Twistors in TGD

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Abstract

In twistor Grassmannian approach to $\mathcal{N}=4$ SYM twistors are replaced with supertwistors and the extreme elegance of the description of various helicity states using twistor space wave functions and M^8-H duality suggest that super-twistors are realized at the level of both M^8 and H. M^8 supertwistors are naturally realized at the level of momentum space.

1. Basic problem of twistor approach and mass as a relative notion in TGD framework

In TGD framework M^8-H duality allows to geometrize the notion of super-twistor in the sense that different components of super-field correspond to components of super-octonion each of which corresponds to a space-time surfaces satisfying minimal surface equations with string world sheets as singularities - this is geometric counterpart for masslessness.

In TGD particles are massless in 8-D sense and in general massive in 4-D sense but 4-D twistors are needed also now so that a modification of twistor approach is needed. The incidence relation for twistors suggests the replacement of the usual twistors with either non-commutative quantum twistors or with octo-twistors. Quantum twistors could be associated with the space-time level description of massive particles and octo-twistors with the description at imbedding space level. A possible alternative interpretation of quantum spinors is in terms of quantum measurement theory with finite measurement resolution in which precise eigenstates as measurement outcomes are replaced with universal probability distributions defined by quantum group. This has also application in TGD inspired theory of consciousness.

Twistor lift of TGD involves representation of space-time surfaces as 6-surfaces in twistor space of H having structure of S^2 bundle over space-time surface resulting in dimensional reduction. These 6-surfaces would be holomorphic and thus minimal surfaces represented in terms of polynomials having same degree as the corresponding M^8 octonionic polynomial by number theoretic universality.

2. Criticizing the notion of twistor space of M^4

I have assumed that what I call geometric twistor space of M^4 is simply $M^4\times S^2$. One can however consider standard twistor space CP_3 with metric signature (3,-3) as an alternative. This option reproduces the nice results of the earlier approach but the philosophy is different: there is no fundamental length scale but the hierarchy of causal diamonds (CDs) predicted by zero energy ontology (ZEO) gives rise to the breaking of the exact scaling invariance of M^8 picture. M^4 in H would not be be replaced with conformally compactified M^4 (M^4_{conf}) but conformally compactified cd (cd_{conf}) for which a natural identification is as $CP_{2,h}$ obtained from CP_2 by replacing second complex coordinate replaced with hypercomplex coordinate. The sizes of twistor spaces of cd_{conf} using CP_2 size as unit would reflect the hierarchy of size scales for CDs. The consideration on the twistor space of M^8 in similar picture leads to the identification of corresponding twistor space as HP_3 - quaternionic variant of CP_3: the counterpart of CD_8 would be HP_2.

The outcome of octo-twistor approach together with M^8-H duality leads to a nice picture view about twistorial description of massive states based on quaternionic generalization of twistor (super-)Grassmannian approach with twistor space identificed as $HP_{3,h}$, the quaternionic variant of $CP_{3,h}$. A radically new view is that descriptions in terms of massive and massless states are alternative options, and correspond to two different alternative twistorial descriptions and leads to the interpretation of p-adic thermodynamics as completely universal massivation mechanism having nothing to do with dynamics.

As a side product emerges a deeper understanding of ZEO based quantum measurement theory and consciousness theory relying on the universal roots of octonionic polynomials of M^8, which are not 4-D but analogs of 6-D branes. By M^8-H duality the finite sub-groups of $SU(2)$ of McKay correspondence appear quite concretely in the description of the measurement resolution of 8-momentum.

The parallel progress in the understanding SUSY in TGD framework in turn led to the identification of the super-counterparts of M^8, H and of twistor spaces modifying dramatically the physical interpretation of SUSY. Super-spinors in twistor space would provide the description of quantum states. Quaternionic super Grassmannians would be involved with M^8 description.

Contents

1 Introduction

This article was inspired by a longer paper "TGD view about McKay Correspondence, ADE Hierarchy, Inclusions of Hyperfinite Factors, and Twistors". I found it convenient to isolate the part of paper related to twistors. In twistor Grassmannian approach to $\mathcal{N} = 4$ SYM twistors are replaced with supertwistors and the extreme elegance of the description of various helicity states using twistor space wave functions suggests that super-twistors are realized at the level of M^8 geometry. These supertwistors are realized at the level of momentum space.

In TGD framework $M^8 - H$ duality allows to geometrize the notion of super-twistor in the sense that different components of super-field correspond to components of super-octonion each of which corresponds to a space-time surfaces satisfying minimal surface equations with string world sheets as singularities - this is geometric counterpart for masslessness.

1.1 Basic problem of twistor approach and mass as a relative notion in TGD framework

The basic problem of the ordinary twistor approach is that the states must be massless in 4-D sense. In TGD framework particles would be massless in 8-D sense. This leads to alternative descriptions depending on the choice of $M^{\subset} M^8$ and the 4-D mass of the particle depends on the choice of M^4. For M^4_L description $M^4_L \subset M^8$ is chosen so that states are massless in 4-D sense, and the description at momentum space level would be in terms of products of ordinary M^4 twistors and CP_2 twistors. For M^4_T description particles are massive in 4-D sense. How to generalize the twistor description to 8-D case?

The incidence relation for twistors suggests the replacement of the usual twistors with either non-commutative quantum twistors or with octo-twistors. Quantum twistors could be associated with the space-time level description of massive particles and octo-twistors with the description at imbedding space level. A possible alternative interpretation of quantum spinors is in terms of quantum measurement theory with finite measurement resolution in which precise eigenstates as measurement outcomes are replaced with universal probability distributions defined by quantum group. This has also application in TGD inspired theory of consciousness.

1.2 Criticizing the notion of twistor space of M^4

Twistor lift of TGD involves representation of space-time surfaces as 6-surfaces in twistor space of H having structure of S^2 bundle over space-time surface resulting in dimensional reduction. These 6-surfaces would be holomorphic and thus minimal surfaces represented in terms of polynomials having same degree as the corresponding M^8 octonionic polynomial by number theoretic universality.

1. I have assumed that what I call geometric twistor space of M^4 is simply $M^4 \times S^2$. It however turned out that one can consider standard twistor space CP_3 with metric signature (3,-3) as an alternative. This option reproduces the nice results of the earlier approach but the philosophy is different: there is no fundamental length scale but the hierarchy of causal diamonds (CDs) predicted by zero energy ontology (ZEO) gives rise to the breaking of the exact scaling invariance of M^8 picture. This forces to modify $M^8 - H$ correspondence so that it involves map from M^4 to CP_3 followed by a projection to hyperbolic variant of CP_2.

   M^4 in H would not be replaced with conformally compactified M^4 (M^4_{conf}) but conformally compactified cd (cd_{conf}) for which a natural identification is as CP_2 with second complex coordinate replaced with hypercomplex coordinate. The sizes of twistor spaces of cd_{conf} using CP_2 size as unit would reflect the hierarchy of size scales for CDs. The consideration on the twistor space of M^8 in similar picture leads to the identification of corresponding twistor space as HP_3 - quaternionic variant of CP_3: the counterpart of CD_8 would be HP_2.

2. Octotwistors can be expressed as pairs of quaternionic twistors. Octotwistor approach suggests a generalization of twistor Grassmannian approach obtained by replacing the bi-spinors with complexified quaternions and complex Grassmannians with their quaternionic counterparts. Although TGD is not a quantum field theory, this proposal makes sense for cognitive

representations identified as discrete sets of spacetime points with coordinates in the extension of rationals defining the adele [L2] implying effective reduction of particles to point-like particles.

3. The outcome of octo-twistor approach together with $M^8 - H$ duality leads to a nice picture view about twistorial description of massive states based on quaternionic generalization of twistor Grassmannian approach. A radically new view is that descriptions in terms of massive and massless states are alternative options, and correspond to two different alternative twistorial descriptions and leads to the interpretation of p-adic thermodynamics as completely universal massivation mechanism having nothing to do with dynamics. As a side product emerges a deeper understanding of ZEO based quantum measurement theory and consciousness theory relying on the universal roots of octonionic polynomials of M^8, which are not 4-D but analogs of 6-D branes. By $M^8 - H$ duality the finite sub-groups of $SU(2)$ of McKay correspondence appear quite concretely in the description of the measurement resolution of 8-momentum.

1.3 What super-twistors are in TGD framework

What about super-twistors in TGD framework?

1. The parallel progress in the understanding SUSY in TGD framework [L5] in turn led to the identification of the super-counterparts of M^8, H and of twistor spaces modifying dramatically the physical interpretation of SUSY. Super-spinors in twistor space would provide the description of quantum states. Super-Grassmannians would be involved with the construction of scattering amplitudes. Quaternionic super Grassmannians would be involved with M^8 description.

2. The great surprise from physics point of view is that in fermionic sector only quarks are allowed by $SO(1,7)$ triality and that anti-leptons are local 3-quark composites identifiable as spartners of quarks. Gauge bosons, Higgs and graviton would be also spartners and assignable to super-coordinates of imbedding space expressible as super-polynomials of quark oscillator operators. Super-symmetrization means also quantization of fermions allowing local many-quark states.

3. SUSY breaking would be caused by the same universal mechanism as ordinary massivation of massless states. The mass formulas would be supersymmetric but the choice of p-adic prime identifiable as ramified prime of extension of rationals would depend on the state of super-multiplet. ZEO would make possible symmetry breaking without symmetry breaking as Wheeler might put it.

What about the interpretation of quantum twistors? They could make sense as 4-D space-time description analogous to description at space-time level. Now one can consider generalization of the twistor Grassmannian approach in terms of quantum Grassmannians.

2 Could standard view about twistors work at space-time level after all?

While asking what super-twistors in TGD might be, I became critical about the recent view concerning what I have called geometric twistor space of M^4 identified as $M^4 \times S^2$ rather than CP_3 with hyperbolic metric. The basic motivations for the identification come from M^8 picture in which there is number theoretical breaking of Poincare and Lorentz symmetries. Second motivation was that M^4_{conf} - the conformally compactified M^4 - identified as group $U(2)$ [B1] (see `http://tinyurl.com/y35k5wwo`) assigned as base space to the standard twistor space CP_3 of M^4, and having metric signature (3,-3) is compact and is stated to have metric defined only modulo conformal equivalence class.

As found in the previous section, TGD strongly suggests that M^4 in $H = M^4 \times CP_2$ should be replaced with hyperbolic variant of CP_2, and it seems to me that these spaces are not identical.

Amusingly, $U(2)$ and CP_2 are fiber and base in the representation of $SU(3)$ as fiber space so that the their identification does not seem plausible.

On can however ask whether the selection of a representative metric from the conformal equivalence class could be seen as breaking of the scaling invariance implied also by ZEO introducing the hierarchy of CDs in M^8. Could it be enough to have M^4 only at the level of M^8 and conformally compactified M^4 at the level of H? Should one have $H = cd_{conf} \times CP_2$? What cd_{conf} would be: is it hyperbolic variant of CP_2?

2.1 Getting critical

The only way to make progress is to become very critical now and then. These moments of almost despair usually give rise to a progress. At this time I got very critical about the TGD inspired identification of twistor spaces of M^4 and CP_2 and their properties.

2.1.1 Getting critical about geometric twistor space of M^4

Let us first discuss the recent picture and how to modify it so that it is consistent with the hierarchy of CDs. The key idea is that the twistor space and its base space represents CD so that one obtains scale hierarchy of twistor spaces as a realization of broken scale invariance giving rise to the p-adic length scale hierarchy.

1. I have identified the twistor space of M^4 simply as $T(M^4) = M^4 \times S^2$. The interpretation would be at the level of octonions as a product of M^4 and choices of M^2 as preferred complex sub-space of octonions with S^2 parameterizing the directions of spin quantization axes. Real octonion axis would correspond to time coordinate. One could talk about the space of of light-like directions. Light-like vector indeed defines M^2. This view could be defended by the breaking of both translation and Lorentz invariance in the octonionic approach due to the choice of M^2 and by the fact that it seems to work.

   Remark: $M^8 = M^4 \times E^4$ is complexified to M_c^8 by adding a commuting imaginary unit i appearing in the extensions of rationals and ordinary M^8 represents its particular sub-space. Also in twistor approach one uses often complexified M^4.

2. The objection is that it is ordinary twistor space identifiable as CP_3 with (3,-3) signature of metric is what works in the construction of twistorial amplitudes. CP_3 has metric as compact space and coset space. Could this choice of twistor space make sense after all as geometric twistor space?

   Here one must pause and recall that the original key idea was that Poincare invariance is symmetry of TGD for $X^4 \subset M^4 \times CP_2$. Now Poincare symmetry has been transformed to a symmetry acting at the level of M^8 in the moduli space of octonion structures defined by the choice of the direction of octonionic real axis reducing Poincare group to $T \times SO(3)$ consisting of time translations and rotations. Fixing of M^2 reducrs the group to $T \times SO(2)$ and twistor space can be seen as the space for selections of quantization axis of energy and spin.

3. But what about the space H? The first guess is $H = M^4_{conf} \times CP_2$. According to [B1] (see `http://tinyurl.com/y35k5wwo`) one has $M^4_{conf} = U(2)$ such that $U(1)$ factor is time- like and $SU(2)$ factor is space-like. One could understand $M^4_{conf} = U(2)$ as resulting by addition and identification of metrically 2-D light-cone boundaries at $t = \pm\infty$. This is topologically like compactifying E^3 to S^3 and gluing the ends of cylinder $S^3 \times D^1$ together to the $S^3 \times S^1$.

   The conformally compactified Minkowski space M^4_{conf} should be analogous to base space of CP_3 regarded as bundle with fiber S^2. The problem is that one cannot imagine an analog of fiber bundle structure in CP_3 having $U(2)$ as base. The identification $H = M^4_{conf} \times CP_2$ does not make sense.

4. In ZEO based breaking of scaling symmetry it is CD that should be mapped to the analog of M^4_{conf} - call it cd_{conf}. The only candidate is $cd_{conf} = CP_2$ with one hypercomplex coordinate. To understand why one can start from the following picture. The light-like boundaries of

CD are metrically equivalent to spheres. The light-like boundaries at $t = \pm\infty$ are identified as in the case of M^4_{conf}. In the case of CP_2 one has 3 homologically trivial spheres defining coordinate patches. This suggests that cd_{conf} is simply CP_2 with second complex coordinate made hypercomplex. M^4 and E^4 differ only by the signature and so would do cd_{conf} and CP_2.

The twistor spheres of CP_3 associated with points of M^4 intersect at point if the points differ by light-like vector so that one has and singular bundle structure. This structure should have analog for the compactification of CD. CP_3 has also bundle structure $CP_3 \to CP_2$. The S^2 fibers and base are homologically non-trivial and complex analogs of mutually orthogonal line and plane and intersect at single point. This defines the desired singular bundle structure via the assignment of S^2 to each point of CP_2.

The M^4 points must belong to the interior of cd and this poses constraints on the distance of M^4 points from the tips of cd. One expects similar hierarchy of cds at the level of momentum space.

5. In this picture $M^4_{conf} = U(2)$ could be interpreted as a base space for the space of CDs with fixed direction of time axis identified as direction of octonionic real axis associated with various points of M^4 and therefore of M^4_{conf}. For Euclidian signature one would have base and fiber of the automorphism sub-group $SU(3)$ regarded as $U(2)$ bundle over CP_2: now one would have CP_2 bundle over $U(2)$. This is perhaps not an accident, and one can ask whether these spaces could be interpreted as representing local trivialization of $SU(3)$ as $U(2) \times CP_2$. This would give to metric cross terms between $U(2)$ and CP_2.

6. The proposed identification can be tested by looking whether it generalizes. What the twistor space for entire M^8 would be? $cd = CD_4$ is replaced with CD_8 and the discussion of the preceding chapter demonstrated that the only possible identification of the twistor space is now is as the 12-D hyperbolic variant of HP_3 whereas $CD_{8,conf}$ would correspond to 8-D hyperbolic variant of HP_2 analogous to hyperbolic variant of CP_2.

The outcome of these considerations is surprising.

1. One would have $T(H) = CP_3 \times F$ and $H = CP_{2,H} \times CP_2$ where $CP_{2,H}$ has hyperbolic metric with metric signature $(1, -3)$ having M^4 as tangent space so that the earlier picture can be understood as an approximation. This would reduce the construction of preferred extremals of 6-D Kähler action in $T(H)$ to a construction of polynomial holomorphic surfaces and also the minimal surfaces with singularities at string world sheets should result as bundle projection. Since $M^8 - H$ duality must respect algebraic dynamics the maximal degree of the polynomials involved must be same as the degree of the octonionic polynomial in M^8.

2. The hyperbolic variant Kähler form and also spinor connection of hyperbolic CP_2 brings in new physics beyond standard model. This Kähler form would serve as the analog of Kähler form assigned to M^4 earlier, and suggested to explain the observed CP breaking effects and matter antimatter asymmetry for which there are two explanations [L5].

Some comments about the Minkowskian signature of the hyperbolic counterparts of CP_3 and CP_2 are in order.

1. Why the metric of CP_3 could not be Euclidian just as the metric of F? The basic objection is that propagation of fields is not possible in Euclidian signature and one completely loses the earlier picture provided by $M^4 \times CP_2$. The algebraic dynamics in M^8 picture can hardly replace it.

2. The map assigning to the point M^4 a point of CP_3 involves Minkowskian sigma matrices but it seems that the Minkowskian metric of CP_3 is not explicitly involved in the construction of scattering amplitudes. Note however that the antisymmetric bi-spinor metric for the spin 1/2 representation of Lorentz group and its conjugate bring in the signature. $U(2,2)$ as representation of conformal symmetries suggests $(2,2)$ signature for 8-D complex twistor space with 2+2 complex coordinates representing twistors.

The signature of CP_3 metric is not explicitly visible in the construction of twistor amplitudes but analytic continuations are carried out routinely. One has also complexified M^4 and M^8 and one could argue that the problems disappear. In the geometric situation the signatures of the subspaces differ dramatically. As already found, analytic continuation could allow to define the variants of twistor spaces elegantly by replacing a complex coordinate with a hyperbolic one.

Remark: For E^4 CP_3 is Euclidian and if one has $E^4_{conf} = U(2)$, one could think of replacing the Cartesian product of twistor spaces with $SU(3)$ group having $M^4_{conf} = U(2)$ as fiber and CP_2 as base. The metric of $SU(3)$ appearing as subgroup of quaternionic automorphisms leaving $M^4 \subset M^8$ invariant would decompose to a sum of M^4_{conf} metric and CP_2 metric plus cross terms representing correlations between the metrics of M^4_{conf} and CP_2. This is probably mere accident.

2.1.2 $M^8 - H$ duality and twistor space counterparts of space-time surfaces

It seems that by identifying $CP_{3,h}$ as the twistor space of M^4, one could develop $M^8 - H$ duality to a surprisingly detailed level from the conditions that the dimensional reduction guaranteed by the identification of the twistor spheres takes place and the extensions of rationals associated with the polynomials defining the space-time surfaces at M^8- and twistor space sides are the same. The reason is that minimal surface conditions reduce to holomorphy meaning algebraic conditions involving first partial derivatives in analogy with algebraic conditions at M^8 side but involving no derivatives.

1. The simplest identification of twistor spheres is by $z_1 = z_2$ for the complex coordinates of the spheres. One can consider replacing z_i by its Möbius transform but by a coordinate change the condition reduces to $z_1 = z_2$.

2. At M^8 side one has either $RE(P) = 0$ or $IM(P) = 0$ for octonionic polynomial obtained as continuation of a real polynomial P with rational coefficients giving 4 conditions (RE/IM denotes real/imaginary part in quaternionic sense). The condition guarantees that tangent/normal space is associative.

   Since quaternion can be decomposed to a sum of two complex numbers: $q = z_1 + Jz_2$ $RE(P) = 0$ correspond to the conditions $Re(RE(P)) = 0$ and $Im(RE(P)) = 0$. $IM(P) = 0$ in turn reduces to the conditions $Re(IM(P)) = 0$ and $Im(IM(P)) = 0$.

3. The extensions of rationals defined by these polynomial conditions must be the same as at the octonionic side. Also algebraic points must be mapped to algebraic points so that cognitive representations are mapped to cognitive representations. The counterparts of both $RE(P) = 0$ and $IM(P) = 0$ should be satisfied for the polynomials at twistor side defining the same extension of rationals.

4. $M^8 - H$ duality must map the complex coordinates $z_{11} = Re(RE)$ and $z_{12} = Im(RE)$ ($z_{21} = Re(IM)$ and $z_{22} = Im(IM)$) at M^8 side to complex coordinates u_{i1} and u_{i2} with $u_{i1}(0) = 0$ and $u_{i2}(0) = 0$ for $i = 1$ or $i = 2$, at twistor side.

   Roots must be mapped to roots in the same extension of rationals, and no new roots are allowed at the twistor side. Hence the map must be linear: $u_{i1} = a_i z_{i1} + b_i z_{i2}$ and $u_{i2} = c_i z_{i1} + d_i z_{i2}$ so that the map for given value of i is characterized by SL(2,Q) matrix $(a_i, b_i; c_i, d_i)$.

5. These conditions do not yet specify the choices of the coordinates (u_{i1}, u_{i2}) at twistor side. At CP_2 side the complex coordinates would naturally correspond to Eguchi-Hanson complex coordinates (w_1, w_2) determined apart from color $SU(3)$ rotation as a counterpart of $SU(3)$ as sub-group of automorphisms of octonions.

   If the base space of the twistor space $CP_{3,h}$ of M^4 is identified as $CP_{2,h}$, the hyper-complex counterpart of CP_2, the analogs of complex coordinates would be (w_3, w_4) with w_3 hypercomplex and w_4 complex. A priori one could select the pair (u_{i1}, u_{i2}) as any pair $(w_{k(i)}, w_{l(i)})$, $k(i) \neq l(i)$. These choices should give different kinds of extremals: such as CP_2 type extremals, string like objects, massless extremals, and their deformations.

String world sheet singularitees and world-line singularities as their light-like boundaries at the light-like orbits of partonic 2-surfaces are conjectured to characterize preferred extremals as surfaces of H at which there is a transfer of canonical momentum currents between Kähler and volume degrees of freedom so that the extremal is not simultaneously an extremal of both Kähler action and volume term as elsewhere. What could be the counteparts of these surfaces in M^8?

1. The interpretation of the pre-images of these singularities in M^8 should be number theoretic and related to the identification of quaternionic imaginary units. One must specify two non-parallel octonionic imaginary units e^1 and e^2 to determine the third one as their cross product $e^3 = e^1 \times e^2$. If e^1 and e^2 are parallel at a point of octonionic surface, the cross product vanishes and the dimension of the quaternionic tangent/normal space reduces from $D = 4$ to $D = 2$.

2. Could string world sheets/partonic 2-surfaces be images of 2-D surfaces in M^8 at which this takes place? The parallelity of the tangent/normal vectors defining imaginary units e_i, $i = 1, 2$ states that the component of e_2 orthogonal to e_1 vanishes. This indeed gives 2 conditions in the space of quaternionic units. Effectively the 4-D space-time surface would degenerate into 2-D at string world sheets and partonic 2-surfacesa as their duals. Note that this condition makes sense in both Euclidian and Minkowskian regions.

3. Partonic orbits in turn would correspond surfaces at which the dimension reduces to D=3 by light-likeness - this condition involves signature in an essential manner - and string world sheets would have 1-D boundaries at partonic orbits.

2.1.3 Getting critical about implicit assumptions related to the twistor space of CP_2

One can also criticize the earlier picture about implicit assumptions related the twistor spaces of CP_2.

1. The possibly singular decomposition of F to a product of S^2 and CP_2 would has a description similar to that for CP_3. One could assign to each point of CP_2 base homologically non-trivial sphere intersecting it orthogonally.

2. I have assumed that the twistor space $T(CP_2) = F = SU(3)/U(1) \times U(1)$ allows Kaluza-Klein type metric meaning that the metric decomposes to a sum of the metrics assignable to the base CP_2 and fiber S^2 plus cross terms representing interaction between these degrees of freedom. It is easy to check that this assumption holds true for Hopf fibration $S^3 \to S^2$ having circle $U(1)$ as fiber (see `http://tinyurl.com/qbvktsx`). If Kaluza-Klein picture holds true, the metric of F would decompose to a sum of CP_2 metric and S^2 metric plus cross terms representing correlations between the metrics of CP_2 and S^2.

3. One should demonstrate that $F = SU(3)/U(1) \times U(1)$ has metric with the expected Kaluza-Klein property. One can represent $SU(3)$ matrices as products XYZ of 3 matrices. X represents a point of base space CP_2 as matrix, Y represents the point of the fiber $S^2 = U(2)/U(1) \times U(1)$ of F in similar manner as $U(2)$ matrix, and the Z represents $U(1) \times U(1)$ element as diagonal matrix [B1](see `http://tinyurl.com/y6c3pp2g`).

   By dropping $U(1) \times U(1)$ matrix one obtains a coordinatization of F. To get the line element of F in these coordinates one could put the coordinate differentials of $U(1) \times U(1)$ to zero in an expression of $SU(3)$ line element. This should leave sum of the metrics of CP_2 and S^2 with constant scales plus cross terms. One might guess that the left- and righ-invariance of the $SU(3)$ metric under $SU(3)$ implies KK property.

Also CP_3 should have the KK structure if one wants to realize the breaking of scaling invariance as a selection of the scale of the conformally compactified M^4. In absence of KK structure the space-time surface would depend parametrically on the point of the twistor sphere S^2.

2.2 The nice results of the earlier approach to M^4 twistorialization

The basic nice results of the earlier picture should survive in the new picture.

1. Central for the entire approach is twistor lift of TGD replacing space-time surfaces with 6-D surfaces in 12-D $T(M^4) \times T(CP_2)$ having space-time surfaces as base and twistor sphere S^2 as fiber. Dimensional reduction identifying twistor spheres of $T(M^4)$ an $T(CP_2)$ and makes these degrees of freedom non-dynamical.

2. Dimensionally reduced action 6-D Kähler action is sum of 4-D Kähler action and a volume term coming from S^2 contribution to the induced Kähler form. On interpretation is as a generalization of Maxwell action for point like charge by making particle a 3-surface.

   The interpretation of volume term is in terms of cosmological constant. I have proposed that a hierarchy of length scale dependent cosmological constants emerges. The hierarchy of cosmological constants would define the running length scale in coupling constant evolution and would correspond to a hierarchy of preferred p-aic length scales with preferred p-adic primes identified as ramified primes of extension of rationals.

3. The twistor spheres associated $M^4 \times S^2$ and F were assumed to have same radii and most naturally same Euclidian signature: this looks very nice since there would be only single fundamental length equal to CP_2 radius determining the radius of its twistor sphere. The vision to be discussed would be different. There would be no fundamental scale and length scales would emerge through the length scale hierarchy assignable to CDs in M^8 and mapped to length scales for twistor spaces.

   The identification of twistor spheres with same radius would give only single value of cosmological constant and the problem of understanding the huge discrepancy between empirical value and its naive estimate would remain. I have argued that the Kähler forms and metrics of the two twistor spheres can be rotated with respect to each other so that the induced metric and Kähler form are rotated with respect to each other, and the magnetic energy density assignable to the sum of the induced Kähler forms is not maximal.

   The definition of Kähler forms involving preferred coordinate frame would gives rise to symmetry breaking. The essential element is interference of real Kähler forms. If the signatures of twistor spheres were opposite, the Kähler forms differ by imaginary unit and the interference would not be possible.

   Interference could give rise to a hierarchy of values of cosmological constant emerging as coefficient of the Kähler magnetic action assignable to $S^2(X^4)$ and predict length scale dependent value of cosmological constant and resolve the basic problem related to the extremely small value of cosmological constant.

4. One could criticize the allowance of relative rotation as adhoc: note that the resulting cosmological constant becomes a function depending on S^2 point. For instance, does the rotation really produce preferred extremals as minimal surfaces extremizing also Kähler action except at string world sheets? Each point of S^2 would correspond to space-time surface X^4 with different value of cosmological constant appearing as a parameter. Moreover, non-trivial relative rotation spoils the covariant constancy and $J^2(S^2) = -g(S^2)$ property for the S^2 part of Kähler form, and that this does not conform with the very idea of twistor space.

5. One nice implication would be that space-time surfaces would be minimal surfaces apart from 2-D string world sheet singularities at which there is a transfer of canonical momentum currents between Kähler and volume degrees of freedom. One can also consider the possibility that the minimal surfaces correspond to surfaces give as roots of 3 polynomials of hypercomplex coordinate of M^2 and remaining complex coordinates.

   Minimal surface property would be direct translation of masslessness and conform with the twistor view. Singular surfaces would represent analogs of Abelian currents. The universal dynamics for minimal surfaces would be a counterpart for the quantum criticality. At M^8 level the preferred complex plane M^2 of complexified octonions would represent the singular string world sheets and would be forced by number theory.

Masslessness would be realized as generalized holomorphy (allowing hyper-complexity in M^2 plane) as proposed in the original twistor approach but replacing holomorphic fields in twistor space with 6-D twistor spaces realized as holomorphic 6-surfaces.

2.3 ZEO and twistorialization as manners to introduce scales in M^8 physics

M^8 physics as such has no scales. One motivation for ZEO is that it brings in the scales as sizes of causal diamonds (CDs).

2.3.1 ZEO generates scales in M^8 physics

Scales are certainly present in physics and must be present also in TGD Universe.

1. In TGD Universe CP_2 scale plays the role of fundamental length scale, there is also the length scale defined by cosmological constant and the geometric mean of these two length scales defining a scale of order 10^{-4} meters emerging in the earlier picture and suggesting a biological interpretation.

   The fact that conformal inversion $m^k \rightarrow R^2 m^k/a^2$, $a^2 = m^k m_k$ is a conformal transformation mapping hyperboloids with $a \geq R$ and $a \leq R$ to each other, suggests that one can relate CP_2 scale and cosmological scale defined by Λ by inversion so that cell length scale would define one possible radius of cd_{conf}.

2. In fact, if one has $R(cd_{conf}) = x \times R(CP_2)$ one obtains by repeated inversions a hierarchy $R(k) = x^k R$ and for $x = \sqrt{p}$ one obtains p-adic length scale hierarchy coming as powers of $\sqrt{p}$, which can be also negative. This suggests a connection with p-adic length scale hypothesis and connections between long length scale and short length scale physics: they could be related by inversion. One could perhaps see Universe as a kind of Leibnizian monadic system in which monads reflect each other with respect to hyperbolic surfaces $a = constant$. This would conform with the holography.

3. Without additional assumptions there is a complete scaling invariance at the level of M^8. The scales could come from the choice of 8-D causal diamond CD_8 as intersection of 8-D future and past directed light-cones inducing choice of cd in M^4. CD serves as a correlate for the perceptive field of a conscious entity in TGD inspired theory of consciousness and is crucial element of zero energy ontology (ZEO) allowing to solve the basic problem of quantum measurement theory.

2.3.2 Twistorial description of CDs

Could the map of the surfaces of 4-surfaces of M^8 to $cd_{conf} \times CP_2$ by a modification of $M^8 - H$ correspondence allow to describe these scales? If so, compactification via twistorialization and $M^8 - H$ correspondence would be the manner to describe these scales as something emergent rather than fundamental.

1. The simplest option is that the scale of cd_{conf} corresponds to that of CD_8 and CD_4. One should also understand what CP_2 scale corresponds. The simplest option is that CP_2 scale defines just length unit since it is difficult to imagine how this scale could appear at M^8 level. cd_{conf} scale squared would be multiple or CP_2 scale squared, say prime multiple of it, and assignable to ramified primes of extension of rationals. Inversions would produce further scales. Inversion would allow kind of hologram like representation of physics in long length scales in arbitrary short length scales and vice versa.

2. The compactness of cd_{conf} corresponds to periodic time assignable to over-critical cosmologies starting with big bang and ending with big crunch. Also CD brings in mind over-critical cosmology, and one can argue that the dynamics at the level of cd_{conf} reflects the dynamics of ZEO at the level of M^8.

2.3.3 Modification of H and M^8-H correspondence

It is often said that the metric of M^4_{conf} is defined only modulo conformal scaling factor. This would reflect projectivity. One can however endow projective space CP_3 with a metric with isometry group $SU(2,2)$ and the fixing of the metric is like gauge choice by choosing representative in the projective equivalence class. Thus CP_3 with signature (3,-3) might perhaps define geometric twistor space with base cd_{conf} rather than M^4_{conf} very much like the twistor space $T(CP_2)=F=SU(3)/U(1)\times U(1)$ at the level. Second projection would be to M^4 and map twistor sphere to a point of M^4. The latter bundle structure would be singular since for points of M^4 with light-like separation the twistor spheres have a common point: this is an essential feature in the construction of twistor amplitudes.

New picture requires a modification of the view about H and about M^8-H correspondence.

1. H would be replaced with $cd_{conf}\times CP_2$ and the corresponding twistor space with $CP_3\times F$. M^8-H duality involves the decomposition $M^2\subset M^4\subset M^8=M^4\times CP_2$, where M^4 is quaternionic sub-space containing preferred place M^2. The tangent or normal space of X^4 would be characterized by a point of CP_2 and would be mapped to a point of CP_2 and the point of CP_2 - or rather point plus the space S^2 or light-like vectors characterizing the choices of M^2 - would mapped to the twistor sphere S^2 of CP_3 by the standard formulas.

   $S^2(cd_{conf})$ would correspond to the choices of the direction of preferred octonionic imaginary unit fixing M^2 as quantization axis of spin and $S^2(CP_2)$ would correspond to the choice of isospin quantization axis: the quantization axis for color hyperspin would be fixed by the choice of quaternionic $M^4\subset M^8$. Hence one would have a nice information theoretic interpretation.

2. The M^4 point mapped to twistor sphere $S^2(CP_3)$ would be projected to a point of cd_{conf} and define M^8-H correspondence at the level of M^4. This would define compactification and associate two scales with it. Only the ratio $R(cd_{conf})/R(CP_2)$ matters by the scaling invariance at M^8 level and one can just fixe the scale assignable to $T(CP_2)$ and call it CP_2 length scale.

One should have a concrete construction for the hyperbolic variants of CP_n.

1. One can represent Minkowski space and its variants with varying signatures as sub-spaces of complexified quaternions, and it would seem that the structure of sub-space must be lifted to the level of the twistor space. One could imagine variants of projective spaces CP_n, $n=2,3$ as and HP_n, $n=2,3$. They would be obtained by multiplying imaginary quaternionic unit I_k with the imaginary unit i commuting with quaternionic units. If the quaternions λ involved with the projectivization $(q_1,...,q_n)\equiv\lambda(q_1,...,q_n)$ are ordinary quaternions, the multiplication respects the signature of the subspace. By non-commutativity of quaternions one can talk about left- and right projective spaces.

2. One would have extremely close correspondence between M^4 and CP_2 degrees of freedom reflecting the M^8-H correspondence. The projection $CP_3\to CP_2$ for E^4 would be replaced with the projection for the hyperbolic analogs of these spaces in the case of M^4. The twistor space of M^4 identified as hyperbolic variant of CP_3 would give hyperbolic variant of CP_2 as conformally compactified cd. The flag manifold $F=SU(3)/U(1)\times U(1)$ as twistor space of CP_2 would also give CP_2 as base space.

The general solution of field equations at the level of $T(H)$ would correspond to holomorphy in general sense for the 6-surfaces defined by 3 vanishing conditions for holomorphic functions - 6 real conditions. Effectively this would mean the knowledge of the exact solutions of field equations also at the level of H: TGD would be an integrable theory. Scattering amplitudes would in turn constructible from these solutions using ordinary partial differential equations [L5].

1. The first condition would identify the complex coordinates of $S^2(cd_{conf})$ and $S^2(CP_2)$: here one cannot exclude relative rotation represented as a holomorphic transformation but for $R(cd_{conf})\gg R(CP_2)$ the effect of the rotation is small.

2. Besides this there would be vanishing conditions for 2 holomorphic polynomials. The coordinate pairs corresponding to $M^2 \subset M^4$ would correspond to hypercomplex behavior with hyper complex coordinate $u = \pm t - z$. t and z could be assigned with $U(1)$ fibers of Hopf fibrations $SU(2) \to S^2$.

3. The octonionic polynomial $P(o)$ of degree $n = h_{eff}/h_0$ with rational coefficients fixes the extension of rationals and since the algebraic extension should be same at both sides, the polynomials in twistor space should have same degree. This would give enormous boos concerning the understanding of the proposed cancellation of fermionic Wick contractions in SUSY scattering amplitudes forced by number theoretic vision [L5].

2.3.4 Possible problems related to the signatures

The different signatures for the metrics of the twistor spheres of cd_{conf} and CP_2 can pose technical problems.

1. Twistor lift would replace X^4 with 6-D twistor space X^6 represented as a 6-surface in $T(M^4)\times T(CP_2)$. X^6 is defined by dimensional reduction in which the twistor spheres $S^2(cd_{conf})$ and $S^2(CP_2)$ are identified and define the twistor sphere $S^2(X^4)$ of X^6 serving as a fiber whereas space-time surface X^4 serves as a base. The simplest identification is as $(\theta,\phi)_{S^2(M^4)} = (\theta,\phi)_{S^2(CP_2)}$: the same can be done for the complex coordinates $z_{S^2(M^4_{conf})} = z_{S^2(CP_2))})$. An open question is whether a Möbius transformation could relate the complex coordinates. The metrics of the spheres are of opposite sign and differ only by the scaling factors $R^2(cd_{conf})$ and $R^2(CP_2)$.

2. For cd_{conf} option the signatures of the 2 twistor spheres would be opposite (time-like for cd_{conf}). For $R(cd_{conf})/R(CP_2) = 1$. $J^2 = -g$ is the only consistent option unless the signature of space is not totally positive or negative and implies that the Kähler forms of the two twistor spheres differ by i. The magnetic contribution from $S^2(X^4)$ would give rise to an infinite value of cosmological constant proportional to $1/\sqrt{g_2}$, which would diverge $R(cd_{conf})/R(CP_2) = 1$. There is however no need to assume this condition as in the original approach.

2.4 Hierarchy of length scale dependent cosmological constants in twistorial description

At the level of M^8 the hierarchy of CDs defines a hierarchy of length scales and must correspond to a hierarchy of length scale dependent cosmological constants. Even fundamental scales would emerge.

1. If one has $R(cd_{conf})/R(CP_2) >> 1$ as the idea about macroscopic cd_{conf} would suggest, the contribution of $S^2(cd_{conf})$ to the cosmological constant dominates and the relative rotation of metrics and Kähler form cannot affect the outcome considerably. Therefore different mechanism producing the hierarchy of cosmological constants is needed and the freedom to choose rather freely the ratio $R(cd_{conf})/R(CP_2)$ would provide the mechanism. What looked like a weakness would become a strength.

2. $S^2(cd_{conf}$ would have time-like metric and could have large scale. Is this really acceptable? Dimensional reduction essential for the twistor induction however makes $S^2(cd_{conf})$ non-dynamical so that time-likeness would not be visible even for large radii of $S^2(cd_{conf})$ expected if the size of cd_{conf} can be even macroscopic. The corresponding contribution to the action as cosmological constant has the sign of magnetic action and also Kähler magnetic energy is positive. If the scales are identical so that twistor spheres have same radius, the contributions to the induced metric cancel each other and the twistor space becomes metrically 4-D.

3. At the limit $R(cd_{conf}) \to RCP_2)$ cosmological constant coming from magnetic energy density diverges for $J^2 = -G$ option since it is proportional to $1/\sqrt{g_2}$. Hence the scaling factors must be different. The interpretation is that cosmological constant has arbitrarily large values near

CP_2 length scale. Note however that time dependence is replaced with scale dependence and space-time sheets with different scales have only wormhole contacts.

It would seem that this approach could produce the nice results of the earlier approach. The view about how the hierarchy of cosmological constants emerges would change but the idea about reducing coupling constant evolution to that for cosmological constant would survive. The interpretation would be in terms of the breaking of scale invariance manifesting as the scales of CDs defining the scales for the twistor spaces involved. New insights about p-adic coupling constant evolution emerge and one finds a new "must" for ZEO. $H = M^4 \times CP_2$ picture would emerge as an approximation when cd_{conf} is replaced with its tangent space M^4. The consideration of the quaternionic generalization of twistor space suggests natural identification of the conformally compactified twistor space as being obtained from CP_2 by making second complex coordinate hyperbolic. This need not conform with the identification as $U(2)$.

3 How to generalize twistor Grassmannian approach in TGD framework?

One should be able to generalize twistor Grassmannian approach in TGD framework. The basic modification is replacement of 4-D light-like momenta with their 8-D counterparts. The octonionic interpretation encourages the idea that twistor approach could generalize to 8-D context. Higher-dimensional generalizations of twistors have been proposed but the basic problem is that the index raising and lifting operations for twistors do not generalize (see `http://tinyurl.com/y24lkwce`).

1. For octonionic twistors as pairs of quaternionic twistors index raising would not be lost working for M_T option and light-like M^8 momenta can be regarded sums of M^4_T and E^4 parts as also twistors. Quaternionic twistor components do not commute and this is essential for incidence relation requiring also the possibility to raise or lower the indices of twistors. Ordinary complex twistor Grassmannians would be replaced with their quaternionic countparts. The twistor space as a generalization of CP_3 would be 3-D quaternionic projective space $T(M^8) = HP_3$ with Minkowskian signature (6,6) of metric and having real dimension 12 as one might expect.

   Another option realizing non-commutativity could be based on the notion of quantum twistor to be also discussed.

2. Second approach would rely on the identification of $M^4 \times CP_2$ twistor space as a Cartesian product of twistor spaces of M^4 and CP_2. For this symmetries are not broken, $M^4_L \subset M^8$ depends on the state and is chosen so that the projection of M^8 momentum is light-like so that ordinary twistors and CP_2 twistors should be enough. $M^8 - H$ relates varying M^4_L based and M^4_T based descriptions.

3. The identification of the twistor space of M^4 as $T(M^4) = M^4 \times S^2$ can be motivated by octonionic considerations but might be criticized as non-standard one. The fact that quaternionic twistor space HP_3 looks natural for M^8 forces to ask whether $T(M^4) = CP_3$ endowed with metric having signature (3,3) could work in the case of M^4. In the sequel also a vision based on the identification $T(M^4) = CP_3$ endowed with metric having signature (3,3) will be discussed.

3.1 Twistor lift of TGD at classical level

In TGD framework twistor structure is generalized [K10, K12, K8, L4]. The inspiration for TGD approach to twistorialization has come from the work of Nima Arkani-Hamed and colleagues [B11, B5, B6, B8, B14, B12, B2]. The new element is the formulation of twistor lift also at the level of classical dynamics rather than for the scattering amplitudes only [K10, K8, K12, L4].

1. The 4-D light-like momenta in ordinary twistor approach are replaced by 8-D light-like momenta so that massive particles in 4-D sense become possible. Twistor lift of TGD takes

places also at the space-time level and is geometric counterpart for the Penrose's replacement of massless fields with twistors. Roughly, space-time surfaces are replaced with their 6-D twistor spaces represented as 6-surfaces. Space-time surfaces as preferred extremals are minimal surfaces with 2-D string world sheets as singularities. This is the geometric manner to express masslessness. X^4 is simultaneously also extremal of 4-D Kähler action outside singularities: this realizes preferred extremal property.

2. One can say that twistor structure of X^4 is induced from the twistor structure of $H = M^4 \times CP_2$, whose twistor space $T(H)$ is the Cartesian product of geometric twistor space $T(M^4) = M^4 \times CP_1$ and $T(CP_2) = SU(3)/U(1) \times U(1)$. The twistor space of M^4 assigned to momenta is usually taken as a variant of CP_3 with metric having Minkowski signature and both CP_1 fibrations appear in the more precise definition of $T(M^4)$. Double fibration [B13] (see http://tinyurl.com/yb4bt74l) means that one has fibration from $M^4 \times CP_1$ - the trivial CP_1 bundle defining the geometric twistor space to the twistors space identified as complex projective space defining conformal compactification of M^4. Double fibration is essential in the twistorialization of TGD [K9].

3. The basic objects in the twistor lift of classical TGD are 6-D surfaces in $T(H)$ having the structure of twistor space in the sense that they are CP_1 bundles having X^4 as base space. Dimensional reduction to CP_1 bundle effectively eliminates the dynamics in CP_1 degrees of freedom and its only remnant is the value of cosmological constant appearing as coefficient of volume term of the dimensionally reduced action containing also 4-D Kähler action. Cosmological term depends on p-adic length scales and has a discrete spectrum [L4, L3].

CP_1 has also an interpretation as a projective space constructed from 2-D complex spinors. Could the replacement of these 2-spinors with their quantum counterparts defining in turn quantum CP_1 realize finite quantum measurement resolution in M^4 degrees of freedom? Projective invariance for the complex 2-spinors would mean that one indeed has effectively CP_1.

3.2 Octonionic twistors or quantum twistors as twistor description of massive particles

For M^4_T option the particles are massive and the one encounters the problem whether and how to generalize the ordinary twistor description.

3.3 Basic facts about twistors and bi-spinors

It is convenient to start by summarizing the basic facts about bi-spinors and their conjugates allowing to express massless momenta as $p^{aa'} = \lambda^a \tilde{\lambda}^{a'}$ with $\tilde{\lambda}$ defined as complex conjugate of λ and having opposite chirality (see http://tinyurl.com/y6bnznyn).

1. When λ is scaled by a complex number $\tilde{\lambda}$ suffers an opposite scaling. The bi-spinors allow the definition of various inner products

$$
\begin{aligned}
\langle \lambda, \mu \rangle &= \epsilon_{ab} \lambda^a \mu^b \ , \\
\left[\tilde{\lambda}, \tilde{\mu} \right] &= \epsilon_{a'b'} \tilde{\lambda}^{a'} \tilde{\mu}^{b'} \ , \\
p \cdot q &= \langle \lambda, \mu \rangle \left[\tilde{\lambda}, \tilde{\mu} \right] \ , \ (q_{aa'} = \mu_a \tilde{\mu}_{a'}) \ .
\end{aligned} \tag{3.1}
$$

2. Spinor indices are lowered and raised using antisymmetric tensors $\epsilon^{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$. If the particle has spin one can assign it a positive or negative helicity $h = \pm 1$. Positive helicity can be represented by introducing artitrary negative (positive) helicity bispinor μ_a ($\mu_{a'}$) not parallel to λ_a ($\mu_{a'}$) so that one can write for the polarization vector

$$\epsilon_{aa'} = \frac{\mu_a \tilde{\lambda}_{a'}}{\langle \mu, \lambda \rangle} \ , \ \text{positive helicity} \ ,$$
$$\epsilon_{aa'} = \frac{\lambda_a \tilde{\mu}_{a'}}{\left[\tilde{\mu}, \tilde{\lambda}\right]} \ , \ \text{negative helicity} \ . \qquad (3.2)$$

In the case of momentum twistors the μ part is determined by different criterion to be discussed later.

3. What makes 4-D twistors unique is the existence of the index raising and lifting operations using ϵ tensors. In higher dimensions they do not exist and this causes difficulties. For octonionic twistors with quaternionic components possibly only in $D = 8$ the situation changes.

To get a very rough idea about twistor Grassmannian approach idea, consider tree amplitudes of $\mathcal{N} = 4$ SUSY as example and it is convenient to drop the group theory factor $Tr(T_1T_2 \cdots T_n)$. The starting point is the observation that tree amplitude for which more than $n-2$ gluons have the same helicity vanish. MHV amplitudes have exactly $n-2$ gluons of same helicity- taken by a convention to be negative- have extremely simple form in terms of the spinors and reads as

$$A_n = \frac{\langle \lambda_x, \lambda_y \rangle^4}{\prod_{i=1}^{n} \langle \lambda_i, \lambda_{i+1} \rangle} \qquad (3.3)$$

When the sign of the helicities is changed $\langle .. \rangle$ is replaced with [..].

An essential point in what follows is that the amplitudes are expressible in terms of the antisymmetric bi-linears $\langle \lambda_i, \lambda_j \rangle$ making sense also for octotwistors and identifiable as quaternions rather than octonions.

3.3.1 $M^8 - H$ duality and two alternative twistorializations of TGD

$M^8 - H$ duality suggests two alternative twistorializations of TGD.

1. The first approach would be in terms of M^8 twistors suggested by quaternionic light-lineness of 8-momenta. M^8 twistors would be Cartesian products of M^4 and E^4 twistors. One can imagine a straightforward generalization of twistor scattering amplitudes in terms of generalized Grassmannian approach replacing complex Grassmannian with quaternionic Grassmannian, which is a mathematically well-defined notion.

2. Second approach would rely on $M^4 \times CP_2$ twistors, which are products of M^4 twistors and CP_2 twistors: this description works nicely at classical space-time level but at the level of momentum space the problem is how to describe massivation of M^4 momenta using twistors.

3.3.2 Why the components of twistors must be non-commutative?

How to modify the 4-D twistor description of light-like 4-momenta so that it applies to massive 4-momenta?

1. Twistor consists of a pair $(\mu_{\dot{\alpha}}, \lambda^{\alpha})$ of bi-spinors in conjugate representations of $SU(2)$. One can start from the 4-D incidence relations for twistors

$$\mu_{\dot{\alpha}} = p_{\alpha\dot{\alpha}} \lambda^{\alpha} \ .$$

Here $p_{\alpha\dot{\alpha}}$ denotes the representation of four-momentum $p^k \sigma_k$. The antisymmetric permutation symbols $\epsilon^{\alpha\beta}$ and its dotted version define antisymmetric "inner product" in twistor space. By taking the inner product of μ with itself, one obtains the commutation relation $\mu_1\mu_2 - \mu_2\mu_1 = 0$, which is consistent with right-hand side for massless particles with $p_k p^k = 0$.

2. In TGD framework particles are massless only in 8-D sense so that the right hand side in the contraction is in general non-vanishing. In massive case one can replace four-momentum with unit vector. This requires

$$\langle \mu, \mu \rangle = \mu_1 \mu_2 - \mu_2 \mu_1 \neq 0 \ .$$

The components of 2-spinor become non-commutative.

This raises two questions.

1. Could the replacement of complex twistors by quaternionic twistors make them non-commutative and allow massive states?
2. Could non-commutative quantum twistors solve the problem caused by the light-likeness of momenta allowing 4-D twistor description?

3.3.3 Octotwistors or quantum twistors?

One should be able to generalize twistor amplitudes and twistor Grassmannian approach to TGD framework, where particles are massless in 8-D sense and massive in 4-D sense. Could twistors be replaced by octonionic or quantum twistors.

1. One can express mass squared as a product of commutators of components of the twistors λ and $\tilde{\lambda}$, which is essentially the conjugate of λ:

$$p \cdot p \quad = \quad \langle \lambda, \lambda \rangle \left[\tilde{\lambda}, \tilde{\lambda} \right] \ . \tag{3.4}$$

This operator should be non-vanishing for non-vanishing mass squared. Both terms in the product vanish unless commutativity fails so that mass vanishes. The commutators should have the quantum state as its eigenstate.

2. Also 4-momentum components should have well-defined values. Four-momentum has expression $p^{aa'} = \lambda^a \tilde{\lambda}^{a'}$ in massless case. This expression should generalized to massive case as such. Eigenvalue condition and reality of the momentum components requires that the components $p^{aa'}$ are commuting Hermitian operators.

   In twistor Grassmannian approach complex but light-like momenta are possible as analogs of virtual momenta. Also in TGD framework the complexity of Kähler coupling strength allows to consider complex momenta. For twistor lift they however differ from real momenta only by a phase factor associated with the $1/\alpha_K$ associated with 6-D Kähler action.

   Remark: I have considered also the possibility that states are eigenstates only for the longitudinal M^2 projection of 4-momentum with quark model of hadrons serving as a motivation.

   (a) Could this equation be obtained in massive case by regarding λ^a and $\tilde{\lambda}^{a'}$ as commuting octo-spinors and their complex conjugates? Octotwistors would naturally emerge in the description at imbedding space level. I have already earlier considered the notion of octotwistor [K4] [L1]).

   (b) Or could it be obtained for quantum bi-spinors having same states as eigenstates. Could quantum twistors as generalization of the ordinary twistors correspond to the reduction of the description from the level of M^8 or H to at space-time level so that one would have 4-D twistors and massive particles with 4-momentum identifiable as Noether charge for the action principle determining preferred extremals? I have considered also the notion of quantum spinor earlier [K2, K6, K3, K5, K7].

3. In the case of quantum twistors the generalization of the product of the quantities $\langle \lambda_i, \lambda_{i+1} \rangle$ appearing in the formula should give rise to c-number in the case of quantum spinors. Can one require that the quantities $\langle \lambda_i, \lambda_{i+1} \rangle$ or even $\langle \lambda_i, \lambda_j \rangle$ are c-numbers simultaneously? This would also require that $\langle \lambda, \lambda \rangle$ is non-vanishing c-number in massive case: also incidence relation suggest this condition. Could one think λ as an operator such that $\langle \lambda, \lambda \rangle$ has eigenvalue spectrum corresponding to the quantities $\langle \lambda_i, \lambda_{i+1} \rangle$ appearing in the scattering amplitude?

3.4 The description for M^4_T option using octo-twistors?

For option I with massive M^4_T projection of 8-momentum one could imagine twistorial description by using M^8 twistors as products of M^4_T and E^4 twistors, and a rather straightforward generalization of standard twistor Grassmann approach can be considered.

3.4.1 Could twistor Grassmannians be replaced with their quaternionic variants?

The first guess would simply replace $Gr(k,n)$ with $Gr(2k,2n)$ 4-D twistors 8-D twistors. From twistor amplitudes with quaternionic M^8-momenta one could construct physical amplitudes by going from 8-momentum basis to the 4-momentum- basis with wave functions in irreps of $SO(3)$. Life is however not so simple.

1. The notion of ordinary twistor involves in an essential manner Pauli matrices σ_i satisfying the well-known anti-commutation relations. They should be generalized. In fact, σ_0 and $\sqrt{-1}\sigma_i$ can be regarded as a matrix representation for quaternionic units. They should have analogs in 8-D case.

   Octonionic units ie_i indeed provide this analog of sigma matrices. Octonionic units for the complexification of octonions allow to define incidence relation and representation of 8-momenta in terms of octo-spinors. They do not however allow matrix representation whereas time-like octonions allow interpretation as quaternion in suitable bases and thus matrix representation. Index raising operation is essential for twistors and makes dimension $D = 4$ very special. For naive generalizations of twistors to higher dimensions this operation is lost (see `http://tinyurl.com/y24lkwce`).

2. Could one avoid multiplication of more than two octo-twistors in Grassmann amplitudes leading to difficulties with associativity. An important observation is that in the expressions for the twistorial scattering amplitudes only products $\langle\lambda_i,\lambda_j\rangle$ or $[\tilde{\lambda}_i,\tilde{\lambda}_{i+1}]$ but not both occur. These products are associative even if the spinors are replaced by quaternionic spinors.

   These operations are antisymmetric in the arguments, which suggests cross product for quaternions giving rise to imaginary quaternion so that the product of objects would give rise to a product of imaginary quaternions. This might be a problem since a large number of terms in the product would approach to zero for random imaginary quaternions.

   An ad hoc guess would be that scattering probability is proportional to the product of amplitude as product $\langle\lambda_i,\lambda_j\rangle$ and its "hermitian conjugate" with the conjugates $[\tilde{\lambda}_i,\tilde{\lambda}_{i+1}]$ in the reverse order (this does not affect the outcome) so that the result would be real. Scattering amplitude would be more like quaternion valued operator. Could one have a formulation of quantum theory or at least TGD view about quantum theory allowing this?

3. If ordinary massless 4-momenta correspond to quaternionic sigma matrices, twistors can be regarded as pairs of 2-spinors in matrix representation. Octonionic 8-momenta should correspond to pairs of 4-spinors. As already noticed, octonions do not however allow matrix representation! Octonions for a fixed decomposition $M^8 = M^4 \times E^4$ can be however decomposed to linear combination of two quaternions just like complex numbers to a combination of real numbers. These quaternions would have matrix representation and quaternionic analogs of twistor pair $(\mu,\tilde{\lambda})$. One could perhaps formulate the generalization of twistor Grassmann amplitudes using these pairs. This would suggest replacement of complex bi-spinors with complexified quaternions in the ordinary formalism. This might allow to solve problems with associativity if only $\langle\lambda_i,\lambda_j\rangle$ or $[\tilde{\lambda}_i,\tilde{\lambda}_{i+1}]$ appear in the amplitudes.

4. The argument in the momentum conserving delta function $\delta(\lambda_i\tilde{\lambda}_i)$ should be real so that the conjugation with respect to i would not change the argument and non-commutativity would not be problem. In twistor Grassmann amplitudes the argument $C \cdot Z$ of delta momentum conserving function is linear in the components of complex twistor Z. If complex twistor is replaced with quaternionic twistor, the Grassmannian coordinates C in delta functions $\delta(C \cdot Z)$ must be replaced with quaternionic one.

The replacement of complex Grassmannians $Gr_C(k,n)$ with quaternionic Grassmannians $Gr_H(k,n)$ is therefore highly suggestive. Quaternionic Grassmannians (see http://tinyurl.com/y23jsffn) are quotients of symplectic Lie groups $Gr_H(k,n) = U_n(H)/(U_r(H) \times U_{n-r}(H))$ and thus well-defined. In the description using $Gl_H(k,n)$ matrices the matrix elements would be quaternions and $k \times k$ minors would be quaternionic determinants.

Remark: Higher-D projective spaces of octonions do not exist so that in this sense dimension $D = 8$ for imbedding space would be maximal.

3.4.2 Twistor space of M^8 as quaternionic projective space HP_3?

The simplest Grassmannian corresponds to twistor space and one can look what one obtains in this case. One can also try to understand how to cope with the problems caused by Minkowskian signature.

1. In previous section it was found that the modification of H to $H = cd_{conf} \times CP_2$ with $cd_{conf} = CP_{2,h}$ identifiable as CP_2 with Minkowskian signature of metric is strongly suggestive.

2. For E^8 quaternionic twistor space as analog of CP_3 would be its quaternionic variant HP_3 with expected dimension $D = 16 - 4 = 12$. Twistor sphere would be replaced with its quaternionic counterpart $SU(2)_H/U(1)_H$ having dimension 4 as expected. $CD_{8,conf}$ as conformally compactified CD_8 must be 8-D. The space HP_2 has dimension 8 and is analog of CP_2 appearing as analog of base space of CP_3 identified as conformally compactified 4-D causal diamond cd_{conf}. The quaternionic analogy of $M^4_{conf} = U(2)$ identified as conformally compactified M^4 would be $U(2)_H$ having dimension $D = 10$ rather than 8.

   HP_3 and HP_2 might work for E^8 but it seems that the 4-D analog of twistor sphere should have signature (2,-2) whereas base space should have signature (1,-7). Some kind of hyperbolic analogs of these spaces obtained by replacing quaternions with their hypercomplex variant seem to be needed. The same receipe in the twistorialization of M^4 would give cd_{conf} as analog of CP_2 with second complex coordinate made hyperbolic. I have already considered the construction of hyperbolic analogs of CP_2 and CP_3 as projective spaces. These results apply to HP_2 and HP_3.

3. What about octonions? Could one define octonionic projective plane OP_2 and its hyperbolic variants corresponding to various sub-spaces of M^8? Euclidian OP_2 known as Cayley plane exists as discovered by Ruth Moufang in 1933. Octonionic higher-D projective spaces and Grassmannians do not however exist so that one cannot assign OP_3 as twistor spaces.

3.4.3 Can one obtain scattering amplitudes as quaternionic analogs of residue integrals?

Can one obtain complex valued scattering amplitudes (i commuting with octonionic units) in this framework?

1. The residue integral over quaternionic C-coordinates should make sense, and pick up the poles as vanishing points of minors. The outcome of repeated residue integrations should give a sum over poles with complex residues.

2. Residue calculus requires analyticity. The problem is that quaternion analyticity based on a generalization of Cauchy-Riemann equations allows only linear functions. One could define quaternion (and octonion) analyticity in restricted sense using powers series with real coefficients (or in extension involving i commuting with octonion units). The quaternion/octonion analytic functions with real coefficients are closed with respect to sum and product. I have used this definition in the proposed construction of algebraic dynamics for in $X^4 \subset M^8$ [L1].

3. Could one define the residue integral purely algebraically? Could complexity of the coefficients (i) force complex outcome: if pole q_0 is not quaternionically real the function would not allow decompose to $f(q)/(q - q_0)$ with f allowing similar Taylor series at pole. If so, then the formulas of Grassmannian formalism could generalize more or less as such at M^8 level and one could map the predictions to predictions of $M^4 \times CP_2$ approach by analog of Fourier transform transforming these quantum state basis to each other.

This option looks rather interesting and involves the key number theoretic aspects of TGD in a crucial manner.

3.5 Do super-twistors make sense at the level of M^8?

By $M^8 - H$ duality [L1] there are two levels involved: M^8 and H. These levels are encountered both at the space-time level and momentum space level. Do super-octonions and super-twistors make sense at M^8 level?

1. At the level of M^8 the high uniqueness and linearity of octonion coordinates makes the notion of super-octonion natural. By $SO(8)$ triality octonionic coordinates (bosonic octet 8_0), octonionic spinors (fermionic octet 8_1), and their conjugates (anti-fermionic octet 8_{-1}) would for triplet related by triality. A possible problem is caused by the presence of separately conserved B and L. Together with fermion number conservation this would require $\mathcal{N} = 4$ or even $\mathcal{N} = 4$ SUSY, which is indeed the simplest and most beautiful SUSY.

2. At the level of the 8-D momentum space octonionic twistors would be pairs of two quaternionic spinors as a generalization of ordinary twistors. Super octo-twistors would be obtained as generalization of these.

The progress in the understanding of the TGD version of SUSY [L5] led to a dramatic progress in the understanding of super-twistors.

1. In non-twistorial description using space-time surfaces and Dirac spinors in H, imbedding space coordinates are replaced with super-coordinates and spinors with super-spinors. Theta parameters are replaced with quark creation and annihilation operators. Super-coordinate is a super-polynomial consisting of monomials with vanishing total quark number and appearing in pairs of monomial and its conjugate to guarantee hermiticity.

   Dirac spinor is a polynomial consisting of powers of quark creation operators multiplied by monomials similar to those appearing in the super-coordinate. Anti-leptons are identified as spartners ofquarks identified as local 3-quark states. The multi-spinors appearing in the expansions describe as such local many-quark-antiquark states so that super-symmetrization means also second quantization. Fermionic and bosonic states assignable to H-geometry interact since super-Dirac action contains induced metric and couplings to induced gauge potentials.

2. The same recipe works at the level of twistor space. One introduces twistor super-coordinates analogous to super-coordinates of H and M^8. The super YM field of $\mathcal{N} = 4$ SUSY is replaced with super-Dirac spinor in twistor space. The spin degrees of freedom associated with twistor spheres S^2 would bring in 2 additional spin-like degrees of freedom.

   The most plausible option is that the new spin degrees are frozen just like the geometric S^2 degrees of freedom. The freezing of bosonic degrees of freedom is implied by the construction of twistor space of X^4 by dimensional reduction as a 6-D surface in the product of twistor spaces of M^4 and CP_2. Chirality conditions would allow only single spin state for both spheres.

3. Number theoretical vision implies that the number of Wick contractions of quarks and antiquarks cannot be larger than the degree of the octonionic polynomial, which in turn should be same as that of the polynomials of twistor space giving rise to the twistor space of space-time surface as 6-surface. The resulting conditions correspond to conserved currents identifiable as Noether currents assignable to symmetries.

Also Grassmannian is replaced with super-Grassmannian and super-coordinates as matrix elements of super matrices are introduced.

1. The integrand of the Grassmannian integral defining the amplitude can be expanded in Taylor series with respect to theta parameters associated with the super coordinates C as rows of super $G(k,n)$ matrix.

2. The delta function $\delta(C, Z)$ factorizing into a product of delta functions is also expanded in Taylor series to get derivatives of delta function in which only coordinates appear. By partial integration the derivatives acting on delta function are transformed to derivatives acting on integrand already expanded in Taylor series in theta parameters. The integration over the theta parameters using the standard rules gives the amplitudes associated with different powers of theta parameters associated with Z and from this expression one can pick up the scattering amplitudes for various helicities of external particles.

The super-Grassmannian formalism is extremely beautiful but one must remember that one is dealing with quantum field theory. It is not at all clear whether this kind of formalism generalizes to TGD framework, where particle are 3-surfaces [L1]. The notion of cognitive representation effectively reducing 3-surfaces to a set of point-like particles strongly suggests that the generalization exists.

The progress in understanding of $M^8 - H$ duality throws also light to the problem whether SUSY is realized in TGD and what SUSY breaking does mean. It seems now clear that sparticles are predicted and SUSY remains in the simplest scenario exact but that p-adic thermodynamics causes thermal massivation: unlike Higgs mechanism, this massivation mechanism is universal and has nothing to do with dynamics. This is due to the fact that zero energy states are superpositions of states with different masses. The selection of p-adic prime characterizing the sparticle causes the mass splitting between members of super-multiplets although the mass formula is same for all of them.

The increased undestanding of what twistorialization leads to an improved understanding of what twistor space in TGD could be. It turns out that the hyperbolic variant $CP_{3,h}$ of the standard twistor space CP_3 is a more natural identification than the earlier $M^4 \times S^2$ also in TGD framework but with a scale corresponding to the scale of CD at the level of M^8 so that one obtains a scale hierarchy of twistor spaces. Twistor space has besides the projection to M^4 also a bundle projection to the hyperbolic variant $CP_{2,h}$ of CP_2 so that a remarkable analogy between M^4 and CP_2 emerges. One can formulate super-twistor approach to TGD using the same formalism as will be discussed in this article for the formulation at the level of H. This requires introducing besides 6-D Kähler action and its super-variant also spinors and their super-variants in super-twistor space. The two formulations are equivalent apart from the hierarchy of scales for the twistor space. Also M^8 allows analog of twistor space as quaternionic Grassmannian HP_3 with signature (6,6). What about super- variant of twistor lift of TGD? consider first the situation before the twistorialization.

1. The parallel progress in the understanding SUSY in TGD framework [L5] leads to the identification of the super-counterparts of M^8, H and of twistor spaces modifying dramatically the physical interpretation of SUSY. Super-spinors in twistor space would provide the description of quantum states. Super-Grassmannians would be involved with the construction of scattering amplitudes. Quaternionic super Grassmannians would be involved with M^8 description.

2. In fermionic sector only quarks are allowed by $SO(1, 7)$ triality and that anti-leptons are local 3-quark composites identifiable as spartners of quarks. Gauge bosons, Higgs and graviton would be also spartners and assignable to super-coordinates of imbedding space expressible as super-polynomials of quark oscillator operators. Super-symmetrization means also quantization of fermions allowing local many-quark states.

3. SUSY breaking would be caused by the same universal mechanism as ordinary massivation of massless states. The mass formulas would be supersymmetric but the choice of p-adic prime identifiable as ramified prime of extension of rationals would depend on the state of super-multiplet. ZEO would make possible symmetry breaking without symmetry breaking as Wheeler might put it.

3.5.1 Super-counterpart of twistor lift using the proposed formalism

The construction of super-coordinates and super-spinors [L5] suggests a straightforward formulation of the super variant of twistor lift . One should only replace the super-imbedding space and super-spinors with super-twistor space and corresponding super-spinors and formulate the theory using 6-D super-Kähler action and super-Dirac equation and the same general prescription for constructing S-matrix. Dimensional reduction should give essentially the 4-D theory apart from the variation of the radius of the twistor space predicting variation of cosmological constant. The size scale of CD would correspond to the size scale of the twistor space for M^4 and for CP_2 the size scale would serve as unit and would not vary.

The first step is the construction of ordinary variant of Kähler action and modified Dirac action for 6-D surfaces in 12-D twistor space.

1. Replace the spinors of H with the spinors of 12-D twistor space and assume only quark chirality. By the bundle property of the twistor space one can express the spinors as tensor products of spinors of the twistor spaces $T(M^4)$ and $T(CP_2)$. One can express the spinors of $T(M^4)$ tensor products of spinors of M^4 - and S^2 spinors locally and spinors of $T(CP_2)$ as tensor products of CP_2 - and S^2 spinors locally. Chirality conditions should reduce the number of 2 spin components for both $T(M^4)$ and $T(CP_2)$ to one so that there are no additional spin degrees of freedom.

   The dimensional reduction can be generalized by identifying the two S^2 fibers for the preferred extremals so that one obtains induced twistor structure. In spinorial sector the dimensional reduction must identify spinorial degrees of freedom of the two S^2s by the proposed chirality conditions also make them non-dynamical. The S^2 spinors covariantly constant in S^2 degrees of freedom.

2. Define the spinor structure of 12-D twistor space, define induced spinor structure at 6-D surfaces defining the twistor space of space-time surface. Define the twistor counterpart of the analog of modified Dirac action using same general formulas as in case of H.

Construct next the super-variant of this structure.

1. Introduce second quark oscillator operators labelled by the points of cognitive representation in 12-D twistor space effectively replacing 6-D surface with its discretization and having quantized quark field q as its continuum counterpart. Replace the coordinates of the 12-D twistor space with super coordinates h_s expressed in terms of quark and anti-quark oscillator operators labelled by points of cognitive representation, and having interpretation as quantized quark field q restricted to the points of representation.

2. Express 6-D Kähler action and Dirac action density in terms of super-coordinates h_s. The local monomials of q appear in h_s and therefore also in the expansion of super-variants of modified gamma matrices defined by 6-D ähler action as contractions of canonical momentum currents of the action density L_K with the gamma matrices of 12-D twistor space. In super-Kähler action also the local composites of q giving rise to currents formed from the local composites of 3-quarks and antiquarks and having interpretation as leptons and anti-leptons occur - leptons would be therefore spartners of squarks.

3. Perform super-expansion also for the induced spinor field q_s in terms of monomials of q. $q_s(q)$ obeys super-Dirac equation non-linear in q. But also q should satisfy super-Dirac action as an analog of quantized quark field and non-linearity indeed forces also q to have has super-expansion. Thus both quark field q and super-quark field q_s both satisfy super-Dirac equation.

   The only possibility is $q_s = q$ stating fixed point property under $q \to q_s$ having interpretation in terms of quantum criticality fixing the values of the coefficients of various terms in q_s and in the super-coordinate h_s having interpretation as coupling constants. One has quantum criticality and discrete coupling constant evolution with respect to extension of rationals characterizing adelic physics.

4. Super-Dirac action vanishes for its solutions and the exponent of super-action reduces to exponent of super-Kähler action, whose matrix elements between positive and negative energy parts of zero energy states give S-matrix elements.

   Super-Dirac action has however an important function: the derivatives of quark currents appearing in the super-Kähler action can be transformed to a linear strictly local action of super spinor connection ($\partial_\alpha \rightarrow A_{\alpha,s}$ effectively). Without this lattice discretization would be needed and cognitive representation would not be enough.

To sum up, the super variants of modified gamma matrices of the 6-surface would satisfy the condition $D_{\alpha,s}\Gamma_s^\alpha = 0$ expressing preferred extremal property and guaranteeing super-hermicity of D_s. q_s would obey super-Dirac equation $D_s q_s = 0$. The self-referential identification $q = q_s$ would express quantum criticality of TGD.

4 Could one describe massive particles using 4-D quantum twistors?

The quaternionic generalization of twistors looks almost must. But before this I considered also the possibility that ordinary twistors could be generalized to quantum twistors to describe particle massivation. Quantum twistors could provide space-time level description, which requires 4-D twistors, which cannot be ordinary M^4 twistors. Also the classical 4-momenta, which by QCC would be equal to M^8 momenta, are in general massive so that the ordinary twistor approach cannot work. One cannot of course exclude the possibility that octo-twistors are enough or that M^8_L description is equivalent with space-time description using quantum twistors.

4.1 How to define quantum Grassmannian?

The approach to twistor amplitude relies on twistor Grassmann approach [B7, B4, B3, B10, B11, B2] (see `http://tinyurl.com/yxllwcsn`). This approach should be replaced by replacing Grassmannian $GR(K,N) = Gl(n,C)/Gl(n-m,C) \times Gl(m,C)$ with quantum Grassmannian.

4.1.1 Naive approach to the definition of quantum Grassmannian

Quantum Grassmannian is a notion studied in mathematics and the approach of [A1] (see `http://tinyurl.com/y5q6kv6b`) looks reasonably comprehensible even for physicist. I have already earlier tried to understand quantum algebras and their possible role in TGD [K1]. It is however better to start as ignorant physicist and proceed by trial and error and find whether mathematicians have ended up with something similar.

1. Twistor Grassmannian scattering amplitudes involving k negative helicity gluons involve product of $k \times k$ minors of an $k \times n$ matrix C taken in cyclic order. C defines $k \times n$ coordinates for Grassmannian $Gr(k,n)$ of which part is redundant by the analogs of gauge symmetries $Gl(n-m,C) \times Gl(m,C)$. Here n is the number of external gluons and k the number of negative helicity gluons. The $k \times k$ determinants taken in cyclic order appear in the integrand over Grassmannian. Also the quantum variants of these determinants and integral over quantum Grassmannian should be well-defined and residue calculus gives hopes for achieving this.

2. One should define quantum Grassmannian as algebra according to my physicist's understanding algebra can be defined by starting from a free algebra generated by a set of elements - now the matrix elements of quantum matrix. One poses on these elements relations to get the algebra considered. What could these conditions be in the recent case.

3. A natural condition is that the definition allows induction in the sense that its restriction to quantum sub-matrices is consistent with the general definition of $k \times n$ quantum matrices. In particular, one can identify the columns and rows of quantum matrices as instances of quantum vectors.

4. How to generalize from 2×2 case to $k \times n$ case? The commutation relations for neighboring elements of rows and columns are fixed by induction. In 4×4 corresponding to M^4 twistors one would obtain for $(a_1, ..., a_4)$. $a_i a_{i+1} = q a_{i+1} a_i$ cyclically ($k = 1$ follows $k = 4$).

   What about commutations of a_i and a_{i+k}, $k > 1$. Is there need to say anything about these commutators? In twistor Grassmann approach only connected $k \times k$ minors in cyclic order appear. Without additional relations the algebra might be too large. One could argue that the simplest option is that one has $a_ia_{i+k} = qa_{i+k}a_i$ for k odd $a_ia_{i+k} = q^{-1}a_{i+k}a_i$ for k even. This is required from the consistency with cyclicity. These conditions would allow to define also sub-determinants, which do not correspond to connected $k \times k$ squares by moving the elements to a a connected patch by permutations of rows and columns.

5. What about elements along diagonal? The induction from 2×2 would require the commutativity of elements along right-left diagonals. Only commutativity of the elements along left-right diagonal be modified. Or is the commutativity lost only along directions parallel to left-right diagonal? The problem is that the left-right and right-left directions are transformed to each other in odd permutations. This would suggest that only even permutations are allowed in the definition of determinant

6. Could one proceed inductively and require that one obtains the algebra for 2×2 matrices for all 2×2 minors? Does this apply to all 2×2 minors or only to connected 2×2 minors with cyclic ordering of rows and columns so that top and bottom row are nearest neighbors as also right and left column. Also in the definition of 3×3 determinant only the connected developed along the top row or left column only 2×2 determinants involving nearest neighbor matrix elements appear. This generalizes to $k \times k$ case.

It is time to check how wrong the naive intuition has been. Consider 2×2 matrices as simple example. In this case this gives only 1 condition ($ad - bc = -da + cb$) corresponding to the permutation of rows or columns. Stronger condition suggested by higher-D case would be $ad = da$ and $bc = cb$. The definition of 2×2 in [A1] however gives for quantum 2-matrices $(a, b; c, d)$ the conditions

$$\begin{array}{ll} ac = qca \ , & bd = qda \ , \\ ab = qba \ , cd = qdc \ , & \\ bc = cb \ , & ad - da = (q - q^{-1})bc \ . \end{array} \tag{4.1}$$

The commutativity along left-right diagonal is however lost for $q \neq 1$ so that quantum determinant depends on what row or column is used to expand it. The modification of the commutation relations along rows and columns is what one might expect and wants in order to achieve non-commutativity of twistor components making possible massivation in M^4 sense.

The limit $q \to 1$ corresponds to non-trivial algebra in general and would correspond to $\beta = 4$ for inclusions of HFFs expected to give representations of Kac-Moody algebras. At this limit only massless particles in 4-D sense are allowed. This suggests that the reduction of Kac-Moody algebras to quantum groups corresponds to symmetry breaking associated with massivation in 4-D sense.

4.1.2 Mathematical definition of quantum Grassmannian

It would seem that the proposed approach is reasonable. The article [A2] (see `http://tinyurl.com/yycflgrd`) proposing a definition of quantum determinant explains also the basic interpretation of what the non-commutativity of elements of quantum matrices does mean.

1. The first observation is that the commutation of the elements of quantum matrix corresponds to braiding rather than permutation and this operation is represented by R-matrix. The formula for the action of braiding is

$$R^{ab}_{cd} t^c_e t^d_f = t^a_d t^b_c R^{cd}_{ef} \ . \tag{4.2}$$

Here R-matrix is a solution of Yang-Baxter equaion and characterizes completely the commutation relations between the elements of quantum matrix. The action of braiding is obtained by applying the inverse of R-matrix from left to the equation. By iterating the braidings of nearest neighbors one can deduce what happens in the braiding exchanging quantum matrix elements which are not nearest neighbors. What is nice that the R-matrix would fix the quantum algebra, in particular quantum Grassmannian completely.

2. In the article the notion of quantum determinant is discussed and usually the definition of quantum determinant involves also the introduction of metric g^{ab} allowing the raising of the indices of the permutation symbol. One obtains formulas relating metric and R-matrix and restricting the choice of the metric. Note however that if ordinary permutation symbol is used there is no need to introduce the metric.

The definition quantum Grassmannian proposed does not involve hermitian conjugates of the matrices involved. One can define the elements of Grassmannian and Grassmannian residue integrals without reference to complex conjugation: could one do without hermitian conjugates? On the other hand, Grassmannians have complex structure and Kähler structure: could this require hermitian conjugates and commutation relations for these?

4.2 Two views about quantum determinant

If one wants to define quantum matrices in $Gr(k,n)$ so that quantal twistor-Grassmann amplitudes make sense, the first challenge is to generalize the notion of $k \times k$ determinant.

One can consider two approaches concerning the definition of quantum determinant.

1. The first guess is that determinant should not depend on the ordering of rows or columns apart from the standard sign factor. This option fails unless one modifies the definition of permutation symbol.

2. The alternative view is that permutation symbol is ordinary and there is dependence on the row or column with respect to which one develops. This dependence would however disappear in the scattering amplitudes. If the poles and corresponding residues associated with the $k \times k$-minors of the twistor amplitude remain invariant under the permutation, this is not a problem. In other words, the scattering amplitudes are invariant under braid group. This is what twistor Grassmann approach implies and also TGD predict.

For the first option quantum determinant would be braiding invariant. The standard definition of quantum determinant is discussed in detail in [A2] (see `http://tinyurl.com/yycflgrd`).

1. The commutation of the elements of quantum matrix corresponds to braiding rather than permutation and as found, this operation is represented by R-matrix.

2. Quantum determinant would change only by sign under the braidings of neighboring rows and columns. The braiding for the elements of quantum matrix would compensate the braiding for quantum permutation symbol. Permutation symbol is assumed to be q-antisymmetric under braiding of any adjacent indices. This requires that permutation $i_k \leftrightarrow i_{k+1}$ regarded as braiding gives a contraction of quantum permutation symbol $\epsilon_{i_1,\ldots 1_k}$ with $R^{ij}_{i_k i_{k+1}}$ plus scaling by some normalization factor λ besides the change of sign.

$$\epsilon_{a_1 \ldots a_k a_{k+1} \ldots a_n} = -\lambda \epsilon_{a_1 \ldots ij \ldots a_n} R^{ji}_{a_k a_{k+1}} \quad . \tag{4.3}$$

   The value of λ can be calculated.

3. The calculation however leads to the result that that quantum determinant $\mathcal{D}$ satisfies $\mathcal{D}^2 = 1$! If the result generalizes for sub-determinants defined by $k \times k$-minors (, which need not be the case) would have determinants satisfying $\mathcal{D}^2 = 1$, and the idea about vanishing of $k \times k$-minor essential for getting non-trivial twistor scattering amplitude as residue would not make sense.

It seems that the braiding invariant definition of quantum determinant, which of course involves technical assumptions) is too restrictive. Does this mean that the usual definition requiring development with respect to preferred row is the physically acceptable option? This makes sense if only the integral but not integrand is invariant under braidings. Braiding symmetry would be analogous to gauge invariance.

4.3 How to understand the Grassmannian integrals defining the scattering amplitudes?

The beauty of the twistor Grassmannian approach is that the residue integrals over quantum $Gr(k,n)$ would reduce to sum over poles (or possibly integrals over higher-D poles). Could residue calculus provide a manner to integrate q-number valued functions of q-numbers? What would be the minimal assumptions allowing to obtain scattering amplitudes as c-numbers?

Consider first what the integrand to be replaced with its quantum version looks like.

1. Twistor scattering amplitudes involve also momentum conserving delta function expressible as $\delta(\lambda_a\tilde{\lambda}^a)$. This sum and - as it seems - also the summands should be c-numbers - in other words one has eigenstates of the operators defining the summands.

2. By introducing Grassmannian space $Gr(k,n)$ with coordinates $C_{\alpha,i}$ (see `http://tinyurl.com/yxllwcsn`), one can linearize $\delta(\lambda_a\tilde{\lambda}^a)$ to a product of delta functions $\delta(C\cdot Z) = \delta(C\cdot\tilde{\lambda}) \times \delta(C^{\perp}\cdot\lambda)$ (I have not written the delta function is Grassmann parameters related to super coordinates). Z is the n-vector formed by the twistors associated with incoming particles.

   The $4\times k$ components of $C_{\alpha,k}Z^k$ should be c-numbers at least when they vanish. One should define quantum twistors and quantum Grassmannian and pose the constraints on the poles.

How to achieve the goal? Before proceeding it is good to recall the notion of non-commutative geometry (see `http://tinyurl.com/yxrcr8xv`). Ordinary Riemann geometry can be obtained from exterior algebra bundle, call it E. The Hilbert space of square integrable sections in E carries a representation of the space of continuous functions $C(M)$ by multiplication operators. Besides this there is unbounded differential operator D, which so called signature operator and defined in terms of exterior derivative and its dual: $D = d + d^*$. This spectral triple of algebra, Hilbert space, and operator D allows to deduce the Riemann geometry.

The dream is that one could assign to non-commutative algebras non-commutative spaces using this spectral triple. The standard q-p quantization is example of this: one obtains now Lagrange manifolds as ordinary commutative manifolds.

Consider now the situation in the case of quantum Grassmannian.

1. In the recent case the points defining the poles of the function - it might be that the eventual poles are not a set of discrete points but a higher-dimensional object - would form the commutative part of non-commutative quantum space. In this space the product of quantum minors would become ordinary number as also the argument $C\cdot Z$ of the delta function. This commutative sub-space would correspond to a space in which maximum number of minors vanish and residues reduce to c-numbers.

   Thus poles of the integrand of twistor amplitude would correspond to eigenstates for some $k\times k$ minors of Grassmannian with a vanishing eigenvalue. The residue at the pole at given step in the recursion pole by pole need not be c-number but the further residue integrals should eventually lead to a c-number or c-number valued integrand.

2. The most general option would be that the conditions hold true only in the sense that some $k\times k$ minors for $k\geq 2$ are c-numbers and have a vanishing eigenvalue but that smaller minors need not have this property. Also $C_{\alpha,k}Z^k$ should be c-numbers and vanish. Residue calculus would give rise to lower-D integrals in step-wise manner.

   The simplest and most general option is that one can speak only about eigenvalues of $k\times k$ minors. At pole it is enough to have one minor for which eigenvalue vanishes whereas other minors could remain quantal. In the final reduction the product of all non-vanishing $k\times k$ minors appearing in cyclic order in the integrand should have a well-defined c-number as eigenvalue. Does this allow the appearance of only cyclic minors.

   A stronger condition would be that all non-vanishing minors reduce to their eigenvalues. Could it be that only the n cyclic minors can commute simultaneously and serve as analogs of q-coordinates in phase space? The complex dimension of $G_C(n,k)$ is $d=(n-k)k$. If the space spaced by minors corresponds to Lagrangian manifold with real dimension not larger than d, one has $k\leq d=(n-k)k$. This gives $k\leq n/2(1+\sqrt{1-2/n})$ For $k=2$ this gives

$k \leq n/2$. For $n \to \infty$ one has $k \leq n/2+1$. For $k > n/2$ one can change the roles of positive and negative helicities. It has been found that in certain sense the Grassmannian contributing to the twistor amplitude is positive.

The notion of positivity found to characterize the part of Grassmannian contributing to the residue integral and also the minors and the argument of delta function [B9](see `http://tinyurl.com/yd9tf2ya`) would suggest that it is also real sub-space in some sense and this finding supports this picture.

The delta function constraint forcing $C \cdot Z$ to zero must also make sense. $C \cdot Z$ defines $k \times 6$ matrix and also now one must consider eigenvalues of $C \cdot Z$. Positivity suggest reality also now. Z adds $4 \times n$ degrees of freedom and the number $6 \times k$ of additional conditions is smaller than $4 \times n$. $6k \leq 4 \times n$ combined with $k \leq n/2$ gives $k \leq n/2$ so that the conditions seems to be consistent.

3. The c-number property for the cyclic minors could define the analog of Lagrangian manifold for the phase space or Kähler manifold. One can of course ask, whether Kähler structure of $Gr(k,n)$ could generalize to quantum context and give the integration region as a sub-manifold of Lagrangian manifold of $Gr(k,n)$ and whether the twistor amplitudes could reduce to integral over sub-manifold of Lagrangian manifold of ordinary $Gr(k,n)$.

To sum up, I have hitherto thought that TGD allows to get rid of the idea of quantization of coordinates. Now I have encountered this idea from totally unexpected perspective in an attempt to understand how 8-D masslessness and its twistor description could relate to 4-D one. Grassmannians are however very simple and symmetric objects and have natural coordinates as $k \times n$ matrices interpretable as quantum matrices. The notion of quantum group could find very concrete application as a solution to the basic problem of the standard twistor approach. Therefore one can consider the possibility that they have quantum counterparts and at least the residue integrals reducing to c-numbers make sense for quantum Grassmannians in algebraic sense.

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