Does the notion of polynomial of infinite order make sense?

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Abstract

$M^8 - H$ duality relates number theoretical and geometric visions of physics in the TGD framework. At the level of M^8 polynomials with rational coefficients would define the space-time surfaces as the "roots" of their complexified octonionic continuations. The basic dynamical principle states that they have associative normal spaces. In principle, analytic functions with rational Taylor coefficients are also possible and can give rise to transcendental extensions. A longstanding question has been whether it makes sense to talk about polynomials of infinite degree.

It turns out that if the polynomials of infinite degree exist, they must correspond to composites for an infinite number of polynomials. This follows from the fact that both finite and infinite Galois groups must be profinite so that an infinite Galois group is a Galois group of ...extensions of extensions.....of rationals.

The example in which the polynomials of form $P = P \circ R$ where Q is an infinite composite of a single polynomial Q vanishing at origin and having it as a critical point has as a basin of attraction a set having Julia set as boundary. All points in the basin of attraction for origin are roots at the limit. All points in the basin of attraction for origin are roots at the limit so that completion of rationals to complex numbers would result.

Contents

1 Introduction

TGD motivates the question whether the notion of a polynomial of infinite degree could make sense. In the following I consider this question from the point of view of a physicist and start from the vision about physics as generalized number theory.

1.1 Background and motivations for the idea

M^8-H-duality ($H = M^4 \times CP_2$) states that space-time surfaces defined as 4-D roots of complexified octonionic polynomials so that they have quaternionic normal space, can be mapped to 4-surfaces in H [L1, L2, L6].

The octonionic polynomials are obtained by algebraic continuation of ordinary real polynomials with rational coefficients although one can also consider algebraic coefficients.

This construction makes sense also for analytic functions with rational (or algebraic) coefficients. For the twistor lift of TGD, cosmological constant Λ emerges via the coefficient of a volume term of the action containing also Kähler action. This leads to an action consisting of Kähler action with both CP_2 and M^4 terms having very interesting and physically attractive properties, such as spin glass degeneracy. $\Lambda = 0$ would correspond to an infinite volume limit making the QFT description possible as an approximate description. Also the thermodynamic limit could correspond to this limit.

Irreducible polynomials of rational coefficients give rise to algebraic extensions characterized by the Galois group and these notions are central in adelic vision.

I do not know of any deep reason preventing analytic functions with rational Taylor coefficients. These would make possible transcendental extensions. For instance, the product $\prod_p(e^x - p))$ for some subset of primes p would give as roots transcendental numbers $log(p)$. The Galois group would be however trivial although the extension is infinite. Second example is provided by trigonometric functions $sin(x)$ and $cos(x)$ with roots coming as multiples of $n\pi$ and $(2n+1)\pi/2$. This might be necessary in order to have Fourier analysis. The translations by a multiple of π for x act permuted roots but do not leave rational numbers rational so that the interpretation as a Galois group is not possible so that also now Galois group would be trivial.

A long standing question has been whether there exist analytic functions which could be regarded as polynomials of infinite order by posing some conditions to the Taylor coefficients. If so, one might hope that the notion of Galois group could make sense also now, and one might perhaps obtain a unified view about transcendental extensions of rationals.

1. For polynomials as roots of octonionic polynomials space-time surfaces are finite and located inside finite-sized causal diamond (CD).

   In the TGD Universe cosmological constant Λ depends on the p-adic length scale and approaches zero at infinite length scale. At the $\Lambda = 0$ limit, which corresponds also to QFT and thermodynamical limits, space-time surfaces would have infinite size. Only Kähler action with M^4 and CP_2 parts and having ground state degeneracy analogous to spin glass degeneracy would be present.

2. The octonionic algebraic continuations of analytic functions with rational coefficients and subject to restrictions guaranteeing that the notion of prime function makes sense, would define space-time surfaces as their roots.

3. Prime analytic functions defining space-time surfaces would in some sense be polynomials of infinite degree and could be even characterized by the Galois group. For real polynomials complex conjugations for the roots is certainly this kind of symmetry.

   These functions should have Taylor series at origin, which is a special point for octonionic polynomials with rational (or perhaps even algebraic) coefficients. The selection of origin as a preferred point relates directly to the condition eliminating possible problems due to the loss of associativity and commutativity.

   The prime property is possible only if the set of these polynomials fails to have a field property (so that the inverse of any element would be well-defined) since for fields one does not have the notion of prime. The field property is lost if the allowed functions vanish at origin so that one cannot have a Taylor series at origin and the inverse diverges at origin.

The vanishing at origin guarantees that the functional composite $f \circ g$ of f and g has the roots of g. Roots are inherited as algebraical complexity as a kind of evolution increases. In TGD inspired biology, the roots of polynomials are analogous to genes and the conservation of roots in the function composition would be analogous to the conservation of genes.

1.2 Attractor basin of fractal as set of roots

It turns out that if the polynomials of infinite degree exist, they must correspond to composites for an infinite number of polynomials. This follows from the fact that both finite and infinite Galois groups must be profinite so that an infinite Galois group is a Galois group of ...extensions of extensions.....of rationals.

The example in which the polynomials of form $P = P \circ R$ where Q is an infinite composite of a single polynomial Q vanishing at origin and having it as a critical point has as a basin of attraction a set having Julia set as boundary. All points in the basin of attraction for origin are roots at the limit.All points in the basin of attraction for origin are roots at the limit so that algebraic completion of rationals to complex numbers would result.

Profiniteness suggests an interpretation of this set in terms of p-adic topology or a product of a subset of p-adic number fields somehow determined by the number theoretic properties of Q. Algebraic completions of p-adic topology could also be in question. p-Adic number fields are indeed profinite and as additive groups can act as infinite Galois groups permuting the zeros. The action of p-adic translations could leave the basin of attraction invariant.

2 What is it to be a polynomial of infinite degree?

In the following the conditions on the notion of a polynomial of infinite degree are discussed.

2.1 Conditions for the prime analytic function

Could one make anything concrete about this idea? What kind of functions f could serve as analogs of polynomials of infinite degree with transcendental roots. The question whether any analytic function with rational coefficients vanishing at origin can have a possibly unique decomposition to prime analytic functions will not be discussed in the sequel?

1. Suppose that the analytic prime decomposes to a product over monomials $x - x_i$ with transcendental roots x_i such that the Taylor series has rational coefficients. This requires an infinite Taylor series.

2. One obtains an infinite number of conditions. Each power x^n in f has a rational coefficient f_n equal to the sum over all possible products $\prod_{k=1}^{n} x_{i_k}$ of n transcendental roots x_{i_k}. This gives an infinite number of conditions and each condition involves an infinite number of roots. If the number N of transcendental roots is finite as it is for polynomials, each term involves a finite number of products and the conditions imply that the roots are algebraic. The number of transcendental roots must therefore be infinite. At least formally, these conditions make sense.

3. The sums of products are generalized symmetric functions of transcendental roots and should have rational values equal to x_n. This generalizes the corresponding condition for ordinary polynomials. Symmetric functions for S_n have S_n as a group of symmetries. For a Galois extension of a polynomial of order n, the Galois group is a subgroup of S_n. This suggests that the Galois group is a subgroup of S_∞. S_∞ as the simple A_∞ as a subgroup of even permutations. The simple groups are analogs of primes for finite groups and one can hope that this is true for infinite and discrete groups [L5].

There are infinitely many ways to represent an algebraic extension in terms of a polynomial and the same is true for transcendental extensions with the rationality condition.

1. Consider a general decomposition of the polynomial of an infinite order to a product of monomials with roots spanning the transcendental extension. Could a suitable representation of extension as an infinite polynomial allow rational coefficients f_n for the function $\sum f_n x^n$ defined by the infinite product?

2. f_n is the sum over all possible products of roots obtained by dropping n different roots from the product of all roots which should be finite and equal to one for the generalization of monic polynomials. Therefore there is an infinite sum of terms, which are inverses of finite products and therefore transcendental but one can hope that the infinite number of the summands allows the rationality condition to be satisfied.

2.2 Profinite groups and Galois extensions as inverse limits

Infinite groups indeed appear as Galois groups of infinite extensions. Absolute Galois groups, say Galois groups of algebraic numbers, provide the basic example.

1. There exists a natural topology, known as Krull topology, which turns Galois group to a profinite group (totally disconnected, Hausdorff topological group) (`https://en.wikipedia.org/wiki/Profinite_group`), which is also Stone space (`https://en.wikipedia.org/wiki/Stone_space`).

2. Profinite groups are not countably infinite but are effectively finite just as hyper-finite factors of type II_1 are finite-dimensional: they appear naturally in the TGD framework [K2, K1]. Profinite groups have Haar measure giving them a finite volume. Profinite groups behave in many respects like finite groups (compact groups also behave in this manner as far representations are considered). Profiniteness is possessed by products, closed subgroups, and the coset groups associated with the closed normal subgroups.

3. Every profinite infinite group is a Galois group for an infinite extension for some field K but one cannot control which field K is realized for a given profinite group [A1]. Additive p-adics groups and their products appear as Galois groups of an infinite extension for some field K. The Galois theory of infinite field extensions involves profinite groups obtained as Galois groups for the inverse limits of finite field extensions $..F_n \rightarrow F_{n+1} \rightarrow$.

4. This kind of iterated extensions are of special interest in the TGD framework and an infinite extension would be obtained at the limit [L4]. The naive expectation is that the polynomial of infinite degree is a limit of a composite $...P_n \circ P_{n-1}.. \circ P_1$ of rational polynomials. The number of infinite extensions obtained in this manner would be infinite.

   An interesting question is under what conditions the limiting infinite polynomial exists as an analytic function and whether the Taylor coefficients are rational or in some extension of rationals. The naive intuition is that the inverse limit preserves rationality.

5. The identification as the iterate $...P_n \circ P_{n-1} \circ P_1$ is indeed suggestive. Infinite cyclic extension defined at the limit by the polynomial x^N, $N = \infty$, to be discussed below, has this kind of interpretation. The Galois group of this kind of extension is however not simple.

   Remark: The polynomials in question are not irreducible: the composite of N polynomials has x^N as a factor.

6. Is the infinite-D extension obtained as an inverse limit transcendental or algebraic? In the TGD framework the condition that the polynomial $P_1 \circ P_2$ has the roots of P_1 as roots implies the loss of the field property of analytic functions making the notion of analytic prime possible. The roots of the infinite polynomial contain all roots of finite polynomials appearing in the sequence. This would suggest that the extension is not transcendental. Giving up the property $P_i(0) = 0$ also leads to a loss of root inheritance.

For finite-dimensional Galois extensions, there exists an infinite number of polynomials generating the extension and one can consider families of extensions parametrized by a set of rational parameters. The Galois group does not change under small variations of parameters [L5]. If the inverse limit based on an infinite composite of polynomials makes sense, the situation could be the same for possibly existing rational polynomials of infinite order? The study of infinite Galois groups could provide insights on the problem.

2.3 Could infinite extensions of rationals with a simple Galois group exist?

Simple Galois groups have no normal subgroups and are of special interest as the building bricks of extensions by functional composition of polynomials. The infinite Galois groups obtained as inverse limit have however an infinite hierarchy of normal subgroups and simple argument suggests that the extensions are algebraic. Could infinite-D transcendental extensions defined by an analytic function with rational coefficients and with a simple infinite Galois group, exist?

If inverse limit is essential for profiniteness for infinite groups, then simple infinite groups are excluded as Galois groups. Indeed, the topology of an infinite simple group G cannot be profinite. The Krull topology has as a basis for open sets all cosets of normal subgroups H of finite index (the number of cosets gH is finite). Simple group has no normal subgroups except a trivial group consisting of a unit element and the group itself. The only open sets would be the empty set and G itself.

In fact, there is also a theorem stating that every Galois group is profinite (see `https://cutt.ly/wQ2W1Of`). All finite groups are profinite in discrete topology. This theorem however excludes infinite simple Galois groups. If one allows only polynomials with P(0)=0, the conservation of algebraic roots suggests that infinite polynomials with transcendental roots are not possible.

The condition for the failure of the field property however leaves the iterates of polynomials for which only the highest polynomial in the infinite sequence of functional compositions vanishes at origin. These infinite polynomials could have transcendental roots.

2.4 Two examples

In the following two examples are considered to test whether the notion of a polynomial of infinite order might work.

2.4.1 Infinite cyclic extensions

The natural question is whether the transcendental roots be regarded as limits of roots for a polynomial with rational coefficients at the limit when the degree N approaches infinity. The above arguments suggest that the limits involve an infinite function composition.

Consider as an example cyclic extension defined by a polynomial X^N, which can be regarded as a composite of polynomials x^{p_i} for $\prod p_i = N$. This is perhaps the simplest possible extension than one can imagine.

1. The roots are now powers of roots of unity. The notion of the root of unity as $e^{i2\pi/N}$ does not make sense at the limit $N \to \infty$. One can however consider the roots $e^{i2\pi M/N}$ and its powers such that the limit $M/N \to \alpha$ is irrational. The powers of $exp(in\alpha)$ give a dense subset of the circle S^1 consisting of irrational points. Note that one obtains an infinite number of extensions labelled by irrational values of α.

2. The polynomial should correspond to the limit $P_N(x) = x^N - 1$, $N \to \infty$. For each finite value of N, one has $P_N(x) = \prod_{n=1}^{N}(x - U^n) - 1$, $U = e^{i2\pi/N}$. The reduction to $P = x^N - 1$ follows from the vanishing of all terms involving lower powers of x than x^N.

3. If these conditions hold true at the limit $N \to \infty$, one obtains the same result. The coefficient of x^N equals to 1 trivially. The coefficient of x^{N-1} is the sum over all roots and should vanish. This is also assumed in Fourier analysis $\sum_n e^{i\alpha n} = 0$ for irrational α. For $\alpha = 0$ the sum equals to $N = \infty$ identified as Dirac delta function. The lower terms give conditions expected to reduce to this condition. This can be explicitly checked for f_1

4. The Galois group is in this case the cyclic group $U_{\infty,\alpha}$ defined by the powers of U_α.

2.4.2 Infinite iteration yields continuum or roots

The iterations of polynomials define an $N \rightarrow \infty$ limit, which can be handled mathematically whereas for an arbitrary sequence of polynomials in the functional composition it is difficult to say anything about the possible emergence of transcendental roots. Note however that the $lim_{N\rightarrow\infty}(1+1/N)^N = e$ shows that transcendentals can appear as limits of rationals. I have considered iterations of polynomials and approach to chaos from the point of view of M^8-H duality in [L3].

Consider polynomials $P_N = Q_N \circ R$, where R with $Q(0)=0$ is fixed polynomial and $Q_N = Q^{\circ N}$ is the N:th integrate of some irreducible polynomial Q with $Q(0)=0$ and $dQ/dz(0)=0$. Origin is a fixed critical point of Q and the attractor towards which the points in the attractor basin of origin end up in the iteration and become roots of P_∞ and are roots at this limit. For the real points in the intersection of the positive real axis and attractor basin are roots at this limit so that one has a continuum of roots. The set of roots consists of a continuous segment $[0,T)$ and a discrete set coming from the Julia set defining the boundary of the attractor basin.

Profiniteness suggests an interpretation of this set in terms of p-adic topology or a product of a subset of p-adic topologies somehow determined by the number theoretic properties of Q. p-Adic number fields are indeed profinite and as additive groups can act as infinite Galois groups permuting the zeros. The action of p-adic translations could indeed leave the basin of attraction invariant.

In the TGD framework these roots correspond to values of M^4 time (or energy!) in M^8 mapped to the actual time values in H by M^8-H duality. I have referred to them as "very special moments in the life of self" with a motivation coming from TGD inspired theory of consciousness [L1, L2]. One might perhaps say that at this limit subjective time consisting of these moments becomes continuous in the interval $[0,T]$.

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