p-Adicizable discrete variants of classical Lie groups and coset spaces in TGD framework

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Abstract

p-Adization of quantum TGD is one of the long term projects of TGD. The notion of finite measurement resolution reducing to number theoretic existence in p-adic sense is the fundamental notion. p-Adic geometries replace discrete points of discretization with p-adic analogs of monads of Leibniz making possible to construct differential calculus and formulate p-adic variants of field equations allowing to construct p-adic cognitive representations for real space-time surfaces.

This leads to a construction for the hierarchy of p-adic variants of imbedding space inducing in turn the construction of p-adic variants of space-time surfaces. Number theoretical existence reduces to conditions demanding that all ordinary (hyperbolic) phases assignable to (hyperbolic) angles are expressible in terms of roots of unity (roots of e).

The construction reduces to the construction of p-adicizable discrete subgroups of classical Lie groups. The construction starts from SU(2) and U(1) and proceeds iteratively. Remarkably, the finite discrete p-adicizable subgroups of SU(2) correspond to those appearing in the hierarchy of inclusions of hyperfinite factors and include the groups assignable to Platonic solids. One can seen them as cognitively especially simple finite p-adicizable objects providing p-adic approximation of sphere. The Platonic solids have analogs also for larger classical Lie groups.

1 Introduction

In TGD framework p-adicization and adelization are carried out at all levels of geometry: imbedding space, space-time and WCW. Adelization at the level of state spaces requires that it is common from all sectors of the adele and has as coefficient field an extension of rationals allowing both real and p-adic interpretations: the sectors of adele give only different views about the same quantum state.

In the sequel the recent view about the p-adic variants of imbedding space, space-time and WCW is discussed. The notion of finite measurement resolution reducing to number theoretic existence in p-adic sense is the fundamental notion. p-Adic geometries replace discrete points of discretization with p-adic analogs of monads of Leibniz making possible to construct differential calculus and formulate p-adic variants of field equations allowing to construct p-adic cognitive representations for real space-time surfaces.

This leads to a beautiful construction for the hierarchy of p-adic variants of imbedding space inducing in turn the construction of p-adic variants of space-time surfaces. Number theoretical existence reduces to conditions demanding that all ordinary (hyperbolic) phases assignable to (hyperbolic) angles are expressible in terms of roots of unity (roots of e).

For $SU(2)$ one obtains as a special case Platonic solids and regular polygons as preferred p-adic geometries assignable also to the inclusions of hyperfinite factors [K4, K2]. Platonic solids represent idealized geometric objects ofthe p-adic world serving as a correlate for cognition as contrast to the geometric objects of the sensory world relying on real continuum.

In the case of causal diamonds (CDs) - the construction leads to the discrete variants of Lorentz group $SO(1,3)$ and hyperbolic spaces $SO(1,3)/SO(3)$. The construction gives not only the p-adicizable discrete subgroups of $SU(2)$ and $SU(3)$ but applies iteratively for all classical Lie groups meaning that the counterparts of Platonic solids are countered also for their p-adic coset spaces. Even the p-adic variants of WCW might be constructed if the general recipe for the construction of finite-dimensional symplectic groups applies also to the symplectic group assignable to $\Delta CD \times CP_2$.

The emergence of Platonic solids is very remarkable also from the point of view of TGD inspired theory of consciousness and quantum biology. For a couple of years ago I developed a model of music harmony [K3] [L1] relying on the geometries of icosahedron and tetrahedron. The basic observation is that 12-note scale can be represented as a closed curve connecting nearest number points (Hamiltonian cycle) at icosahedron going through all 12 vertices without self intersections. Icosahedron has also 20 triangles as faces. The idea is that the faces represent 3-chords for a given harmony characterized by Hamiltonian cycle. Also the interpretation terms of 20 amino-acids identifiable and genetic code with 3-chords identifiable as DNA codons consisting of three letters is highly suggestive.

One ends up with a model of music harmony predicting correctly the numbers of DNA codons coding for a given amino-acid. This however requires the inclusion of also tetrahedron. Why icosahedron should relate to music experience and genetic code? Icosahedral geometry and its dodecahedral dual as well as tetrahedral geometry appear frequently in molecular biology but its appearance as a preferred p-adic geometry is what provides an intuitive justification for the model of genetic code. Music experience involves both emotion and cognition. Musical notes could code for the points of p-adic geometries of the cognitive world. The model of harmony in fact generalizes. One can assign Hamiltonian cycles to any graph in any dimension and assign chords and harmonies with them. Hence one can ask whether music experience could be a form of p-adic geometric cognition in much more general sense.

The geometries of biomolecules brings strongly in mind the geometry p-adic space-time sheets. p-Adic space-time sheets can be regarded as collections of p-adic monad like objects at algebraic space-time points common to real and p-adic space-time sheets. Monad corresponds to p-adic units with norm smaller than unit. The collections of algebraic points defining the positions of monads and also intersections with real space-time sheets are highly symmetric and determined by the discrete p-adicizable subgroups of Lorentz group and color group. When the subgroup of the rotation group is finite one obtains polygons and Platonic solids. Bio-molecules typically consists of this kind of structures - such as regular hexagons and pentagons - and could be seen as cognitive representations of these geometries often called sacred! I have proposed this idea long time ago and the discovery of the recipe for the construction of p-adic geometries gave a justification for this idea.

2 p-Adic variants of causal diamonds

To construct p-adic variants of space-time surfaces one must construct p-adic variants of the imbedding space. The assumption that the p-adic geometry for the imbedding space induces p-adic geometry for sub-manifolds implies a huge simplification in the definition of p-adic variants of preferred extremals. The natural guess is that real and p-adic space-time surfaces gave algebraic points as common: so that the first challenge is to pick the algebraic points of the real space-time surface. To define p-adic space-time surface one needs field equations and the notion of p-adic continuum and by assigning to each algebraic point a p-adic continuum to make it monad, one can solve p-adic field equations inside these monads.

The idea of finite measurement resolution suggests that the solutions of p-adic field equations inside monads are arbitrary. Whether this is consistent with the idea that same solutions of field equations can be interpreted either p-adically or in real sense is not quite clear. This would be guaranteed if the p-adic solution has same formal representation as the real solution in the vicinity of given discrete point - say in terms of polynomials with rational coefficients and coordinate variables which vanish for the algebraic point.

Real and p-adic space-time surfaces would intersect at points common to all number fields for given adele: cognition and sensory worlds intersect not only at the level of WCW but also at the level of space-time. I had already considered giving up the latter assumption but it seems to be necessary at least for string world sheets and partonic 2-surfaces if not for entire space-time surfaces.

2.1 General recipe

The recipe would be following.

1. One starts from a discrete variant of $CD \times CP_2$ defined by an appropriate discrete symmetry groups and their subgroups using coset space construction. This discretization consists of points in finite-dimensional extension of p-adics induced by an extension of rationals. These points are assumed to be in the intersection of reality and p-adicities at space-time level - that is common for real and p-adic space-time surfaces. Cognitive representations in the real world are thus discrete and induced by the intersection. This is the original idea which I was ready to give up as the vision about discretization at WCW level allowing to solve all problems related to symmetries emerged. At space-time level the p-adic discretization reduces symmetry groups to their discrete subgroups: cognitive representations unavoidably break the symmetries. What is important the distance between discrete p-adic points labelling monads is naturally their real distance. This fixes metrically real-p-adic/sensory-cognitive correspondence.

2. One replaces each point of this discrete variant $CD \times CP_2$ with p-adic continuum defined by an algebraic extension of p-adics for the adele considered so that differentiation and therefore also p-adic field equations make sense. The continuum for given discrete point of $CD_d \times CP_{2,d}$ defines kind of Leibnizian monad representing field equations p-adically. The solution decomposes to p-adically differentiable pieces and the global solution of field equations makes sense since it can be interpreted in terms of pseudo-constants. p-Adicization means discretization but with discrete points replaced with p-adic monads preserving also the information about local behavior. The loss of well-ordering inside p-adic monad reflects its loss due to the finiteness of measurement resolution.

3. The distances between monads correspond to their distances for real variant of $CD \times CP_2$. Are there natural restrictions on the p-adic sizes of monads? Since p-adic units are in question that size in suitable units is $p^{-N} < 1$. It would look natural that the p-adic size of the is smaller than the distance to the nearest monad. The denser the discretization is, the larger the value of N would be. The size of the monad decreases at least like $1/p$ and for large primes assignable to elementary particles ($M_{127} = 2^{127} - 1$) is rather small. The discretizations of the subgroups share the properties of the group invariant geometry of groups so that they are to form a regular lattice like structure with constant distance to nearest neighbors. At the imbedding level therefore p-adic geometries are extremely symmetric. At the level of space-time geometries only a subset of algebraic points is picked and the symmetry tends to be lost.

2.2 CD degrees of freedom

Consider first CD degrees of freedom.

1. For M^4 one has 4 linear coordinates. Should one p-adicize these or should one discretize CDs defined as intersections of future and past directed light-cones and strongly suggested

by ZEO. CD seems to represent the more natural option. The construction of a given CD suggests that one should replace the usual representation of manifold as a union of overlapping regions with intersection of two light-cones with coordinates related in the intersection as in the case of ordinary manifold: $\cup \rightarrow \cap$.

2. For a given light-cone one must introduce light-cone proper time a, hyperbolic angle η and two angle coordinates (θ, ϕ). Light-cone proper time a is Lorentz invariant and corresponds naturally to an ordinary p-adic number of more generally to a p-adic number in algebraic extension which does not involve phases.

   The two angle coordinates (θ, ϕ) parameterizing S^2 can be represented in terms of phases and discretized. The hyperbolic coordinate can be also discretized since e^p exists p-adically, and one obtains a finite-dimensional extension of p-adic numbers by adding roots of e and its powers. e is completely exceptional in that it is p-adically an algebraic number.

3. This procedure gives a discretization in angle coordinates. By replacing each discrete value of angle by p-adic continuum one obtains also now the monad structure. The replacement with continuum means the replacement

$$U_{m,n} \equiv exp(i2\pi m/n) \rightarrow U_{m,n} \times exp(i\phi) \ , \tag{2.1}$$

   where ϕ is p-adic number with norm $p^{-N} < 1$ It can also belong to an algebraic extension of p-adic numbers. Building the monad is like replacing in finite measurement the representative point of measurement resolution interval with the entire interval. By finite measurement resolution one cannot fix the order inside the interval. Note that one obtains a hierarchy of subgroups depending on the upper bound p^{-n} for the modulus. For $p \ mod \ 4 = 1$ imaginary unit exist as ordinary p-adic number and for $p \ mod \ 4 = 3$ in an extension including $\sqrt{-1}$.

4. For the hyperbolic angle one has

$$E_{m,n} \equiv exp(m/n) \rightarrow E_{m,n} \times exp(\eta) \tag{2.2}$$

   with the ordinary p-adic number η having norm $p^{-N} < 1$. Lorentz symmetry is broken to a discrete subgroup: this could be interpreted in terms of finite cognitive resolution. Since e^p is p-adic number also hyperbolic angle has finite number of values and one has compactness in well-defined sense although in real context one has non-compactness.

   In cosmology this discretization means quantization of redshift and thus recession velocities. A concise manner to express the discretization to say that the cosmic time constant hyperboloids are discrete variants of Lobatchevski spaces $SO(3,1)/SO(3)$. The spaces appear naturally in TGD inspired cosmology.

5. The coordinate transformation relating the coordinates in the two intersecting coordinate patches maps hyperbolic and ordinary phases to each other as such. Light-cone proper time coordinates are related in more complex manner. $a_+^2 = t^2 - r^2$ and $a_-^2 = (t-T)^2 - r^2$ are related by $a_+^2 - a_-^2 = 2tT - T^2 = 2a_+ cosh(\eta)T - T^2$.

   This leads to a problem unless one allows a_+ and a_- to belong to an algebraic extension containing the roots of e making possible to define hyperbolic angle. The coordinates $a_\pm$ can also belong to a larger extension of p-adic numbers. The expectation is that one obtains an infinite hierarchy of algebraic extensions of rationals involving besides the phases also other non-Abelian extension parameters. It would seem that the Abelian extension for phases and the extension for a must factorize somehow. Note also that the the expression of a_+ in terms of a_- given by

$$a_+ = -cosh(\eta)T \pm \sqrt{sinh^2(\eta)T^2 + a_-^2} \ . \tag{2.3}$$

This expression makes sense p-adically for all values of a_- if one can expand the square root as a converning power series with respect to a_-. This is true if $a_-/sinh(\eta)T$ has p-adic norm smaller than 1.

6. What about the boundary of CD which corresponds to a coordinate singularity? It seems that this must be treated separately. The boundary has topology $S^2 \times R_+$ and S^2 can be p-adicized as already explained. The light-like radial coordinate $r = asinh(\eta)$ vanishes identically for finite values of $sinh(\eta)$. Should one regard r as ordinary p-adic number? Or should one think that entire light-one boundary corresponds to single point $r = 0$? The discretization of r is in powers of a roots of e is very natural so that each power $E_{m,n}$ corresponds to a p-adic monad. If now powers $E_{m,n}$ are involved, one obtains just the monad at $r = 0$.

   The construction of quantum TGD leads to the introduction of powers $exp(log(r/r_0)s)$, where s is zero of Riemann Zeta [K1]. These make sense p-adically if $u = log(r/r_0)$ has p-adic norm smaller than unity and s makes sense p-adically. The latter condition demanding that the zeros are algebraic numbers is quite strong.

3 Construction for $SU(2)$, $SU(3)$, and classical Lie groups

In the following the detailed construction for $SU(2)$, $SU(3)$, and classical Lie groups will be sketched.

3.1 Subgroups of $SU(2)$ having p-adic counterparts

In the case $U(1)$ the subgroups defined by roots of unity reduce to a finite group Z_n. What can one say about p-adicizable discrete subgroups of $SU(2)$?

1. To see what happens in the case of $SU(2)$ one can write $SU(2)$ element explicitly in quaternionic matrix representation

$$(\theta, n) \equiv cos(\theta)Id + sin(\theta)\sum_i n_i I_i \ . \qquad (3.1)$$

   Here Id is quaternionic real unit and I_i are quaternionic imaginary units. $n = (n_1, n_2, n_2)$ is a unit vector representable as $(cos(\phi), sin(\phi)cos(\psi), sin(\phi)sin(\psi))$. This representation exists p-adically if the phases $exp(i\theta)$, $exp(i(\phi)$ and $exp(i\psi)$ exist p-adically so that they must be roots of unity.

   The geometric interpretation is that n defines the direction of rotation axis and θ defines the rotation angle.

2. This representation is not the most general one in p-adic context. Suppose that one has two elements of this kind characterized by (θ_i, n_i)such that the rotation axes are different. From the multiplication table of quaternions one has for the product (θ_{12}, n_{12}) of these

$$cos(\theta_{12}) \quad = \quad cos(\theta_1)cos(\theta_2) - sin(\theta_1)sin(\theta_2)n_1 \cdot n_2 \ . \qquad (3.2)$$

   This makes sense p-adically if the inner product $cos(\chi) \equiv n_1 \cdot n_2$ corresponds to root of unity in the extension of rationals used. Therefore the angle between the rotation axes is number theoretically quantized in order that p-adicization works.

   One can solve θ_{12} from the above equation in real context but in the general case it does not correspond to $U_{m,n}$. This is not however a problem from p-adic point of view. The reduction to a root of unity is true only in some special cases. For $n_1 = n_2$ the group generated by the products reduces a discrete $Z_n \subset U(1)$ generated by a root of unity. If n_1 and n_2 are orthogonal the angle between rotation axes corresponds trivially to a root of unity. In this case one has the isometries of cube. For other Platonic solids the angles between rotation axes associated with various $U(1)$ subgroups generating the entire sub-group are fixed by their geometries. The rotation angles correspond to $n = 3$ for tetrahedron and icosahedron and $n = 5$ dodecahedron and for $n = 3$. There is also duality between cube and octahedron and icosahedron and dodecahedron.

3. Platonic solids can be geometrically seen as discretized variants of $SU(2)$ and it seems that they correspond to finite discrete subgroups of $SU(2)$ defining $SU(2)_d$. Platonic sub-groups appear in the hierarchy of Jones inclusions. The other finite subgroups of $SU(2)$ appearing in this hierarchy act on polygons of plane and being generated by Z_n and rotations around the axes of plane and would naturally correspond to discrete U(1) sub-groups of $SU(2)$ and in a well-defined sense to a degenerate situation. By Mc-Kay correspondence all these groups correspond to ADE type Lie groups. These subgroups define finite discretizations of $SU(2)$ and S^2. p-Adicization would lead directly to the hierarchy of inclusions assigned also with the hierarchy of sub-algebras of super-symplectic algebra characterized by the hierarchy of Planck constants.

4. There are also p-adicizable discrete subgroups, which are infinite. By taking two rotations with angles which correspond to root of unity with rotation axes, whose mutual angle corresponds to root of unity one can generate an infinite discrete subgroup of $SU(2)$ existing in p-adic sense. More general discrete U(1) subgroups are obtained by taking n rotation axes with mutual angles corresponding to roots of unity and generating the subgroup from these. In case of Platonic solids this gives a finite subgroup.

3.2 Construction of p-adicizable discrete subgroups of CP_2

The construction of p-adic CP_2 proceeds along similar lines.

1. In the original ultra-naive approach the local p-adic metric of CP_2 is obtained by a purely formal replacement of the ordinary metric of CP_2 with its p-adic counterpart and it defines the CP_2 contribution to induced metric. This makes sense since Kähler function is rational function and components of CP_2 metric and spinor connection are rational functions. This allows to formulate p-adic variants of field equations. This description is however only local. It says nothing about global aspects of CP_2 related to the introduction of algebraic extension of p-adic numbers.

   One should be able to realize the angle coordinates of CP_2 in a physically acceptable manner. The coordinates of CP_2 can be expressed by compactness in terms of trigonometric functions, which suggests a realization of them as phases for the roots of unity. The number of points depends on the Abelian extension of rationals inducing that of p-adics which is chosen. This gives however only discrete version of p-adic CP_2 serving as a kind of spine. Also the flesh replacing points with monads is needed.

2. A more profound approach constructs the algebraic variants of CP_2 as discrete versions of the coset space $CP_2 = SU(3)/U(2)$. One restricts the consideration to an algebraic subgroup of $SU(3)_d$ with elements, which are 3×3 matrices with components, which are algebraic numbers in the extension of rationals. Since they are expressible in terms of phases one can express them in terms of roots of unity. In the same manner one identifies $U(2)_d \subset SU(3)_d$. $CP_{2,d}$ is the coset space $SU(3)_D/U(2)_d$ of these. The representative of a given coset is a point in the coset and expressible in terms of roots of unity.

3. The construction of the p-adicizable subgroups of $SU(3)$ suggests a generalization. Since $SU(3)$ is 8-D and Cartan algebra is 2-D the coset space is 6-dimensional flag-manifold $F = SU(3)/U(1) \times U(1)$ with coset consisting of elements related by automorphism $g \equiv hgh^{-1}$. F defines the twistor space of CP_2 characerizing the choices for the quantization axes of color quantum numbers. The points of F should be expressible in terms of phase angles analogous to the angle defining rotation axis in the case of SU(2).

   In the case of $SU(2)$ n U(1) subgroups with specified rotation axes with p-adically existing mutual angles are considered. The construction as such generates only $SU(2)_d$ subgroup which can be trivially extended to $U(2)_d$. The challenge is to proceed further.

Cartan decomposition of the Lie algebra (see https://en.wikipedia.org/wiki/Cartan_decomposition) seems to provide a solution to the problem. In the case of $SU(3)$ it corresponds to the decomposition to U(2) sub-algebra and its complement. One could use the decomposition $G = KAK$ where K is maximal compact subgroup. A is exponentiation of the maximal Abelian subalgebra, which is 3-dimensional for CP_2. By Abelianity the p-adicization of A in terms of roots of unity simple. The image of A in G/K is totally geodesic sub-manifold. In the recent case one has $G/Ki = CP_2$ so that the image of A is geodesic sphere S^2. This decomposition implies the representation using roots of unity. The construction of discrete p-adicizable subgroups of $SU(n)$ for $n > 3$ would continue iteratively.

4. Since the construction starts from $SU(2)$, $U(1)$, and Abelian groups, and proceeds iteratively it seems that Platonic solids have counterparts for all classical Lie groups containing $SU(2)$. Also level p-adicizable discrete coset spaces have analogous of Platonic solids.

The results imply that $CD \times CP_2$ is replaced by a discrete set of p-adic monads at a given level of hierarchy corresponding to the finite cognitive resolution.

3.3 Generalization to other groups

The above argument demonstrates that p-adicization works iteratively for $SU(n)$ and thus for $U(n)$. For finite-dimensional symplectic group $Sp(n, R)$ the maximal compact sub-group is $U(n)$ so that that KAK construction should work also now. $SO(n)$ can be regarded as subgroup of $SU(n)$ so that the p-adiced discretrized variants of maximal compact subgroups should be constructible and KAK give the groups. The inspection of the table of the Wikipedia article (see https://en.wikipedia.org/wiki/Classical_group) encourages the conjecture that the construction of $SU(n)$ and $U(n)$ generalizes to all classical Lie groups.

This construction could simplify enormously also the p-adicization of WCW and the theory would discretize even in non-compact degrees of freedom. The non-zero modes of WCW correspond to the symplectic group for $\delta M^4 \times CP_2$, and one might hope that the p-adicization works also at the limit of infinite-dimensional symplectic group with $U(\infty)$ taking the role of K.

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