The relationship between U-matrix and M-matrices

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June 20, 2019

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Abstract

S-matrix is the key notion in quantum field theories. In Zero Energy Ontology (ZEO) this notion must be replaced with the triplet U-matrix, M-matrix, and S-matrix. U-matrix realizes unitary time evolution in the space for zero energy states realized geometrically as dispersion in the moduli space of causal diamonds (CDs) leaving second boundary (passive boundary) of CD and states at it fixed.

This process can be seen as the TGD counterpart of repeated state function reductions leaving the states at passive boundary unaffected and affecting only the member of state pair at active boundary (Zeno effect). In TGD inspired theory of consciousness self corresponds to the sequence these state function reductions. M-matrix describes the entanglement between positive and negative energy parts of zero energy states and is expressible as a hermitian square root H of density matrix multiplied by a unitary matrix S, which corresponds to ordinary S-matrix, which is universal and depends only the size scale n of CD through the formula $S(n) = S^n$. M-matrices and H-matrices form an orthonormal basis at given CD and H-matrices would naturally correspond to the generators of super-symplectic algebra.

The first state function reduction to the opposite boundary corresponds to what happens in quantum physics experiments. The relationship between

U- and S-matrices has remained poorly understood. In this article this relationship is analyzed by starting from basic principles. One ends up to formulas allowing to understand the architecture of U-matrix and to reduce its construction to that for S-matrix having interpretation as exponential of the generator L_{-1} of the Virasoro algebra associated with the super-symplectic algebra.

1 Introduction

S-matrix is the key notion in quantum field theories. In Zero Energy Ontology (ZEO) this notion must be replaced with the triplet U-matrix, M-matrix, and S-matrix. U-matrix realizes unitary time evolution in the space for zero energy states realized geometrically as dispersion in the moduli space of causal diamonds (CDs) leaving second boundary (passive boundary) of CD and states at it fixed.

This process can be seen as the TGD counterpart of repeated state function reductions leaving the states at passive boundary unaffected and affecting only the member of state pair at active boundary (Zeno effect) [K4]. In TGD inspired theory of consciousness self corresponds to the sequence of these state function reductions [K6, K1, K5]. M-matrix describes the entanglement between positive and negative energy parts of zero energy states and is expressible as a hermitian square root H of density matrix multiplied by a unitary matrix S, which corresponds to ordinary S-matrix, which is universal and depends only the size scale n of CD through the formula $S(n) = S^n$. M-matrices and H-matrices form an orthonormal basis at given CD and H-matrices would naturally correspond to the generators of super-symplectic algebra.

The first state function reduction to the opposite boundary corresponds to what happens in quantum physics experiments. The relationship between U- and S-matrices has remained poorly understood.

The original view about the relationship was a purely formal guess: M-matrices would define the orthonormal rows of U-matrix. This guess is not correct physically and one must consider in detail what U-matrix really means.

1. First about the geometry of CD [K7]. The boundaries of CD will be called passive and active: passive boundary correspond to the boundary at which repeated state function reductions take place and give rise to a sequence of unitary time evolutions U followed by localization in the moduli of CD each. Active boundary corresponds to the boundary for which U induces delocalization and modifies the states at it.

   The moduli space for the CDs consists of a discrete subgroup of scalings for the size of CD characterized by the proper time distance between the tips and the sub-group of Lorentz boosts leaving passive boundary and its tip invariant and acting on the active boundary only. This group is assumed to be represented unitarily by matrices Λ forming the same group for all values of n.

   The proper time distance between the tips of CDs is quantized as integer multiples of the minimal distance defined by CP_2 time: $T = nT_0$. Also in quantum jump in which the size scale n of CD increases the increase corresponds to integer multiple of T_0. Using the logarithm of proper time, one can interpret this in terms of a scaling parametrized by an integer. The possibility to interpret proper time translation as a scaling is essential for having a manifest Lorentz invariance: the ordinary definition of S-matrix introduces preferred rest system.

2. The physical interpretation would be roughly as follows. M-matrix for a given CD codes for the physics as we usually understand it. M-matrix is product of square root of density matrix and S-matrix depending on the size scale of CD and is the analog of thermal S-matrix. State function at the opposite boundary of CD corresponds to what happens in the state function reduction in particle physics experiments. The repeated state function reductions at same boundary of CD correspond to TGD version of Zeno effect crucial for understanding consciousness. Unitary U-matrix describes the time evolution zero energy states due to the increase of the size scale of CD (at least in statistical sense). This process is dispersion in the moduli space of CDs: all possible scalings are allowed and localization in the space of moduli of CD localizes the active boundary of CD after each unitary evolution.

In the following I will proceed by making questions. One ends up to formulas allowing to understand the architecture of U-matrix and to reduce its construction to that for S-matrix having interpretation as exponential of the generator L_1 of the Virasoro algebra associated with the super-symplectic algebra.

2 Questions and answers

2.1 What one can say about M-matrices?

1. The first thing to be kept in mind is that M-matrices act in the space of zero energy states rather than in the space of positive or negative energy states. For a given CD M-matrices are products of hermitian square roots of hermitian density matrices acting in the space of zero energy states and universal unitary S-matrix $S(CD)$ acting on states at the active end of CD (this is also very important to notice) depending on the scale of CD:

$$M^i = H^i \circ S(CD) \ .$$

   Here "$\circ$" emphasizes the fact that S acts on zero energy states at active boundary only. H^i is hermitian square root of density matrix and the matrices H^i must be orthogonal for given CD from the orthonormality of zero energy states associated with the same CD. The zero energy states associated with different CDs are not orthogonal and this makes the unitary time evolution operator U non-trivial.

2. Could quantum measurement be seen as a measurement of the observables defined by the Hermitian generators H^i? This is not quite clear since their action is on zero energy states. One might actually argue that the action of this kind of observables on zero energy states does not affect their vanishing

net quantum numbers. This suggests that H^i carry no net quantum numbers and belong to the Cartan algebra. The action of S is restricted at the active boundary of CD and therefore it does not commute with H^i unless the action is in a separate tensor factor. Therefore the idea that S would be an exponential of generators H^i and thus commute with them so that H^i would correspond to sub-spaces remaining invariant under S acting unitarily inside them does not make sense.

3. In TGD framework symplectic algebra actings as isometries of WCW is analogous to a Kac-Moody algebra with finite-dimensional Lie-algebra replaced with the infinite-dimensional symplectic algebra with elements characterized by conformal weights [K3, K2]. There is a temptation to think that the H^i could be seen as a representation for this algebra or its sub-algebra. This algebra allows an infinite fractal hierarchy of sub-algebras of the super-symplectic algebra isomorphic to the full algebra and with conformal weights coming as n-ples of those for the full algebra. In the proposed realization of quantum criticality the elements of the sub-algebra characterized by n act as a gauge algebra. An interesting question is whether this sub-algebra is involved with the realization of M-matrices for CD with size scale n. The natural expectation is that n defines a cutoff for conformal weights relating to finite measurement resolution.

2.2 How does the size scale of CD affect M-matrices?

1. In standard quantum field theory (QFT) S-matrix represents time translation. The obvious generalization is that now scaling characterized by integer n is represented by a unitary S-matrix that is as n:th power of some unitary matrix S assignable to a CD with minimal size: $S(CD) = S^n$. $S(CD)$ is a discrete analog of the ordinary unitary time evolution operator with n replacing the continuous time parameter.

2. One can see M-matrices also as a generalization of Kac-Moody type algebra. Also this suggests $S(CD) = S^n$, where S is the S-matrix associated with the minimal CD. S becomes representative of phase $exp(i\phi)$. The inner product between CDs of different size scales can n_1 and n_2 can be defined as

$$\begin{aligned} \langle M^i(m), M^j(n)\rangle &= Tr(S^{-m}\circ H^iH^j\circ S^n)\times\theta(n-m) \ , \\ \theta(n) = 1 \ \text{for} \ n\geq 0 \ &, \ \theta(n)=0 \ \text{for} \ n<0 \ . \end{aligned} \quad (2.1)$$

Here I have denoted the action of S-matrix at the active end of CD by "$\circ$" in order to distinguish it from the action of matrices on zero energy states which could be seen as belonging to the tensor product of states at active and passive boundary.

It turns out that unitarity conditions for U-matrix are invariant under the translations of n if one assumes that the transitions obey strict arrow of time expressed by $n_j - n_i \geq 0$. This simplifies dramatically unitarity conditions. This gives orthonormality for M-matrices associated with identical CDs. This inner product could be used to identify U-matrix.

3. How do the discrete Lorentz boosts affecting the moduli for CD with a fixed passive boundary affect the M-matrices? The natural assumption is that the discrete Lorentz group is represented by unitary matrices λ: the matrices M^i are transformed to $M^i \circ \lambda$ for a given Lorentz boost acting on states at active boundary only.

   One cannot completely exclude the possibility that S acts unitarily at both ends of zero energy states. In this case the scaling would be interpreted as acting on zero energy states rather than those at active boundary only. The zero energy state basis defined by M_i would depend on the size scale of CD in more complex manner. This would not affect the above formulas except by dropping away the "$\circ$".

Unitary U must characterize the transitions in which the moduli of the active boundary of causal diamond (CD) change and also states at the active boundary (paired with unchanging states at the passive boundary) change. The arrow of the experienced flow of time emerges during the period as state function reductions take place to the fixed ("passive") boundary of CD and do not affect the states at it. Note that these states form correlated pairs with the changing states at the active boundary. The physically motivated question is whether the arrow of time emerges statistically from the fact that the size of CD tends to increase in average sense in repeated state function reductions or whether the arrow of geometric time is strict. It turns out that unitarity conditions simplify dramatically if the arrow of time is strict.

2.3 What can one say about U-matrix?

1. Just from the basic definitions the elements of a unitary matrix, the elements of U are between zero energy states (M-matrices) between two CDs with possibly different moduli of the active boundary. Given matrix element of U should be proportional to an inner product of two M-matrices associated with these CDs. The obvious guess is as the inner product between M-matrices

$$\begin{aligned} U^{ij}_{m,n} &= \langle M^i(m,\lambda_1), M^j(n,\lambda_2)\rangle \\ &= Tr(\lambda_1^{\dagger} S^{-m}\circ H^i H^j \circ S^n \lambda_2) \\ &= Tr(S^{-m}\circ H^i H^j \circ S^n \lambda_2 \lambda_1^{-1})\theta(n-m) \ . \end{aligned} \tag{2.2}$$

Here the usual properties of the trace are assumed. The justification is that the operators acting at the active boundary of CD are special case of operators acting non-trivially at both boundaries.

2. Unitarity conditions must be satisfied. These conditions relate S and the hermitian generators H^i serving as square roots of density matrices. Unitarity conditions $UU^\dagger = U^\dagger U = 1$ is defined in the space of zero energy states and read as

$$\sum_{j_1 n_1} U^{ij_1}_{mn_1} (U^\dagger)^{j_1 j}_{n_1 n} = \delta^{i,j}\delta_{m,n}\delta_{\lambda_1,\lambda_2} \tag{2.3}$$

   To simplify the situation let us make the plausible hypothesis contribution of Lorentz boosts in unitary conditions is trivial by the unitarity of the representation of discrete boosts and the independence on n.

3. In the remaining degrees of freedom one would have

$$\sum_{j_1,k\geq Max(0,n-m)} Tr(S^k \circ H^i H^{j_1}) Tr(H^{j_1} H^j \circ S^{n-m-k}) = \delta^{i,j}\delta_{m,n} \ . \tag{2.4}$$

   The condition $k \geq Max(0,n-m)$ reflects the assumption about a strict arrow of time and implies that unitarity conditions are invariant under the proper time translation $(n,m) \rightarrow (n+r, m+r)$. Without this condition n back-wards translations (or rather scalings) to the direction of geometric past would be possible for CDs of size scale n and this would break the translational invariance and it would be very difficult to see how unitarity could be achieved. Stating it in a general manner: time translations act as semigroup rather than group.

4. Irreversibility reduces dramatically the number of the conditions. Despite this their number is infinite and correlates the Hermitian basis and the unitary matrix S. There is an obvious analogy with a Kac-Moody algebra at circle with S replacing the phase factor $exp(in\phi)$ and H^i replacing the finite-dimensional Lie-algebra. The conditions could be seen as analogs for the orthogonality conditions for the inner product. The unitarity condition for the analog situation would involve phases $exp(ik\phi_1) \leftrightarrow S^k$ and $exp(i(n-m-k)\phi_2) \leftrightarrow S^{n-m-k}$ and trace would correspond to integration $\int d\phi_1$ over ϕ_1 in accordance with the basic idea of non-commutative geometry that trace corresponds to integral. The integration of ϕ_i would give $\delta_{k,0}$ and $\delta_{m,n}$. Hence there are hopes that the conditions might be satisfied. There is however a clear distinction to the Kac-Moody case since S^n does not in general act in the orthogonal complement of the space spanned by H^i.

5. The idea about reduction of the action of S to a phase multiplication is highly attractive and one could consider the possibility that the basis of H^i can be chosen in such a manner that H^i are eigenstates of of S. This would reduce the unitarity constraint to a form in which the summation over k can be separated from the summation over j_1.

$$\sum_{k\geq Max(0,n-m)} exp(iks_i - (n-m-k)s_j)\sum_{j_1} Tr(H^iH^{j_1})Tr(H^{j_1}H^j) = \delta^{i,j}\delta_{m,n} \quad (2.5)$$

The summation over k should gives a factor proportional to δ_{s_i,s_j}. If the correspondence between H^i and eigenvalues s_i is one-to-one, one obtains something proportional to $\delta(i,j)$ apart from a normalization factor. Using the orthonormality $Tr(H^iH^j) = \delta^{i,j}$ one obtains for the left hand side of the unitarity condition

$$exp(is_i(n-m))\sum_{j_1} Tr(H^iH^{j_1})Tr(H^{j_1}H^j) = exp(is_i(n-m))\delta_{i,j} \ . \quad (2.6)$$

Clearly, the phase factor $exp(is_i(n-m))$ is the problem. One should have Kronecker delta $\delta_{m,n}$ instead. One should obtain behavior resembling Kac-Moody generators. H^i should be analogs of Kac-Moody generators and include the analog of a phase factor coming visible by the action of S.

2.4 How to obtain unitarity correctly?

It seems that the simple picture is not quite correct yet. One should obtain somehow an integration over angle in order to obtain Kronecker delta.

1. A generalization based on replacement of real numbers with function field on circle suggests itself. The idea is to the identify eigenvalues of generalized Hermitian/unitary operators as Hermitian/unitary operators with a spectrum of eigenvalues, which can be continuous. In the recent case S would have as eigenvalues functions $\lambda_i(\phi) = exp(is_i\phi)$. For a discretized version ϕ would have has discrete spectrum $\phi(n) = 2\pi k/n$. The spectrum of λ_i would have n as cutoff. Trace operation would include integration over ϕ and one would have analogs of Kac-Moody generators on circle.

2. One possible interpretation for ϕ is as an angle parameter associated with a fermionic string connecting partonic 2-surface. For the super-symplectic generators suitable normalized radial light-like coordinate r_M of the light-cone boundary (containing boundary of CD) would be the counterpart of angle variable if periodic boundary conditions are assumed.

   The eigenvalues could have interpretation as analogs of conformal weights. Usually conformal weights are real and integer valued and in this case it is necessary to have generalization of the notion of eigenvalues since otherwise the exponentials $exp(is_i)$ would be trivial. In the case of super-symplectic algebra I have proposed that the generating elements of the algebra have conformal weights given by the zeros of Riemann zeta. The spectrum of conformal weights for the generators would consist of linear combinations of the zeros of zeta with integer coefficients. The imaginary parts of the conformal weights could appear as eigenvalues of S.

3. It is best to return to the definition of the U-matrix element to check whether the trace operation appearing in it can already contain the angle integration. If one includes to the trace operation appearing the integration over ϕ it gives $\delta_{m,n}$ factor and U-matrix has elements only between states assignable to the same causal diamond. Hence one must interpret U-matrix elements as functions of ϕ realized factors $exp(i(s_n - s_m)\phi)$. This brings strongly in mind operators defined as distributions of operators on line encountered in the theory of representations of non-compact groups such as Lorentz group. In fact, the unitary representations of discrete Lorentz groups are involved now.

4. The unitarity condition contains besides the trace also the integrations over the two angle parameters ϕ_i associated with the two U-matrix elements involved. The left hand side of the unitarity condition reads as

$$\begin{aligned} \sum_{k\geq Max(0,n-m)} & I(ks_i)I((n-m-k)s_j) \times \sum_{j_1} Tr(H^iH^{j_1})Tr(H^{j_1}H^j) \\ = & \delta^{i,j}\delta_{m,n} \quad , \quad I(s) = \frac{1}{2\pi} \times \int d\phi exp(is\phi) = \delta_{s,0} \quad . \end{aligned} \tag{2.7}$$

   Integrations give the factor $\delta_{k,0}$ eliminating the infinite sum obtained otherwise plus the factor $\delta_{n,m}$. Traces give Kronecker deltas since the projectors are orthonormal. The left hand side equals to the right hand side and one achieves unitarity. It seems that the proposed ansatz works and the U-matrix can be reduced by a general ansatz to S-matrix.

5. It should be made clear that the use of eigenstates of S is only a technical trick, the physical states need not be eigenstates. If the active parts of zero energy states where eigenstates of S, U-matrix would not have matrix elements between different H^i and projection operator could not change during time evolution.

2.5 What about the identification of S?

1. S should be exponential of time the scaling operator whose action reduces to a time translation operator along the time axis connecting the tips of CD and realized as scaling. In other words, the shift $t/T_0 = m \rightarrow m+n$ corresponds to a scaling $t/T_0 = m \rightarrow km$ giving $m+n = km$ in turn giving $k = 1+n/m$. At the limit of large shifts one obtains $k \simeq n/m \rightarrow \infty$, which corresponds to QFT limit. nS corresponds to $(nT_0)\times(S/T_0) = TH$ and one can ask whether QFT Hamiltonian could corresponds to $H = S/T_0$.

2. It is natural to assume that the operators H^i are eigenstates of radial scaling generator $L_0 = ir_M d/dr_M$ at both boundaries of CD and have thus well-defined conformal weights. As noticed the spectrum for super-symplectic algebra could also be given in terms of zeros of Riemann zeta.

3. The boundaries of CD are given by the equations $r_M = m^0$ and $r_M = T - m_0$, m_0 is Minkowski time coordinate along the line between the tips of CD and T is the distance between the tips. From the relationship between r_M and m_0 the action of the infinitesimal translation $H \equiv i\partial/\partial_{m^0}$ can be expressed as conformal generator $L_{-1} = i\partial/\partial_{rM} = r_M^{-1} L_0$. Hence the action is non-diagonal in the eigenbasis of L_0 and multiplies with the conformal weights and reduces the conformal weight by one unit. Hence the action of U can change the projection operator. For large values of conformal weight the action is classically near to that of L_0: multiplication by L_0 plus small relative change of conformal weight.

4. Could the spectrum of H be identified as energy spectrum expressible in terms of zeros of zeta defining a good candidate for the super-symplectic radial conformal weights. This certainly means maximal complexity since the number of generators of the conformal algebra would be infinite. This identification might make sense in chaotic or critical systems. The functions $(r_M/r_0)^{1/2+iy}$ and $(r_M/r_0)^{-2n}$, $n > 0$, are eigenmodes of r_M/dr_M with eigenvalues $(1/2 + iy)$ and $-2n$ corresponding to non-trivial and trivial zeros of zeta.

   There are two options to consider. Either L_0 or iL_0 could be realized as a hermitian operator. These options would correspond to the identification of mass squared operator as L_0 and approximation identification of Hamiltonian as iL_1 as iL_0 making sense for large conformal weights.

   (a) Suppose that $L_0 = r_M d/dr_M$ realized as a hermitian operator would give harmonic oscillator spectrum for conformal confinement. In p-adic mass calculations the string model mass formula implies that L_0 acts essentially as mass squared operator with integer spectrum. I have proposed conformal confinent for the physical states net conformal weight is real and integer valued and corresponds to the sum over negative integer valued conformal weights corresponding to the trivial zeros and sum over real parts of non-trivial zeros with conformal weight equal to 1/2. Imaginary parts of zeta would sum up to zero.

   (b) The counterpart of Hamiltonian as a time translation is represented by $H = iL_0 = ir_M d/dr_M$. Conformal confinement is now realized as the vanishing of the sum for the real parts of the zeros of zeta: this can be achieved. As a matter fact the integration measure dr_M/r_M brings implies that the net conformal weight must be 1/2. This is achieved if the number of non-trivial zeros is odd with a judicious choice of trivial zeros. The eigenvalues of Hamiltonian acting as time translation operator could correspond to the linear combination of imaginary part of zeros of zeta with integer coefficients. This is an attractive hypothesis in critical systems and TGD Universe is indeed quantum critical.

2.6 What about quantum classical correspondence?

Quantum classical correspondence realized as one-to-one map between quantum states and zero modes has not been discussed yet.

1. M-matrices would act in the tensor product of quantum fluctuating degrees of freedom and zero modes. The assumption that zero energy states form an orthogonal basis implies that the hermitian square roots of the density matrices form an orthonormal basis. This condition generalizes the usual orthonormality condition.

2. The dependence on zero modes at given boundary of CD would be trivial and induced by 1-1 correspondence $|m\rangle \rightarrow z(m)$ between states and zero modes assignable to the state basis $|m_{\pm}$ at the boundaries of CD, and would mean the presence of factors $\delta_{z_+,f(m_+)} \times \delta_{z_-,f(n_-)}$ multiplying M-matrix $M^i_{m,n}$.

To sum up, it seems that the architecture of the U-matrix and its relationship to the S-matrix is now understood and in accordance with the intuitive expectations the construction of U-matrix reduces to that for S-matrix and one can see S-matrix as discretized counterpart of ordinary unitary time evolution operator with time translation represented as scaling: this allows to circumvent problems with loss of manifest Poincare symmetry encountered in quantum field theories and allows Lorentz invariance although CD has finite size. What came as surprise was the connection with stringy picture: strings are necessary in order to satisfy the unitary conditions for U-matrix. Second outcome was that the connection with super-symplectic algebra suggests itself strongly. The identification of hermitian square roots of density matrices with Hermitian symmetry algebra is very elegant aspect discovered already earlier. A further unexpected result was that U-matrix is unitary only for strict arrow of time (which changes in the state function reduction to opposite boundary of CD).

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