Are Preferred Extremals Quaternion-Analytic in Some Sense?

M. Pitkänen
Email: matpitka6@gmail.com.
http://tgdtheory.com/.

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Abstract

A generalization of 2-D conformal invariance to its 4-D variant is strongly suggestive in TGD framework, and leads to the idea that for preferred extremals of action space-time regions have (co-)associative/(co-)quaternionic tangent space or normal space. The notion of $M^8 - H$ correspondence allows to formulate this idea more precisely. The beauty of this notion is that it does not depend on the signature of Minkowski space M^4 representable as sub-space of of complexified quaternions M_c^4, which in turn can be seen as sub-space of complexified octonions M_c^8.

The 4-D generalization of conformal invariance suggests strongly that the notion of analytic function generalizes somehow. This notion is however not so straightforward even in Euclidian signature, and the generalization to Minkowskian signature brings in further problems. The Cauchy-Riemann-Fuerter conditions make however sense also in Minkowskian quaternionic situation and the problem is whether they allow the physically expected solutions. One should also show that the possible generalization is consistent with (co)-associativity.

In this article these problems are considered. Also a comparison with Igor Frenkel's ideas about hierarchy of Lie algebras, loop, algebras and double look algebras and their quantum variants is made: it seems that TGD as a generalization of string models replacing string world sheets with space-time surfaces gives rise to the analogs of double loop algebras and they quantum variants and Yangians. The straightforward generalization of double loop algebras seems to make sense only at the light-like boundaries of causal diamonds and at light-like orbits of partonic 2-surfaces but that in the interior of space-time surface the simple form of the conformal generators is not preserved. The twistor lift of TGD in turn corresponds nicely to the heuristic proposal of Frenkel for the realization of double loop algebras.

1 Introduction

A generalization of 2-D conformal invariance to its 4-D variant is strongly suggestive in TGD framework, and leads to the idea that for preferred extremals of action space-time regions have (co-)associative/(co-)quaternionic tangent space or normal space. The notion of $M^8 - H$ correspondence allows to formulate this idea more precisely. The beauty of this notion is that it does not depend on the signature of Minkowski space M^4 representable as sub-space of of complexified quaternions M_c^4, which in turn can be seen as sub-space of complexified octonions M_c^8. The 4-D generalization of conformal invariance suggests strongly that the notion of analytic function generalizes somehow. This notion is however not so straightforward even in Euclidian signature and the generalization to Minkowskian signature brings in further problems. One should also show that the possible generalization is consistent with (co)-associativity.

1.1 Minimal vision

Before continuing it is good to bring in mind the minimal assumptions and general vision.

1. If $M^8 - H$ duality [K1] holds true, the space-time surface $X^4 \subset M^8 = M^4 \times E^4$ is quaternionic surface in the sense that it have quaternionic tangent space and contains preferred $M^2 \subset M^4$ as part of their tangent space or more generally the 2-D hyper-complex subspaces $M^2(x)$ define and integrable distribution defining 2-D surface.

2. Quaternionicity in geometric sense in M^8 alone *implies* the interpretation as a 4-D surface in $H = M^4 \times CP_2$. There is *no need* to assume quaternionicity in geometric sense in H although it cannot be excluded and would have strong implications [K1]. This one should remember in order to avoid drowning to an inflation of speculations.

It is not at all clear what quaternion analyticity in Minkowskian signature really means or whether it is even possible. The skeptic inside me has a temptation to conclude that the direct extrapolation of quaternion analyticity from Euclidian to Minkowskian signature for space-time surfaces in H is not necessary and might be even impossible. On the other hand, the properties of the known extremals strongly suggest its analog. Quaternion analyticity could however appear at the tangent space level for various generalized conformal algebras transformed to double loop algebras for the proposed realization of the quaternion analyticity.

1.2 Problems in Minkowskian signature

At what level quaternion analyticity could appear in TGD framework?

The obvious ideas coming in mind are appropriately defined quaternionic and octonion analyticity as analog of analyticity for analytic functions $f(z)$ playing central role in conformal field theories. I have used a considerable amount of time to consider these possibilities but had to give up the idea about octonion analyticity could somehow allow to preferred extemals. Quaternion analyticity however made a comeback as I learned about the formulation of Fuerter for the Cauchy-Riemann conditions satisfied by the real and imaginary parts of complex analytic functions from two articles (CRF conditions). The first article [A2] was about so called triholomorphic maps between 4-D almost quaternionic manifolds. The article gave as a reference an article [A1] about quaternionic analogs of Cauchy-Riemann conditions discussed by Fueter long ago (somehow I have managed to miss Fueter's work just like I missed Hitchin's work about twistorial uniqueness of M^4 and CP_2), and also a new linear variant of these conditions, which seems especially interesting from TGD point of view as will be found.

Consider first the situation in Euclidian signature. The disappointment is that the solutions of the CRF conditions does not allow powers series of quaternion as solutions.To see what is involved introduce quaternionic units $1, I, J, K$ with $K = IJ$ and write $q = t + Iz + Jx + Ky$ as $q = t + Iz + (x + IY)J = u + vJ$. Here u and v are commuting complex coordinates.

One class of solutions reduces to ordinary analytic maps $(f_1, f_2) : E^2 \times E^2 \rightarrow E^2 \times E^2$: $f = f_1(u) + f_2(v)J$. A more general class of solutions is of form $f = \sum u^m v^n a_{mn}$, $u = t + Iz$, $v = x + Iy$. a_{mn} are quaternionic coefficients of Taylor series and act from right - this is essential since quaternions are not commutative. u and v are ordinary complex variables and CRF conditions state $\partial_{\overline{u}} + J\partial\overline{v}f = 0$. It is essential that the action is from left and also that J acts from left. One can speak of left-analyticity!

These articles consider quaternion analyticity only for the Euclidian signature of the metric. What about Minkowskian signature? The first thing to notice is that CRF conditions are well-defined also in the Minkowskian signature and must have solutions. Does Minkowskian variant of quaternion analyticity appear only in the formulation of conformal algebras and replace loop algebra with double loop algebra in some sense (roughly $z^m \rightarrow u^m v^n$, $u = t - z$, $v = x + iy$)? Or does it appear in some form also at the level of preferred extremals for which geometric form of quaternionicity is expected to appear - at least at the level of M^8?

There are difficulties with the most obvious guesses.

1. One would like to generalize the notion of the ordinary analyticity to Minkowskian signature. Massless wave equation in M^2 is the natural application to start with. The solutions are of form $f(t-z)+g(t+z)$ and represent waves moving with light velocity in opposite directions. The analog of conformal map of M^2 preserving hyperbolic angles instead of ordinary angles is defined as $T-Z = f(y-z), T+Z = f(T+Z)$ One can introduce hypercomplex coordinate $u = t + iIz$ by introduced additional imaginary unit i besides I. These units commute. The solutions are obtained as power series of $f(u)$ by picking up the coefficients of 1 and iI.

2. This is nice but fails for Minkowskian quaternions. The problem is that the powers of $u = t + iIz + iJx + iKy$ do not represent points of M^4 but more general points of M^4_c: for instance, the product $iI \times iJ = K$ is in the complement of M^4. CRF equations have solutions

also now. For instance $f = f_1(u) + f_2(v)iJ$, $u = t + iIz$, $v = x + Iy$ are possible. The more general solutions $f = \sum u^m v^n a_{mn}$ are not however possible unless one complexifies M^4.

The known solutions of field equations however demonstrate clearly that one has a generalization of ordinary analyticity. For instance, so called massless extremals are solutions of field equations for which CP_2 coordinates are arbitrary functions of a light-like coordinate u of M^2 and coordinate defined v defined in plane E^2 orthogonal to M^2 (for more detailed discussion see [K2]) This defines decomposition $M^4 = M^2 \times E^2$ corresponding to the physical and unphysical polarizations. It is not clear whether Minkowskian variant of CRF conditions could allow this kind of solutions. In fact, this picture leads to the notion of Hamilton-Jacobi structure generalizing the notion of complex structure from E^4 to M^4. The challenge is to see whether this picture is consistent with some form of quaternion analyticity.

The consistency of quaternion analyticity with the associativity/co-associativity of tangent space of space-time surface making sense in both Euclidian and Minkowskian signature remains an open problem. The geometric form of (co-)quaternionicity (or (co-)associativity) requires that the associator $a(bc) - (ab)c$ for any 3 tangent space vectors vanishes. These conditions involve products of 3 partial derivates of imbedding space coordinates. For co-associativity this holds true in the normal space. One must remember that associativity/co-associativity conditions might be needed only in M^8 but could make sense also for H. Same applies to quaternion analyticity.

In the following I will not consider in more detail Minkowskian form of quaternion analyticity: this is discussed in the section "Quaternions and TGD" of [K2]. Suffice it to say that I do not understand it completely. Instead, I will consider possible connection with the vision of Igor Frenkel, one of the great Russian mathematicians, whose vision inspired by mathematics resembles strong the vision of TGD inspired by physics of TGD Universe.

2 Can the known extremals satisfy the realistic form of quaternion-analyticity?

To test the consistency the realistic form of quaternion analyticity, at the level of M^8 or even H, the best thing to do is to look whether quaternion analyticity is possible for the known extremals for the twistor lift of Kähler action.

Twistor lift drops away most vacuum extremals from consideration and leaves only minimal surfaces. The surviving vacuum extremals include CP_2 type extremals with light-like geodesic rather than arbitrary light-like curve as M^4 projection. Vacuum extremals expressible as graph of map from M^4 to a Lagrangian sub-manifold of CP_2 remain in the spectrum only if they are also minimal surfaces: this kind of minimal surfaces are known to exist.

Massless extremals (MEs) with 2-D CP_2 projection remain in the spectrum. Cosmic strings of form $X^2 \times Y^2 \subset M^4 \times CP_2$ such that X^2 is string world sheet (minimal surface) and Y^2 complex sub-manifold of CP_2 are extremals of both Kähler action and volume term. One can also check whether Hamilton-Jacobi structure of M^4 and of Minkowskian space-time regions and complex structure of CP_2 have natural counterparts in the quaternion-analytic framework.

1. Consider first cosmic strings. In this case the quaternionic-analytic map from $X^4 = X^2 \times Y^2$ to $M^4 \times CP_2$ with octonion structure would be map X^4 to 2-D string world sheet in M^4 and Y^2 to 2-D complex manifold of CP_2. This could be achieved by using the linear variant of CRF condition. The map from X^4 to M^4 would reduce to ordinary hyper-analytic map from X^2 with hyper-complex coordinate to M^4 with hyper-complex coordinates just as in string models. The map from X^4 to CP_2 would reduce to an ordinary analytic map from X^2 with complex coordinates. One would not leave the realm of string models.

2. For the simplest massless extremals (MEs) CP_2 coordinates are arbitrary functions of light-like coordinate $u = k \cdot m$, k constant light-like vector, and of $v = \epsilon \cdot m$, ϵ - a polarization vector orthogonal to k. The interpretation as classical counterpart of photon or Bose-Einstein condensate of photons is obvious. There are good reasons to expect that this ansatz generalizes by replacing the variables u and v with coordinate along the light-like and space-like coordinate lines of Hamilton-Jacobi structure [K3]. The non-geodesic motion of photons with

light-velocity and variation of the polarization direction would be due to the interactions with the space-time sheet to which it is topologically condensed.

Now space-time surface would have naturally M^4 coordinates and the map $M^4 \rightarrow M^4$ would be just identity map satisfying the radial CRF condition. Can one understand CP_2 coordinates in terms of the realistic form of quaternion- analyticity? The dependence of CP_2 coordinates on $u = t - x$ only can be formulated as CFR condition $\partial_{\overline{u}} s^k = 0$ and this could is expected to generalize in the formulation using the geometric representation of quaternionic imaginary units at both sides. The dependence on light-light coordinate u follows from the translationally invariant CRF condition.

The dependence on the real coordinate $v = t - z$ does not conform with the proposed ansatz since the dependence is naturally on complex coordinate w assignable to the polarization plane of form $z = f(w)$. This would give dependence on 2 transversal coordinates and CP_2 projection would be 3-D rather than 2-D. One can of course ask whether this dependence is actually present for preferred extremals? Could the polarization vector be complex local polarization vector orthogonal to the light-like vector? In quantum theory complex polarization vectors are used routinely and become oscillator operators in second quantization and in TGD Universe MEs indeed serve as space-time correlates for photons or their BE condensates.

If this picture makes sense, one must modify the ansatz for the preferred extremals with Minkowskian signature. The E^4 and coordinates and perhaps even real CP_2 coordinates can depend on light-like coordinate u.

3. Vacuum extremals with Lagrangian manifold as (in the generic case 2-D) CP_2 projection survive if they are minimal surfaces. This property should guarantee the realistic form of quaternion analyticity. Hyper-quaternionic structure for space-time surface using Hamilton-Jacobi structure is the first guess. CP_2 should allow a quaternionic coordinate decomposing to a pair of complex coordinates such that second complex coordinate is constant for 2-D Lagrangian manifold and second parameterizes it. Any 2-D surface allows complex structure defined by the induced metric so that there are good hopes that these coordinates exist. The quaternion-analytic map would map in the most general case is trivial for both hypercomplex and complex coordinate of M^4 but the quaternionic Taylor coefficients reduce to real numbers to that the image is 2-D.

4. For CP_2 type vacuum extremals surviving as extremals the M^4 projection is light-like geodesic with $t + z = 0$ with suitable choice of light-like coordinates in M^2. $t - z$ can arbitrary function of CP_2 coordinates. Associativity of the normal space is the only possible option now.

   One can fix the coordinates of X^4 to be complex coordinates of CP_2 so that one gets rid of the degeneracy due to the choice of coordinates. M^4 allows hyper-quaternionic coordinates and Hamilton-Jacobi structures define different choices of hyper-quaternionic coordinates. Now the second light- like coordinate would vary along random light-like geodesics providing slicing of M^4 by 3-D surfaces. Hamilton-Jacobi structure defines at each point a plane $M^2(x)$ fixed by the light-like vector at the point and the 2-D orthogonal plane. In fact 4-D coordinate grid is defined.

5. In the naive generalization CRF conditions are linear. Whether this is the case in the formulation using the geometric representation of the imaginary units is not clear since the quaternionic imaginary units depend on the vielbein of the induced 3-metric (note however that the SO(3) gauge rotation appearing in the conditions could allow to compensate for the change of the tensors in small deformations of the spaced-time surface). If linearity is real and not true only for small perturbations, one could have linear superpositions of different types of solutions, which looks strange. Could the superpositions describe perturbations of say cosmic strings and massless extremals?

6. According to [A1] both forms of the algebraic CRF conditions generalize to the octonionic situation and right multiplication of powers of octonion by Taylor coefficients plus linearity imply that there are no problems with associativity. This inspires several questions.

Could octonion analytic maps of imbedding space allow to construct new solutions from the existing ones? Could quaternion analytic maps applied at space-time level act as analogs of holomorphic maps and generalize conformal gauge invariance to 4-D context?

3 Quaternion analyticity and generalized conformal algebras

The realistic quaternionic analyticity should apply at the level of conformal algebras for conformal algebra is replaced with a direct sum of 2-D conformal and hyper-conformal algebra assignable to string world sheets and partonic 2-surfaces. This would conform with SH and the considerations above.

It is however too early to exclude the possibility that the powers z^n of conformal algebras are replaced by $u^m z^n$ ($u = t - z$ and $w = x + iy$) for symmetries restricted to the light-like boundaries of CD and to the light-like orbits of partonic 2-surfaces. This preferred form at boundaries would be essential for reducing degrees of freedom implied by SSA and SKMA gauge conditions. In the interior of space-time surfaces this simple form would be lost.

This would realize the Minkowskian analog of double loop algebras suggested by 4-dimensionality. This option is encouraged by the structure of super-symplectic algebra and generalization of Kac-Moody algebras for light-like orbits of partonic 2-surfaces. Again one must however remember that these algebras should have a realization at the level of M^8 but might not be necessary at the level of H.

1. The basic vision of quantum TGD is that string world sheets are replaced with 4-D surfaces and this forces a generalization of the notion of conformal invariance and one indeed obtains generalized conformal invariances for both the light-like boundaries of CD and for the light-like 3-surfaces defining partonic orbits as boundaries between Minkowskian and Euclidian space-time regions. One can very roughly say that string string world sheets parameterized by complex coordinate are replaced by space-time surfaces parameterized by two complex coordinates. Quaternion analyticity in the sense discussed would roughly conform with this picture.

2. The recent work with the Yangians [K8] and so called n-structures related to the categorification of TGD [K7] suggest that double loop algebras for which string world sheets are replaced with 4-D complex spaces. Quantum groups and Yangians assignable to Kac-Moody algebras rather than Lie algebras should be also central. Also double quantum groups depending on 2 parameters with so called elliptic R-matrix seem to be important. This physical intuition agrees with the general vision of Russian mathematician Igor Frenkel, who is one of the pioneers of quantum groups. For the article summarizing the work of Frenkel see `http://tinyurl.com/y7eego8c`). This article tells also about the work of Frenkel related to quaternion analyticity, which he sees to be of physical relevance but as a mathematicians is well aware of the fact that the non-commutativity of quaternions poses strong interpretation problems and means the loss of many nice properties of the ordinary analyticity.

3. The twistor lift of TGD suggest similar picture [K5, K4, K6]. The 6-D twistor space of space-time surface would be 6-surface in the product $T(M^4) \times T(CP_2)$ of geometric twistor spaces of M^4 and CP_2 and have induced twistor structure. The detailed analysis of this statement strongly suggests that data given at surfaces with dimension not higher than $D = 2$ should fix the preferred extremals. For the twistor lift action contains besides Kähler action also volume term. Asymptotic solutions are extremals of both Kähler action and minimal surfaces and all non-vacuum extremals of Kähler action are minimal surfaces so that the only change is that vacuum extremals of Kähler action must be restricted to be minimal surfaces.

The article about the work of Igor Frenkel (see `http://tinyurl.com/y7eego8c`) explains the general mathematics inspired vision about 3-levelled hierarchy of symmetries.

1. At the lowest level are Lie algebras. Gauge theories are prime example about this level.

2. At the second level loop algebras and quantum groups (defined as deformations of enveloping algebra of Lie algebra) and also Yangians. Loop algebras correspond to string models and quantum groups to TQFTs formulated at 3-D spaces.

3. At the third level are double loop algebras, quantum variants of loop algebras (also Yangians), and double quantum quantum groups - deformations of Lie algebras for which the R-matrix is elliptic function and depends on 2 complex parameters.

The conjecture of Frenkel (see `http://tinyurl.com/y7eego8c`) based on mathematical intuition is that these levels are actually the only ones. The motivation for this claim is 2-dimensionality making possible braiding and various quantum algebras. The set of poles for the R-matrix forms Abelian group with respect to addition in complex plane and can have rank equal to 0, 1, or (single pole, poles along line, lattice of poles). Higher ranks are impossible in $D = 2$.

In TGD framework physical intuition leads to a similar vision.

1. The dimension $D = 4$ for space-time surface and the choice $H = M^4 \times CP_2$ have both number theoretical and twistorial motivations [K8]. The replacement of point like particle with partonic 2-surface implies that TGD corresponds to the third level since loop algebras are replaced with their double loop analogs. 4-dimensionality makes also possible 2-braids and reconnections giving rise to a new kind of topological physics.

   The double loop group would represent the most dynamical level and its singly and doubly quantized variants correspond to a reduction in degrees of freedom, which one cannot exclude in TGD.

   The interesting additional aspect relates to the adelic physics [L1] implying a hierarchy of physics labelled by extensions of rationals. For cognitive representations the dynamics is discretized [K7]. Light-like 3-surfaces as partonic orbits are part of the picture and Chern-Simons term is naturally associated with them. TGD as almost topological QFT has been one of the guiding ideas in the evolution of TGD.

2. Double loop algebras represent unknown territory of mathematical physics. Igor Frenkel has also considered a possible realization of double loop algebras (see `http://tinyurl.com/y7eego8c`). He starts from the work of Mickelson (by the way, my custos in my thesis defence in 1982!) giving a realization of loop algebras: the idea is clearly motivated by WZW model which is 2-D conformal field theory with action containing a term associated with a 3-ball having 2-sphere as boundary.

   Mickelson starts from a circle represented as a boundary of a disk at which the physical states of CFT are realized. CFT is obtained by gluing together two disks with the boundary circles identified. The sphere in turn can be regarded as a boundary of a ball. The proposal of Frenkel is to complexify all these structures: circle becomes a Riemann surface, disk becomes 4-D manifold possessing complex structure in some sense, and 3-ball becomes 6-D complex manifold in some sense conjectured to be Calabi-Yau manifold.

3. The twistor lift of TGD leads to an analogous proposal. Circle is replaced with partonic 2-surfaces and string world sheets. 2-D complex surface is replaced with space-time region with complex structure or Hamilton-Jacobi structure [K3] and possessing twistor structure. 6-D Calabi-Yau manifold is replaced with the 6-D twistor space of space-time surface (sphere bundle over space-time surface) represented as 6-surface in 12-D Cartesian product $T(H) = T(M^4) \times T(CP_2)$ of the geometric twistor spaces of M^4 and CP_2.

   Twistor structure is induced and this is conjectured to determine the dynamics to be that for the preferred extremals of Kähler action plus volume term. This vision would generalize Penrose's original vision by eliminating gauge fields as primary dynamical variables and replacing there dynamics with the geometrodynamics of space-time surface.

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