

Langlands Program and TGD: Years Later

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Abstract

Langlands correspondence is for mathematics what unified theories are for physics. The number theoretic vision about TGD has intriguing resemblances with number theoretic Langlands program. There is also geometric variant of Langlands program. I am of course amateur and do not have grasp about the mathematical technicalities and can only try to understand the general ideas and related them to those behind TGD. Physics as geometry of WCW ("world of classical worlds") and physics as generalized number theory are the two visions about quantum TGD: this division brings in mind geometric and number theoretic Langlands programs. This motivates re-consideration of Langlands program from TGD point of view. I have written years ago a chapter about this earlier but TGD has evolved considerably since then so that it is time for a second attempt to understand what Langlands is about.

By Langlands correspondence the representations of $G \rtimes Gal$ and G should correspond to each other. This suggests that the representations of G should have G-spin such that the dimension of this representation is same as the representation of non-commutative Galois group. This would conform with the vision about physics as generalized number theory. Could this be the really deep physical content of Langlands correspondence?

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1 Introduction

Langlands correspondence is for mathematics what unified theories are for physics. The number theoretic vision about TGD has intriguing resemblances with number theoretic Langlands program [A7, A2] (see <http://tinyurl.com/z6tew2e>). There is also geometric variant of Langlands program [A3, A1, A4, A6] (see https://en.wikipedia.org/wiki/Geometric_Langlands_correspondence). I am of course amateur and do not have grasp about the mathematical technicalities and can only try to understand the general ideas and related them to those behind TGD. Physics as geometry of WCW ("world of classical worlds") and physics as generalized number theory are the two visions about quantum TGD: this division brings in mind geometric and number theoretic Langlands programs. This motivates re-consideration of Langlands program from TGD point of view. I have written years ago a chapter about this [K4] but TGD has evolved considerably since then so that it is time for a second attempt to understand what Langlands is about.

1.1 Langlands program briefly

The basic concept in number theoretical Langlands program is algebraic extension L/Q of rational numbers Q and more generally, an extension L/K of algebraic extension of Q called global number field. K can denote also other number fields. If K corresponds to reals or complex numbers or to p-adic numbers or their extension, it is called local. Also extensions of finite fields and function fields can be considered. Already gives idea about the generality of Langlands program.

1. Algebraic extension of rational numbers can be constructed by finding the roots of an irreducible n :th order monic polynomial of real argument (coefficients are integers and the coefficients of the highest power is unity so that modulo p reduction conserves the degree) see <http://tinyurl.com/gwrhgate>) and extending Q by them so that one obtains algebraically n -dimensional number field as an algebraic extension of Q . Denote the extension of rationals Q defined by irreducible polynomial P by L . Galois group $Gal(L/K)$ consists of the automorphisms of this structure mapping sums into sums, products into products, and rationals of K into rationals and its order is the dimension of the extension.

One can combine several extensions of this kind by extending with corresponding roots and can construct algebraic numbers by combining all extensions of this kind. The Galois group of algebraic numbers is known as absolute Galois group and enormously complex. Absolute Galois group Gal_{abs} (see <http://tinyurl.com/gvcywrns>) has the Galois groups $Gal(L/K)$ of irreducible polynomials as subgroups.

2. Algebraic numbers have infinite algebraic dimension and can be regarded as an extension of any global field K and has factor groups $Gal_{abs}/Gal(K)$ as Galois group. One has restriction homomorphisms from $(Gal_{abs}/Gal(K))Gal(K)$ to $Gal(K)$ and imbedding homomorphisms of $Gal(K)$ to Gal_{abs} . One can construct representations of Galois groups in various groups such as classical Lie groups and algebraic groups and this kind of representations give information about number theory. The distinctions between Lie groups and algebraic groups are very delicate and not of practical significance for a physicist.

The term algebraic matrix group G tells that the matrices satisfy some algebraic conditions specifying a subgroup of general linear group. One can specify the number field for matrix elements by using the notion $G(K)$. In TGD framework discrete subgroups of matrix groups with values in algebraic extension of rationals are highly interesting.

3. Langlands program extends also the ring of integers associated with global number field to the ring of adeles (see <http://tinyurl.com/gt6j9me>) associated with global number field K inducing extensions of p-adic number fields. Adeles correspond to the Cartesian product of non-vanishing positive reals R_+ and of the p-adic integers for the algebraic extensions of p-adic number fields induced by K . Adeles contain as a multiplicative subgroup the group

of ideles, which apart from finite number of exceptional primes have p-adic norm equal to 1. This is essential for the existence of non-vanishing multiplicative inverse of adèle.

The great vision of Langlands resting on the work carried out by number theorists during centuries is that there is a deep connection between number theory and representation theory for Lie groups and reductive algebraic groups. Originally groups $GL(n)$ were considered already by Artin as providing representations of non-Abelian Galois groups but Langlands proposed a generalization to reductive algebraic groups. To my best - not so impressive - understanding both classical Lie groups and algebraic groups are reductive.

By Langlands correspondence the representations of $G \times Gal$ and G should correspond to each other. The analogy with the representations of Lorentz group suggests that the representations of G should have “spin” for some compact subgroup of G acting from left or right such that the dimension of this representation is same as the representation of non-commutative Galois group.

Automorphic functions are indeed typically functions in G , which reduce to a function invariant under left and/or right action of a compact or even discrete subgroups H_1 and H_2 or more generally, belong to a finite-dimensional unitary representation of $H_1 \times H_2$ in $H_1 \backslash G / H_2$. Therefore they can be said to have $H_1 \times H_2$ quantum numbers analogous to spin if interpreted as “field modes” in the space of double cosets $H_1 g H_2$. This would conform with the vision about physics as generalized number theory. If I have understood correctly, the question is whether a finite-dimensional representation of H_1 or H_2 could correspond to a finite-dimensional representation of Galois group at the number theory side.

Langlands formulated a correspondence between so called **a**) admissible infinite-dimensional automorphic representations for a reductive group $G(K)$ and **b**) representations of Galois groups in its Langlands dual $G_L(C)$ (complex non-compact group). Infinite-dimensionality requires non-compactness for $G(R)$ since compact groups have only finite-dimensional unitary irreducible representations. Here K is either local (archimedean (real or complex) or non-archimedean (p-adic number field or its extension) or global number field (algebraic extensions of rationals) so that the approach is extremely general.

Archimedean fields represent relatively simple situation. Non-archimedean fields are much more difficult and global fields extremely difficult and to my understanding very few proofs exist. For algebraic extension of rationals adèle ring is obtained as Cartesian product of p-adic integers with extension induced by the extension of rationals. If K is itself non-Archimedean field, the notion of adèle ring does not seem to make sense as such: should the extension define an extension of rationals in turn inducing an extension of other p-dic number fields?

1.2 A modest attempt for an overview

I try to give an overall view about Langlands conjecture.

1. G is reductive group (includes semisimple Lie groups) in given algebraic extension K of rationals, and can be extended to adelic group $G(A)$, where A denotes the adèle formed by non-vanishing reals and integers for extensions of p-adic number fields induced by K . $G_L(C)$ is complex group and provides a representation of Galois group of K : one speaks of homomorphisms of Galois group to $G_L(C)$.
2. Langlands started from the representations of Galois group in group $GL(n, K)$ and later generalized to arbitrary reductive Lie group $G(K)$. Here K is arbitrary number field, which could be global number field (algebraic extension of rationals) or real or complex variant of G or a variant of G for p-adic number field or its extension induced by algebraic extension of rationals. The representations in real and p-adic number fields are combined to adelic representation and could be seen as infinite tensor product. For global number fields $G(K)$ (extensions of rationals) is discrete and does not allow the analytic machinery requiring Lie groups: just these are of special interest in TGD framework.
3. Since $G(K)$ is discrete for global fields K , one wants to simplify things by replacing K with what is called separable closure \bar{K} of K analogous to complex numbers. This also allows to have infinite-dimensional representations. $G(\bar{K})$ allows Lie-group and Lie-algebra structure so that the machinery of Lie algebras can be used.

One can assign Galois group $Gal(\overline{K}/K)$ to the extension of K to \overline{K} . If K is a finite-dimensional extension of rationals this Galois group (absolute Galois group) is extremely complex object and is known to possess topology highly reminiscent of p-adic topologies. \overline{K} corresponds to complex algebraic numbers for the algebraic extensions of rationals. For p-adic number fields the fact that all polynomials effectively reduce to polynomials of degree not larger than $p-1$, \overline{K} and $Gal(\overline{K}/K)$ are considerably simpler entities (see <http://tinyurl.com/mkqhp5n>). The transition to \overline{K} does not delete the information about K also the adèle structure keeps information about K .

4. In $G(\overline{K})$ one can speak about Lie algebra and its root system. One assigns to this root system a co-root system and in terms of it defines the connected component $G_L^0(C)$ of Langlands dual as a complex group. To keep information about the algebraic extension, one extends $G_L^0(C)$ to the semi-direct product $G_L^0(C) \times Gal(K)$. The Galois group of finite-dimensional extension K acting appears and preserves information about the extension. It would seem that the representations of this group must be constructed from products of representations of $Gal(K)$ and $GL^0(C)$ so that additional discrete degrees of freedom appear. Kind of Galois covering of $G_L^0(C)$ serves as Langlands dual for $G(\overline{K})$.
5. This correspondence involves reductive algebraic group G and its Langlands dual G_L interpreted as complex group (see <http://tinyurl.com/zts4rqf>). G_L has as its roots co-roots of G :

$$\alpha \rightarrow \alpha' = 2\alpha/(\alpha, \alpha)$$

so that the dimension of Cartan algebra and number of roots is same but the angles between some roots have changed:

$$(\alpha', \beta') = 4(\alpha, \beta)/(\alpha, \alpha)(\beta, \beta) .$$

All simply laced Lie groups (ADE groups) with $(\alpha, \alpha) = 2$ are self-dual as also G_2 and F_4 and $GL(n)$.

The root systems B_n and C_n are mapped to each other so that $SO(2N+1)$ is dual to $Sp(N)$ whereas $SO(2n)$ is self dual as D_n type group. Connected Lie groups are dual to adjoint type Lie groups: for instance $SU(N)$ is dual to $SU(N)/Z_n$. One could try to understand the complexification of the dual from the fact that the natural representation of the roots of polynomial is as points of complex plane and Galois group therefore naturally acts in complex plane. Why the type of the group is changed looks however mysterious.

6. Information about K is not lost at the group theory side since adèle group contains information about K . Also the separable closure \overline{K} for p-adic number fields and their extensions is not equal to the algebraic closure since separable closure contains only separable extensions (minimal polynomial has only roots with multiplicity one).

Langlands conjecture states that the automorphic forms - so called Artin's L-functions - defined by the homomorphisms from Galois group $Gal(K)$ to $G_L^0(C)$ extended to a semi-direct product with the Galois or its modification Weil group (see <http://tinyurl.com/hk74sw7>) to be distinguished from Weyl group in Lie-algebra theory co-incide with the automorphic forms assignable to "good" representations of $G(\overline{K})$, which correspond to group theory side of the duality - group theoretic L-functions.

Connections of Langlands program with physics have been found already at the level of gauge theories and in string models. Electric-magnetic duality discovered by Montonen involves gauge group and its Langlands dual and there are reasons to expect that electric-magnetic duality - weak form of electric-magnetic self-duality in TGD framework [K2] - could have important implications for the understanding Langlands duality.

Witten, Frenkel and many other leading mathematicians and theoretical physicists have been developing geometric Langlands program [A3, A1, A4, A6]. Geometric Langlands is considerably simpler (simplicity is relative notion here!) than its number theoretical counterpart since the

monstrous automorphism group of algebraic numbers (by definition mapping products to products and sums to sums) with the fundamental group of Riemann surface with punctures. Kac-Moody algebras and the monodromy groups as representations of fundamental group of Riemann surface are essentially involved.

1.3 Why number theoretic vision about TGD could have something to do with Langlands program?

Due to the technicalities involved it is impossible for a physicist like me to understand Langlands program at technical level. TGD is however proposed to be a unified theory of physics and it would not be surprising if some connections would exist.

1. The number theoretic universality [K15] is one of the basic principles of TGD with motivations coming from both p-adic mass calculations [K12] and mathematical description of cognition in TGD inspired theory of consciousness [K6, K1]. This principle states that physics is adelic and the physics in real and various p-adic sectors is obtained by a kind of analytic continuation from physics for algebraic extensions of rationals. The analogy with Langlands program is obvious and suggests strongly a connection with number theoretic Langlands.
2. In TGD framework Kac-Moody algebras generalize to super-symplectic algebra, which is immensely more complex than Kac-Moody algebras and has strong number theoretic flavor (for instance, conformal weights could relate closely to the zeros of Riemann zeta). Could super-symplectic algebra be for number theoretic Langlands what Kac-Moody is for geometric Langlands (see <http://tinyurl.com/j7tdho6> and <http://tinyurl.com/zj81f2w>)?
3. Discretizations based on algebraic extension are a corner stone of TGD view about space-time relying on the notion of finite measurement resolution. Discretization means replacement of Lie group G by finite discrete subgroup assignable to algebraic extension K of rationals. The discretization are at the level of imbedding space and their existence as coset spaces relies heavily on the symmetries of imbedding space.

One can perform completion for the points of discretization to what one might call monads [L1]. In real context they are analogous to the open sets defining charts of manifold. In p-adic sectors monads are disjoint and consist of p-adic integers. The field equations for Kähler action (or its modification suggested by twistorialization containing extremely small volume term) are satisfied inside monads.

Galois group of K act as dynamical symmetry group transforming discretizations to each other so that one has kind of covering space structure at the level of WCW with sheets correspond to points of Galois group. This suggests that the counterparts of symmetries with elements in the extensions of rationals combined to semi-direct product with Galois group are crucial in TGD and that Galois groups act as symmetry groups having action very similar to that for fundamental groups.

Note that also the isometry group G of imbedding space restricted to $G(K)$ acts as discrete symmetries so that space-time surfaces (and 3-surfaces at boundaries of causal diamonds, string world sheets, and partonic 2-surfaces) provide a representation space for these groups. $G(K)$ act also on the induced spinor fields which can be assumed to have components in K (or \overline{K}).

4. The geometric realization for the hierarchy of Planck constants [K13] is proposed to be in terms of coverings of space-time surfaces for which ends at the boundaries of CD correspond to singular covering with all sheets co-inciding. Could Galois group define this covering. This would require that Galois maps the discretization of 3-surface to itself at boundaries of CD. The stronger condition that it maps the ends points to itself seems too strong.

A further conjecture is that the hierarchy of Planck constants corresponds to the hierarchy of inclusions of hyperfinite factors (HFFs) having canonical representation in terms of second quantized induced spinors needed to define WCW gamma matrices and WCW spinors. The

inclusions are known to correspond discrete subgroups of $SU(2)$ and labelled by ADE diagrams, which by McKay correspondence correspond to Dynkin diagrams for ADE type Kac-Moody groups (see <http://tinyurl.com/jyjplzc>). The conjecture is that the Kac-Moody groups form a hierarchy of dynamical symmetries as remnants of symplectic symmetries due the infinite number of conditions stating the vanishing for a subset of symplectic Noether charges. These would be self-dual under Langlands duality.

Since the representations of $G \rtimes Gal$ and G should correspond to each other, the representations of G should have G -spin such that the dimension of this representation is same as the representation of non-commutative Galois group. This would conform with the vision about physics as generalized number theory. Could this be the really deep physical content of Langlands correspondence?

2 More detailed view about Langlands correspondence

Langlands correspondence [A7, A2] (see <http://tinyurl.com/z6tew2e>) has group theoretical and number theoretical sides and in the following I try to summarize what I have vaguely understood about these aspects.

2.1 Group theory side of Langlands correspondence

Consider first the group theory side. I want to confess that the following explanations are just a collection of physicist's impressions and probably too much for the patience of mathematician.

First the view of physicist about what the representations of $G(\overline{K})$ might be.

1. These groups have representations defined by functions in some complex analytic manifolds (say complex groups) and more general representations involving the analog of classical field representing particle with spin which are defined in Minkowski space and so that the action of Lorentz group $G = SO(1, 3)$ on field is well-defined and spin characterizes the representation of field under rotation group $SO(3) \subset SO(1, 3)$. The field corresponds to well defined mass and satisfied d'Alembert equation representing Casimir operator for $SO(1, 3)$. At the level of momentum space one has representation of $SO(1, 3)$ at mass shell, that is coset space $H_3 = SO(1, 3)/SO(3)$, 3-D hyperbolic space.

More generally, the field can live in group manifold G or its coset space G/H and have spin in the sense that this field transforms as finite-dimensional representation of a sub-group $H \subset G$. The so called automorphic representations are in question: the action of group element $h \in H$ to the field $f(g)$ is given by $f(hg) = D_h(g)f(g)$. Here $D_h(g)$ is finite-D representation matrix which is easily found to satisfy so called co-cycle property: $D_{h_1 h_2}(g) = D_{h_1}(h_2 g) D_{h_2}(g)$. For 1- representations this equation holds for functions defining Abelian representation of H . Also now the analog of d'Alembert equation satisfied by free particle in field theory is assumed: one has eigenfunctions of the Casimir operator of the group: this requires that one consider Lie group. The interpretation would be that one has spinning particle in the coset space G/H .

2. The trace of the representation matrix $D_h(g)$ as function of group element is a fundamental character of the representation invariant under automorphisms $h \rightarrow ghg^{-1}$ of the group and is known as character of group representation. For instance, for rotation group character depends on rotation angle only, not on the direction of the rotation axis. Now the matrix $D_h(e)$ defines a character as a function in sub-group H which can be discrete.

Automorphic forms characterize the group representations in question. The following definition from Wikipedia (see <http://tinyurl.com/gquturl> and <http://tinyurl.com/hsy8ewf>) resembles the description anticipated above except that I am not sure whether G -spin is allowed or whether only the analogs of scalar fields are considered.

Suppose f is function in complex manifold X in which group Γ acts. f is automorphic form if one has

$$f(\gamma(x)) = j_\gamma(x)f(x) ,$$

where $j_\gamma(x)$ is everywhere non-vanishing holomorphic function called factor of automorphy. Factor of automorphy is cocycle for the action of G meaning that one has from the definition

$$j_{\gamma_1\gamma_2}(x) = j_{\gamma_1}(\gamma_2(x))j_{\gamma_2}(x) .$$

Product of automorphic forms is automorphic with factor of automorphy given by the product of the factors. Automorphic forms form a vector space for a given factor of automorphy. If Γ is a lattice in Lie group then factor of automorphy for Γ corresponds to a line bundle on the quotient G/Γ . For instance, Γ can be a subgroup of $SL(2, R)$ acting on upper half complex plane. One can generalize the definition by replacing complex functions f with vector valued functions. In this case j corresponds to a representation matrix for Γ .

The complex analytic manifold X is often topological group G having Γ as its discrete subgroup. Hence automorphic form corresponds to a collection of functions $j_\gamma(g)$ of functions in G . As a special case one obtains modular forms for $PSL(2, R)$ and Γ a modular subgroup $PSL(2, Z)$ or one of its congruence subgroups with diagonal elements 1 modulo prime and diagonal elements zero modulo prime. In adelic approach these congruence subgroup can be treated at once using adeles.

Automorphic form could be at least formally defined also as a vector valued function f in G . Components of vector can be said to define analogs of component of a field with G -“spin”. In the case of non-compact groups this representation would be by its finite dimension non-unitary but in principle this is possible (the unitary representations of Poincare group with spin are good example).

1. The vector transforms under $\gamma \in \Gamma$ according to a given factor j of automorphy which is matrix in general case. I do not know whether it is allowed to be matrix in case of non-Abelian Galois groups.
2. It is an eigenfunction of Casimir operators of G .
3. Satisfies some conditions on growth at infinity.

Automorphic functions can be defined in terms of Hecke characters (the analogy with Riemann zeta) and Hecke characters can in turn be defined for the unitary representations of group G , which is in general non-compact. The basic idea is to start from the representation of finite and compact groups in terms of group algebra endowed with sum (quantum superpositions of wave functions in group) and convolution (product induced by group product) and generalize to non-compact case. One can also require invariance under left and/or right action by some sub-group so that one obtains functions in coset spaces $H_1 \backslash G / H_2$. One can consider functions in G , which are invariant under the left action of H_1 and right action of H_2 . More generally, the functions could belong to irreducible unitary representations of H_1 and H_2 - physicist would perhaps say that the classical field “field” in double coset space has H_1 and H_2 “spin”. Obviously the number of possibilities is endless.

In the simplest case these functions are constant in doublet cosets $H_1 g H_2$ and one can construct them by taking a function $f(g)$ in G and forming a sum of the values $f(g_1 g h_2)$ normalized suitably to give a kind of averaging. If the group in question is continuous group one can perform integration using left/right-invariant Haar measure. One can identify the action of Hecke operator as the formation of this average and identify eigen functions and eigen values of Hecke operator. One can generalize the Hecke operator to an operator producing function that belongs to a representation of $H_1 \times H_2$ and defining also now eigenfunctions. This leads an elegant mathematics. The upper complex plane identifiable as $SL(2, R)/SO(2)$ defines a coset space and posing left invariance of a complex analytic function $f(z)$ under $SL(2, Z)$ or its subgroup acting as Möbius transformations one obtains Hecke operators and Hecke characters defining examples of automorphic functions. The coefficients of the Fourier expansion of eigen function are eigen values of Hecke operator.

The group $SL(2, C)$ - double covering of Lorentz group $SO(1, 3)$ is of special interested both number theoretically and geometrically. In this case the group H is typically discrete subgroup Γ of $SL(2, C)$ and the coset space $\Gamma \backslash SL(2, C) / SU(2)$. In this case the “spin” could correspond to a finite-D representation of Γ , which should be unitary. There are additional more technical conditions to be satisfied for the representation to be unitary. Often non-compact groups such as $GL(n, F)$ for an arbitrary algebraic number field is considered. Algebraic extensions of rationals, p-adic number fields, reals, complex numbers. The generality of the approach is stunning.

2.2 Number theoretical side of Langlands correspondence

On the number theoretic side the challenge is to find representations of Galois groups and their extensions to Weil groups. Also these lead to the notion of automorphic function. Here I can only give some notices about the historical development of the ideas leading to the vision of Langlands.

1. The story begins from the study of the simplest possible algebraic extensions defined by root of integer and characterized by this integer, call it n . These extensions are known as quadratic extensions and have Abelian Galois group consisting of 2 elements. One can generalize the notions of integer and prime to corresponding ideals for any algebraic extensions and the general phenomenon is that rational prime (ideals) can either stable, split to a product of different prime ideals of the extension, or ramify in which case higher powers of prime ideals of extension can appear. For instance, in the extension $Q(\sqrt{-1})$ number 2 ramifies to $(2) = (1+i)^2$ (note that 2 and $2i$ differing by unit are equivalent as representatives of ideal), primes $p \pmod{4} = 1$ split and primes $p \pmod{4} = 3$ are stable.

The physical analogy for splitting is that proton as elementary particle is in improved resolution a bound state of 3 quarks.

2. Quadratic reciprocity (see <http://tinyurl.com/njpnx69>) can be seen starting point of the developments leading to Langlands conjecture. For instance, Euler, Legendre, and Gauss have made contributions here. One considers the question when prime q is square modulo prime p that is quadratic residue modulo p : $q = x^2 \pmod{p}$, prime. Define Legendre symbol (p/q) to be 1 if q is quadratic residue modulo p and -1 if this not the case. Quadratic reciprocity states

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}.$$

This law allows to relate (p/q) and (q/p) in the four cases corresponding to $p \pmod{4} \in \{1, 3\}$, $q \pmod{4} \in \{1, 3\}$. Legendre symbol is relevant for quadratic extensions of rationals since its value tells whether a given prime q ramifies in p -adic number field Q_p or not.

Quadratic reciprocity generalizes to cubic, quadratic, quartic reciprocities and Eisenstein reciprocity (see <http://tinyurl.com/huxm68w> generalizes this law to higher powers. There is also reciprocity theorem for cyclotomic extensions (see <http://tinyurl.com/z43cb5u> and <http://tinyurl.com/gm3sbzj>) which are Abelian as also quadratic extensions. Artin's reciprocity (see <http://tinyurl.com/j8ngckh>) is a further generalization.

The next step was the emergence of class field theory applying to Abelian extensions L/K of global field K . The goal was to describe L/K in terms of arithmetics of K : this includes finite Abelian extensions of KK , realization of $Gal(L/K)$ and describe the decomposition of prime ideal from K to L (see <http://tinyurl.com/z3s4kjn>). Local number fields integrated into adèle provide the needed tool by reducing the arithmetics to modulo p arithmetics. This can be seen as an application of Hasse principle (see <http://tinyurl.com/jkh3auq>).

1. A typical problem is the splitting of primes of K to primes of the extension L/K which has been already described. One would like to understand what happens for a given prime in terms of information about K . The splitting problem can be formulated also for the extensions of the local fields associated with K induced by L/K .
2. Consider what happens to a prime ideal p of K in L/K . In general p decomposes to product $p = \prod_{i=1}^g P_i^{e_i}$ of powers of prime ideals P_i of L . For $e_i > 1$ ramification is said to occur. The finite field K/p is naturally imbeddable to the finite field L/P_j defining its extension. The degree of the residue field extension $(L/P_i)/(K/p)$ is denoted by f_i and called inertia degree of P_i over p . The degree of L/K equals to $[L : K] = \sum e_i f_j$.

If the extension is Galois extension (see <http://tinyurl.com/zu5ey96>), one has $e_i = e$ and $f_i = f$ giving $[L : K] = efg$. The subgroups of Galois group $Gal(L/K)$ known as decomposition group D_i and inertia group I_i are important. The Galois group of F_i/F equals to D_i/I_i .

For Galois extension the Galois group $Gal(L/K)$ leaving p invariant acts transitively on the factors P_i permuting them with each other. Decomposition group D_i is defined as the subgroup of $Gal(L/K)$ taking P_i to itself.

The subgroup of $Gal(L/K)$ inducing identity isomorphism of P_i is called inertia group I_i and is independent of i . I_i induces automorphism of $F_i = L/P_i$. $Gal(F_i/F)$ is isomorphic to D_i/I_i . The orders of I_i and D_i are e and ef respectively. The theory of Frobenius elements identifies the element of $Gal(F_i/F) = D_i/I_i$ as generator of cyclic group $Gal(F_i/F)$ for the finite field extension F_i/F . Frobenius element can be represented and defines a character.

3. Quadratic extensions $Q(\sqrt{n})$ are simplest Abelian extensions and serve as a good starting point (see <http://tinyurl.com/zofhmb8>) the discriminant $D = n$ for $p \bmod 4 = 1$ and $D = 4n$ otherwise characterizes splitting and ramification. Odd prime p of the extension not dividing D splits if and only if D quadratic residue modulo p . p ramifies if D is divisible by p . Also the theorem by Kronecker and Weber stating that every Abelian extension is contained in cyclotomic extension of Q is a helpful result (cyclotomic polynomials has as its roots all n roots of unity for given n)

Even in quadratic extensions L of K the decomposition of ideal of K to a product of those of extension need not be unique so that the notion of prime generalized to that of prime ideal becomes problematic. This requires a further generalization. One ends up with the notion of ideal class group (see <http://tinyurl.com/hasyllh>): two fractional ideals I_1 and I_2 of L are equivalent if there are elements a and b such that $aI_1 = bI_2$. For instance, if given prime of K has two non-equivalent decompositions $p = \pi_1\pi_2$ and $p = \pi_3\pi_4$ of prime ideal p associated with K to prime ideals associated with L , then π_2 and π_3 are equivalent in this sense with $a = \pi_1$ and $b = \pi_4$. The classes form a group J_K with principal ideals defining the unit element with product defined in terms of the union of product of ideals in classes (some products can be identical). Factorization is non-unique if the factor J_K/P_K - ideal class group - is non-trivial group. $Q(\sqrt{-5})$ gives a representative example about non-unique factorization: $2 \times 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ (the norms are 4×9 and 6×6 for the two factorizations so that they cannot be equivalent).

This leads to class field theory (see <http://tinyurl.com/zdnw7j3> and <http://tinyurl.com/z3s4kjn>).

1. In class field theory one considers Abelian extensions with Abelian Galois group. The theory provides a one-to-one correspondence between finite abelian extensions of a fixed global field K and appropriate classes of ideals of K or open sub-groups of the idele class group of K . For example, the Hilbert class field, which is the maximal unramified abelian extension of K , corresponds to a very special class of ideals for K .
2. Class field theory introduces the adèle formed by reals and p -adic number fields Q_p or their extensions induced by algebraic extension of rationals. The motivation is that the very tough problem for global field K (algebraic extension of rationals) defines much simpler problems for the local fields Q_p and the information given by them allows to deduce information about K . This because the polynomials of order n in K reduce effectively to polynomials of order $n \bmod p^k$ in Q_p if the coefficients of the polynomial are smaller than p^k . One reduces monic irreducible polynomial f characterizing extension of Q to a polynomial in finite field F_p . This allows to find the extension Q_p induced by f .

An irreducible polynomial in global field need not be irreducible finite field and therefore can have multiple roots: this corresponds to a ramification. One identifies the primes p for which complete splitting (splitting to first ordinary monomials) occurs as unramified primes.

3. Class field theory also includes a reciprocity homomorphism, which acts from the idele class group of a global field K , i.e. the quotient of the ideles by the multiplicative group of K , to the Galois group of the maximal abelian extension of K . Wikipedia article masks the statement “Each open subgroup of the idele class group of K is the image with respect to the norm map from the corresponding class field extension down to K ”. Unfortunately, the content of this statement is difficult to comprehend with physicist’s background in number theory.

Number theoretic Langlands program is the next step in the process and could be seen as an extension of class field theory to the case of non-Abelian extensions. The following must be understood as an attempt of a physicist to understand what is involved. In my attempts to understand the formulas a valuable guideline is that they should involve only information about the number field K . Hecke character and L-function defined by Dirichlet series are basic notions besides notions of ideal generalizing of the notion of integer, and the notions of adèle and idele (invertible adèle). I must admit that I am still unable to understand how reciprocity theorems identifying two kinds of characters lead to the concrete form of reciprocity.

1. Hecke character (see <http://tinyurl.com/hxg6l9e>) is a generalization of Dirichlet character for Z/kZ (see <http://tinyurl.com/jqtp5cv>) giving rise to Dirichlet L-functions (see <http://tinyurl.com/zsssrms>) generalizing Riemann Zeta and defined as

$$L(\chi, s) = \sum_{n>0} \chi(n)n^{-s} .$$

Hecke character is defined for idele class group rather than Galois group and can be seen as a character of idele group trivial in principal ideles. The conductor of Hecke character χ is defined as the largest ideal m such that χ is a Hecke character mod m .

The L-function associated with the Hecke character is an analog of Riemann zeta. There is sum over ideals not divided by m and weighted by Hecke character analogous to that over integers in Riemann zeta and its variants. The number $n > 0$ in Riemann zeta is replaced by the ideal norm $N(I)$ of ideal I , which is the finite size of the quotient ring R/I , where R is the ring of integers associated with K . One sums only over ideals not divisible by m . Hence the formula for the Dirichlet series defining L-function reads as

$$L(\chi, s) = \sum_{(I,m)=1} \chi(I)N(I)^{-s} . \quad (2.1)$$

Note that the character could be replaced with a character defined for the adelic extensions of group and the L-function also now carries information about ideles and therefore about K .

2. Already Artin's reciprocity (see <http://tinyurl.com/j8ngckh>) introduced the representations of group $GL(1, F)$, where F is global or local field. Artin proved that the L-functions associated with the characters of Galois group and with ideal class group were identical. The homomorphisms of Abelian Galois group to $GL(1, C)$ define so called Artin's L-functions in (analogous to Riemann zeta) in terms of characters of Galois group. These make sense also for non-Abelian extensions. Hecke characters defined as characters for the representations of the ideal class group give rise to the generalizations of Dirichlet L-functions analogous to Riemann zeta. Artin's reciprocity states that these two kind of L-functions are identical. For non-Abelian extensions higher-dimensional representation of Galois group are possible and this inspires the idea the introduction of $Gl(n, C)$ and is higher-D representations defining L-functions as so called automorphic forms.
3. Langlands conjecture (see <http://tinyurl.com/mkqhp5n>) generalizes Artin's approach to non-Abelian case. This requires non-Abelian infinite-dimensional representations possible for $Gl(n, F)$ and the theory of infinite-dimensional group representations becomes a tool of number theorist.

Langlands generalizes $Gl(n, F)$ to arbitrary reductive algebraic groups $G(F)$ and extends these groups to their adelic variants $G(A)$ bringing in ideles appearing also in Artin's L-function associated with the homomorphisms of Galois (Weil) group to the non-abelian case. These give rise to Artin's to L-functions for the semi-direct product of the dual G_L with Galois (Weil group) and the conjecture is that the automorphic forms for G for admissible representations co-incide with these.

The characters of the idele group are replaced with those for the “good” automorphic representations $G(\bar{K})$ defined by the Eq. 2.1. The summation over ideals of K follows automatically from the fact that the representations are for the adelic variant of G . It carries also information about Weyl group since one considers separable closures.

Langlands postulates also functoriality [A5] (see <http://tinyurl.com/zts4rqf>) making category theory so powerful. This allows to deduce from the existence of homomorphism between two groups G information about the relationship between representations of the dual group.

To sum up, I cannot claim of understanding much about this at the level of details. I however realize that the number theoretic vision relates in a highly interesting manner to Langlands theory and comparison might provide fresh insights to TGD and maybe even to Langlands theory by suggesting concrete physical identifications of groups associated with the Langlands correspondence and also suggesting a purely geometric action for the Galois groups via the adelic manifold concept.

3 TGD and Langlands correspondence

In the sequel I compare first Langlands program with TGD, which also involves both number theoretic and geometric visions and after that consider more detailed ideas.

3.1 Comparing the motivations

There are important similarities and also differences between the mathematical machineries used in Langlands approach and in TGD. Also motivations are different.

3.1.1 Motivation for number theoretical universality

In Langlands approach reductive algebraic groups are allowed with matrix elements in various number fields (number theoretical universality). Classical Lie groups with matrix elements in some number field are algebraic groups. The basic motivation is generality. One studies algebraic groups over field K , which can be archimedean local field (reals or complex numbers), non-archimedean local field (finite extension of p-adic number field induced by extension of rationals), or global field (extension of rationals). One introduces also the separable closure \bar{K} of K making possible to use the machinery of Lie groups and algebras. Separability means that only the roots of polynomials with different roots appear in extension. For p-adic number fields the separable closure is rather intricate notion. For algebraic extensions of rationals it correspond to algebraic numbers.

TGD view:

1. In TGD framework number theoretic universality implies that algebraic extensions of rationals define kind of intersection of reality and p-adicities. Therefore the discrete counterparts of Lie groups with matrix elements in the extensions of Q are of special importance in TGD. Langlands program includes these and are the most difficult ones.
2. If the hypothesis about ADE hierarchy assignable inclusions of HFFs [K11] holds true and has direct connection with $h_{eff}/h = n$ phases, all ADE Lie groups are allowed as dynamical symmetry groups and one achieves almost the same generality as in the case of Langlands correspondence. The maximal separable extensions for global and local fields make these fields analogous to complex numbers so that Lie-algebraic machinery can be used.
3. What is new that TGD suggest the allowance of all extensions of rationals inducing finite-dimensional extensions of p-adic number fields. In TGD context the extension of rationals can include also powers of a root of e since e^p is ordinary p-adic number and root of e induces finite-D extension of p-adic numbers (finite-dimensionality of extension is natural from the point of view of cognition). For non-compact groups the discretization of hyperbolic angles in this manner in p-adic context corresponds to the use of roots of unity for ordinary angles. One can say that the matrices with adèle valued elements act in what might be called extension of the world of sensory experience to involve also cognition. That e^p is ordinary p-adic number suggests that non-compact groups are effectively compact in p-adic context.

The Galois group of the extension by e^q the map $\sigma(e) = 1/e$ generates automorphism mapping rationals to rationals. The linear maps induced by $f(e) = e^k$, K integer are homomorphism since they map sums into sums and products into products but are not bijections except for $k = \pm 1$. One can wonder whether these maps could define analogs of automorphisms defining analog of inclusion hierarchy for hyper-finite factors (HFFs) [K11].

3.1.2 Motivation for p-adic number fields

In Langlands approach one motivation for including p-adic number fields is Hasse principle (see <http://tinyurl.com/jkh3auq>): in the case of p-adic number fields the notion of algebraic number is not so stunningly complex as for rationals. The reason is that for p-adic units polynomials reduce effectively to polynomials with degree $n \bmod p < p$ with integer coefficients in the range $[0, p - 1]$. This implies a huge simplification. The main reason for the mathematical applications of p-adic numbers is just this.

TGD view: From the viewpoint of TGD inspired theory of consciousness the motivation is the need to describe cognition mathematically. Cognition indeed simplifies: 2-adic cognition represents the largest possible simplification and cognitive evolution means increase of p as well as the increase of the dimension of algebraic extension of rationals (perhaps also that induced by root of e). It was however p-adic mass calculations assuming that mass squared is thermal in p-adic thermodynamics, which led to the p-adic physics [K12, K5].

3.1.3 Motivation for adelicization

In Langlands approach adelicization means treatment of all number fields simultaneously. p-Adic number fields are combined to form kind of Cartesian product called adeles. Only p-adic integers are allowed and it is natural to pose the additional condition that apart from a finite number of exceptions these integers are p-adic units. Automorphic representations can be seen as infinite tensor products of representations associated with the number fields defining the adèle.

TGD view:

1. The adelic view is used in different sense in TGD framework. Infinite tensor product of representations would create serious problems related to the physical interpretation in TGD framework since it seems that real and p-adic representations are only different views from the same number theoretically universal thing in the intersection of real and various p-adic sectors. One could say that the subgroups of algebraic groups with the matrix elements in the extension of rationals are in the intersection of real and various p-adic group theories.
2. The notion of p-adic manifold relies on the same idea. The discretization in algebraic extension of rationals is in the intersection and to each discrete point one can assign a monad which is real or p-adic and in which field equations such as those satisfied by preferred extremals of Kähler action are satisfied. One could perhaps say that these discrete algebraic points give rise to a number theoretically universal “spine” or back-bone of the space-time surface or any adelic geometry.

Real continua around these points would give rise to the flesh around these bones (sensory representations). Also mind is needed and p-adic monads realized as p-adic integers would give it (cognitive representations). The definition of p-adic geometry works nicely for coset spaces [L1] and induction procedure allows to define adelic geometries for space-time surface using discretization consisting of algebraic imbedding space points. The interpretation is in terms of finite measurement resolution and the hierarchy of algebraic extensions of rationals defines an infinite hierarchy of resolutions.

This physical picture would suggest a generalization of the notion of geometry by fusing real and p-adic variants of the manifold to adelic geometry. In group theory this would mean a hierarchy of groups assignable to algebraic extensions of rationals with discrete group elements of discrete subgroups accompanied by monads defining the neighborhood of group element in archimedean or non-archimedean sense. These monads would make sense also in real context.

3. One could see the variants of group G in various number fields as completions of the number theoretically universal core part of G define in an extension of rationals common to all local number fields. Each point in the discretization would correspond to real or p-adic monad or for standard notion of manifold to an open neighborhood.

What is new is that the system of open sets would correspond to the discretization having interpretation in terms of finite measurement resolution and the discrete subgroup could have direct physical meaning. For instance, Lorentz boosts would be quantized to velocities $\beta = \tanh(n/m)$, $n \in \mathbb{Z}$ and this velocity quantization could be seen in cosmology. There is indeed evidence for the quantization of redshifts [E1, E2]: possible TGD based explanations are discussed in [K8].

4. For instance, group $SO(2)$ represented by matrices

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

could be replaced with group for which $\theta = k2\pi/n$ so that one has roots of unity and one would have in p-adic context union of these group elements multiplied by a genuine p-adic Lie group with trigonometric functions replaced by the p-adic counterparts.

The group $SO(1,1)$ represented by the matrices

$$\begin{pmatrix} \cosh(\eta) & \sinh(\eta) \\ \sinh(\eta) & \cosh(\eta) \end{pmatrix}$$

could be replaced with the group obtained by quantizing η in the manner already described and multiplying this group with the p-adic Lie group with hyperbolic functions replaced with their p-adic counterparts. Since e^p is ordinary p-adic number the number of discrete points of the monad would be finite and one would have analog of compactness for a group which is non-compact in real context.

3.1.4 Motivation for global fields

In Langlands approach the motivation for considering groups with matrix elements in global number fields is purely mathematical and Galois group is studied as number theoretical symmetry.

TGD view: In TGD framework the discretization of the imbedding space in terms of points belonging to algebraic extension of rationals (or that including also the root of e) and inducing corresponding discretization of space-time surface means that Galois group of the extension acts as a physical symmetry group inducing an orbit of discretizations.

1. Does this mean that isometry group and symmetry groups with elements in K must be combined to semi-direct product with Galois group? One would have analogs of particle multiplets defined by irreducible representations of Galois group. Would this bring in a kind of number theoretic spin as additional degree of freedom? These particle like entities would emerge in number theoretical evolution as increase of the algebraic extension of rationals.
2. Or is an interpretation as a discrete orbital degree of freedom more appropriate? The singular n -fold coverings assignable to space-time surface associated with $h_{eff}/h = n$ phases identified in terms of dark matter could have natural interpretation as Galois coverings. Singularity means that the sheets of the covering co-incide at the ends of space-time surface at light-like boundaries of the causal diamond (CD). The action of Galois group becomes trivial if the points at the ends of space-time are rational. One can consider also Galois groups which are associated with a given extension of an extensions and same picture would hold true. This identification of $h_{eff}/h = n$ would imply very strong correlation between number theory and dark matter phases.

3.2 TGD inspired ideas related to number theoretic Langlands correspondence

The question is whether TGD might allow to get new perspective to the Langlands duality. TGD certainly suggests number theoretical view about quantum physics as also view about quantum physics as infinite-dimensional geometry of WCW.

There are many notions which could relate to Langlands correspondence.

1. The notion of p-adic or monadic geometry [L1] emerges as a realization for finite measurement resolution at space-time level based on discretization in terms of algebraic extension of rationals and having Galois group as symmetry group. This geometry is also adelic geometry. Also the semi-direct products of various symmetry groups restricted to an extension of rationals and their semi-direct products with Galois group emerge naturally in this framework. Could this be the physical counterpart for the semi-direct product of G_L with Galois group? Complexification and replacement of K with separable closure are carried out for technical reasons in Landlands approach. Could automorphic functions have physical meaning in TGD framework? In principle \overline{K} makes sense also in TGD framework. Could one think that one just restricts the automorphic functions from $G(\overline{K})$ to $G(K)$ and from $G_L(C)$ to $G_L(k)$ by imbedding K to C as number theoretic universality suggests? Could one continue the universal automorphic functions from the discrete spine of the adelic geometry to the interior of the monads by using their form in \overline{K} defining formulas?
2. Dark matter phases labelled by hierarchy of Planck constants and proposed to correspond singular coverings of space-time surface. Could the hierarchy of extensions of rationals correspond to this hierarchy. Could these coverings be Galois coverings becoming singular at the 3-D ends of space-time surface about boundaries of CD so that Galois leaves the corresponding 3-surface invariant by mapping it to itself or even leaving it invariant in point-wise manner?
3. Inclusions of hyperfinite factors (HFFs) [K11] are proposed to realize for finite measurement resolution quantum level in TGD framework. McKay correspondence (see <http://tinyurl.com/z48d92t>) suggesting that ADE Lie groups of Kac-Moody groups act as dynamical Lie groups identifiable as remnants of symplectic symmetries acting as isometries of WCW [K3, K14]. Could the hierarchy of extensions of rationals correspond to this hierarchy?
4. Weak form of electric-magnetic duality [K2] as self-duality reflecting self-duality of CP_2 and leading to ask whether Langlands duality reduces to self-duality for various symmetry groups of TGD.
5. Symplectic group defines the isometries of “world of classical worlds” (WCW) [K3, K2] and it is difficult to avoid the idea that the generalization of Kac-Moody algebra defined by symplectic group is crucial for the physical realization of Langlands correspondence in TGD framework.

3.2.1 Galois groups as symmetry groups in number theoretic vision

I have already earlier proposed that Galois groups could act as physical symmetries in TGD framework.

1. Number theoretic vision about TGD leads to the notion of adelic geometry involving both real, algebraic and various p-adic geometries giving meaning a generalization of manifold based on finite measurement resolution. Extensions of rationals inducing finite-dimensional extensions of p-adic numbers are central. The outcome is what might be called monadology. p-Adic space-time geometries make sense as induced geometries with discretizations defining points labelling the monads induced from the discretization of the imbedding space. Strong form of holography allows reduction to the level string world sheets and partonic 2-surfaces serving as space-time genes (also gauge equivalence classes light-like orbits of partonic 2-surface labelled by Galois group might be involved).

What is remarkable is that the Galois group of extension defines a symmetry group for discretizations in physical sense giving from given set of monads a new one. The roots of a polynomial behind the extension label a set of n surfaces defining a kind of covering for one of the sheets and Galois group acts in this set defining a covering space.

2. Do the sheets of the covering define disjoint space-time surfaces or do they form single connected space-time surface as in the case of Riemann surface for $z^{1/n}$? Could both options be involved? In the latter case there should be 3-regions at which the space-time sheets are glued together to give a singular covering. Either these 3-surfaces or even the points at these 3-surfaces could be fixed points of Galois group.
3. Also the hierarchy of Planck constants is associated with the emergence of coverings of space-time surface. These coverings are singular in the sense that the sheets co-incide at the eds of space-time surface at the boundaries of causal diamond (CD: there is scale hierarchy of CDs).

Could these coverings be Galois coverings defined by the orbit of discretized space-time surface under Galois group of extension of rationals? Could $n = h_{eff}/h$ - tentatively identified as the number of sheets of covering - correspond to the dimension of the Galois group of the algebraic extension? More generally, if the Galois group is Galois group for an extension L of K which itself can be extension, singularity requires that the reduction must take place to K at the ends [K17]. One can imagine two options.

- Option a): The discrete points of the 4-surface reduce to rational points (or points of K) at its 3-D ends at boundaries of CD and perhaps also at light-like orbits of partonic 2-surfaces? One variant of this option is that reduction occurs only at partonic 2-surfaces and string world sheets or strings at the ends.
- Option b): Galois group leave only the discretized 3-surface invariant and maps its points along it?

One can invent an objection against Option a). It is essential that the points of discretization have the same interpretation in real and p-adic senses. Hence the points should be expressible solely in terms of the algebraic numbers defining the extension: say roots of unity and powers for the roots of unity but not involving integers larger than 1 with varying p-adic norm. If integers appear then the p-adic norm of point can differ from unity. Points of unit circle or points of sphere with trigonometric functions of angles expressible solely in terms of roots of unity (Platonic solids) are representative examples. This does not allow the reduction of points of 3-surfaces at the ends of CD to rational points of H .

Option b) looks more attractive. Galois groups would act as dynamical symmetries of dark matter. Although the action on 3-surfaces at the ends of CD would be trivial, the action on the modes of induced spinor fields could be trivial also at the ends of CD.

Note that Galois covering is not the only interpretation for the covering that I have proposed: I have considered also an identification based on twistor lift of the space-time surface to its 6-D twistor space in the product of twistor spaces of M^4 and CP_2 which are twistorially unique in that they allow Kähler structure [K17].

4. The action of Galois group as a symmetry group acting geometrically on adelic geometries brings in mind the Belyi's theorem stating that Riemann surfaces describable as dessin d'enfants - "*child's drawings*" providing a combinatorial representation of Riemann surface as graph - can be defined as algebraic curves over the field of algebraic numbers.

The mysterious absolute group has therefore has a geometric interaction on these Riemann surfaces allowing representation in terms of dessin d'enfant (see <http://tinyurl.com/zy393e3>). Now the Galois group of algebraic extension would have analogous representation on the discretization using points with coordinates in extension of rationals induced by the corresponding discretization for imbedding space (actually causal diamond (CD)) defining the analog of dessin d'enfant.

This raises several questions.

1. This picture brings in mind the notion of virtual particle. At boundaries of CD the 3-surface would be on mass shell in the sense of being fixed point of Galois group and inside the CD it could be off-mass shell number theoretically although field equations for preferred extremal would be satisfied. Could this correspond to the non-determinism of Kähler action? Should one sum in the construction of scattering amplitudes over the surfaces at Galois orbit as in path integral?
2. Or should one regard the entire many-sheeted covering as the basic entity? I have indeed proposed that one can perform second quantization for the n -sheeted cover associated with $h_{eff}/h = n$ by adding fermions to different sheets of this cover and obtain this manner states with fractional quantum numbers with fractionization by factor $1/n$.
3. The interpretation as discrete gauge invariance with gauge fixing as a choice of single representative from Galois orbit does not look attractive. Note however that I have discussed the possibility of a huge generalization of M-theory dualities relating Calabi-Yau's and their mirrors as a generalization of old-fashioned string model duality [K17]: space-time surface could be seen as space-time correlates for computations connecting initial and final collections of algebraic objects with algebraic operations taking place at the vertices at which the Euclidian space-time regions representing lines of scattering diagram meet along their 3-D ends. This symmetry can be also seen as discrete analog of gauge symmetry involving the analog of gauge choice.

3.2.2 $h_{eff}/h = n$ hierarchy, hierarchy of inclusions of HFFs and McKay correspondence, and hierarchy of extensions of rationals

The relationship between dark matter hierarchy as a hierarchy $h_{eff}/h = n$ phases, hierarchy of inclusions of HFFs, McKay correspondence, and hierarchy of extensions of rationals and corresponding hierarchy of Galois groups is highly interesting and has been already touched.

1. I have proposed that dark matter hierarchy corresponds to a hierarchy of inclusions of HFFs [K11] giving rise to a hierarchy of ADE Lie groups or Kac-Moody as effective symmetry groups. By McKay correspondence (see <http://tinyurl.com/z48d92t>) ADE groups correspond to finite discrete sub-groups of $SU(2)$ in one-one correspondence with Dynkin diagrams assignable to ADE type Kac-Moody algebras. This leads to ask whether the inclusion hierarchy is accompanied by a hierarchy of ADE type Kac-Moody algebras or Lie algebras.
2. ADE type Lie or Kac-Moody groups self-dual under Langlands correspondence could emerge as remnant of the symplectic symmetries (a sub-algebra of full symplectic algebra $Sympl$ isomorphic to it and its commutator with $Sympl$ have vanishing Noether charges). It could be assignable to string world sheets carrying the modes of induced spinor fields as dynamical symmetries. The duals of these ADE type groups are essentially identical with them and could combine with Galois groups to form semi-direct products.
3. One has a fractal hierarchy of sub-algebras of isomorphic sub-algebras of the symplectic algebra with conformal weights coming as n_1 -multiples of the full algebra. Could n_1 satisfy $n_1 = h_{eff}/h = n$ with n identifiable as the dimension of algebraic extension of rationals? Or could one have $n_1 = ord(G)$, where $ord(G)$ is the order of the Galois group having n as a factor?

3.2.3 Weak form of electric-magnetic duality and Langlands correspondence

The first question about Langlands correspondence is why $G_L \rtimes Gal$ corresponds to G and what this precisely means.

1. One can extend Galois group and symmetry group (say Poincare or Lorentz group acting on discretized space-time surface or on 2-surface or on induced spinor field) to their semi-direct product: group multiplication law would be $(t_1, g_1)(t_2, g_2) = (t_1 t_2, g_1 t_1(g_2))$: this group would be the analog of $G_L \times Gal(K)$. The finite-dimensional representations of Galois group clearly give rise to what might be called number theoretic spin.

2. The innocent question of a physicist familiar with the unitary representations of Poincare group defined by fields with spin is whether the dimension of Galois representation for G_L could correspond to dimension for the representation for the spin associated with the representation of the dual G in analogy with Langlands correspondence. If the idea about hierarchy of Planck constants makes sense, strings and partonic 2-surfaces at the ends of space-time surface at boundaries of CD would correspond to G since the action of Galois would be trivial on them and G_L effectively reduces to G . String world sheets and light-like orbits of partonic 2-surfaces would correspond to G_L and Galois group would bring in additional degrees of freedom. The action of G_L on the induced spinors with components in field K would be however non-trivial. This could serve as a motivation for the introduction of n -D representations of G formed by many-fermion states.

Second basic mystery relates to the duality $G - G_L$ with G and G_L . Why the groups G and G_L different? Or are they same in TGD framework?

1. G and G_L are essentially the same for Lorentz group, Poincare group, color group, for the holonomy group of spinor connection and for ADE groups possibly accompanying the hierarchy of inclusions of HFFs. One might also expect that the situation remains the same for Kac-Moody groups. "Essentially" means for color group $G = SU(3)$ one has $G_L = SU(3)/Z_3$.

Whether the situation is same for the infinite-dimensional symplectic group assignable to the boundary of CD, is not clear since finite-dimensional symplectic groups are dual to odd-dimensional rotational groups. In fact, the infinite number of vanishing conditions for symplectic charges is expected to reduce it effectively to finite-dimensional Lie group or Kac-Moody group.

2. In TGD framework wormhole throats with identical electric and magnetic fluxes serves as the building bricks of elementary particles. Weak form of electric-magnetic duality is self-duality restricted to the light-like orbits of partonic 2-surfaces defining boundary conditions and inspired by the fact that electric and magnetic Kähler charge for CP_2 are identical. One can assign magnetic fluxes to partonic 2-surfaces and electric fluxes to the boundaries defined by the orbits of partonic 2-surfaces. One can define weighted fluxes for Hamiltonians of $\delta M_{\pm}^4 \times CP_2$ as this kind of fluxes and obtain analogs of magnetic and electric representations classically. Trivial form of duality would mean that these representations are identical. Weak form of electric magnetic duality in this form suggests Langlands self-duality.

One can assign magnetic and electric fluxes also to string world sheets. If one assumes weak form of self-duality also for them, electric and magnetic fluxes are identical also for them.

One must be here cautious since algebraic discretization is involved and fluxes are defined only by assuming a continuation to continuous surface. This is indeed provided by the interiors of the monads assignable to the discrete points. In p-adic context the definition of flux as integral can be problematic.

$G = G_L$ does not trivialize Langlands correspondence.

1. If one considers semi-direct product of $G_L \rtimes Gal(K)$ and representation of G without the addition of Galois group as a semi-direct factor then the situation is non-trivial even for $G_L = G$. In Langlands program one must indeed use semi-direct product since the action of Galois group in $G_L(C)$ is usually trivial. For $G(\bar{K})$ the action of $Gal(K)$ is non-trivial in both global and local fields so that the inclusion of $Gal(K)$ as semi-direct factor would not be needed. Adeles contain however also positive reals and the action of $Gal(K)$ is trivial. This suggests that one must in the double coset representations in $H_1 \backslash G / H_2$ an irreducible unitary representation to either H_1 or H_2 and that this representation corresponds to the higher-dimensional representation of non-Abelian Galois group. If so, the representation of $G_L \rtimes Gal(K)$ could factor to a product of a representation of G_L invariant under $Gal(K)$ with a finite-dimensional representation of non-Abelian $Gal(K)$ and would correspond to a representation of G in $H_1 \backslash G / H_2$ with H_i - "spin" in analogy with representations of Lorentz group.

The reduction of the quantum numbers assignable to Lie groups to number theory would be of course in accordance with the vision about physics as generalized number theory and could be perhaps seen as the deep physical content of Langlands correspondence.

2. The relationship to the fractionization of quantum numbers occurring for anyons is interesting. The covering analogous to that for $z^{1/n}$ gives an idea about the situation. Using single sheet with coordinate z one would obtain $1/n$ fractionization of spin at this sheet since 2π rotation leads to different sheet and only $n \times 2\pi$ rotation must leave the state unaffected. If one uses w as coordinate the range of angle coordinate is 2π - no fractionization [K7]. In TGD framework fractionization would mean that spin fractionizes for the rotation generator assignable to M^4 but does not so for the rotation generator assignable to the space-time surface X^4 . Spin fractionization is associated with magnetic monopoles (maybe 2-sheeted coverings forced by the fact that monopole flux must flow to another space-time sheet through wormhole contact) so that there might be a connection.

3.2.4 $M^8 - M^4 \times CP_2$ duality, classical number fields, and Langlands correspondence

Quaternions and octonions seem to relate closely to the basic structure TGD [K10]: $M^4 \times CP_2$ allows octonionic structure in tangent space and space-time surfaces as preferred extremals could correspond to quaternionic/co-quaternionic surfaces with tangent space/normal space being quaternionic/associative. Also the notion of quaternion analyticity makes sense [K17]. The interesting question concerns the properties of various automorphism groups under Langlands duality. G_2 acting as automorphisms of octonions, its subgroup $SU(3)$ preserving preferred imaginary unit of octonions, and the covering group $SU(2)$ of the group $SO(3)$ of quaternionic automorphisms are self dual. $SO(3)$ has $SL(1, R)$ (I use $SL(n, R)$ to mean the same as $SL(2n, R)$ by some authors) as Langlands dual but the complexified groups are same so that one has self-duality also now.

For years ago I proposed what I called $M^8 - M^4 \times CP_2$ duality [K10, K15] and have not been able to kill this proposal. M^8 can be seen as tangent space of $M^4 \times CP_2$ and can be interpreted as subspace of complexified octonions. The idea is that 4-surfaces of M^8 with the property that tangent space at each point is associative (co-associative) or equivalently quaternionic (co-quaternionic) and containing in their tangent space $M^2 \subset M^8 = M^2 \times E^6$ are mappable to surfaces in $M^4 \times CP_2$. The point of CP_2 would parameterize the tangent space as subspace of E^6 and transform as $3 + \bar{3}$ under $SU(3)$ automorphisms. That the coordinates for time= constant section of M^8 transform either as 7-D G_2 representation whereas the points of 7-D hyperboloid transform as $7 - D$ representation of $SO(7)$ suggest some kind of duality.

The isometry group of M^8 is $SO(1, 7)$ and decomposes for a fixed $M^8 = M^4 \times E^4$ decomposition to $SO(1, 3) \times SO(4)$. The automorphism group of M^8 identified in terms of octonions is G_2 and $SU(3)$ is the automorphism group associated with $M^6 = M^2 \times E^6$ decomposition and acts as isometries of CP_2 . There is infinite number of different octonion structures corresponding to the choices of subspaces $M^2 \times E^6$ parameterized by $SO(1, 7)/SO(1, 1) \times SO(6)$ having dimension $D = 28 - 1 - 15 = 12$. Note that all groups involved are self-dual in Langlands correspondence.

The notions of p-adic octonions and quaternions do not make sense: the reason is that the norm of non-vanishing quaternion/octonion can be vanishing. This can be case also for p-adic analog of complex numbers if -1 is square of p-adic number as it is for $p \bmod 4 = 1$. This does not allow definition of p-adic Hilbert space. This difficulty is not present if one restricts the consideration to points of algebraic extension interpreted as p-adic numbers. In this case one can construct versions of G_2 and $SU(3)$ by replacing real numbers with global field. Also the action of Galois group is well-defined on space-time surface so that one can form semi-direct sum of these groups with Galois group. G_2 and $SU(3)$ are self-dual.

3.2.5 Could supersymplectic algebra be for number theoretic Langlands what Kac-Moody algebra is for geometric Langlands

Super-symmetric symplectic algebra [K3, K2] and conformal algebra of light-cone boundary is much more complex structure than Kac-Moody algebras and central in TGD. The reason is that effective 2-dimensionality of the light-cone boundary of four-dimensional Minkowski space leads to huge extension of the ordinary conformal symmetries.

1. Supersymplectic algebra has the structure of conformal algebra. The analog of complex coordinate for the is the light-like radial coordinate r of light-cone boundary. Radial conformal weights can be complex numbers and numbers $s = 1/2 + iy$ are favored since they give rise to the analogs of plane waves. Light-cone boundary having the structure $S^2 \times R_+$ metrically with R_+ corresponding to null direction. Therefore there is also an extension of conformal algebra of sphere S^2 . For this extension one has ordinary conformal weight assignable to S^2 and radial conformal weight assignable to R_+ . The physical role of this algebra which is actually also isometry algebra has remained unclear. What is however clear that dimension for M^4 makes it mathematically completely unique.
2. I have proposed that the conformal weights for the generators of the symplectic algebra could correspond to poles of fermionic zeta function $\zeta_F(s) = \zeta(s)/\zeta(2s)$ [K16]. The number of generators of the algebra could be infinite so that it would be extremely complex as compared to the Kac-Moody algebras. Unitarity demands that for physical states the imaginary part of the total conformal weight which is essentially the sum of zeros of zeta is real. This implies conformal confinement and that physical states have integer or half-integer valued total conformal weights as for the ordinary super-conformal algebras.
3. A further conjecture is that for the zeros $s = 1/2 + iy$ of Riemann zeta p^{iy} is root of unity [K16]. This conjecture is motivated by the findings suggesting that the zeros form a quasicrystal meaning that the Fourier transforms for the function located at zeros is of similar form.

Kac-Moody algebras are important for geometric Langlands based on fundamental group.

1. So called critical representations for Kac-Moody algebra are involved. For them the central extension parameter equals to $k = -c_\psi^g/2$, where c_ψ^g is Casimir operator for the adjoint representation. Negativity of k implies non-unitarity. The Virasoro generators in the associated Sugawara representation for Virasoro algebra would have infinite normalization constant $N = 1/2\beta$, $\beta = k + c_\psi^g/2 = 0$ and it would not be well-defined. Physically critical Kac-Moody representation does not seem interesting.
2. A formal generalization of Sugawara construction of representation of Virasoro algebra from that of symplectic algebra mimicking Kac-Moody case does not seem to work. The normalization factor $k + c_\psi^g/2$ dividing the quadratic expression of L_n in terms of Kac-Moody generators diverges if Casimir diverges and the outcome is ill-defined. Quadratic Casimir in adjoint representation is expressible in terms of structure constants as $f^{ABC}f_{ABC}$. Structure constants now Glebsch-Gordans for the representations of $SO(3) \times SU(3)$. Obviously the symplectic counterpart for the sum $f^{ABC}f_{ABC}$ for the Casimir operator of Lie group diverges so that Sugawara construction fails. This is of course not a real problem since Sugawara construction fails in any case for the critical weight needed in Kac-Moody algebra approach to geometric Langlands.

Could super-symplectic algebra help to understand number theoretic Langlands?

1. The conditions defining preferred extremal state vanishing of almost all symplectic Noether charges and suggest that symplectic group reduces effectively to a finite-D Lie group or Kac-Moody group. These groups form a hierarchy and could be assigned to inclusions of HFFs identifiable as ADE groups dictated by the inclusion (essentially self-dual under Langlands correspondence). These Kac-Moody groups could also have natural action at strings identified as boundaries of string world sheets. Also for these Kac-Moody groups critical representations lack physical interpretation.
2. Number theoretic discretization requires the consideration of discrete subgroup of ADE Lie group obtained by restriction to global field rather than Lie algebra. One could restrict the L-functions of automorphic representations to the subgroup of complex Lie group $G(C)$ having the group $G(K)$ associated with the global number field K .
3. This picture suggests tha the extension of Galois group in extension E/K has counterpart for the Lie groups appearing in ADE hierarchy realized at the level of Lie algebras: perhaps by adding n generators to the Cartan algebra.

3.3 Could geometric and number theoretic Langlands relate to each other?

One can see the analogy between Galois group and fundamental group also in the following manner (see the blog posting of Peter Woit at <http://tinyurl.com/hlgrrjk>). Primes are analogous to prime polynomials from which one can construct more complex polynomials as products. Rational numbers are analogous to rational functions defined as ratios of polynomials. This suggests an analogy between number theoretic Langlands and geometric Langlands for which rationals and their extensions are replaced by rational functions. One manner to interpret this analogy is to see ordinary rationals as kind of functions. Second manner is to see rational functions as generalizations of rationals. The latter interpretation looks more attractive to me.

There are indeed strong analogies between Galois groups and fundamental groups. Covering spaces can be assigned with fundamental groups and algebraic extensions of rationals are analogous to coverings: the orbit of a given point under Galois group is analogs to set of copies of the point at the sheets of the covering.

The problem is that fundamental group typically contains Z as a summand, which does not occur for Galois groups. For a punctured plane having Z as fundamental group one can construct infinite covering with trivial homotopy group. If one identifies k :th $k+n$:th sheet the fundamental group is $Z_n = Z/nZ$. For Q_p one expects reduction of fundamental group to Z_m , $m = n \pmod p$. This encourage speculative ideas related to the connection of number-theoretic and geometric Langlands.

3.3.1 Adelic geometries and the realization of fundamental group in terms of Galois group

Could geometric Langlands reduce to number theoretic Langlands in some cases? This would mean representation of fundamental group as Galois group of algebraic extension.

1. The notion of the adelic geometry involving algebraic discretization in both real and p-adic sectors with discretized points accompanied by locally smooth neighborhoods in which field equations for Kähler action are satisfied would suggests this. For a given discretization in terms of points of extension one obtains set of discretizations by applying Galois group and Galois group acts as symmetry group permuting the sheets of the covering covering space: the number of sheets as dimension of extension divides the order of Galois group.
2. If the n -fold singular coverings assigned with $h_{eff}/h = n$ corresponds to a Galois coverings, the sheets of covering reduce to single one at the singular ends of space-time surface at the lightlike boundaries of CD and one obtains a space analogous to the base space of covering and having homotopy group given by Galois group. Therefore the representations of Galois group would become representations of fundamental group for the adelic geometry. The action of this group would be non-trivial on spinors also at the ends of CD.

The analogy with fundamental group suggests that there are two manners to consider the situation. The images of the discrete adelic geometry under Galois group define the covering for which fundamental group is trivial. The restriction to single space-time sheet at the orbit under Galois group would mean the restriction to base space with non-trivial fundamental group given by Galois group. For the first option Galois group would permute the sheets of covering and define dynamical symmetry. For the second option non-trivial homotopy would correspond to these degrees of freedom. These two descriptions might define the core of number theoretic Langlands duality having interpretation also as geometric duality.

3.3.2 Does the hierarchy of infinite primes generalize number theoretic Langlands?

In TGD framework one can see the analogy from other direction. The construction of infinite primes leads to a repeated second quantization of arithmetic quantum field theory with bosonic and fermionic single particle states labelled by primes [K9]. At the lowest level ordinary primes label the single particle states and at the first level one obtains infinite primes as Fock states.

Infinite primes can be mapped to monomials of single variable with zeros which are rational numbers. One obtains also infinite primes analogous of bound states as analogs of irreducible polynomials of single variables: now the zeros correspond to algebraic numbers.

One can continue the second quantization by taking these infinite primes as labels of single particle states and repeating the procedure. Now one can map the infinite primes to polynomials of two variables. This process can be continued ad infinitum.

The variables appearing in irreducible polynomials assignable to the hierarchy of infinite primes are formal variables and it is not clear it makes sense to interpret them as coordinates for some space. If this were the case, one might consider connecting with Geometric Langlands associated with these space with generalization of number theoretic Langlands.

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