Could the precursors of perfectoids emerge in TGD?

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Abstract

The work of Peter Scholze based on the notion of perfectoid has raised a lot of interest in the community of algebraic geometers. One application of the notion relates to the attempt to generalize algebraic geometry by replacing polynomials with analytic functions satisfying suitable restrictions. Also in TGD this kind of generalization might be needed at the level of $M^4 \times CP_2$ whereas at the level of M^8 algebraic geometry might be enough. The notion of perfectoid as an extension of p-adic numbers Q_p allowing all p:th roots of p-adic prime p is central and provides a powerful technical tool when combined with its dual, which is function field with characteristic p.

Could perfectoids have a role in TGD? The infinite-dimensionality of perfectoid is in conflict with the vision about finiteness of cognition. For other p-adic number fields Q_q, $q \neq p$ the extension containing p:th roots of p would be however finite-dimensional even in the case of perfectoid. Furthermore, one has an entire hierarchy of almost-perfectoids allowing powers of p^m:th roots of p-adic numbers. The larger the value of m, the larger the number of points in the extension of rationals used, and the larger the number of points in cognitive representations consisting of points with coordinates in the extension of rationals. The emergence of almost-perfectoids could be seen in the adelic physics framework as an outcome of evolution forcing the emergence of increasingly complex extensions of rationals.

1 Introduction

In algebraic-geometry community the work of Peter Scholze [?] (see http://tinyurl.com/y7h2sms7) introducing the notion of perfectoid related to p-adic geometry has raised a lot of interest. There are two excellent popular articles about perfectoids: the first article in AMS [A1] (see http://tinyurl.com/ydx38vk4) and second one in Quanta Magazine (see http://tinyurl.com/yc2mxxqh). I had heard already earlier about the work of Scholze but was too lazy to even attempt to understand what is buried under the horrible technicalities of modern mathematical prose. Rachel Francon re-directed my attention to the work of Scholze (see http://tinyurl.com/yb46oza6). The work of Scholze is interesting also from TGD point of view since the construction of p-adic geometry is a highly non-trivial challenge in TGD.

1. One should define first the notion of continuous manifold but compact-open characteristic of p-adic topology makes the definition of open set essential for the definition of topology problematic. Even single point is open so that hopes about p-adic manifold seem to decay to dust. One should pose restrictions on the allowed open sets and p-adic balls with radii coming as powers of p are the natural candidates. p-Adic balls are either disjoint or nested: note that also this is in conflict with intuitive picture about covering of manifold with open sets. All this strangeness originates in the special features of p-adic distance function known as ultra-metricity. Note however that for extensions of p-adic numbers one can say that the Cartesian products of p-adic 1-balls at different genuinely algebraic points of extension along particular axis of extension are disjoint.

2. At level of M^8 the p-adic variants of algebraic varieties defined as zero loci of polynomials do not seem to be a problem. Equations are algebraic conditions and do not involve derivatives like partial differential equations naturally encountered if Taylor series instead of polynomials

are allowed. Analytic functions might be encountered at level of $H = M^4 \times CP_2$ and here p-adic geometry might well be needed.

The idea is to define the generalization of p-adic algebraic geometry in terms of p-adic function fields using definitions very similar to those used in algebraic geometry. For instance, generalization of variety corresponds to zero locus for an ideal of p-adic valued function field. p-Adic ball of say unit radius is taken as the basic structure taking the role of open ball in the topology of ordinary manifolds. This kind of analytic geometry allowing all power series with suitable restrictions to function field rather than allowing only polynomials is something different from algebraic geometry making sense for p-adic numbers and even for finite fields.

3. One would like to generalize the notion of analytic geometry even to the case of number fields with characteristic p (p-multiple of element vanishes), in particular for finite fields F_p and for function fields $F_p[t]$. Here one encounters difficulties. For instance, the factorial $1/n!$ appearing as normalization factor of forms diverges if p divides it. Also the failure of Frobenius homomorphism to be automorphism for $F_p[t]$ causes difficulties in the understanding of Galois groups.

The work of Scholze has led to a breakthrough in unifying the existing ideas in the new framework provided by the notion of perfectoid. The work is highly technical and involves infinite-D extension of ordinary p-adic numbers adding all powers of all roots p^{1/p^m}, $m = 1, 2....$ Formally, an extension by powers of p^{1/p^∞} is in question.

This looks strange at first but it guarantees that all p-adic numbers in the extension have p:th roots, one might say that one forms a p-fold covering/wrapping of extension somewhat analogous to complex numbers. This number field is called perfectoid since it is perfect meaning that Frobenius homomorphism $a \rightarrow a^p$ is automorphism by construction. $Frob$ is injection always and by requiring that p:th roots exist always, it becomes also a surjection.

This number field has same Galois groups for all of its extensions as the function field $G[t]$ associated with the union of function fields $G = F_p[t^{1/p^m}]$. Automorphism property of $Frob$ saves from the difficulties with the factorization of polynomials and p-adic arithmetics involving remainders is replaced with purely local modulo p arithmetics.

2 The notion of perfectoid

In this section describe my understanding about the motivations and the basic idea of Scholze.

2.1 About motivations of Scholze

Scholze has several motivations for this work. Since I am not a mathematician, I am unable to really understand all of this at deep level but feel that my duty as user of this mathematics is at least to try!

1. Diophantine equations is a study of polynomial equations in several variables, say $x^2+2xy+y=0$. The solutions are required to be integer valued: in the example considered $x = y = 0$ and $x = -y = -1$ is such a solution. For integers the study of the solution is very difficult and one approach is to study these equations modulo p that is reduced the equations to finite field G_p for any p. The equations simplify enormously since ane has $a^p = a$ in F_p. This identity in fact defines so called Frobenius homomorphism acting as automorphism for finite fields. This holds true also for more complex fields with characteristic p say the ring $F_p[t]$ of power series of t with coefficients in F_p.

   The powers of variables, say x, appearing in the equation is reduced to at most x^{p-1}. One can study the solutions also in p-adic number fields. The idea is to find first whether finite field solution, that is solution modulo p, does exist. If this is the case, one can calculate higher powers in p. If the series contains finite number of terms, one has solution also in the sense of ordinary integers.

2. One of the related challenges is the generalization of the notion of variety to a geometry defined in arbitrary number field. One would like to have the notion of geometry also for finite

fields, and for their generalizations such as $F_p[t]$ characterized by characteristic p ($px = 0$ holds true for any element of the field). For fields of characteristic 1 - extensions of rationals, real, and p-adic number fields) $xp = 0$ not hold true for any $x \neq 0$. Any field containing rationals as sub-field, being thus local field, is said to have characteristic equal to 1. For local fields the challenge is relatively easy.

3. The situation becomes more difficult if one wants a generalization of differential geometry. In differential geometry differential forms are in a key role. One wants to define the notion of differential form in fields of characteristic p and construct a generalization of cohomology theory. This would generalize the notion of topology to p-adic context and even for finite fields of finite character. A lot of work has been indeed done and Grothendieck has been the leading pioneer.

   The analogs of cohomology groups have values in the field of p-adic numbers instead of ordinary integers and provide representations for Galois groups for the extensions of rationals inducing extensions of p-adic numbers and finite fields.

   In ordinary homology theory non-contractible sub-manifolds of various dimensions correspond to direct summands Z (group of integers) for homology groups and by Poincare duality those for cohomology groups. For Galois groups Z is replaced with Z_N. N depends on extension to which Galois group is associated and if N is divisible by p one encounters technical problems.

   There are many characteristic p- and p-adic cohomologies such as etale cohomology, chrystalline cohomology, algebraic de-Rham cohomology. Also Hodge theory for complex differential forms generalizes. These cohomologies should be related by homomorphism and category theoretic thinking the proof of the homomorphism requires the construction of appropriate functor between them.

   The integrals of forms over sub-varieties define the elements of cohomology groups in ordinary cohomology and should have p-adic counterparts. Since p-adic numbers are not well-ordered, definite integral has no straightforward generalization to p-adic context. One might however be able to define integrals analogous to those associated with differential forms and depending only on the topology of sub-manifold over which they are taken. These integrals would be analogous to multiple residue integrals, which are the crux of the twistor approach to scattering amplitudes in super-symmetric gauge theories. One technical difficulty is that for a field of finite characteristic the derivative of X^p is pX^{p-1} and vanishes. This does not allow to define what integral $\int X^{p-1}dX$ could mean. Also $1/n!$ appears as natural normalization factor of forms but if p divides it, it becomes infinite.

2.2 Attempt to understand the notion of perfectoid

Consider now the basic ideas behind the notion of perfectoid.

1. For finite finite fields F_p Frobenius homomorphism $a \rightarrow a^p$ is automorphism since one has $a^p = a$ in modulo p arithmetics. A field with this property is called perfect and all local fields are perfect. Perfectness means that an algebraic number in any extension L of perfect field K is a root of a separable minimal polynomial. Separability means that the number of roots in the algebraic closure of K of the polynomial is maximal and the roots are distinct.

2. All fields containing rationals as sub-fields are perfect. For fields of characteristic p $Frob$ need not be a surjection so that perfectness is lost. For instance, for $F_p[t]$ $Frob$ is trivially injection but surjective property is lost: $t^{1/p}$ is not integer power of t.

   One can however extend the field to make it perfect. The trick is simple: add to to $F_p[t]$ all fractional powers t^{1/p^n} so that all p:th roots exist and $Frob$ becomes and automorphism. The automorphism property of $Frob$ allows to get rid of technical problems related to a factorization of polynomials. The resulting extension is infinite-dimensional but satisfies the perfectness property allowing to understand Galois groups, which play key role in various cohomology theories in characteristic p.

3. Let $K = Q_p[p^{1/p^\infty}]$ denote the infinite-dimensional extension of p-adic number field Q_p by adding all powers of p^m:th roots for all all $m = 1, 2, ...$ This is not the most general option: K could be also only a ring. The outcome is perfect field although it does not of course have Frobenius automorphism since characteristic equals to 1.

   One can divide K by p to get K/p as the analog of finite field F_p as its infinite-dimensional extension. K/p allows all p:th roots by construction and $Frob$ is automorphism so that K/p is perfect by construction.

   The structure obtained in this manner is closely related to a perfect field with characteristic p having same Galois groups for all its extensions. This object is computationally much more attractive and allows to prove theorems in p-adic geometry. This motivates the term perfectoid.

4. One can assign to K another object, which is also perfectoid but has characteristic p. The correspondence is as follows.

   (a) Let F_p be finite field. F_p is perfect since it allows trivially all p:th roots by $a^p = a$. The ring $F_p[t]$ is however not prefect since t^{1/p^m} is not integer power of t. One must modify $F_p[t]$ to obtain a perfect field. Let $G_m = F_p[t^{1/p^m}]$ be the ring of formal series in powers of t^{1/p^m} defining also function field. These serious are called t-adic and one can define t-adic norm.

   (b) Define t-adic function field K_b called the **tilt** of K as

   $$K_b = \cup_{m=1,...}(K/p)[t^{1/pm}][t] \quad .$$

   One has all possible power series with coefficients in K/p involving all roots t^{1/p^m}, $m = 1, 2, ...$, besides powers of positive integer powers of t. This function field has characteristic p and all roots exist by construction and $Frob$ is automorphism. K_b/t is perfect meaning that the minimal polynomials for the for given analog of algebraic number in any of its extensions allows separable polynomial with maximal number of roots in its closure.

   This sounds rather complicated! In any case, K_b/t has same number theoretical structure as $Q_p[p^{1/p^\infty}]/p$ meaning that Galois groups for all of its extensions are canonically isomorphic to those for extensions of K. Arithmetics modulo p is much simpler than p-adic arithmetic since products are purely local and there is no need to take care about remainders in arithmetic operations, this object is much easier to handle.

   Note that also p-adic number fields fields Q_p as also $F_p = Q_p/p$ are perfect but the analog of $K_b = F_b[t]$ fails to be perfect.

2.3 Second attempt to understand the notions of perfectoid and its tilt

This subsection is written roughly year after the first version of the text. I hope that it reflects a genuine increase in my understanding.

1. Scholze introduces first the notion of perfectoid. This requires some background notions. The characteristic p for field is defined as the integer p for which $px = 0$ (p is prime) for all elements x. Frobenius homomorphism (Frob familiarly) is defined as $Frob : x \to x^p$. For a field of characteristic p $Frob$ is an algebra homomorphism mapping product to product and sum to sum: this is very nice and relatively easy to show even by a layman like me.

2. Perfectoid is a field having either characteristic $p = 0$ (reals, p-adics for instance) or for which $Frob$ is a surjection meaning that $Frob$ maps at least one number to a given number x.

3. For finite fields $Frob$ is identity: $x^p = x$ as proved already by Fermat. For reals and p-adic number fields with characteristic p=0 it maps all elements to unit element and is not a surjection. Field is perfect if it has either $p = 0$ (reals, p-adics) or if Frobenius is surjection. Finite fields are obviously perfectoids too.

Scholze introduces besides perfectoids K also what he calls tilt K_b of the perfectoid. K_b is infinite-D extension of p-adic numbers by iterated p:th roots p-adic numbers: the units of the extension correspond to the roots p^{1/p^k}. They are something between p-adic number fields and reals and leads to theorems giving totally new insights to arithmetic geometry. Unfortunately, my technical skills in mathematics are hopelessly limited to say anything about these theorems.

1. As we learned during the first student year of mathematics, real numbers can be defined as Cauchy sequences of rationals converging to a real number, which can be also algebraic number or transcendental. The elements in the tilt K_b would be this kind of sequences.

2. Scholze starts from (say) p-adic numbers and considers infinite sequence of iterates of $1/p$:th roots. At given step $x \rightarrow x^{1/p}$. This gives the sequence $(x, x^{1/p}, x^{1/p^2}, x^{1/p^3}, ...)$ identified as an element of the tilt K_b. At the limit one obtains $1/p^\infty$ root of x.

   Remark: For finite fields each step is trivial ($x^p = x$) so that nothing interesting results: one has $(x, x, x, x, ...)$

   (a) For p-adic number fields the situtation is non-trivial. $x^{1/p}$ exists as p-adic number for all p-adic numbers with unit norm having $x = x_0 + x_1 p + ...$ In the lowest order $x \simeq x_0$ the root is just x since x is effectively an element of finite field in this approximation. One can develop the $x^{1/p}$ to a power series in p and continue the iteration. The sequence obtained defines an element of tilt K_b of field K, now p-adic numbers.

   (b) If the p-adic number x has norm p^n, $n \neq 0$ and is therefore not p-adic unit, the root operation makes sense only if one performs an extension of p-adic numbers containing all the roots p^{1/p^k} . These roots define one particular kind of extension of p-adic numbers and the extension is infinite-dimensional since all roots are needed. One can approximate K_b by taking only finite number iterated roots.

3. The tilt is said to be fractal: this is easy to understand from the presence of the iterated p:th root. Each step in the sequence is like zooming. One might say that p-adic scale becomes p:th root of itself. In TGD the p-adic length scale L_p is proportional to $p^{1/2}$: does the scaling mean that the p-adic length scale would defined hierarchy of scales proportional to $p^{1/2kp}$: root of itself and approach the CP_2 scale since the root of p approaches unity. Tilts as extensions by iterated roots would improve the length scale resolution.

One day later after writing this I got the feeling that I might have vaguely understood one more important thing about the tilt of p-adic number field: changing of the characteristic 0 of p-adic number field to characteristics $p > 0$ of the corresponding finite field for its tilt (thanks for Ulla for the links). What could this mean?

1. Characteristic p (p is the prime labelling p-adic number field) means $px = 0$. This property makes the mathematics of finite fields extremely simple: in the summation one need not take care of the residue as in the case of reals and p-adics. The tilt of the p-adic number field would have the same property! In the infinite sequence of the p-adic numbers coming as iterated p:th roots of the starting point p-adic number one can sum each p-adic number separately. This is really cute if true!

2. It seems that one can formulate the arithmetics problem in the tilt where it becomes in principle as simple as in finite field with only p elements! Does the existence of solution in this case imply its existence in the case of p-adic numbers? But doesn't the situation remain the same concerning the existence of the solution in the case of rational numbers? The infinite series defining p-adic number must correspond a sequence in which binary digits repeat with some period to give a rational number: rational solution is like a periodic solution of a dynamical system whereas non-rational solution is like chaotic orbit having no periodicity? In the tilt one can also have solutions in which some iterated root of p appears: these cannot belong to rationals but to their extension by an iterated root of p.

The results of Scholze could be highly relevant for the number theoretic view about TGD in which octonionic generalization of arithmetic geometry plays a key role since the points of space-time surface with coordinates in extension of rationals defining adele and also what I call cognitive representations determining the entire space-time surface if $M^8 - H$ duality holds true (space-time surfaces would be analogous to roots of polynomials). Unfortunately, my technical skills in mathematics needed are hopelessly limited.

TGD inspires the question is whether this kind of extensions could be interesting physically. At the limit of infinite dimension one would get an ideal situation not realizable physically if one believes that finite-dimensionality is basic property of extensions of p-adic numbers appearing in number theoretical quantum physics (they would related to cognitive representations in TGD). Adelic physics [L2] involves all finite-D extensions of rationals and the extensions of p-adic number fields induced by them and thus also cutoffs of extensions of type K_b- which I have called precursors of K_b.

2.4 How this relates to Witt vectors?

Witt vectors provide an alternative representation of p-adic arithmetics of p-adic integers in which the sum and product are reduced to purely local digit-wise operations for each power of p for the components of Witt vector so that one need not worry about carry pinary digit.

1. The idea is to consider the sequence consisting pinary cutoffs to p-adic number $xmodp^n$ and identify p-adic integer as this kind of sequence as n approaches infinity. This is natural approach when one identifies finite measurement resolution or cognitive resolution as a cutoff in some power of p^n. One simply forms the numbers $X_n = x\ mod\ p^{n+1]}$: for numbers $1, ..., p-1$ they are called Teichmueller representatives and only they are needed to construct the sequences for general x. One codes this sequence of pinary cutoffs to Witt vector.

2. The non-trivial observation made by studying sums of p-adic numbers is that the sequence $X_0, X_1, X_2, ...$ of approximations define a sequence of components of Witt vector as $W_0 = X_0$, $W_1 = X_0^p + pX_1$, $W_2 = X_0^(p^2) + pX_1^p + p^2X_2$, ... or more formally $W_n = Sum_{i<n} p^i Z X_i^[p^(n-i)]$.

3. The non-trivial point is that Witt vectors form a commutative ring with local digit-wise multiplication and sum modulo p: there no carry digits. Effectively one obtains infinite Cartesian power of finite field F_p. This means a great simplification in arithmetics. One can do the arithmetics using Witt vectors and deduce the sum and product from their product.

4. Witt vectors are universal. In particular, they generalize to any extension of p-adic numbers. Could Witt vectors bring in something new from physics point of view? Could they allow a formulation for the hierarchy of pinary cutoffs giving some new insights? For instance, neuro-computationalist might ask whether brain could perform p-adic arithmetics using a linear array of modules (neurons or neuron groups) labelled by $n = 1, 2, ...$ calculates sum or product for component W_n of Witt vector? No transfer of carry bits between modules would be needed. There is of course the problem of transforming p-adic integers to Witt vectors and back - it is not easy to imagine a natural realization for a module performing this transformation. Is there any practical formulation for say p-adic differential calculus in terms of Witt vectors?

I would seem that Witt vectors might relate in an interesting manner to the notion of perfectoid. The basic result proved by Petter Scholtze is that the completion $\cup_n Q_p(p^{1/p^n})$ of p-adic numbers by adding p^n:th roots and the completion of Laurent series $F_p((t))$ to $\cup_n F_p((t^{1/p^n}))$ have isomorphic absolute Galois groups and in this sense are one and same thing. On the other hand, p-adic integers can be mapped to a subring of $F_p(t)$ consisting of Taylor series with elements allowing interpretation as Witt vectors.

3 TGD view about p-adic geometries

As already mentioned, it is possible to define p-adic counterparts of n-forms and also various p-adic cohomologies with coefficient field taken as p-adic numbers and these constructions presumably make sense in TGD framework too. The so called rigid analytic geometry is the standard proposal for what p-adic geometry might be.

The very close correspondence between real space-time surfaces and their p-adic variants plays realized in terms of cognitive representations [?] [L3, L1] plays a key role in TGD framework and distinguishes it from approaches trying to formulate p-adic geometry as a notion independent of real geometry.

Ordinary approaches to p-adic geometry concentrate the attention to single p-adic prime. In the adelic approach of TGD one considers both reals and all p-adic number fields simultaneously.

Also in TGD framework Galois groups take key role in this framework and effectively replace homotopy groups and act on points of cognitive representations consisting of points with coordinates in extension of rationals shared by real and p-adic space-time surfaces. One could say that homotopy groups at level of sensory experience are replaced by Galois at the level of cognition. It also seems that there is very close connection between Galois groups and various symmetry groups. Galois groups would provide representations for discrete subgroups of symmetry groups.

In TGD framework there is strong motivation for formulating the analog of Riemannian geometry of $H = M^4 \times CP_2$ for p-adic variants of H. This would mean p-adic variant of Kähler geometry. The same challenge is encountered even at the level of "World of Classical Worlds" (WCW) having Kähler geometry with maximal isometries. p-Adic Riemann geometry and n-forms make sense locally as tensors but integrals defining distances do not make sense p-adically and it seems that the dream about global geometry in p-adic context is not realizable. This makes sense: p-adic physics is a correlate for cognition and one cannot put thoughts in weigh or measure their length.

3.1 Formulation of adelic geometry in terms of cognitive representations

Consider now the key ideas of adelic geometry and of cognitive representations.

1. The king idea is that p-adic geometries in TGD framework consists of p-adic balls of possibly varying radii p^n assignable to points of space-time surface for which the preferred imbedding space coordinates are in the extension of rationals. At level of M^8 octonion property fixes preferred coordinates highly uniquely. At level of H preferred coordinates come from symmetries.

   These points define a cognitive representation and inside p-adic points the solution of field equations is p-adic variant of real solution in some sense. At M^8 level the field equations would be algebraic equations and real-p-adic correspondence would be very straightforward. Cognitive representations would make sense at both M^8 level and H level.

   Remark: In ordinary homology theory the decomposition of real manifold to simplexes reduces topology to homology theory. One forgets completely the interiors of simplices. Could the cognitive representations with points labelling the p-adic balls could be seen as analogous to decompositions to simplices. If so, homology would emerge as something number theoretically universal. The larger the extension of rationals, the more precise the resolution of homology would be. Therefore p-adic homology and cohomology as its Poincare dual would reduce to their real counterparts in the cognitive resolution used.

2. $M^8 - H$ correspondence would play a key role in mapping the associative regions of space-time varieties in M^8 to those in H. There are two kinds of regions. Associative regions in which polynomials defining the surfaces satisfy criticality conditions and non-associative regions. Associative regions represent external particles arriving in CDs and non-associative regions interaction regions within CDs.

3. In associative regions one has minimal surface dynamics (geodesic motion) at level of H and coupling parameters disappear from the field equations in accordance with quantum criticality. The challenge is to prove that $M^8 - H$ correspondence is consistent with the minimal surface dynamics n H. The dynamics in these regions is determined in M^8 as zero loci of polynomials satisfying quantum criticality conditions guaranteeing associativity and is deterministic also in p-adic sectors since derivatives are not involved and pseudo constants depending on finite number of pinary digits and having vanishing derivative do not appear. $M^8 - H$ correspondence guarantees determinism in p-adic sectors also in H.

4. In non-associative regions $M^8 - H$ correspondence does not make sense since the tangent space of space-time variety cannot be labelled by CP_2 point and the real and p-adic H

counterparts of these regions would be constructed from boundary data and using field equations of a variational principle (sum of the volume term and Kähler action term), which in non-associative regions gives a dynamics completely analogous to that of charged particle in induced Kähler field. Now however the field characterizes extended particle itself.

Boundary data would correspond to partonic 2-surfaces and string world sheets and possibly also the 3-surfaces at the ends of space-time surface at boundaries of CD and the light-like orbits of partonic 2-surfaces. At these surfaces the 4-D (!) tangent/normal space of space-time surface would be associative and could be mapped by $M^8 - H$ correspondence from M^8 to H and give rise to boundary conditions.

Due to the existence of p-adic pseudo-constants the p-adic dynamics determined by the action principle in non-associative regions inside CD would not be deterministic in p-adic sectors. The interpretation would be in terms of freedom of imagination. It could even happen that boundary values are consistent with the existence of space-time surface in p-adic sense but not with the existence of real space-time surfaces. Not all that can be imagined is realizable.

At the level of M^8 this vision seems to have no obvious problems. Inside each ball the same algebraic equations stating vanishing of $IM(P)$ (imaginary part of P in quaternionic sense) hold true. At the level of H one has second order partial differential equations, which also make sense also p-adically. Besides this one has infinite number of boundary conditions stating the vanishing of Noether charges assignable to sub-algebra super-symplectic algebra and its commutator with the entire algebra at the 3-surfaces at the boundaries of CD. Are these two descriptions really equivalent?

During writing I discovered an argument, which skeptic might see as an objection against $M^8 - H$ correspondence.

1. M^8 correspondence maps the space-time varieties in M^8 in non-local manner to those in $H = M^4 \times CP_2$. CP_2 coordinates characterize the tangent space of space-time variety in M^8 and this might produce technical problems. One can map the real variety to H and find the points of the image variety satisfying the condition and demand that they define the "spine" of the p-adic surface in p-adic H.

2. The points in extensions of rationals in H need not be images of those in M^8 but should this be the case? Is this really possible? M^4 point in $M^4 \times E^4$ would be mapped to $M^4 \subset M^4 \times CP_2$: this is trivial. 4-D associative tangent/normal space at m containing preferred M^2 would be characterized by CP_2 coordinates: this is the essence of $M^8 - H$ correspondence. How could one guarantee that the CP_2 coordinates characterizing the tangent space are really in the extension of rationals considered? If not, then the points of cognitive representation in H are not images of points of cognitive representation in M^8. Does this matter?

3.2 Are almost-perfectoids evolutionary winners in TGD Universe?

One could take perfectoids and perfectoid spaces as a mere technical tool of highly refiner mathematical cognition. Since cognition is basic aspect of TGD Universe, one could also ask perfectoids or more realistically, almost-perfectoids, could be an outcome of cognitive evolution in TGD Universe?

1. p-Adic algebraic varieties are defined as zero loci of polynomials. In the octonionic M^8 approach identifying space-time varieties as zero loci for RE or IM of octonionic polynomial (RE and IM in quaternionic sense) this allows to define p-adic variants of space-time surfaces as varieties obeying same polynomial equations as their real counterparts provided the coefficients of octonion polynomials obtainable from real polynomials by analytic continuation are in an extension of rationals inducing also extension of p-adic numbers.

   The points with coordinates in the extension of rationals common to real and p-adic variants of M^8 identified as cognitive representations are in key role. One can see p-adic space-time surfaces as collections of "monads" labelled by these points at which Cartesian product of 1-D p-adic balls in each coordinate degree. The radius of the p-adic ball can vary. Inside each ball the same polynomial equations are satisfied so that the monads indeed reflect other monads.

Kind of algebraic hologram would be in question consisting of the monads. The points in extension allow to define ordinary real distance between monads. Only finite number of monads would be involved since the number of points in extension tends to be finite. As the extension increases, this number increases. Cognitive representations become more complex: evolution as increase of algebraic complexity takes place.

2. Finite-dimensionality for the allowed extensions of p-adic number fields is motivated by the idea about finiteness of cognition. Perfectoids are however infinite-dimensional. Number theoretical universality demands that on only extensions of p-adics induced by those of rationals are allowed and defined extension of the entire adele. Extensions should be therefore be induced by the same extension of rationals for all p-adic number fields.

   Perfectoids correspond to an extension of Q_p apparently depending on p. This dependence is in conflict with number theoretical universality if real. This extension could be induced by corresponding extension of rationals for all p-adic number fields. For p-adic numbers Q_q $q \neq p$ all equation $a^{p^n} = x$ reduces to $a^n = x \ mod \ p$ and this in term to $a^m = x \ mod \ p$, $m = n \ mod \ p$. Finite-dimensional extension is needed to have all roots of required kind! This extension is therefore finite-D for all $q \neq p$ and infinite-D for p.

3. What about infinite-dimensionality of the extension. The real world is rarely perfect and our thoughts about it even less so, and one could argue that we should be happy with almost-perfectoids! "Almost" would mean extension induced by powers of p^{1/p^m} for large enough m, which is however not infinite. A finite-dimensional extension approaching perfectoid asymptotically is quite possible!

4. One could see the almost perfectoid as an outcome of evolution and perfectoid as the asymptotic states. High dimension of extension means that p-adic numbers and extension of rationals have large number of common numbers so that also cognitive representations contain a large number of common points. Maybe the p-adic number fields, which are evolutionary winners, have managed to evolve to especially high-dimensional almost-perfectoids! Note however that also the roots of e can be considered as extensions of rationals since corresponding p-adic extensions are finite-dimensional. Similar evolution can be considered also now.

   To get some perspective mote that for large primes such as $M_{127} = 2^{127} - 1$ characterizing electron the lowest almost perfectoid would give powers of $M_{127}^{1/M_{127}} = (2^{127} - 1)^{1/(2^{127}-1)} \sim 1 + log(2)2^{-120}$! The lattice of points in extension is extremely dense near real unit. The density of of points in cognitive representations near this point would be huge. Note that the length scales comes as negative powers of two, which brings in mind p-adic length scale hypothesis [K1].

Although the octonionic formulation in terms of polynomials (or rational functions identifying space-time varieties as zeros or poles of $RE(P)$ or $IM(P)$ is attractive in its simplicity, one can also consider the possibility of allowing analytic functions of octonion coordinate obtained from real analytic functions. These define complex analytic functions with commutative imaginary unit used to complexify octonions. Could meromorphic functions real analytic at real axis having only zeros and poles be allowed? The condition that all p-adic variants of these functions exist simultaneously is non-trivial. Coefficients must be in the extension of rationals considered and convergence poses restrictions. For instance, e^x converges only for $|x|_p < 1$. These functions might appear at the level of H.

REFERENCES

Mathematics

[A1] Batt B. What is a perfectoid space. 2014.

Books related to TGD

[K1] Pitkänen M. *p-Adic length Scale Hypothesis*. Online book. Available at: http://www.tgdtheory.fi/tgdhtml/padphys.html, 2013.

Articles about TGD

[L1] Pitkänen M. Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry? Available at: http://tgdtheory.fi/public_html/articles/ratpoints.pdf, 2017.

[L2] Pitkänen M. Philosophy of Adelic Physics. Available at: http://tgdtheory.fi/public_html/articles/adelephysics.pdf, 2017.

[L3] Pitkänen M. Philosophy of Adelic Physics. In *Trends and Mathematical Methods in Interdisciplinary Mathematical Sciences*, pages 241–319. Springer.Available at: https://link.springer.com/chapter/10.1007/978-3-319-55612-3_11, 2017.