Super-number fields: does physics emerge from the notion of number?

M. Pitkänen
Email: matpitka6@gmail.com.
http://tgdtheory.com/.

October 2, 2017

Abstract

A proposal that all physics emerges from the notion of number field is made. The first guess for the number field in question would be complexified octonions for which inverse exists except at complexified light-cone boundary: this has interpretation in terms of propagation of signals with light-velocity 8-D sense. The emergence of fermions however requires super-octonions as super variant of number field. Rather surprisingly, it turns out that super-number theory makes perfect sense. One can define the inverse of super-number if it has non-vanishing ordinary part and also the notion of primeness makes sense and construct explicitly the super-primes associated with ordinary primes. The prediction of new number piece of theory can be argued to be a strong support for the integrity of TGD.

Contents

1 Introduction

The notion of emergence is fashionable in the recent day physics, in particular, he belief is that 3-space emerges in some manner. In the sequel I consider briefly the standard view about emergence idea from TGD point of view, then suggest that the emergence in the deepest sense requires emergence of physics from the notion of number and that complexified octonions [K1] [L3, L4, L1, L2] are the most plausible candidate in this respect. After that I will show that number theory generalizes to super-number theory: super-number fields make sense and one can define the notion of super-prime. Every new step of progress creates worry about consistency with the earlier work, now the work done during last months with physics as octonionic algebraic geometry and also this aspect is discussed.

1. The notion of holography is behind the emergence of 3-space and implies that the notion of 2-space is taken as input. This could be justified by conformal invariance.

2. The key idea is that 3-space emerges somehow from entanglement. There is something that must entangle and this something must be labelled by points of space: one must introduce a discretised space. Then one must do some handwaving to make it 3-D - perhaps by arguing that holography based on 2-D holograms is unique by conformal invariance. The next hand-wave would replace this as a 3-D continuous space at infrared limit.

3. How to get space-time and how to get general coordinate invariance? How to get the symmetries of standard model and special relativity? Somehow all this must be smuggled into the theory when the audience is cheated to direct its attention elsewhere. This Münchausen trick requires a professional magician!

4. One attempt could take as starting point what I call strong form of holography (SH) in which 2-D data determine 4-D physics. Just like 2-D real analytic function determines analytic function of two complex variables in spacetime of 2 complex dimensions by analytic continuation (this hints strongly to quaternions). This is possible if conformal invariance is generalized to that for light-like 3-surfaces such as light-cone boundary. But the emergence magician should do the same without these.

   In TGD one could make this even simpler. Octonionic polynomials and rational functions are obtained from real polynomials of real variable by octonion-analytic continuation. And since polynomials and rational functions P_1/P_2 are in question their values at finite number of discrete points determined them if the orders of P_1 and P_2 are known!

   If one accepts adelic hierarchy based on extensions of rationals the coefficients of polynomials are in extensions of rationals and the situation simplifies further. The criticality conditions guaranteeing associativity for external particles is one more simplification: everything b becomes discrete. The physics at fundamental level could be incredibly simple: discrete number of points determines space-time surfaces as zero loci for $RE(P)$ or $IM(P)$ (octonions are decomposed to two quaternions gives $RE(o)$ and $IM(o)$).

   How this is mapped to physics leading to standard model emerging from the formulation in $M \times CP_2$ This map exists - I call it $M^8 - H$ duality - and takes space-time varieties in Minkowskian sector of complexified octonions to a space-time surface in $M^4 \times CP_2$ coding for standard model quantum numbers and classical fields.

How to get all this without bringing in octonionic imbedding space: this is the challenge for the emergence-magician! I am afraid this this trick is impossible. I will however propose a deeper for what emergence is. It would not be emergence of space-time and all physics from entanglement but from the notion of number, which is at the base of all mathematics. This view led to a discovery of the notion of super-number field, a completely new mathematical concept, which should show how deep the idea is.

2 Does physics emerge from the notion of number field?

Concerning emergence one can start from a totally different point of view. Even if one gets rid of space as something fundamental from Hilbert sapce and entanglement, one has not reached the most fundamental level. Structures like Hilbert space, manifold, etc. are not fundamental mathematical structures: they require the notion of number field. Number field is the fundamental notion.

Could entire physics emerge from the notion of number field alone: space-time, fermions, standard model interactions, gravitation? There are good hopes about this in TGD framework if one accepts $M^8 - H$ duality and physics as octonionic algebraic geometry! One could however argue that fermions do not follow from the notion of number field alone. The real surprise was that formalizing this more precisely led to a realization that the very notion of number field generalizes to what one could call super-number field!

2.1 Emergence of physics from complexified octonionic algebraic geometry

Consider first the situation for number fields postponing the addition of attribute "super" later.

1. Number field endowed with basic arithmetic operations $+$, $-$, $\cdot$, $/$ is the basic notion for anyone wanting to make theoretical physics. There is a rich repertoire of number fields. Finite fields, rationals and their extensions, real numbers, complex numbers, quaternions, and octonions. There also p-adic numbers and their extensions induced by extensions of

rationals and fusing into adele forming basic structure of adelic physics. Even the complex, quaternionic, and octonionic rationals and their extensions make sense. p-Adic variants of say octonions must be however restricted to have coefficients belonging to an extension of rationals unless one is willing to give up field property (the p-adic analog of norm squared can vanish in higher p-adic dimensions so that inverse need not exist).

There are also function fields consisting of functions with local arithmetic operations. Analytic functions of complex variable provides the basic example. If function vanishes at some point its inverse element diverges at the same point. Function fields are derived objects rather than fundamental.

2. Octonions are the largest classical number field and are therefore the natural choice if one wants to reduce physics to the notion of number. Since one wants also algebraic extensions of rationals, it is natural to introduce the notion of complexified octonion by introducing an additional imaginary unit - call it i, commuting with the 7 octonionic imaginary units I_k. One obtains complexified octonions.

   That this is not a global number field anymore turns out to be a blessing physically. Complexified octonion z_kE^k has $z_k = z_k + iy_k$. The complex valued norm of octonion is given by $z_0^2 + ...z_7^2$ (there is no conjugation involved. The norm vanishes at the complex surface $z_0^2 + ...z_7^2 = 0$ defining a 7-D surface in 7-D O_c (the dimension is defined in complex sense). At this surface - complexified light-cone boundary - number field theory property fails but is preserved elsewhere since one can construct the inverse of octonion.

   At the real section M^8 (8-D Minkowski space with one real (imaginary) coordinate and 7 imaginary (real) coordinates the vanishing takes place also. This surface corresponds to the 7-D light-cone boundary of 8-D Minkowskian light-cone. This suggests that light-like propagation is basically due to the complexification of octonions implying local failure of the number field property. Same happens also in other real sections with $0 < n < 8$ real coordinates and $0 < m = 8 - n < 8$ imaginary coordinates and one obtains variant of light-cone with different signatures. Euclidian signature corresponding to $m = 0$ or $m = 8$ is an exception: light-cone boundary reduces to single point in this case and one has genuine number field - no propagation is possible in Euclidian signature.

   Similar argument applies in the case of complexified quaternions Q_c and complexified complex numbers $z_1 + z_2I \in C_c$, where I is octonionic imaginary unit. For Q_c one obtains ordinary 3-D light-cone boundary in real section and 1-D light-cone boundary in the case of C_c. It seems that physics demands complexification! The restriction to real sector follows from the requirement that norm squared reduces to a real number. All real sectors are possible and I have already considered the question whether this should be taken as a prediction of TGD and whether it is testable.

2.2 Super-octonionic algebraic geometry

There is also a natural generalization of octonionic TGD to super-octonionic TGD based on octonionic triality. $SO(1,7)$ allows besides 8-D vector representations also spinor representations 8_c and $\overline{8}_c$. This suggests that super variant of number field of octonions might make sense. One would have $o = o_8 + o_{c,8} + \overline{0}_{c,8}$.

1. Should one combine o_8, $o_{c,8}$ and $\overline{o}_{c,8}$ to a coordinate triplet $(o_8, o_{c,8}, \overline{o}_{c,8})$ as done in super-symmetric theories to construct super-fields? The introduction of super-fields as primary dynamical variables is a good idea now since the very idea is to reduce physics to algebraic geometry at the level of M^8. Polynomials of super-octonions defining space-time varieties as zero loci for their real or imaginary part in quaternionic sense could however take the role of super fields. Space-time surface would correspond to zero loci for $RE(P)$ or $IM(P)$.

2. The idea about super-octonions should be consistent with the idea that we live in a complexified number field. How to define the notion of super-octonion? The tensor product $8 \otimes 8_c$ contains 8_c and $8 \otimes 8_{\overline{c}}$ contains $8_{\overline{c}}$ and one can use Glebsch-Gordan coefficients to contract o and θ_c and o and $\overline{\theta}_{c,n}$. The tensor product of 8_c and $8_{\overline{c}}$ defined using structure constants defining octonion product gives 8. Therefore one must have

$$o_s = o + \Psi_c \times \theta_{\overline{c}} + \Psi_{\overline{c}} \times \theta_c \ , \quad (2.1)$$

where the products are octonion products. Super parts of super-coordinates would not be just Grassmann numbers but octonionic products of Grassmann numbers with octonionic spinors in 8_c and $\overline{8}_c$. This would bring in the octonionic analogs of spinor fields into the octonionic geometry.

This seems to be consistent with super field theories since octonionic polynomials and even rational functions would give the analogs of super-fields. What TGD would provide would be an algebraic geometrization of super-fields.

3. What is the meaning of the conditions $RE(P) = 0$ and $IM(P) = 0$ for super-octonions? Does this condition hold true for all $d_G = 2^{16}$ super components of $P(o_s)$ or is it enough to pose the condition only for the octonionic part of $P(o)$? In the latter case Ψ_c and $\Psi_{\overline{c}}$ would be free and this does not seem sensical and does not conform with octonionic super-symmetry. Therefore the first option will be studied in the sequel.

If super-octonions for a super variant of number field so that also inverse of super-octonion is well-defined, then even rational functions of complexified super-octonions makes sense and poles have interpretation in terms of 8-D light-fronts (partonic orbits at level of H). The notion must make sense also for other classical number fields, finite fields, rationals and their extensions, and p-adic numbers and their extensions. Does this structure form a generalization of number field to a super counter part of number field? The easiest manner to kill the idea is to check what happens in the case of reals.

1. The super-real would be of form $s = x + y\theta$, $\theta^2 = 0$. Sum and product are obviously well-defined. The inverse is also well-defined and given by $1/s = (x - y\theta))/x^2$. Note that for complex number $x + iy$ the inverse would be $\overline{z}/z\overline{z} = (x - yi)/(x^2 + y^2)$. The formula for super-inverse follows from the same formula as the inverse of complex number by defining conjugate of super-real s as $\overline{s} = x - y\theta$ and the norm squared of s as $|s|^2 = s\overline{s} = x^2$. The formula however makes sense only if the ordinary part of super-number is non-vanishing: $x \neq 0$ so that one must weaken the notion of number field somewhat. Pure super-numbers are with vanishing ordinary part have zero norm.

   One can identify super-integers as $N = m + n\theta$. One can also identify super-real units as number of unit norm. Any number $1_n = 1 + n\theta$ has unit norm and the norms form an Abelian group under multiplication: $1_m 1_n = 1_{m+n}$. Similar non-uniqueness of units occurs also for algebraic extensions of rationals.

2. Could one have super variant of number theory? Can one identify super-primes? Super-norm satisfies the usual defining property $|xy| = |x||y|$. Super-prime is defined only apart from the multiplicative factor 1_m giving not contribution to the norm. This is not a problem but a more rigorous formulation leads to the the replacement of primes with prime ideals labelled by primes already in the ordinary number theory.

   If the norm of super-prime is ordinary prime it cannot decompose to a product of super-primes. Not all super-primes having given ordinary prime as norm are however independent. If super-primes $p + n\theta$ and $p + m\theta$ differ by a multiplication with unit $1_r = 1 + r\theta$, one has $n - m = pr$. Hence there are only p super-primes with norm p and they can be taken $p_s = p + k\theta$, $k \in \{0, p - 1\}$. A structure analogous to a cyclic group Z_p emerges.

   Note that also θ is somewhat analogous to prime although its norm is vanishing.

3. Just for fun, one an ask what is the super counterpart of Riemann Zeta. Riemann zeta can be regarded as an analog of thermodynamical partition function reducing to a product for partition functions for bosonic systems labelled by primes p. The contribution from prime p is factor $1/(1-p^{-s})$. p^{-s} is analogous to Boltzmann weight $N(E)exp(-E/T)$, where $N(E)$ is number of states with energy E. The degeneracy of states labelled by prime p is for ordinary primes $N(p) = 1$. For super-primes the degeneracy is $N(p) = p$ and the weight becomes

$1/(1-N(p)p^{-s}) = 1/(1-p^{-s+1})$. Super Riemann zeta is therefore $zeta(s-1)$ having critical line at $s = 3/2$ rather than at $s = 1/2$ and trivial zeros at real points $s = -1, -3, -5$, rather than at $s = -2, -4, -6, ...$

A more rigorous argument starts from the definition of norm for super-numbers. Norm must be multiplicative so that it must be defined as a determinant of the matrix of the linear map defined by the multiplication with super-number. For super-number $m + n\theta$ this matrix is given by $(m, 0; n, m)$ with determinant m^2. For primes the determinant reduces to p^2. Nothing prevents from defining the norm as square roots of this determinant so that one has $N(p) = p$. Pure super numbers have vanishing norm.

There are good reasons to expect that the above arguments work also for algebraic extensions of super-rationals and in fact for all number fields, even for super-variants of complex numbers, quaternions and octonions. This because the conditions for invertibility reduce to that for real numbers. One would have a generalization of number theory to super-number theory! Net search gives no references to anything like this. Perhaps the generalization has not been noticed because the physical motivation has been lacking. M^8-H duality would imply that entire physics, including fermion statistics, standard model interactions and gravitation reduces to the notion of number in accordance with number theoretical view about emergence.

2.3 Is this picture consistent with the previous work?

All beautiful moments of discovery are soon spoiled by worried questions popping up into mind. Is this picture about super-symmetry consistent with the view about super-octonionic surfaces developed during the last months? Usually the first formulation is not the final one.

In the definition of the first variant of super-octonions [L4] I followed the standard idea about what super-coordinates assuming that the super-part of super-octonion is just an anti-commuting Grassmann number without any structure: I just replaced o with $o + \theta_k E^k + \overline{\theta}_k E^k$ regarding θ_k as anticommiting coordinates. Now θ_k receives octonionic coefficient: $\theta_k \rightarrow o_k\theta_k$. θ_k is now analogous to unit vector.

1. In the original picture, the vanishing condition for RE or IM for a particular monomial of theta parameters gives 4 complex conditions for 8 complex coordinates of O_c. Each equation for 2^{8+8} monomials of Grassmann parameters was assumed to be independent and give separate space-time variety. This gives 2^{16} separate space-time varieties coming from the coefficients of non-vanishing monomials of Grassmann numbers spanning a Grassmann algebra of 16-D space having dimension $d = 2^{16}$ and each was assumed to describe geometrically a many-fermion state with particular fermion numbers.

2. Now the situation is different since one assigns independent octonionic coordinates to anti-commuting degrees of freedom. One has linear space with partially anti-commutative basis. O_c is effectively replaced with O_c^3 so that one has 8+8+8=24-dimensional Cartesian product (it is amusing that the magic dimension 24 for physical polarizations of bosonic string models emerges).

3. What is the number of equations in the new picture? For N super-coordinates one has 2^N separate monomials analogous to many-fermion states. Now one has $N = 8 + 8 = 16$ and this gives 2^{16} monomials! In the general case $RE = 0$ or $IM = 0$ gives 4 equations for each of the $d_G = 2^{16}$ monomials: the number of equations $RE = 0$ or $IM = 0$ is 4×2^{16} and exceeds the number $d_O = 24$ of octonion valued coordinates. In the original interpretation these equations were regarded as independent and gave different space-time variety for each many-fermion state.

   In the new framework these equations cannot be treated independently. One has 24 octonionic coordinates and 2^{16} equations. In the generic case there are no solutions. This is actually what one hopes since otherwise one would have a state involving superposition of many-fermion states with several fermion numbers.

The freedom to pose constraints on the coefficients of Grassmann parameters however allows to reduce degrees of freedom. All coefficients must be however expressible as products of $3 \times 8 = 24$ components of super-octonion.

1. One can have solutions for which both 8_c part and $\overline{8}_c$ parts vanish. This gives the familiar 4 equations for 8 variables and 4-surfaces.

2. Consider first options, which fail. If 8_c- or $\overline{8}_c$ part vanishes one has $d_G = 2^8$ and $4 \times d_G = 4 \times 64$ equations for $dO_= 8+8 = 16$ variables having no solutions in the generic case. The restriction of 8_c to its 4-D quaternionic sub-space would give $d_O = 4$ and $4d_G = 4 \times 2^4 = 64$ conditions and 16 variables. The reduction to complex sub-space $z_1 + z_2 I$ of super-octonions would give $d_O = 2^2$ and $4 \times 2^2 = 16$ conditions for $8 + 2 = 10$ variables.

3. The restriction to 1-D sub-space of super-octonions would give $4 \times 2^1 = 8$ conditions and $8 + 1 = 9$ variables. Could the solution be interpreted as 1-D fermionic string assignable to the space-like boundary of space-time surface at the boundary of CD? Skeptic inside me asks whether this could mean the analog of $\mathcal{N} = 1$ SUSY, which is not consistent with H picture.

   Second possibility is restriction to light-like subspace for which powers of light-like octonion reduce effectively to powers of real coordinate. Fermions would be along light-lines in M^8 and along light-like curves in H. The powers of super-octonion have super-part, which belongs to the 1-D super-space in question: only single fermion state is present besides scalar state.

4. There are probably other solutions to the conditions but the presence of fermions certainly forces a localization of fermionic states to lower-dimensional varieties. This is what happens also in H picture. During years the localization of fermion to string worlds sheets and their boundaries has popped up again and again from various arguments. Could one hope that super-number theory provides the eventual argument.

   But how could one understand string world sheets in this framework? If they do not carry fermions at H-level, do they appear naturally as 2-D structures in the ordinary sense?

To sum up, although many details must be checked and up-dated, super-number theory provides and extremely attractive approach promising ultimate emergence as a reduction of physics to the notion of number. When physical theory leads to a discovery of new mathematics, one must take it seriously.

REFERENCES

Books related to TGD

[K1] Pitkänen M. Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry? In *TGD as a Generalized Number Theory*. Online book. Available at: `http://tgdtheory.fi/public_html/tgdnumber/tgdnumber.html#ratpoints`, 2017.

Articles about TGD

[L1] Pitkänen M. About enumerative algebraic geometry in TGD framework. Available at: `http://tgdtheory.fi/public_html/articles/GWinvariant.pdf`, 2017.

[L2] Pitkänen M. Do Riemann-Rochtheorem and Atyiah-Singer index theorem have applications in TGD? Available at: `http://tgdtheory.fi/public_html/articles/Riemann-Roch.pdf`, 2017.

[L3] Pitkänen M. Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry?: part I. Available at: `http://tgdtheory.fi/public_html/articles/ratpoints1.pdf`, 2017.

[L4] Pitkänen M. Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry?: part II. Available at: `http://tgdtheory.fi/public_html/articles/ratpoints2.pdf`, 2017.