or the phase $exp(i\phi)$ instead. If one wants to do Fourier analysis on circle one must introduce roots $U_{n,N} =$

$exp(in2\pi/N)$ of unity. This means discretization of the circle. Introducing all roots $U_{n,p} = exp(i2\pi n/p)$, such that $p$ divides $N$, one can represent all $U_{k,n}$ up to $n = N$. Integration is naturally replaced with sum by using discrete Fourier analysis on circle. Note that the roots of unity can be expressed as products of powers of roots of unity $exp(in2\pi/p^k)$, where $p^k$ divides $N$.

2. There is a number theoretical delicacy involved. By Fermat's theorem $a^{p-1}\ mod\ p = 1$ for $a = 1, ...p-1$ for a given p-adic prime so that for any integer $M$ divisible by a factor of $p-1$ the $M$: th roots of unity exist as ordinary p-adic numbers. The problem disappears if these values of $M$ are excluded from the discretization for a given value of the p-adic prime. The manner to achieve this is to assume that $N$ contains no divisors of $p-1$ and is consistent with the notion of finite measurement resolution. For instance, $N = p^n$ is an especially natural choice guaranteeing this.

3. The p-adic integral defined as a Fourier sum does not reduce to a mere discretization of the real integral. In the real case the Fourier coefficients must approach to zero as the wave vector $k = n2\pi/N$ increases. In the p-adic case the condition consistent with the notion of finite measurement resolution for angles is that the p-adic valued Fourier coefficients approach to zero as $n$ increases. This guarantees the p-adic convergence of the discrete approximation of the integral for large values of $N$ as $n$ increases. The map of p-adic Fourier coefficients to real ones by canonical identification could be used to relate p-adic and real variants of the function to each other.

This finding would suggests that p-adic geometries -in particular the p-adic counterpart of $CP_2$, are discrete. Variables which have the character of a radial coordinate are in natural manner p-adically continuous whereas phase angles are naturally discrete and described in terms of algebraic extensions. The conclusion is disappointing since one can quite well argue that the discrete structures can be regarded as real. Is there any manner to escape this conclusion?

1. Exponential function $exp(ix)$ exists p-adically for $|x|_p \leq 1/p$ but is not periodic. It provides representation of p-adic variant of circle as group $U(1)$. One obtains actually a hierarchy of groups $U(1)_{p,n}$ corresponding to $|x|_p \leq 1/p^n$. One could consider a generalization of phases as products $Exp_p(N, n2\pi/N + x) = exp(in2\pi n/N)exp(ix)$ of roots of unity and exponent functions with an imaginary exponent. This would assign to each root of unity p-adic continuum interpreted as the analog of the interval between two subsequent roots of unity at circle. The hierarchies of measurement resolutions coming as $2\pi/p^n$ would be naturally accompanied by increasingly smaller p-adic groups $U(1)_{p,n}$.

2. p-Adic integration would involve summation plus possibly also an integration over each p-adic variant of discretization interval. The summation over the roots of unity implies that the integral of $\int exp(inx)dx$ would appear for $n = 0$. Whatever the value of this integral is, it is compensated by a normalization factor guaranteeing orthonormality.

3. If one interprets the p-adic coordinate as p-adic integer without the identification of points differing by a multiple of $n$ as different points the question whether one should require p-adic continuity arises. Continuity is obtained if $U_n(x+mp^m) = U_n(x)$ for large values of $m$. This is obtained if one has $n = p^k$. In the spherical geometry this condition is not needed and would mean quantization of angular momentum as $L = p^k$, which does not look natural. If representations of translation group are considered the condition is natural and conforms with the spirit of the p-adic length scale hypothesis.

The hyperbolic counterpart of circle corresponds to the orbit of point under Lorentz group in two 2-D Minkowski space. Plane waves are replaced with exponentially decaying functions of the coordinate $\eta$ replacing phase angle. Ordinary exponent function $exp(x)$ has unit p-adic norm when it exists so that it is not a suitable choice. The powers $p^n$ existing for p-adic integers however approach to zero for large values of $x = n$. This forces discretization of $\eta$ or rather the hyperbolic phase as powers of $p^x$, $x = n$. Also now one could introduce products of $Exp_p(nlog(p) + z) = p^n exp(x)$ to achieve a p-adic continuum. Also now the integral over the discretization interval is compensated by orthonormalization and can be forgotten. The integral of exponential function would reduce to a sum $\int Exp_p dx = \sum_k p^k = 1/(1-p)$. One can also introduce finite-dimensional but non-algebraic extensions of p-adic numbers allowing $e$ and its roots $e^{1/n}$ since $e^p$ exists p-adically.

### 5.4.2 Plane with translational and rotational symmetries

Consider first the situation by taking translational symmetries as a starting point. In this case Cartesian coordinates are natural and Fourier analysis based on plane waves is what one wants to define. As in the previous case, this can be done using roots of unity and one can also introduce p-adic continuum by using the p-adic variant of the exponent function. This would effectively reduce the plane to a box. As already noticed, in this case the quantization of wave vectors as multiples of $1/p^k$ is required by continuity.

One can take also rotational symmetries as a starting point. In this case cylindrical coordinates $(\rho, \phi)$ are natural.

1. Radial coordinate can have arbitrary values. If one wants to keep the connection $\rho = \sqrt{x^2+y^2}$ with the Cartesian picture square root allowing extension is natural. Also the values of radial coordinate proportional to odd power of $p$ are problematic since one should introduce $\sqrt{p}$: is this extension internally consistent? Does this mean that the points $\rho \propto p^{2n+1}$ are excluded so that the plane decomposes to annuli?

2. As already found, angular momentum eigen states can be described in terms of roots of unity and one could obtain continuum by allowing also phases defined by p-adic exponent functions.

3. In radial direction one should define the p-adic variants for the integrals of Bessel functions and they indeed might make sense by algebraic continuation if one consistently defines all functions as Fourier expansions. Delta-function renormalization causes technical problems for a continuum of radial wave vectors. One could avoid the problem by using exponentially decaying variants of Bessel function in the regions far from origin, and here the already proposed description of the hyperbolic counterparts of plane waves is suggestive.

4. One could try to understand the situation also using Cartesian coordinates. In the case of sphere this is achieved by introducing two coordinate patches with Cartesian coordinates. Pythagorean phases are rational phases (orthogonal triangles for which all sides are integer valued) and form a dense set on circle. Complex rationals (orthogonal triangles with integer valued short sides) define a more general dense subset of circle. In both cases it is difficult to imagine a discretized version of integration over angles since discretization with constant angle increment is not possible.

### 5.4.3 The case of sphere and more general symmetric space

In the case of sphere spherical coordinates are favored by symmetry considerations. For spherical coordinates $sin(\theta)$ is analogous to the radial coordinate of plane. Legedre polynomials expressible as polynomials of $sin(\theta)$ and $cos(\theta)$ are expressible in terms of phases and the integration measure $sin^2(\theta)d\theta d\phi$ reduces the integral of $S^2$ to summation. As before one can introduce also p-adic continuum. Algebraic cutoffs in both angular momentum $l$ and $m$ appear naturally. Similar cutoffs appear in the representations of quantum groups and there are good reasons to expect that these phenomena are correlated.

Exponent of Kähler function appears in the integration over WCW . From the expression of Kähler gauge potential given by $A_\alpha = J_\alpha{}^\theta \partial_\theta K$ one obtains using $A_\alpha = cos(\theta)\delta_{\alpha,\phi}$ and $J_{\theta\phi} = sin(\theta)$ the expression $exp(K) = sin(\theta)$. Hence the exponent of Kähler function is expressible in terms of spherical harmonics.

The completion of the discretized sphere to a p-adic continuum- and in fact any symmetric space- could be performed purely group theoretically.

1. Exponential map maps the elements of the Lie-algebra to elements of Lie-group. This recipe generalizes to arbitrary symmetric space $G/H$ by using the Cartan decomposition $g = t + h$, $[h,h] \subset h$, $[h,t] \subset t$, $[t,t] \subset h$. The exponentiation of $t$ maps $t$ to $G/H$ in this case. The

exponential map has a p-adic generalization obtained by considering Lie algebra with coefficients with p-adic norm smaller than one so that the p-adic exponent function exists. As a matter fact, one obtains a hierarchy of Lie-algebras corresponding to the upper bounds of the p-adic norm coming as $p^{-k}$ and this hierarchy naturally corresponds to the hierarchy of angle resolutions coming as $2\pi/p^k$. By introducing finite-dimensional transcendental extensions containing roots of $e$ one obtains also a hierarchy of p-adic Lie-algebras associated with transcendental extensions.

2. In particular, one can exponentiate the complement of the $SO(2)$ sub-algebra of $SO(3)$ Lie-algebra in p-adic sense to obtain a p-adic completion of the discrete sphere. Each point of the discretized sphere would correspond to a p-adic continuous variant of sphere as a symmetric space. Similar construction applies in the case of $CP_2$. Quite generally, a kind of fractal or holographic symmetric space is obtained from a discrete variant of the symmetric space by replacing its points with the p-adic symmetric space.

3. In the N-fold discretization of the coordinates of M-dimensional space $t$ one $(N-1)^M$ discretization volumes which is the number of points with non-vanishing $t$-coordinates. It would be nice if one could map the p-adic discretization volumes with non-vanishing $t$-coordinates to their positive valued real counterparts by applying canonical identification. By group invariance it is enough to show that this works for a discretization volume assignable to the origin. Since the p-adic numbers with norm smaller than one are mapped to the real unit interval, the p-adic Lie algebra is mapped to the unit cell of the discretization lattice of the real variant of $t$. Hence by a proper normalization this mapping is possible.

The above considerations suggest that the hierarchies of measurement resolutions coming as $\Delta\phi = 2\pi/p^n$ are in a preferred role. One must be however cautious in order to avoid too strong assumptions. The following arguments however support this identification.

1. The vision about p-adicization characterizes finite measurement resolution for angle measurement in the most general case as $\Delta\phi = 2\pi M/N$, where $M$ and $N$ are positive integers having no common factors. The powers of the phases $exp(i2\pi M/N)$ define identical Fourier basis irrespective of the value of $M$ unless one allows only the powers $exp(i2\pi kM/N)$ for which $kM < N$ holds true: in the latter case the measurement resolutions with different values of $M$ correspond to different numbers of Fourier components. Otherwise the measurement resolution is just $\Delta\phi = 2\pi/p^n$. If one regards $N$ as an ordinary integer, one must have $N = p^n$ by the p-adic continuity requirement.

2. One can also interpret $N$ as a p-adic integer and assume that state function reduction selects one particular prime (no superposition of quantum states with different p-adic topologies). For $N = p^n M$, where $M$ is not divisible by $p$, one can express $1/M$ as a p-adic integer $1/M = \sum_{k\geq 0} M_k p^k$, which is infinite as a real integer but effectively reduces to a finite integer $K(p) = \sum_{k=0}^{N-1} M_k p^k$. As a root of unity the entire phase $exp(i2\pi M/N)$ is equivalent with $exp(i2\pi R/p^n)$, $R = K(p)M \mod p^n$. The phase would non-trivial only for p-adic primes appearing as factors in $N$. The corresponding measurement resolution would be $\Delta\phi = R2\pi/N$. One could assign to a given measurement resolution all the p-adic primes appearing as factors in $N$ so that the notion of multi-p p-adicity would make sense. One can also consider the identification of the measurement resolution as $\Delta\phi = |N/M|_p = 2\pi/p^k$. This interpretation is supported by the approach based on infinite primes [K11].

### 5.4.4 What about integrals over partonic 2-surfaces and space-time surface?

One can of course ask whether also the integrals over partonic 2-surfaces and space-time surface could be p-adicized by using the proposed method of discretization. Consider first the p-adic counterparts of the integrals over the partonic 2-surface $X^2$.

1. WCW Hamiltonians and Kähler form are expressible using flux Hamiltonians defined in terms of $X^2$ integrals of $JH_A$, where $H_A$ is $\delta CD \times CP_2$ Hamiltonian, which is a rational function of the preferred coordinates defined by the exponentials of the coordinates of the sub-space $t$ in the appropriate Cartan algebra decomposition. The flux factor $J = \epsilon^{\alpha\beta} J_{\alpha\beta}\sqrt{g_2}$ is scalar and does not actually depend on the induced metric.

2. The notion of finite measurement resolution would suggest that the discretization of $X^2$ is somehow induced by the discretization of $\delta CD \times CP_2$. The coordinates of $X^2$ could be taken to be the coordinates of the projection of $X^2$ to the sphere $S^2$ associated with $\delta M^4_\pm$ or to the homologically non-trivial geodesic sphere of $CP_2$ so that the discretization of the integral would reduce to that for $S^2$ and to a sum over points of $S^2$.

3. To obtain an algebraic number as an outcome of the summation, one must pose additional conditions guaranteeing that both $H_A$ and $J$ are algebraic numbers at the points of discretization (recall that roots of unity are involved). Assume for definiteness that $S^2$ is $r_M = constant$ sphere. If the remaining preferred coordinates are functions of the preferred $S^2$ coordinates mapping phases to phases at discretization points, one obtains the desired outcome. These conditions are rather strong and mean that the various angles defining $CP_2$ coordinates -at least the two cyclic angle coordinates- are integer multiples of those assignable to $S^2$ at the points of discretization. This would be achieved if the preferred complex coordinates of $CP_2$ are powers of the preferred complex coordinate of $S^2$ at these points. One could say that $X^2$ is algebraically continued from a rational surface in the discretized variant of $\delta CD \times CP_2$. Furthermore, if the measurement resolutions come as $2\pi/p^n$ as p-adic continuity actually requires and if they correspond to the p-adic group $G_{p,n}$ for which group parameters satisfy $|t|_p \leq p^{-n}$, one can precisely characterize how a p-adic prime characterizes the real partonic 2-surface. This would be a fulfilment of one of the oldest dreams related to the p-adic vision.

A even more ambitious dream would be that even the integral of the Kähler action for preferred extremals could be defined using a similar procedure. The conjectured slicing of Minkowskian space-time sheets by string world sheets and partonic 2-surfaces encourages these hopes.

1. One could introduce local coordinates of $H$ at both ends of CD by introducing a continuous slicing of $M^4 \times CP_2$ by the translates of $\delta M^4_\pm \times CP_2$ in the direction of the time-like vector connecting the tips of CD. As space-time coordinates one could select four of the eight coordinates defining this slicing. For instance, for the regions of the space-time sheet representable as maps $M^4 \rightarrow CP_2$ one could use the preferred $M^4$ time coordinate, the radial coordinate of $\delta M^4_+$, and the angle coordinates of $r_M = constant$ sphere.

2. Kähler action density should have algebraic values and this would require the strengthening of the proposed conditions for $X^2$ to apply to the entire slicing meaning that the discretized space-time surface is a rational surface in the discretized $CD \times CP_2$. If this condition applies to the entire space-time surface it would effectively mean the discretization of the classical physics to the level of finite geometries. This seems quite strong implication but is consistent with the preferred extremal property implying the generalized Bohr rules. The reduction of Kähler action to 3-dimensional boundary terms is implied by rather general arguments. In this case only the effective algebraization of the 3-surfaces at the ends of CD and of wormhole throats is needed [K6]. By effective 2-dimensionality these surfaces cannot be chosen freely.

3. If Kähler function and WCW Hamiltonians are rational functions, this kind of additional conditions are not necessary. It could be that the integrals of defining Kähler action flux Hamiltonians make sense only in the intersection of real and p-adic worlds assumed to be relevant for the physics of living systems.

### 5.4.5 Tentative conclusions

These findings suggest following conclusions.

1. Exponent functions play a key role in the proposed p-adicization. This is not an accident since exponent functions play a fundamental role in group theory and p-adic variants of real geometries exist only under symmetries- possibly maximal possible symmetries- since otherwise the notion of Fourier analysis making possible integration does not exist. The inner product defined in terms of integration reduce for functions representable in Fourier basis to sums and can be carried out by using orthogonality conditions. Convolution involving integration reduces to a product for Fourier components. In the case of imbedding space and WCW these conditions are satisfied but for space-time surfaces this is not possible.

2. There are several manners to choose the Cartan algebra already in the case of sphere. In the case of plane one can consider either translations or rotations and this leads to different p-adic variants of plane. Also the realization of the hierarchy of Planck constants leads to the conclusion that the extended imbedding space and therefore also WCW contains sectors corresponding to different choices of quantization axes meaning that quantum measurement has a direct geometric correlate.

3. The above described 2-D examples represent symplectic geometries for which one has natural decomposition of coordinates to canonical pairs of cyclic coordinate (phase angle) and corresponding canonical conjugate coordinate. p-Adicization depends on whether the conjugate corresponds to an angle or non-compact coordinate. In both cases it is however possible to define integration. For instance, in the case of $CP_2$ one would have two canonically conjugate pairs and one can define the p-adic counterparts of $CP_2$ partial waves by generalizing the procedure applied to spherical harmonics. Products of functions expressible using partial waves can be decomposed by tensor product decomposition to spherical harmonics and can be integrated. In particular inner products can be defined as integrals. The Hamiltonians generating isometries are rational functions of phases: this inspires the hope that also WCW Hamiltonians also rational functions of preferred WCW coordinates and thus allow p-adic variants.

4. Discretization by introducing algebraic extensions is unavoidable in the p-adicization of geometrical objects but one can have p-adic continuum as the analog of the discretization interval and in the function basis expressible in terms of phase factors and p-adic counterparts of exponent functions. This would give a precise meaning for the p-adic counterparts of the imbedding space and WCW if the latter is a symmetric space allowing coordinatization in terms of phase angles and conjugate coordinates.

5. The intersection of p-adic and real worlds would be unique and correspond to the points defining the discretization.

# 6 Generalization Of Riemann Geometry

Geometrization of physics program requires Riemann geometry and its variants such as Kähler geometry in the p-adic context. The notion of the p-adic space-time surface and its relationship to its real counterpart should be also understood. In this section the basic problems and ideas related to these challenges are discussed.

## 6.1 P-Adic Riemannian Geometry Depends On CognitiveRepresentation

p-Adic Riemann geometry is a direct formal generalization of the ordinary Riemann geometry. In the minimal purely algebraic generalization one does not try to define concepts like arch length and volume involving definite integrals but simply defines the p-adic geometry via the metric identified as a quadratic form in the tangent space of the p-adic manifold. Canonical identification would make it possible to define p-adic variant of Riemann integral formally allowing to calculate arc lengths and similar quantities but looks like a trick. The realization that the p-adic variant of harmonic analysis makes it possible to define definite integrals in the case of symmetric space became possible only after a detailed vision about what quantum TGD is [K15] had emerged.

Symmetry considerations dictate the p-adic counterpart of the Riemann geometry for $M^4_+ \times CP_2$ to a high degree but not uniquely. This non-uniqueness might relate to the distinction between different cognitive representations. For instance, in the case of Euclidian plane one can introduce linear or cylindrical coordinates and the manifest symmetries dictating the preferred coordinates correspond to translational and rotational symmetries in these two cases and give rise to different p-adic variants of the plane. Both linear and cylindrical coordinates are fixed only modulo the action of group consisting of translations and rotations and the degeneracy of choices can be interpreted in terms of a choice of quantization axes of angular momentum and momenta.

The most natural looking manner to define the p-adic counterpart of $M^4$ is by using a p-adic completion for a subset of rational points in coordinates which are preferred on physical basis. In case of $M^4$ linear Minkowski coordinates are an obvious choice but also the counterparts of Robertson-Walker coordinates for $M^4_+$ defined as $[t,(z,x,y)] = a\times[cosh(\eta), sinh(\eta)(cos(\theta), sin(\theta)cos(\phi), sin(\theta)sin($ expressible in terms of phases and their hyperbolic counterparts and transforming nicely under the Cartan algebra of Lorentz group are possible. p-Adic variant is obtained by introducing finite measurement resolution for angle and replacing angle range by finite number of roots of unity. Same applies to hyperbolic angles.

Rational $CP_2$ could be defined as a coset space $SU(3,Q)/U(2,Q)$ associated with complex rational unitary $3\times3$-matrices. $CP_2$ could be defined as coset space of complex rational matrices by choosing one point in each coset $SU(3,Q)/U(2,Q)$ as a complex rational $3\times3$-matrix representable in terms of Pythagorean phases [A2] and performing a completion for the elements of this matrix by multiplying the elements with the p-adic exponentials $exp(iu)$, $|u|_p < 1$ such that one obtains p-adically unitary matrix.

This option is not very natural as far as integration is considered. $CP_2$ however allows the analog of spherical coordinates for $S^2$ expressible in terms of angle variables alone and this suggests the introduction of the variant of $CP_2$ for which the coordinate values correspond to roots of unity. Completion would be performed in the same manner as for rational $CP_2$. This non-uniqueness need not be a drawback but could reflect the fact that the p-adic cognitive representation of real geometry are geometrically non-equivalent. This means a refinement of the principle of General Coordinate Invariance taking into account the fact that the cognitive representation of the real world affects the world with cognition included in a delicate manner.

## 6.2 P-Adic Imbedding Space

The construction of both quantum TGD and p-adic QFT limit requires p-adicization of the imbedding space geometry. Also the fact that p-adic Poincare invariance throws considerable light to the p-adic length scale hypothesis suggests that p-adic geometry is really needed. The construction of the p-adic version of the imbedding space geometry and spinor structure relies on the symmetry arguments and to the generalization of the analytic formulas of the real case almost. The essential element is the notion of finite measurement resolution leading to discretization in large and to p-adicization below the resolution scale. This approach leads to a highly nontrivial generalization of the symmetry concept and p-adic Poincare invariance throws light to the p-adic length scale hypothesis. An important delicacy is related to the identification of the fundamental p-adic length scale, which corresponds to the unit element of the p-adic number field and is mapped to the unit element of the real number field in the canonical identification mapping p-adic mass squared to its real counterpart.

### 6.2.1 The identification of the fundamental p-adic length scale

The fundamental p-adic length scale corresponds to the p-adic unit $e = 1$ and is mapped to the unit of the real numbers in the canonical identification. The correct physical identification of the fundamental p-adic length scale is of crucial importance since the predictions of the theory for p-adic masses depend on the choice of this scale.

In TGD the "radius" $R$ of $CP_2$ is the fundamental length scale ($2\pi R$ is by definition the length of the $CP_2$ geodesics). In accordance with the idea that p-adic QFT limit makes sense only above length scales larger than the radius of $CP_2$ $R$ is of same order of magnitude as the p-adic length scale defined as $l = \pi/m_0$, where $m_0$ is the fundamental mass scale and related to the "cosmological constant" $\Lambda$ ($R_{ij} = \Lambda s_{ij}$) of $CP_2$ by

$$m_0^2 = 2\Lambda . \quad (6.1)$$

The relationship between $R$ and $l$ is uniquely fixed:

$$R^2 = \frac{3}{m_0^3} = \frac{3}{2\Lambda} = \frac{3l^2}{\pi^2} . \quad (6.2)$$

Consider now the identification of the fundamental length scale.

1. One must use $R^2$ or its integer multiple, rather than $l^2$, as the fundamental p-adic length scale squared in order to avoid the appearance of the p-adically ill defined $\pi$: s in various formulas of $CP_2$ geometry.

2. The identification for the fundamental length scale as $1/m_0$ leads to difficulties.

   (a) The p-adic length for the $CP_2$ geodesic is proportional to $\sqrt{3}/m_0$. For the physically most interesting p-adic primes satisfying $p\ mod\ 4 = 3$ so that $\sqrt{-1}$ does not exist as an ordinary p-adic number, $\sqrt{3} = i\sqrt{-3}$ belongs to the complex extension of the p-adic numbers. Hence one has troubles in getting real length for the $CP_2$ geodesic.

   (b) If $m_0^2$ is the fundamental mass squared scale then general quark states have mass squared, which is integer multiple of $1/3$ rather than integer valued as in string models.

3. These arguments suggest that the correct choice for the fundamental length scale is as $1/R$ so that $M^2 = 3/R^2$ appearing in the mass squared formulas is p-adically real and all values of the mass squared are integer multiples of $1/R^2$. This does not affect the real counterparts of the thermal expectation values of the mass squared in the lowest p-adic order but the effects, which are due to the modulo arithmetics, are seen in the higher order contributions to the mass squared. As a consequence, one must identify the p-adic length scale $l$ as

$$l \equiv \pi R \ ,$$

   rather than $l = \pi/m_0$. This is indeed a very natural identification. What is especially nice is that this identification also leads to a solution of some longstanding problems related to the p-adic mass calculations. It would be highly desirable to have the same p-adic temperature $T_p = 1$ for both the bosons and fermions rather than $T_p = 1/2$ for bosons and $T_p = 1$ for fermions. For instance, black hole elementary particle analogy as well as the need to get rid of light boson exotics suggests this strongly. It indeed turns out possible to achieve this with the proposed identification of the fundamental mass squared scale.

### 6.2.2 p-Adic counterpart of $M^4_+$

The construction of the p-adic counterpart of $M^4_+$ seems a relatively straightforward task and should reduce to the construction of the p-adic counterpart of the real axis with the standard metric. As already noticed, linear Minkowksi coordinates are physically and mathematically preferred coordinates and it is natural to construct the metric in these coordinates.

There are some quite interesting delicacies related to the p-adic version of the Poincare invariance. Consider first translations. In order to have imaginary unit needed in the construction of the ordinary representations of the Poincare group one must have $p\ mod\ 4 = 3$ to guarantee that $\sqrt{-1}$ does not exist as an ordinary p-adic number. It however seems that the construction of the representations is at least formally possible by replacing imaginary unit with the square root of some other p-adic number not existing as a p-adic number.

It seems that only the discrete group of translations allows representations consisting of orthogonal plane waves. p-Adic plane waves can be defined in the lattice consisting of the multiples of $x_0 = m/n$ consisting of points with p-adic norm not larger that $|x_0|_p$ and the points $p^n x_0$ define fractally scaled-down versions of this set. In canonical identification these sets corresponds to volumes scaled by factors $p^{-n}$.

A physically interesting question is whether the Lorentz group should contain only the elements obtained by exponentiating the Lie-algebra generators of the Lorentz group or whether also large Lorentz transformations, containing as a subgroup the group of the rational Lorentz transformations, should be allowed. If the group contains only small Lorentz transformations, the quantization volume of $M^4_+$ (say the points with coordinates $m^k$ having p-adic norm not larger than one) is also invariant under Lorentz transformations. This means that the quantization of the theory in the p-adic cube $|m^k| < p^n$ is a Poincare invariant procedure unlike in the real case.

The appearance of the square root of $p$, rather than the naively expected $p$, in the expression of the p-adic length scale can be understood if the p-adic version of $M^4$ metric contains $p$ as a scaling factor:

$$\begin{aligned} ds^2 &= pR^2 m_{kl} dm^k dm^l \ , \\ R &\leftrightarrow 1 \ , \end{aligned} \tag{6.2}$$

where $m_{kl}$ is the standard $M^4$ metric $(1,-1,-1,-1)$. The p-adic distance function is obtained by integrating the line element using p-adic integral calculus and this gives for the distance along the k: th coordinate axis the expression

$$s = R\sqrt{p}m^k \ . \tag{6.3}$$

The map from p-adic $M^4$ to real $M^4$ is canonical identification plus a scaling determined from the requirement that the real counterpart of an infinitesimal p-adic geodesic segment is same as the length of the corresponding real geodesic segment:

$$m^k \rightarrow \pi(m^k)_R \ . \tag{6.4}$$

The p-adic distance along the k: th coordinate axis from the origin to the point $m^k = (p-1)(1+p+p^2+...) = -1$ on the boundary of the set of the p-adic numbers with norm not larger than one, corresponds to the fundamental p-adic length scale $L_p = \sqrt{p}l = \sqrt{p}\pi R$:

$$\sqrt{p}((p-1)(1+p+...))R \rightarrow \pi R\frac{(p-1)(1+p^{-1}+p^{-2}+...)}{\sqrt{p}} = L_p \ . \tag{6.4}$$

What is remarkable is that the shortest distance in the range $m^k = 1,..m-1$ is actually $L/\sqrt{p}$ rather than $l$ so that p-adic numbers in range span the entire $R_+$ at the limit $p \rightarrow \infty$. Hence p-adic topology approaches real topology in the limit $p \rightarrow \infty$ in the sense that the length of the discretization step approaches to zero.

### 6.2.3 The two variants of $CP_2$

As noticed, $CP_2$ allows two variants based on rational discretization and on the discretiation based on roots of unity. The root of unity option corresponds to the phases associated with $1/(1+r^2) = tan^2(u/2) = (1-cos(u))/(1+cos(u))$ and implies that integrals of spherical harmonics can be reduced to summations when angular resolution $\Delta u = 2\pi/N$ is introduced. In the p-adic context, one can replace distances with trigonometric functions of distances along zig zag curves connecting the points of the discretization. Physically this notion of distance is quite reasonable since distances are often measured using interferometer.

In the case of rational variant of $CP_2$ one can proceed by defining the p-adic counterparts of $SU(3)$ and $U(2)$ and using the identification $CP_2 = SU(3)/U(2)$. The p-adic counterpart of $SU(3)$ consists of all $3\times 3$ unitary matrices satisfying p-adic unitarity conditions (rows/columns are mutually orthogonal unit vectors) or its suitable subgroup: the minimal subgroup corresponds to the exponentials of the Lie-algebra generators. If one allows algebraic extensions of the p-adic numbers, one obtains several extensions of the group. The extension allowing the square root of a p-adically real number is the most interesting one in this respect since the general solution of the unitarity conditions involves square roots.

The subgroup of $SU(3)$ obtained by exponentiating the Lie-algebra generators of $SU(3)$ normalized so that their non-vanishing elements have unit p-adic norm, is of the form

$$SU(3)_0 = \{x = exp(\sum_k it_k X_k) \ ; \ |t_k|_p < 1\} = \{x = 1+iy \ ; \ |y|_p < 1\} \ . \tag{6.5}$$

The diagonal elements of the matrices in this group are of the form $1 + O(p)$. In order $O(p)$ these matrices reduce to unit matrices.

Rational $SU(3)$ matrices do not in general allow a representation as an exponential. In the real case all $SU(3)$ matrices can be obtained from diagonalized matrices of the form

$$h = diag\{exp(i\phi_1), exp(i\phi_2), exp(exp(-i(\phi_1 + \phi_2)\} \ . \quad (6.6)$$

The exponentials are well defined provided that one has $|\phi_i|_p < 1$ and in this case the diagonal elements are of form $1 + O(p)$. For $p\ mod\ 4 = 3$ one can however consider much more general diagonal matrices

$$h = diag\{z_1, z_2, z_3\} \ ,$$

for which the diagonal elements are rational complex numbers

$$z_i = \frac{(m_i + in_i)}{\sqrt{m_i^2 + n_i^2}} \ ,$$

satisfying $z_1z_2z_3 = 1$ such that the components of $z_i$ are integers in the range $(0, p-1)$ and the square roots appearing in the denominators exist as ordinary p-adic numbers. These matrices indeed form a group as is easy to see. By acting with $SU(3)_0$ to each element of this group and by applying all possible automorphisms $h \to ghg^{-1}$ using rational $SU(3)$ matrices one obtains entire $SU(3)$ as a union of an infinite number of disjoint components.

The simplest (unfortunately not physical) possibility is that the "physical" $SU(3)$ corresponds to the connected component of $SU(3)$ represented by the matrices, which are unit matrices in order $O(p)$. In this case the construction of $CP_2$ is relatively straightforward and the real formalism should generalize as such. In particular, for $p\ mod\ 4 = 3$ it is possible to introduce complex coordinates $\xi_1, \xi_2$ using the complexification for the Lie-algebra complement of $su(2) \times u(1)$. The real counterparts of these coordinates vary in the range $[0, 1)$ and the end points correspond to the values of $t_i$ equal to $t_i = 0$ and $t_i = -p$. The p-adic sphere $S^2$ appearing in the definition of the p-adic light cone is obtained as a geodesic sub-manifold of $CP_2$ ($\xi_1 = \xi_2$ is one possibility). From the requirement that real $CP_2$ can be mapped to its p-adic counterpart it is clear that one must allow all connected components of $CP_2$ obtained by applying discrete unitary matrices having no exponential representation to the basic connected component. In practice this corresponds to the allowance of all possible values of the p-adic norm for the components of the complex coordinates $\xi_i$ of $CP_2$.

The simplest approach to the definition of the $CP_2$ metric is to replace the expression of the Kähler function in the real context with its p-adic counterpart. In standard complex coordinates for which the action of $U(2)$ subgroup is linear, the expression of the Kähler function reads as

$$\begin{aligned} K &= log(1 + r^2) \ , \\ r^2 &= \sum_i \bar{\xi}_i \xi_i \ . \end{aligned} \quad (6.6)$$

p-Adic logarithm exists provided $r^2$ is of order $O(p)$. This is the case when $\xi_i$ is of order $O(p)$. The definition of the Kähler function in a more general case, when all possible values of the p-adic norm are allowed for $r$, is based on the introduction of a p-adic pseudo constant $C$ to the argument of the Kähler function

$$K = log(\frac{1 + r^2}{C}) \ .$$

$C$ guarantees that the argument is of the form $\frac{1+r^2}{C} = 1 + O(p)$ allowing a well-defined p-adic logarithm. This modification of the Kähler function leaves the definition of Kähler metric, Kähler form and spinor connection invariant.

A more elegant manner to avoid the difficulty is to use the exponent $\Omega = exp(K) = 1 + r^2$ of the Kähler function instead of Kähler function, which indeed well defined for all coordinate values. In terms of $\Omega$ one can express the Kähler metric as

$$g_{k\bar{l}} = \frac{\partial_k \partial_{\bar{l}} \Omega}{\Omega} - \frac{\partial_k \Omega \partial_{\bar{l}} \Omega}{\Omega^2} \quad . \tag{6.7}$$

The p-adic metric can be defined as

$$s_{i\bar{j}} = R^2 \partial_i \partial_{\bar{j}} K = R^2 \frac{(\delta_{i\bar{j}} r^2 - \bar{\xi}_i \xi_j)}{(1+r^2)^2} \quad . \tag{6.7}$$

The expression for the Kähler form is the same as in the real case and the components of the Kähler form in the complex coordinates are numerically equal to those of the metric apart from the factor of $i$. The components in arbitrary coordinates can be deduced from these by the standard transformation formulas.

## 6.3 Topological Condensate As A Generalized Manifold

The ideas about how p-adic topology emerges from quantum TGD have varied. The first belief was that p-adic topology is only an effective topology of real space-time sheets. This belief turned out to be not quite correct. p-Adic topology emerges also as a genuine topology of the space-time and p-adic regions could be identified as correlates for cognition and intentionality. The vision about quantum TGD as a generalized number theory provides possible solutions to the basic problems associated with the precise definition of topological condensate.

### 6.3.1 Generalization of number concept and fusion of real and p-adic physics

The unification of real physics of material work and p-adic physics of cognition leads to the generalization of the notion of number field. Reals and various p-adic number fields are glued along their common rationals (and common algebraic numbers too) to form a fractal book like structure. Allowing all possible finite-dimensional extensions of p-adic numbers brings additional pages to this "Big Book".

This generalization leads to a generalization of the notion of manifold as a collection of a real manifold and its p-adic variants glued together along common rationals (see **Fig.** `http://tgdtheory.fi/appfigures/book.jpg` or **Fig. ??** in the appendix of this book ). The precise formulation involves of course several technical problems. For instance, should one glue along common algebraic numbers and Should one glue along common transcendentals such as $e^p$? Are algebraic extensions of p-adic number fields glued together along the algebraics too?

This notion of manifold implies a generalization of the notion of imbedding space. p-Adic transcendentals can be regarded as infinite numbers in the real sense and thus most points of the p-adic space-time sheets would be at infinite distance and real and p-adic space-time sheets would intersect in a discrete set consisting of rational points. This view in which cognition would be literally cosmic phenomena is in a sharp contrast with the often held belief that p-adic topology emerges below Planck length scale.

It took some time to end up with this vision. The first picture was based on the notion of real and p-adic space-time sheets glued together by using canonical identification or some of its variants but led to insurmountable difficulties since p-adic topology is so different from real topology. One can of course ask whether one can speak about p-adic counterparts of notions like boundary of 3-surface or genus of 2-surface crucial for TGD based model of family replication phenomenon. It seems that these notions generalize as purely algebraically defined concepts which supports the view that p-adicization of real physics must be a purely algebraic procedure.

### 6.3.2 How large p-adic space-time sheets can be?

Space-time region having finite size in the real sense can have arbitrarily large size in p-adic sense and vice versa. This raises a rather thought provoking questions. Could the p-adic space-time sheets have cosmological or even infinite size with respect to the real metric but have be p-adically finite? How large space-time surface is responsible for the p-adic representation of my body? Could the large or even infinite size of the cognitive space-time sheets explain why creatures of a finite physical size can invent the notion of infinity and construct cosmological theories? Could it be that pinary cutoff $O(p^n)$ defining the resolution of a p-adic cognitive representation would define the size of the space-time region needed to realize the cognitive representation?

In fact, the mere requirement that the neighborhood of a point of the p-adic space-time sheet contains points, which are p-adically infinitesimally near to it can mean that points infinitely distant from this point in the real sense are involved. A good example is provided by an integer valued point $x = n < p$ and the point $y = x+p^m$, $m > 0$: the p-adic distance of these points is $p^{-m}$ whereas at the limit $m \to \infty$ the real distance goes as $p^m$ and becomes infinite for infinitesimally near points. The points $n+y$, $y = \sum_{k>0} x_k p^k$, $0 < n < p$, form a p-adically continuous set around $x = n$. In the real topology this point set is discrete set with a minimum distance $\Delta x = p$ between neighboring points whereas in the p-adic topology every point has arbitrary nearby points. There are also rationals, which are arbitrarily near to each other both p-adically and in the real sense. Consider points $x = m/n$, $m$ and $n$ not divisible by $p$, and $y = (m/n) \times (1 + p^k r)/(1 + p^k s)$, $s = r + 1$ such that neither $r$ or $s$ is divisible by $p$ and $k >> 1$ and $r >> p$. The p-adic and real distances are $|x - y|_p = p^{-k}$ and $|x - y| \simeq (m/n)/(r + 1)$ respectively. By choosing $k$ and $r$ large enough the points can be made arbitrarily close to each other both in the real and p-adic senses.

The idea about infinite size of the p-adic cognitive space-time sheets providing representation of body and brain is consistent with TGD inspired theory of consciousness, which forces to take very seriously the idea that even human consciousness involves cosmic length scales.

### 6.3.3 What determines the p-adic primes assignable to a given real space-time sheet?

The p-adic realization of the Slaving Principle suggests that various levels of the topological condensate correspond to real matter like regions and p-adic mind like regions labelled by p-adic primes $p$. The larger the length scale, the larger the value of $p$ and the course the induced real topology. If the most interesting values of $p$ indeed correspond Mersenne primes, the number of most interesting levels is finite: at most 12 levels below electron length scale: actually also primes near prime powers of two seem to be physically important.

The intuitive expectation is that the p-adic prime associated with a given real space-time sheet characterizes its effective p-adic topology. As a matter fact, several p-adic effective topologies can be considered and the attractive hypothesis is that elementary particles are characterized by integers defined by the product of these p-adic primes and the integers for particles which can have direct interactions possess common prime factors.

The intuitive view is that those primes are favored for with the p-adic space-time sheet obtained by an algebraic continuation has as many rational or algebraic space-time points as possible in common with the real space-time sheet. The rationale is that if the real space-time sheet is generated in a quantum jump in which p-adic space-time sheet is transformed to a real one, it must have a large number of points in common with the real space-time sheet if the probability amplitude for this process involves a sum over the values of an n-point function of a conformal field theory over all common n-tuples and vanishes when the number of common points is smaller than $n$.

# 7 Appendix: P-Adic Square Root Function And Square Root Allowing Extension Of P-Adic Numbers

The following arguments demonstrate that the extension allowing square roots of ordinary p-adic numbers is 4-dimensional for $p < 2$ and 8-dimensional for $p = 2$.

## 7.1 $P > 2$ Resp. $P = 2$ Corresponds To $D = 4$ Resp. $D = 8$ Dimensional Extension

What is important is that only the square root of ordinary p-adic numbers is needed: the square root need not exist outside the real axis. It is indeed impossible to find a finite-dimensional extension allowing square root for all ordinary p-adic numbers numbers. For $p > 2$ the minimal dimension for algebraic extension allowing square roots near real axis is $D = 4$. For $p = 2$ the dimension of the extension is $D = 8$.

For $p > 2$ the form of the extension can be derived by the following arguments.

1. For $p > 2$ a p-adic number $y$ in the range $(0, p-1)$ allows square root only provided there exists a p-adic number $x \in \{0, p-1\}$ satisfying the condition $y = x^2 \ mod \ p$. Let $x_0$ be the smallest integer, which does not possess a p-adic square root and add the square root $\theta$ of $x_0$ to the number field. The numbers in the extension are of the form $x + \theta y$. The extension allows square root for every $x \in \{0, p-1\}$ as is easy to see. p-adic numbers *mod p* form a finite field $G(p, 1)$ [A4] so that any p-adic number $y$, which does not possess square root can be written in the form $y = x_0 u$, where $u$ possesses square root. Since $\theta$ is by definition the square root of $x_0$ then also $y$ possesses square root. The extension does not depend on the choice of $x_0$.

   The square root of $-1$ does not exist for $p \ mod \ 4 = 3$ [A3] and $p = 2$ but the addition of $\theta$ guarantees its existence automatically. The existence of $\sqrt{-1}$ follows from the existence of $\sqrt{p-1}$ implied by the extension by $\theta$. $\sqrt{(-1+p)-p}$ can be developed in power in powers of $p$ and series converges since the p-adic norm of coefficients in Taylor series is not larger than 1. If $p - 1$ does not possess a square root, one can take $\theta$ to be equal to $\sqrt{-1}$.

2. The next step is to add the square root of $p$ so that the extension becomes 4-dimensional and an arbitrary number in the extension can be written as

$$Z = (x + \theta y) + \sqrt{p}(u + \theta v) \ . \tag{7.1}$$

In $p = 2$ case 8-dimensional extension is needed to define square roots. The addition of $\sqrt{2}$ implies that one can restrict the consideration to the square roots of odd 2-adic numbers. One must be careful in defining square roots by the Taylor expansion of square root $\sqrt{x_0 + x_1}$ since $n$: th Taylor coefficient is proportional to $2^{-n}$ and possesses 2-adic norm $2^n$. If $x_0$ possesses norm 1 then $x_1$ must possess norm smaller than 1/8 for the series to converge. By adding square roots $\theta_1 = \sqrt{-1}, \theta_2 = \sqrt{2}$ and $\theta_3 = \sqrt{3}$ and their products one obtains 8-dimensional extension.

The emergence of the dimensions $D = 4$ and $D = 8$ for the algebraic extensions allowing the square root of an ordinary p-adic number stimulates an obvious question: could one regard space-time as this kind of an algebraic extension for $p > 2$ and the imbedding space $H = M^4_+ \times CP_2$ as a similar 8-dimensional extension of the 2-adic numbers? Contrary to the first expectations, it seems that algebraic dimension cannot be regarded as a physical dimension, and that quaternions and octonions provide the correct framework for understanding space-time and imbedding space dimensions. One could perhaps say that algebraic dimensions are additional dimensions of the world of cognitive physics rather than those of the real physics and there presence could perhaps explain why we can imagine all possible dimensions mathematically.

By construction, any ordinary p-adic number in the extension allows square root. The square root for an arbitrary number sufficiently near to p-adic axis can be defined through Taylor series expansion of the square root function $\sqrt{Z}$ at a point of p-adic axis. The subsequent considerations show that the p-adic square root function does not allow analytic continuation to $R^4$ and the points of the extension allowing a square root consist of disjoint converge cubes forming a structure resembling future light cone in certain respects.

## 7.2 P-Adic Square Root Function For $P > 2$

The study of the properties of the series representation of a square root function shows that the definition of the square root function is possible in certain region around the real p-adic axis. What is nice that this region can be regarded as the p-adic analog (not the only one) of the future light cone defined by the condition

$$N_p(Im(Z)) < N_p(t = Re(Z)) = p^k \ , \tag{7.2}$$

where the real p-adic coordinate $t = Re(Z)$ is identified as a time coordinate and the imaginary part of the p-adic coordinate is identified as a spatial coordinate. The p-adic norm for the four-dimensional extension is analogous to ordinary Euclidian distance. p-Adic light cone consists of cylinders parallel to time axis having radius $N_p(t) = p^k$ and length $p^{k-1}(p-1)$. As a real space (recall the canonical correspondence) the cross section of the cylinder corresponds to a parallelpiped rather than ball.

The result can be understood heuristically as follows.

1. For the four-dimensional extension allowing square root ($p > 2$) one can construct square root at each point $x(k,s) = sp^k$ represented by ordinary p-adic number, $s = 1, ..., p-1$, $k \in Z$. The task is to show that by using Taylor expansion one can define square root also in some neighbourhood of each of these points and find the form of this neighbourhood.

2. Using the general series expansion of the square root function one finds that the convergence region is p-adic ball defined by the condition

$$N_p(Z - sp^k) \leq R(k) \ , \tag{7.3}$$

   and having radius $R(k) = p^d, d \in Z$ around the expansion point.

3. A purely p-adic feature is that the convergence spheres associated with two points are either disjoint or identical! In particular, the convergence sphere $B(y)$ associated with any point inside convergence sphere $B(x)$ is identical with $B(x)$: $B(y) = B(x)$. The result follows directly from the ultra-metricity of the p-adic norm. The result means that stepwise analytic continuation is not possible and one can construct square root function only in the union of p-adic convergence spheres associated with the points $x(k,s) = sp^k$ which correspond to ordinary p-adic numbers.

4. By the scaling properties of the square root function the convergence radius $R(x(k,s)) \equiv R(k)$ is related to $R(x(0,s)) \equiv R(0)$ by the scaling factor $p^{-k}$:

$$R(k) = p^{-k}R(0) \ , \tag{7.4}$$

   so that the convergence sphere expands as a function of the p-adic time coordinate. The study of the convergence reduces to the study of the series at points $x = s = 1, ..., k-1$ with a unit p-adic norm.

5. Two neighboring points $x = s$ and $x = s+1$ cannot belong to the same convergence sphere: this would lead to a contradiction with the basic results of about square root function at integer points. Therefore the convergence radius satisfies the condition

$$R(0) < 1 \ . \tag{7.5}$$

   The requirement that the convergence is achieved at all points of the real axis implies

$$\begin{aligned} R(0) &= \frac{1}{p} \ , \\ R(p^k s) &= \frac{1}{p^{k+1}} \ . \end{aligned} \tag{7.5}$$

   If the convergence radius is indeed this, then the region, where the square root is defined, corresponds to a connected light cone like region defined by the condition $N_p(Im(Z)) = N_p(Re(Z))$ and $p > 2$-adic space time is the p-adic analog of the $M^4$ light-cone. If the convergence radius is smaller, the convergence region reduces to a union of disjoint p-adic spheres with increasing radii.

How the p-adic light cone differs from the ordinary light cone can be seen by studying the explicit form of the p-adic norm for $p > 2$ square root allowing extension $Z = x + iy + \sqrt{p}(u + iv)$

$$\begin{aligned} N_p(Z) &= (N_p(det(Z)))^{\frac{1}{4}} \ , \\ &= (N_p((x^2+y^2)^2 + 2p^2((xv-yu)^2 + (xu-yv)^2) + p^4(u^2+v^2)^2))^{\frac{1}{4}} \ , \end{aligned} \tag{7.4}$$

where $det(Z)$ is the determinant of the linear map defined by a multiplication with $Z$. The definition of the convergence sphere for $x = s$ reduces to

$$N_p(det(Z_3)) = N_p(y^4 + 2p^2y^2(u^2+v^2) + p^4(u^2+v^2)^2)) < 1 \ . \tag{7.5}$$

For physically interesting case $p \ mod \ 4 = 3$ the points $(y, u, v)$ satisfying the conditions

$$\begin{aligned} N_p(y) &\leq \frac{1}{p} \ , \\ N_p(u) &\leq 1 \ , \\ N_p(v) &\leq 1 \ , \end{aligned} \tag{7.4}$$

belong to the sphere of convergence: it is essential that for all $u$ and $v$ satisfying the conditions one has also $N_p(u^2 + v^2) \leq 1$. By the canonical correspondence between p-adic and real numbers, the real counterpart of the sphere $r = t$ is now the parallelpiped $0 \leq y < 1, 0 \leq u < p, 0 \leq v < p$, which expands with an average velocity of light in discrete steps at times $t = p^k$.

## 7.3 Convergence Radius For Square Root Function

In the following it will be shown that the convergence radius of $\sqrt{t+Z}$ is indeed non-vanishing for $p > 2$. The expression for the Taylor series of $\sqrt{t+Z}$ reads as

$$\begin{aligned} \sqrt{t+Z} &= = \sqrt{x}\sum_n a_n \ , \\ a_n &= (-1)^n \frac{(2n-3)!!}{2^n n!} x^n \ , \\ x &= \frac{Z}{t} \ . \end{aligned} \tag{7.3}$$

The necessary criterion for the convergence is that the terms of the power series approach to zero at the limit $n \rightarrow \infty$. The p-adic norm of the $n$: th term is for $p > 2$ given by

$$N_p(a_n) = N_p(\frac{(2n-3)!!}{n!})N_p(x^n) < N_p(x^n)N_p(\frac{1}{n!}) \ . \tag{7.4}$$

The dangerous term is clearly the $n!$ in the denominator. In the following it will be shown that the condition

$$U \equiv \frac{N_p(x^n)}{N_p(n!)} < 1 \ for \ \ N_p(x) < 1 \ , \tag{7.5}$$

holds true. The strategy is as follows:
a) The norm of $x^n$ can be calculated trivially: $N_p(x^n) = p^{-Kn}, K \geq 1$.
b) $N_p(n!)$ is calculated and an upper bound for $U$ is derived at the limit of large $n$.

### 7.3.1 p-Adic norm of $n!$ for $p > 2$

Lemma 1: Let $n = \sum_{i=0}^{k} n(i)p^i$, $0 \le n(i) < p$ be the p-adic expansion of $n$. Then $N_p(n!)$ can be expressed in the form

$$\begin{aligned} N_p(n!) &= \prod_{i=1}^{k} N(i)^{n(i)} \ , \\ N(1) &= \frac{1}{p} \ , \\ N(i+1) &= N(i)^{p-1}p^{-i} \ . \end{aligned} \tag{7.4}$$

An explicit expression for $N(i)$ reads as

$$N(i) = p^{-\sum_{m=0}^{i} m(p-1)^{i-m}} \ . \tag{7.5}$$

Proof: $n!$ can be written as a product

$$\begin{aligned} N_p(n!) &= \prod_{i=1}^{k} X(i,n(i)) \ , \\ X(k,n(k)) &= N_p((n(k)p^k)!) \ , \\ X(k-1,n(k-1)) &= N_p( \prod_{i=1}^{n(k-1)p^{k-1}} (n(k)p^k+i)) = N_p((n(k-1)p^{k-1})!) \ , \\ X(k-2,n(k-2)) &= N_p( \prod_{i=1}^{n(k-2)p^{k-2}} (n(k)p^k+n(k-1)p^{k-1}+i) \ ,) \\ &= N_p((n(k-2)p^{k-2})!) \ , \\ X(k-i,n(k-i)) &= N_p((n(k-i)p^{k-i})!) \ . \end{aligned} \tag{7.1}$$

The factors $X(k,n(k))$ reduce in turn to the form

$$\begin{aligned} X(k,n(k)) &= \prod_{i=1}^{n(k)} Y(i,k) \ , \\ Y(i,k) &= \prod_{m=1}^{p^k} N_p(ip^k+m) \ . \end{aligned} \tag{7.1}$$

The factors $Y(i,k)$ in turn are indentical and one has

$$\begin{aligned} X(k,n(k)) &= X(k)^{n(k)} \ , \\ X(k) &= N_p(p^k!) \ . \end{aligned} \tag{7.1}$$

The recursion formula for the factors $X(k)$ can be derived by writing explicitely the expression of $N_p(p^k!)$ for a few lowest values of $k$:
1) $X(1) = N_p(p!) = p^{-1}$.
2) $X(2) = N_p(p^2!) = X(1)^{p-1}p^{-2}$ ( $p^2!$ decomposes to $p-1$ products having same norm as $p!$ plus the last term equal to $p^2$.
i) $X(i) = X(i-1)^{p-1}p^{-i}$

Using the recursion formula repeatedly the explicit form of $X(i)$ can be derived easily. Combining the results one obtains for $N_p(n!)$ the expression

$$
\begin{aligned}
N_p(n!) &= p^{-\sum_{i=0}^{k} n(i)A(i)} \quad , \\
A(i) &= \sum_{m=1}^{i} m(p-1)^{i-m} \quad .
\end{aligned} \tag{7.1}
$$

The sum $A(i)$ appearing in the exponent as the coefficient of $n(i)$ can be calculated by using geometric series

$$
\begin{aligned}
A(i) &= (\frac{p-1}{p-2})^2(p-1)^{i-1}(1+\frac{i}{(p-1)^{i+1}} - \frac{(i+1)}{(p-1)^i}) \quad , \\
&\leq (\frac{p-1}{p-2})^2(p-1)^{i-1} \quad .
\end{aligned} \tag{7.1}
$$

### 7.3.2 Upper bound for $N_p(\frac{x^n}{n!})$ for $p > 2$

By using the expressions $n = \sum_i n(i)p^i$, $N_p(x^n) = p^{-Kn}$ and the expression of $N_p n!$ as well as the upper bound

$$
A(i) \leq (\frac{p-1}{p-2})^2(p-1)^{i-1} \quad . \tag{7.2}
$$

For $A(i)$ one obtains the upper bound

$$
N_p(\frac{x^n}{n!}) \leq p^{-\sum_{i=0}^{k} n(i)p^i(K-(\frac{(p-1)}{(p-2)})^2(\frac{(p-1)}{p})^{i-1})} \quad . \tag{7.2}
$$

It is clear that for $N_p(x) < 1$ that is $K \geq 1$ the upper bound goes to zero. For $p > 3$ exponents are negative for all values of $i$: for $p = 3$ some lowest exponents have wrong sign but this does not spoil the convergence. The convergence of the series is also obvious since the real valued series $\frac{1}{1-\sqrt{N_p(x)}}$ serves as a majorant.

## 7.4 $P = 2$ Case

In $p = 2$ case the norm of a general term in the series of the square root function can be calculated easily using the previous result for the norm of $n!$:

$$
N_p(a_n) = N_p(\frac{(2n-3)!!}{2^n n!})N_p(x^n) = 2^{-(K-1)n+\sum_{i=1}^{k} n(i)\frac{i(i+1)}{2^{i+1}}} \quad . \tag{7.3}
$$

At the limit $n \to \infty$ the sum term appearing in the exponent approaches zero and convergence condition gives $K > 1$, so that one has

$$
N_p(Z) \equiv (N_p(det(Z)))^{\frac{1}{8}} \leq \frac{1}{4} \quad . \tag{7.4}
$$

The result does not imply disconnected set of convergence for square root function since the square root for half odd integers exists:

$$
\sqrt{s+\frac{1}{2}} = \frac{\sqrt{2s+1}}{\sqrt{2}} \quad , \tag{7.5}
$$

so that one can develop square as a series in all half odd integer points of the p-adic axis (points which are ordinary p-adic numbers). As a consequence, the structure for the set of convergence is just the 8-dimensional counterpart of the p-adic light cone. Space-time has natural binary structure in the sense that each $N_p(t) = 2^k$ cylinder consists of two identical p-adic 8-balls (parallelpipeds as real spaces).

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