

Some layman considerations related to the fundamentals of mathematics

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Abstract

In the sequel some questions related to the fundamentals of mathematics as I understand them as physicist are considered.

1. Gödel's problematics is shown to have a topological analog in real topology, which however disappears in p-adic topology which raises the question whether the replacement of the arithmetics of natural numbers with that of p-adic integers could allow to avoid Gödel's problematics.
2. Number theory looks from the point of view of TGD more fundamental than set theory and inspires the question whether the notion of algebraic number could emerge naturally from TGD. There are two ways to understand the emergence of algebraic numbers: the hierarchy of infinite primes in which ordinary primes are starting point and the arithmetics of Hilbert spaces with tensor product and direct sum replacing the usual arithmetic operations. Extensions of rationals give also rise to cognitive variants of n-D spaces.
3. The notion of empty set looks artificial from the point of view of physicist and a possible cure is to take arithmetics as a model. Natural numbers would be analogous to nonempty sets and integers would correspond to pairs of sets (A, B) , $A \subset B$ or $B \subset A$ with equivalence $(A, B) \equiv (A \cup C, B \cup C)$. Empty set would correspond to pairs (A, A) . In quantum context the generalization of the notion of being member of set $a \in A$ suggests a generalization: being an element in set would generalize to being single particle state which in general is de-localized to the set. Subsets would correspond to many-particle states. The basic operation would be addition or removal of element represented in terms of oscillator operator. The order of elements of set does not matter: this would generalize to bosonic and fermionic many particle states and even braid statistics can be considered. In bosonic case one can have multiple points - kind of Bose-Einstein condensate.
4. One can also start from finite-D Hilbert space and identify set as the collection of labels for the states. In infinite-D case there are two cases corresponding to separable and non-separable Hilbert spaces. The condition that the norm of the state is finite without infinite normalization constants forces selection of de-localized discrete basis in the case of a continuous set like reals. This inspires the question whether the axiom of choice should be given up. One possibility is that one can have only states localized to finite or at least discrete set of points which correspond points with coordinates in an extension of rationals.

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1 Introduction

I am not a mathematician and therefore should refrain from consideration of anything related to fundamentals of mathematics. In the discussions with Santeri Satama I could not avoid the temptation to break this rule. I however feel that I must confess my sins and in the following I will do this.

1. Gödel's problematics is shown to have a topological analog in real topology, which however disappears in p-adic topology which raises the question whether the replacement of the arithmetics of natural numbers with that of p-adic integers could allow to avoid Gödel's problematics.
2. Number theory looks from the point of view of TGD more fundamental than set theory and inspires the question whether the notion of algebraic number could emerge naturally from TGD. There are two ways to understand the emergence of algebraic numbers: the hierarchy of infinite primes in which ordinary primes are starting point and the arithmetics of Hilbert spaces with tensor product and direct sum replacing the usual arithmetic operations. Extensions of rationals give also rise to cognitive variants of n-D spaces.
3. The notion of empty set looks artificial from the point of view of physicist and a possible cure is to take arithmetics as a model. Natural numbers would be analogous to nonempty sets and integers would correspond to pairs of sets (A, B) , $A \subset B$ or $B \subset A$ with equivalence $(A, B) \equiv (A \cup C, B \cup C)$. Empty set would correspond to pairs (A, A) . In quantum context the generalization of the notion of being member of set $a \in A$ suggests a generalization: being an element in set would generalize to being single particle state which in general is de-localized to the set. Subsets would correspond to many-particle states. The basic operation would be addition or removal of element represented in terms of oscillator operator. The order of elements of set does not matter: this would generalize to bosonic and fermionic many particle states and even braid statistics can be considered. In bosonic case one can have multiple points - kind of Bose-Einstein condensate.
4. One can also start from finite-D Hilbert space and identify set as the collection of labels for the states. In infinite-D case there are two cases corresponding to separable and non-separable Hilbert spaces. The condition that the norm of the state is finite without infinite normalization constants forces selection of de-localized discrete basis in the case of a continuous set like reals. This inspires the question whether the axiom of choice should be given up. One possibility is that one can have only states localized to finite or at least discrete set of points which correspond points with coordinates in an extension of rationals.

2 Geometric analog for Gödel's problematics

Gödel's problematics involves statements which cannot be proved to be true or false or are simultaneously true and false. This problematics has also a purely geometric analog in terms of set theoretic representation of Boolean algebras when real topology is used but not when p-adic topology is used.

The natural idea is that Boolean algebra is realized in terms of open sets such that the negation of statement corresponds to the complement of the set. In p-adic topologies open sets are simultaneously also closed and there are no boundaries: this makes them and - more generally Stone spaces - ideal for realizing Boolean algebra set theoretically. In real topology the complement of open set is closed and therefore not open and one has a problem.

Could one circumvent the problem somehow?

1. If one replaces open sets with their closures (the closure of open set includes also its boundary, which does not belong to the open set) and closed complements of open sets, the analog of Boolean algebra would consist of closed sets. Closure of an open set and the closure of its open complement - statement and its negation - share the common boundary. Statement and its negation would be simultaneously true at the boundary. This strange situation reminds of Russell's paradox but in geometric form.
2. If one replaces the closed complements of open sets with their open interiors, one has only open sets. Now the sphere would represent statement about which one cannot say whether it is true or false. This would look like Gödelian sentence but represented geometrically.

This leads to an already familiar conclusion: p-adic topology is natural for the geometric correlates of cognition, in particular Boolean cognition. Real topology is natural for the geometric correlates of sensory experience.

3. Gödelian problematics is encountered already for arithmetics of natural numbers although naturals have no boundary in the discrete topology. Discrete topology does not however allow well-ordering of natural numbers crucial for the definition of natural number. In the induced real topology one can order them and can speak of boundaries of subsets of naturals. The ordering of natural numbers by size reflects the ordering of reals: it is very difficult to think about discrete without implicitly bringing in the continuum.

For p-adic integers the induced topology is p-adic. Is Gödelian problematics is absent in p-adic Boolean logic in which set and its complement are both open and closed. If this view is correct, p-adic integers might replace naturals in the axiomatics of arithmetics. The new element would be that most p-adic integers are of infinite size in real sense. One has a natural division of them to cognitively representable ones finite also in real sense and non-representable ones infinite in real sense. Note however that rationals have periodic binary expansion and can be represented as pairs of finite natural numbers.

In algebraic geometry Zariski topology in which closed sets correspond to algebraic surfaces of various dimensions, is natural. Open sets correspond to their complements and are of same dimension as the imbedding space. Also now one encounters asymmetry. Could one say that algebraic surfaces characterize "representable" (= "geometrically provable"?) statements as elements of Boolean algebra and their complements the non-representable ones? 4-D space-time (as possibly associative/co-associative) algebraic variety in 8-D octonionic space would be example of representable statement. Finite unions and intersections of algebraic surfaces would form the set of representable statements. This new-to-me notion of representability is somehow analogous to provability or demonstrability.

3 Number theory from quantum theory

Could one define or at least represent the notion of number using the notions of quantum physics? A natural starting point is hierarchy of extensions of rationals defining hierarchy of adeles. Could one obtain rationals and their extensions from simplest possible quantum theory in which one just constructs many particle states by adding or removing particles using creation and annihilation operators?

3.1 How to obtain rationals and their extensions?

Rationals and their extensions are fundamental in TGD. Can one have quantal construction for them?

1. One should construct rationals first. Suppose one starts from the notion of finite prime as something God-given. At the first step one constructs infinite primes as analogs for many-particle states in super-symmetric arithmetic quantum field theory [K1]. Ordinary primes label states of fermions and bosons. Infinite primes as the analogs of free many-particle states correspond to rationals in a natural manner.

2. One obtains also analogs of bound states which are mappable to irreducible polynomials, whose roots define algebraic numbers. This would give hierarchy of algebraic extensions of rationals. At higher levels of the hierarchy one obtains also analogs of prime polynomials with number of variables larger than 1. One might say that algebraic geometry has quantum representation. This might be very relevant for the physical representability of basic mathematical structures.

3.2 Arithmetics of Hilbert spaces

The notions of prime and divisibility and even basic arithmetics emerge also from the tensor product and direct sum for Hilbert spaces. Hilbert spaces with prime dimension do not decompose to tensor products of lower-dimensional Hilbert spaces. One can even perform a formal generalization of the dimension of Hilbert space so that it becomes rational and even algebraic number.

For some years ago I indeed played with this thought but at that time I did not have in mind reduction of number theory to the arithmetics of Hilbert spaces. If this really makes sense, numbers could be replaced by Hilbert spaces with product and sum identified as tensor product and direct sum!

Finite-dimensional Hilbert space represent the analogs of natural numbers. The analogs of integers could be defined as pairs (m, n) of Hilbert spaces with spaces (m, n) and $(m + r, n + r)$ identified (this space would have dimension $m - n$. This identification would hold true also at the level of states. Hilbert spaces with negative dimension would correspond to pairs with $(m - n) < 0$: the canonical representatives for m and $-m$ would be $(m, 0)$ and $(0, m)$. Rationals can be defined as pairs (m, n) of Hilbert spaces with pairs (m, n) and (km, kn) identified. These identifications would give rise to kind of gauge conditions and canonical representatives for m and $1/m$ are $(m, 1)$ and $(1, m)$.

What about Hilbert spaces for which the dimension is algebraic number? Algebraic numbers allow a description in terms of partial fractions and Stern-Brocot (S-B) tree (see <http://tinyurl.com/yb6ldekq> and <http://tinyurl.com/yc6hhboo>) containing given rational number once. S-B tree allows to see information about algebraic numbers as constructible by using an algorithm with finite number of steps, which is allowed if one accepts abstraction as basic aspect of cognition. Algebraic number could be seen as a periodic partial fraction defining an infinite path in S-B tree. Each node along this path would correspond to a rational having Hilbert space analog. Hilbert space with algebraic dimension would correspond to this kind of path in the space of Hilbert spaces with rational dimension. Transcendentals allow identification as non-periodic partial fraction and could correspond to non-periodic paths so that also they could have Hilbert spaces counterparts.

3.3 How to obtain the analogs higher-D spaces?

Algebraic extensions of rationals allow cognitive realization of spaces with arbitrary dimension identified as algebraic dimension of extension of rationals.

1. One can obtain n -dimensional spaces (in algebraic sense) with integer valued coordinates from n -D extensions of rationals. Now the n -tuples defining numbers of extension and differing by permutations are not equivalent so that one obtains n -D space rather than n -D space divided by permutation group S_n . This is enough at the level of cognitive representations and could explain why we are able to imagine spaces of arbitrary dimension although we cannot represent them cognitively.
2. One obtains also Galois group and orbits of set A of points of extension under Galois group as $G(A)$. One obtains also discrete coset spaces G/H and alike. These do not have any direct analog in the set theory. The hierarchy of Galois groups would bring in discrete group theory automatically. The basic machinery of quantum theory emerges elegantly from number theoretic vision.
3. In octonionic approach to quantum TGD one obtains also hierarchy of extensions of rationals since space-time surface correspond zero loci for RE or IM for octonionic polynomials obtained by algebraic continuation from real polynomials with coefficients in extension of rationals [L1].

4 Could quantum set theory make sense?

In the following my view point is that of quantum physicist fascinated by number theory and willing to reduce set theory to what could be called quantum set theory. It would follow from physics as generalised number theory (adelic physics) and have ordinary set theory as classical correlate.

1. From the point of quantum physics set theory and the notion of number based on set theory look somewhat artificial constructs. Nonempty set is a natural concept but empty set and set having empty set as element used as basic building brick in the construction of natural numbers looks weird to me.
2. From TGD point of view it would seem that number theory plus some basic pieces of quantum theory might be more fundamental than set theory. Could set theory emerge as a classical correlate for quantum number theory already considered and could quantum set theory make sense?

4.1 Quantum set theory

What quantum set theory could mean? Suppose that number theory-quantum theory connection really works. What about set theory? Or perhaps its quantum counterpart having ordinary set theory as a classical correlate?

1. A purely quantal input to the notion of set would be replacement of points delocalized states in the set. A generic single particle quantum state as analog of element of set would not be localized to a single element of set. The condition that the state has finite norm implies in the case of continuous set like reals that one cannot have completely localized states. This would give quantal limitation to the axiom of choice. One can have any discrete basis of state functions in the set but one cannot pick up just one point since this state would have infinite norm.

The idea about allowing only say rationals is not needed since there is infinite number of different choices of basis. Finite measurement resolution is however unavoidable. An alternative option is restriction of the domains of wave functions to a discrete set of points. This set can be chosen in very many manners and points with coordinates in extension of rationals are very natural and would define cognitive representation.

2. One can construct also the analogs of subsets as many-particle states. The basic operation would be addition/removal of a particle from quantum state represented by the action of creation/annihilation operator.

Bosonic states would be invariant under permutations of single particle states just like set is the equivalence class for a collection of elements (a_1, \dots, a_n) such that any two permutations are equivalent. Quantum set theory would however bring in something new: the possibility of both bosonic and fermionic statistics. Permutation would change the state by phase factor -1 . One would have fermionic and bosonic sets. For bosonic sets one could have multiplet elements ("Bose-Einstein condensation"): in the theory of surfaces this could allow multiple copies of the same surface. Even braid statistics is possible. The phase factor in permutation could be complex. Even non-commutative statistics can be considered.

Many particle states formed from particles, which are not identical are also possible and now the different particle types can be ordered. One obtains n -ples decomposing to ordered K -ple of n_i -ples, which consist of identical particles and are quantum sets. One could talk about K -sets as a generalization of set as analogs of classical sets with K -colored elements. Group theory would enter into the picture via permutation groups and braid groups would bring in braid statistics. Braids strands would have K colors.

4.2 How to obtain classical set theory?

How could one obtain classical set theory?

1. Many-particle states represented algebraically are detected in lab as sets: this is quantum classical correspondence. This remains to me one of the really mysterious looking aspects in the interpretation of quantum field theory. For some reason it is usually not mentioned at all in popularizations. The reason is probably that popularization deals typically with wave mechanics but not quantum field theory unless it is about Higgs mechanism, which is the weakest part of quantum field theory!
2. From the point of quantum theory empty set would correspond to vacuum. It is not observable as such. Could the situation change in the presence of second state representing the environment? Could the fundamental sets be always *non-empty* and correspond to states with non-vanishing particle number. Natural numbers would correspond to eigenvalues of an observable telling the cardinality of set. Could representable sets be like natural numbers?
3. Usually integers are identified as pairs of natural numbers (m, n) such that integer corresponds to $m - n$. Could the set theoretic analog of integer be a pair (A, B) of sets such that A is subset of B or vice versa? Note that this does not allow pairs with disjoint members. (A, A) would correspond to empty set. This would give rise to sets (A, B) and their “antisets” (B, A) as analogs of positive and negative integers.

One can argue that antisets are not physically realizable. Sets and antisets would have as analogs two quantizations in which the roles of oscillator operators and their hermitian conjugates are changed. The operators annihilating the ground state are called annihilation operators. Only either of these realization is possible but not both simultaneously.

In ZEO one can ask whether these two options correspond to positive and negative energy parts of zero energy states or to the states with state function reduction at either boundary of CD identified as correlates for conscious entities with opposite arrows of geometric time (generalized Zeno effect).

4. The cardinality of set, the number of elements in the set, could correspond to eigenvalue of observable measuring particle number. Many-particle states consisting of bosons or fermions would be analogs for sets since the ordering does not matter. Also braid statistics would be possible.

What about cardinality as a p-adic integer? In p-adic context one can assign to integer m , integer $-m$ as $m \times (p - 1) \times (1 + p + p^2 + \dots)$. This is infinite as real integer but finite as p-adic integer. Could one say that the antiset of m -element as analog of negative integer has cardinality $-m = m(p-1)(1+p+p^2+..)$. This number does not have cognitive representation since it is not finite as real number but is cognizable.

One could argue that negative numbers are cognizable but not cognitively representable as cardinality of set? This representation must be distinguished from cognitive representations as a point of imbedding space with coordinates in extension of rationals. Could one say that antisets and empty set as its own antiset can be cognized but cannot be cognitively represented?

Nasty mathematician would ask whether I can really start from Hilbert space of state functions and deduce from this the underlying set. The elements of set itself should emerge from this as analogs of completely localized single particle states labelled by points of set. In the case of finite-dimensional Hilbert space this is trivial. The number of points in the set would be equal to the dimension of Hilbert space. In the case of infinite-D Hilbert space the set would have infinite number of points.

Here one has two views about infinite set. One has both separable (infinite-D in discrete sense: particle in box with discrete momentum spectrum) and non-separable (infinite-D in real sense: free particle with continuous momentum spectrum) Hilbert spaces. In the latter case the completely localized single particle states would be represented by delta functions divided by infinite normalization factors. They are routinely used in Dirac’s bra-ket formalism but problems emerge in quantum field theory.

A possible solution is that one weakens the axiom of choice and accepts that only discrete points set (possibly finite) are cognitively representable and one has wave functions localized to discrete set of points. A stronger assumption is that these points have coordinates in extension of

rationals so that one obtains number theoretical universality and adeles. This is TGD view and conforms also with the identification of hyper-finite factors of type II_1 as basic algebraic objects in TGD based quantum theory as opposed to wave mechanics (type I) and quantum field theory (type III). They are infinite-D but allow excellent approximation as finite-D objects.

This picture could relate to the notion of non-commutative geometry, where set emerges as spectrum of algebra: the points of spectrum label the ideals of the integer elements of algebra.

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