Could one Define Dynamical Homotopy Groups in WCW?

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Abstract

I learned that Agostino Prastaro has done highly interesting work with partial differential equations, also those assignable to geometric variational principles such as Kähler action in TGD. I do not understand the mathematical details but the key idea is a simple and elegant generalization of Thom's cobordism theory, and it is difficult to avoid the idea that the application of Prastaro's idea might provide insights about the preferred extremals, whose identification is now on rather firm basis.

One could also consider a definition of what one might call dynamical homotopy groups as a genuine characteristics of WCW topology. The first prediction is that the values of conserved classical Noether charges correspond to disjoint components of WCW. Could the natural topology in the parameter space of Noether charges zero modes of WCW metric) be p-adic and realize adelic physics at the level of WCW? An analogous conjecture was made on basis of spin glass analogy long time ago. Second surprise is that the only the 6 lowest dynamical homotopy/homology groups of WCW would be non-trivial. The Kähler structure of WCW suggets that only Π_0, Π_2, and Π_4 are non-trivial.

The interpretation of the analog of Π_1 as deformations of generalized Feynman diagrams with elementary cobordism snipping away a loop as a move leaving scattering amplitude invariant conforms with the number theoretic vision about scattering amplitude as a representation for a sequence of algebraic operation can be always reduced to a tree diagram. TGD would be indeed topological QFT: only the dynamical topology would matter.

1 Introduction

Agostino Prastaro - working as professor at the University of Rome - has done highly interesting work with partial differential equations, also those assignable to geometric variational principles such as Kähler action in TGD [A3, A4]. I do not understand the mathematical details but the key idea is a simple and elegant generalization of Thom's cobordism theory, and it is difficult to avoid the idea that the application of Prastaro's idea might provide insights about the preferred extremals, whose identification is now on rather firm basis [K6].

One could also consider a definition of what one might call dynamical homotopy groups as a genuine characteristics of WCW topology. The first prediction is that the values of conserved classical Noether charges correspond to disjoint components of WCW. Could this mean that the natural topology in the parameter space of Noether charges zero modes of WCW metric) is p-adic? An analogous conjecture was made on basis of spin glass analogy long time ago. Second surprise is that the only the six lowest dynamical homotopy groups of WCW would be non-trivial. The finite number of these groups dictate by the dimension of imbedding space suggests also an interpretation as analogs of homology groups.

In the following the notion of cobordism is briefly discussed and the idea of Prastaro about assigning cobordism with partial differential equations is discussed.

1.1 About cobordism as a concept

To get some background consider first the notion of cobordism (`https://en.wikipedia.org/wiki/Cobordism`).

1. Thom's cobordism theory [A6] is inspired by the question "When an n-manifold can be represented as a boundary of $n+1$-manifold". One can also pose additional conditions such as continuity, smoothness, orientability, one can add bundles structures and require that they are induced to n-manifold from that of $n+1$-manifold. One can also consider sub-manifolds of some higher-dimensional manifold.

   One can also fix n-manifold M and ask "What is the set of n-manifolds N with the property that there exists $n+1$-manifold W having union of $M \cup N$ as its boundary". One can also allow M to have boundary and pose the same question by allowing also the boundary of connecting $n+1$-manifold W contain also the orbits of boundaries of M and N.

   The cobordism class of M can be defined as the set of manifolds N cobordant with M - that is connectable in this manner. They have same cobordism class since cobordism is equivalence relation. The classes form also a group with respect to disjoint union. Cobordism is much rougher equivalence relation than diffeomorphy or homeomorphy since topology changes are possible. For instance, every 3-D closed un-oriented manifold is a boundary of a 4-manifold! Same is true for orientable cobordisms. Cobordism defines a category: objects are (say closed) manifolds and morphisms are cobordisms.

2. The basic result of Morse, Thom, and Milnor is that cobordism as topology changes can be engineered from elementary cobordisms. One take manifold $M \times I$ and imbeds to its other n-dimensional end the manifold $S^p \times D^q$, $n = p + q$, removes its interior and glues back $D^{p+1} \times S^{q-1}$ along its boundary to the boundary of the resulting hole. This gives n-manifold with different topology, call it N. The outcome is a cobordism connecting M and N unless there are some obstructions.

   There is a connection with Morse theory (`https://en.wikipedia.org/wiki/Morse_theory`) in which cobordism can be seen as a mapping of W to a unit interval such that the inverse images define a slicing of W and the inverse images at ends correspond to M and N.

3. One can generalize the abstract cobordism to that for n-sub-manifolds of a given imbedding space. This generalization is natural in TGD framework. This might give less trivial results since not all connecting manifolds are imbeddable into a given imbedding space. If connecting 4-manifolds connecting 3-manifolds with Euclidian signature (of induced metric) are assumed to have a Minkowskian signature, one obtains additional conditions, which might be too strong (the classical result of Geroch [A2] implies that non-trivial cobordism implies closed time loops - impossible in TGD).

   From TGD point of view this is too strong a condition and in TGD framework space-time surfaces with both Euclidian and Minkowskian signature of the induced metric are allowed. Also cobordisms singular as 4-surfaces are analogous to 3-vertices of Feynman diagrams are allowed.

1.2 Prastaro's generalization of cobordism concept to the level of partial differential equations

I am not enough mathematician in technical sense of the word to develop overall view about what Prastaro has done and I have caught only the basic idea. I have tried to understand the articles [A3, A4] with title "Geometry of PDE's. I/II: Variational PDE's and integral bordism groups" (http://www.sciencedirect.com/science/article/pii/S0022247X05005998 and http://www.sciencedirect.com/science/article/pii/S0022247X05008115), which seem to correspond to my needs. The key idea is to generalize the cobordism concept also to partial differential equations with cobordism replaced with the time evolution defined by partial differential equation. In particular, to geometric variational principles defining as their extremals the counterparts of cobordisms.

Quite generally, and especially so in the case of the conservation of Noether charges give rise to strong selection rules since two n-surfaces with different classical charges cannot be connected by extremals of the variational principle. Note however that the values of the conserved charges depend on the normal derivatives of the imbedding space coordinates at the n-dimensional ends of cobordism. If one poses additional conditions fixing these normal derivatives, the selection rules become even stronger. In TGD framework Bohr orbit property central for the notion of WCW geometry and holography allows to hope that conserved charges depend on 3-surfaces only.

What is so beautiful in this approach that it promises to generalize the notion of cobordism and perhaps also the notions of homotopy/homology groups so that they would apply to partial differential equations quite generally, and especially so in the case of geometric variational principles giving rise to $n+1$-surfaces connecting n-surfaces characterizing the initial and final states classically. TGD with $n = 3$ seems to be an ideal applications for these ideas.

Prastaro also proposes a generalization of cobordism theory to super-manifolds and quantum super-manifolds. The generalization in the case of quantum theory utilizing path integral does not not pose conditions on classical connecting field configurations. In TGD framework these generalizations are not needed since fermion number is geometrized in terms of imbedding space gamma matrices and super(-symplectic) symmetry is realized differently.

2 Dynamical topology in TGD

In the following the motivations for applying Prastaro's ideas in TGD framework are summarized and after that the concreted realization is discussed. The basic outcome is a proposal for the origin of p-adic topology at the level of WCW. It is also proposed that WCW topology might be determined to a high extent by the dynamics of preferred extremals.

2.1 Why Prastaro's idea resonates so strongly with TGD

Before continuing I want to make clear why Prastaro's idea resonates so strongly with TGD.

1. One of the first ideas as I started to develop TGD was that there might be selection rules analogous to those of quantum theory telling which 3-surfaces can be connected by a space-time surface. At that time I still believed in path integral formalism assuming that two 3-surfaces at different time slices with different values of Minkowski time can be connected by any space-time surface for which imbedding space coordinates have first derivatives.

   I soon learned about Thom's theory but was greatly disappointed since no selection rules were involved in the category of abstract 3-manifolds. I thought that possible selection rules should result from the imbeddability of the connecting four-manifold to $H = M^4 \times CP_2$ but my gut feeling was that these rules are more or less trivial since so many connecting 4-manifolds exist and some of them are very probably imbeddable.

   One possible source of selection rules culd have been the condition that the induced metric has Minkowskian signature - one could justify it in terms of classical causality. This restricts strongly topology change in general relativity (http://arxiv.org/pdf/1406.6194v1.pdf). Geroch's classical result [A2] states that non-trivial smooth Lorentz cobordism between compact 3-surfaces implies the existence of closed time loop - not possible in TGD framework. Second non-encouraging result is that scalar field propagating in trouser topology leads to an occurrence of infinite energy burst (http://journals.aps.org/prd/abstract/10.1103/PhysRevD.58.124019).

In the recent formulation of TGD however also Euclidian signature of the induced metric is allowed. For space-time counterparts of 3-particle vertices three space-time surfaces are glued along their smooth 3-D ends whereas space-time surface fails to be everywhere smooth manifold. This picture fits nicely with the idea that one can engineer space-time surfaces by gluing them together along their ends.

2. At that time (before 1980) the discovery of the geometry of the "World of Classical Worlds" (WCW) as a possible solution to the failures of canonical quantization and path integral formalism was still at distance of ten years in future. Around 1985 I discovered the notion of WCW. I made some unsuccessful trials to construct its geometry, and around 1990 finally realized that 4-D general coordinate invariance is needed although basic objects are 3-D surfaces.

   This is realized if classical physics is an exact part of quantum theory - not only something resulting in a stationary phase approximation. Classical variational principle should assign to a 3-surface a physically unique space-time surface - the analog of Bohr orbit - and the action for this surface would define Kähler function defining the Kähler geometry of WCW using standard formula.

   This led to a notion of preferred extremal: absolute minimum of Kähler action was the first guess and might indeed make sense in the space-time regions with Euclidian signature of induced metric but not in Minkowskian regions, which give to the vacuum functional and exponential of Minkowskian Kähler action multiplied by imaginary unit coming from $\sqrt{g}$ - just as in quantum field theories. Euclidian regions give the analog of the free energy exponential of thermodynamics and transform path integral to mathematically well-defined functional integral.

3. After having discovered the notion of preferred extremal, I should have also realized that an interesting generalization of cobordism theory might make sense after all, and could even give rise to the classical counterparts of the selection rules! For instance, conservation of isometry charges defines equivalence classes of 3-surfaces endowed with tangent space data. Bohr orbit property could fix the tangent space data (normal derivatives of imbedding space coordinates) so that conserved classical charges would characterize 3-surfaces alone and thus cobordism equivalence classes and become analogous to topological invariants. This would be in spirit with the attribute "Topological" in TGD!

2.2 What preferred extremals are?

The topology of WCW has remained mystery hitherto - partly due to my very limited technical skills and partly by the lack of any real physical idea. The fact, that p-adic topology seems to be natural at least as an effective topology for the maxima of Kähler function of WCW gave a hint but this was not enough.

I hope that the above summary has made clear why the idea about dynamical cobordism and even dynamical homotopy theory is so attractive in TGD framework. One could even hope that dynamics determines not only Kähler geometry but also the topology of WCW to some extend at least! To get some idea what might be involved one must however first tell about the recent situation concerning the notion of preferred extremal.

1. The recent formulation for the notion of preferred extremal relies on strong form of General Coordinate Invariance (SGCI). SGCI states that two kinds of 3-surfaces can identified as fundamental objects. Either the light-light 3-D orbits of partonic 2-surfaces defining boundaries between Minkowskian and Euclidian space-time regions or the space-like 3-D ends of space-time surfaces at boundaries of CD. Since both choices are equally good, partonic 2-surfaces and their tangent space-data at the ends of space-time should be the most economic choice.

   This eventually led to the realization that partonic 2-surfaces and string world sheets should be enough for the formulation of quantum TGD. Classical fields in the interior of space-time surface would be needed only in quantum measurement theory, which demands classical physics in order to interpret the experiments.

2. The outcome is strong form of holography (SH) stating that quantum physics should be coded by string world sheets and partonic 2-surfaces inside given causal diamond (CD). SH is very much analogous to the AdS/CFT correspondence but is much simpler: the simplicity is made possible by much larger group of conformal symmetries.

   If these 2-surfaces satisfy some consistency conditions one can continue them to 4-D space-time surface inside CD such that string world sheets are surfaces inside them satisfying the condition that charged (possibly all) weak gauge potentials identified as components of the induced spinor connection vanish at the string world sheets and also that energy momentum currents flow along these surfaces. String world sheets carry second quantized free induced spinor fields and fermionic oscillator operator basis is used to construct WCW gamma matrices.

3. The 3-surfaces at the ends of WCW must satisfy strong conditions to guarantee effective 2-dimensionality. Quantum criticality suggests the identification of these conditions. All Noether charges assignable to a sub-algebra of super-symplectic algebra isomorphic to it and having conformal weights which are n-multiples of those of entire algebra vanish/annihilate quantum states. One has infinite fractal hierarchy of broken super-conformal symmetries with the property that the the sub-algebra is isomorphic with the entire algebra. This like a ball at the top of ball at the top of ....

   The speculative vision is that super-symplectic subalgebra with weights coming as n-ples of those for the entire algebra acts as an analog of conformal gauge symmetries on light-like orbits of partonic 2-surfaces, and gives rise to a pure gauge degeneracy whereas other elements of super-symplectic algebra act as dynamical symmetries. The hierarchy of quantum criticalities defines hierarchies of symmetry breakings characterized by hierarchies of sub-algebras for which one n_{i+1} is divisible by n_i. The proposal is that conformal gauge invariance means that the analogs of Bohr orbits are determined only apart from conformal gauge transformations forming to n_i conformal equivalence classes so that effectively one has n_i discrete degrees of freedom assignable to light-like partonic orbits.

4. In this framework manifolds M and N would correspond the 3-surfaces at the boundaries of CD and containing a collection strings carrying induced spinor fields. The connecting 4-surface W would contain string world sheets and the light-like orbits of partonic 2-surfaces as simultaneous boundaries for Minkowskian and Euclidian regions.

A central element in the physical interpretation is the identification of Euclidian space-time regions as space-time correlates of Feynman graphs or twistor graphs - the reader can decide which letter combination is more appropriate. What is common with Feynman/twistor graphs is the presence of analogs of propagator lines and vertices.

1. Propagator line has several meanings depending on whether one considers particles as strings, as single fermion states localizable at the ends of strings, or as Euclidian space-time regions or their light-like boundaries with degenerate induced metric with vanishing determinant. Vertices appear as generalizations of the stringy vertices and as generalization of the vertices of Feynman diagrams in which the incoming 4-surfaces meet along their ends.

   (a) The lines of generalized Feynman graphs defined in topological sense are identified as slightly deformed pieces of CP_2 defining wormhole contacts connecting two Minkowskian regions and having wormhole throats identified as light-like parton orbits as boundaries. Since there is a magnetic monopole flux through the wormhole contacts they must appear as pairs (also larger number is possible) in order that magnetic field lines can close. Elementary particles correspond to pairs of wormhole contacts. At both space-time sheets the throats are connected by magnetic flux tubes carrying monopole flux so that a closed flux tube results having a shape of an extremely flattened square and having wormhole contacts at its ends. It is a matter of taste, whether to call the light-like wormhole throats or their interiors as lines of the generalized Feynman/twistor diagrams.

The light-like orbits of partonic 2-surfaces bring strongly in mind the light-like 3-surfaces along which radiation fields can be restricted - kind of shockwaves at which the signature of the induced space-time metric changes its signature.

(b) String world sheets as orbits of strings are also in an essential role and could be seen as particle like objets. String world sheets could as kind of singular solutions of field equations analogous to characteristics of hyperbolic differential equations. The isometry currents of Kähler action flow along string world sheets and field equations restricted to them are satisfied. As if one would have 2-dimensional solution. $\sqrt{g_4}$ would of course vanishes for genuinely 2-D solution but this one can argue that this is not a problem since $\sqrt{g_4}$ can be eliminated from field equations. String world sheets could serve as 2-D a analoga for a solution of hyperbolic field equations defining expanding wave front localized at 3-D light-like surface.

(c) Propagation in the third sense of word is assignable to the ends of string world sheets at the light-like orbits of partonic 2-surfaces and possibly carrying fermion number. One could say that in TGD one has both fundamental fermions serving as building bricks of elementary particles and strings characterizing interactions between particles. Fermion lines are massless in 8-D sense. By strong form of holography this quantum description has 4-D description space-time description as a classical dual.

2. The topological description of interaction vertices brings in the most important deviation from the standard picture behind cobordism: space-time surfaces are not smooth in TGD framework. One allows topological analogs of 3-vertices of Feynman diagrams realized by connecting three 4-surfaces along their smooth 3-D ends. 3-vertex is also an analog (actually much more!) for the replication in biology. This vertex is *not* the analog of stringy trouser vertex for which space-time surface is continuous whereas 3-surface at the vertex is singular (also trouser vertex could appear in TGD).

   The analog of trouser vertex for string world sheets means splitting of string and fermionic field modes decompose into superposition of modes propagating along the two branches. For instance, the propagation of photon along two paths could correspond to its geometric decay at trouser vertex not identifiable as "decay" to two separate particles.

   For the analog of 3-vertex of Feynman diagram the 3-surface at the vertex is non-singular but space-time surface is singular. The gluing along ends corresponds to genuine 3-particle vertex.

   The view about solution of PDEs generalizes dramatically but the general idea about cobordism might make sense also in the generalized context.

2.3 Could dynamical homotopy/homology groups characterize WCW topology?

The challenge is to at least formulate (with my technical background one cannot dream of much more) the analog of cobordism theory in this framework. One can actually hope even the analog of homotopy/homology theory.

1. To a given 3-surface one can assign its cobordism class as the set of 3-surfaces at the opposite boundary of CD connected by a preferred extremal. The 3-surfaces in the same cobordism class are characterized by same conserved classical Noether charges, which become analogs of topological invariants.

   One can also consider generalization of cobordisms as analogs to homotopies by allowing return from the opposite boundary of CD. This would give rise to first homotopy groupoid. One can even go back and forth several times. These dynamical cobordisms allow to divide 3-surfaces at given boundary of CD in equivalence classes characterized among other things by same values of conserved charges. One can also return to the original 3-surface. This could give rise to the analog of the first homotopy group Π_1.

2. If one takes the homotopy interpretation literally one must conclude that the 3-surfaces with different conserved Noether charges cannot be connected by any path in WCW - they belong

to disjoint components of the WCW! The zeroth dynamical homotopy group Π_0 of WCW would be non-trivial and its elements would be labelled by the conserved Noether charges defining topological invariants!

The values of the classical Noether charges would label disjoint components of WCW. The topology for the space of these parameters would be totally disconnected - no two points cannot be connected by a continuous path. p-Adic topologies are indeed totally disconnected. Could it be that p-adic topology is natural for the conserved classical Noether charges and the sectors of WCW are characetrized by p-adic number fields and their algebraic extensions?

Long time ago I noticed that the 4-D spin glass degeneracy induced by the huge vacuum degeneracy of Kähler action implies analogy between the space of maxima of Kähler function and the energy landscape of spin glass systems [K2]. Ultrametricity (`https://en.wikipedia.org/wiki/Ultrametric_space`) is the basic property of the topology of the spin glass energy landscape. p-Adic topology is ultrametric and the proposal was that the effective topology for the space of maxima could be p-adic.

3. Isometry charges are the most important Noether charges. These Noether charges are very probably not the only conserved charges. Also the generators in the complement of the gauge sub-algebra of symplectic algebra acting as gauge conformal symmetries could be conserved. All these conserved Noether charges would define a parameter space with a natural p-adic topology.

   Since integration is problematic p-adically, one can ask whether only discrete quantum superpositions of 3-surfaces with different classical charges are allowed or whether one should even assume fixed values for the total classical Noether charges appearing in the scattering amplitudes.

   I have proposed this kind of approach for the zero modes of WCW geometry not contributing to the Kähler metric except as parameters. The integration for zero modes is also problematic because there is no metric, which would define the integration measure. Since classical charges do not correspond to quantum fluctuating degrees of freedom they should correspond to zero modes. Hence these arguments are equivalent.

The above argument led to the identification of the analogs of the homotopy group Π_0 and led to the idea about homotopy groupoid/group Π_1. The elements of Π_1 would correspond to space-time surfaces, which run arbitrary number of times fourth and back and return to the initial 3-surface at the boundary of CD. If the two preferred extremals connecting same pair of 3-surfaces can be deformed to each other, one can say that they are equivalent as dynamical homotopies (or cobordisms). What could be the allowed deformations? Are they cobordisms of cobordisms? What this could mean? Could they define the analog of homotopy groupoid Π_2 as foliations of preferred extremals connecting the same 3-surfaces?

1. The number theoretic vision about generalized Feynman diagrams suggests a possible approach. Number theoretic ideas combined with the generalization of twistor approach [K6, K3] led to the vision that generalized Feynman graphs can be identified as sequences or webs of algebraic operations in the co-algebra defined by the Yangian assignable to super-symplectic algebra and acting as symmetries of TGD. Generalized Feynman graphs would represent algebraic computations. Computations can be done in very many different manners and each of them corresponds to a generalized Feynman diagram. These computations transform give same final collection of "numbers" when the initial collection of "numbers" is given. Does this mean that the corresponding scattering amplitudes must be identical?

   If so, a huge generalization of the duality symmetry of the hadronic string models would suggest itself. All computations can be reduced to minimal computations. Accordingly, generalized Feynman diagrams can be reduced to trees by eliminating loops by moving the ends of the loops to same point and snipping the resulting tadpole out! The snipped of tadpole would give a mere multiplicative factor to the amplitude contributing nothing to the scattering rate - just like vacuum bubbles contribute nothing in the case of ordinary Feynman diagrams.

2. How this symmetry could be realized? Could one just assume that only the minimal generalized Feynman diagrams contribute? - not a very attractive option. Or could one hope that only tree diagrams are allowed by the classical dynamics: this was roughly the original vision? The huge vacuum degeneracy of Kähler action implying non-determinism does not encourage this option. The most attractive and most predictive realization conforming with the idea about generalized Feynman diagrammatics as arithmetics would be that all the diagrams differing by these moves give the same result. An analogous symmetry has been discovered for twistor diagrams.

3. Suppose one takes seriously the snipping of a tadpole away from diagram as a move, which does not affect the scattering amplitude. Could this move correspond to an allowed elementary cobordism of preferred extremal? If so, scattering amplitudes would have purely topological meaning as representations of the elements of cobordism classes! TGD would indeed be what it was proposed to be but in much deeper sense than I thought originally. This could also conform with the interpretation of classical charges as topological invariants, realize adelic physics at the level of WCW, and conform with the idea about TGD as almost topological QFT and perhaps generalizing it to topological QFT in generalized sense.

4. One can imagine several interpretations for the snipping operation at space-time level. TGD allows a huge classical vacuum degeneracy: all space-time surfaces having Lagrangian manifold of CP_2 as their CP_2 projection are vacuum extremals of Kähler action. Also all CP_2 extremals having 1-D light-like curve as M^4 projection are vacuum extremals but have non-vanishing Kähler action. This would not matter if one does not have superpositions since multiplicative factors are eliminated in scattering amplitudes. Could the tadpoles correspond to CP_2 type vacuum extremals at space-time level?

   There is also an alternative interpretation. In ZEO causal diamonds (CD) form a hierarchy and one can imagine that the sub-CDs of given CD correspond to quantum fluctuations. Could tadpoles be assigned to sub-CDs of CD be considered+

5. In this manner one could perhaps define elements of homotopy groupoid Π_2 as foliations preferred extremals with same ends - these would be 5-D surfaces. If one has two such 5-D foliations with the same 4-D ends, one can form the reverse of the other and form a closed surface. This would be analogous to a map of S^2 to WCW. If the two 5-D foliations cannot be transformed to each other, one would have something, which might be regarded as a non-trivial element of dynamical homotopy group Π_2.

One can ask whether one could define also the analogs of higher homology or homotopy groupoids and groupoids Π_3 up to Π_5 - the upper bound $n = 5 = 8 - 3$ comes from the fact that foliations of foliations.. can have maximum dimension $D = 8$ and from the dimension of $D = 3$ of basic objects.

1. One could form a foliation of the foliations of preferred extremals as the element of the homotopy groupoid Π_3. Could allowed moves reduce to the snipping operation for generalized Feynman diagrams but performed along direction characterized by a new foliation parameter.

2. The topology of the zero mode sector of WCW parameterized by fixed values of conserved Noether charges as element of Π_0 could be characterized by dynamical homotopy groups Π_n, $n = 1, ..., 5$ - at least partially. These degrees of freedom could correspond to quantum fluctuating degrees of freedom. The Kähler structure of WCW and finite-D analogy suggests that all odd dynamical homotopy groups vanish so that Π_0, Π_2 and Π_4 would be the only non-trivial dynamical homotopy groups. The vanishing of Π_1 would imply that there is only single minimal generalized Feynman diagram contributing to the scattering amplitude. This also true if Feynman diagrams correspond to arithmetic operations.

3. Whether one should call these groups homotopy groups or homology groups is not obvious. The construction means that the foliations of foliations of ... can be seen as images of spheres suggesting "homotopy". The number of these groups is determined by the dimension of imbedding space, which suggests "homology".

4. Clearly, the surfaces defining the dynamical homotopy groups/groupoids would be analogs of branes of M-theory but would be obtained constructing paths of paths of paths... by starting from preferred extremals. The construction of so called n-groups (`https://en.wikipedia.org/wiki/N-group_(category_theory)`) brings strongly in mind this construction.

3 Appendix: About field equations of TGD in jet bundle formulation

Prastaro utilizes jet bundle (`https://en.wikipedia.org/wiki/Jet_bundle`) formulation of partial differential equations (PDEs). This notion allows a very terse formulation of general PDEs as compared to the old-fashioned but much more concrete formulation that I have used. The formulation is rather formula rich and reader might lose easily his/her patience since one must do hard work to learn which formulas follow trivially from the basic definitions.

I will describe this formulation in TGD framework briefly but without explicit field equations, which can be found at [K1]. To my view a representation by using a concrete example is always more reader friendly than the general formulas derived in some reference. I explain my view about the general ideas behind jet bundle formulation with minimal number amount of formulas. The reader can find explicit formulas from the Wikipedia link above.

The basic goal is to have a geometric description of PDE. In TGD framework the geometric picture is of course present from beginning: field patterns as 4-surfaces in field space - somewhat formal geometric objects - are replaced with genuine 4-surfaces in $M^4 \times CP_2$.

3.1 Field equations as conservation laws, Frobenius integrability conditions, and a connection with quaternion analyticity

The following represents qualitative picture of field equations of TGD trying to emphasize the physical aspects. Also the possibility that Frobenius integrability conditions are satisfied and correspond to quaternion analyticity is discussed.

1. Kähler action is Maxwell action for induced Kähler form and metric expressible in terms of imbedding space coordinates and their gradients. Field equations reduce to those for imbedding space coordinates defining the primary dynamical variables. By GCI only four of them are independent dynamical variables analogous to classical fields.

2. The solution of field equations can be interpreted as a section in fiber bundle. In TGD the fiber bundle is just the Cartesian product $X^4 \times CD \times CP_2$ of space-time surface X^4 and causal diamond $CD \times CP_2$. CD is the intersection of future and past directed light-cones having two light-like boundaries, which are cone-like pieces of light-boundary $\delta M^4_\pm \times CP_2$. Space-time surface serves as base space and $CD \times CP_2$ as fiber. Bundle projection Π is the projection to the factor X^4. Section corresponds to the map $x \rightarrow h^k(x)$ giving imbedding space coordinates as functions of space-time coordinates. Bundle structure is now trivial and rather formal.

   By GCI one could also take suitably chosen 4 coordinates of $CD \times CP_2$ as space-time coordinates, and identify $CD \times CP_2$ as the fiber bundle. The choice of the base space depends on the character of space-time surface. For instance CD, CP_2 or $M^2 \times S^2$ (S^2 a geodesic sphere of CP_2), could define the base space. The bundle projection would be projection from $CD \times CP_2$ to the base space. Now the fiber bundle structure can be non-trivial and make sense only in some space-time region with same base space.

3. The field equations derived from Kähler action must be satisfied. Even more: one must have a *preferred* extremal of Kähler action. One poses boundary conditions at the 3-D ends of space-time surfaces and at the light-like boundaries of $CD \times CP_2$.

   One can fix the values of conserved Noether charges at the ends of CD (total charges are same) and require that the Noether charges associated with a sub-algebra of super-symplectic algebra isomorphic to it and having conformal weights coming as n-ples of those for the entire algebra, vanish. This would realize the effective 2-dimensionality required by SH. One must pose boundary conditions also at the light-like partonic orbits. So called weak form of electric-magnetic duality is at least part of these boundary conditions.

It seems that one must restrict the conformal weights of the entire algebra to be non-negative $r \geq 0$ and those of subalgebra to be positive: $mn > 0$. The condition that also the commutators of sub-algebra generators with those of the entire algebra give rise to vanishing Noether charges implies that all algebra generators with conformal weight $m \geq n$ vanish so the dynamical algebra becomes effectively finite-dimensional. This condition generalizes to the action of super-symplectic algebra generators to physical states.

M^4 time coordinate cannot have vanishing time derivative dm^0/dt so that four-momentum is non-vanishing for non-vacuum extremals. For CP_2 coordinates time derivatives ds^k/dt can vanish and for space-like Minkowski coordinates dm^i/dt can be assumed to be non-vanishing if M^4 projection is 4-dimensional. For CP_2 coordinates $ds^k/dt = 0$ implies the vanishing of electric parts of induced gauge fields. The non-vacuum extremals with the largest conformal gauge symmetry (very small n) would correspond to cosmic string solutions for which induced gauge fields have only magnetic parts. As n increases, also electric parts are generated. Situation becomes increasingly dynamical as conformal gauge symmetry is reduced and dynamical conformal symmetry increases.

4. The field equations involve besides imbedding space coordinates h^k also their partial derivatives up to second order. Induced Kähler form and metric involve first partial derivatives $\partial_\alpha h^k$ and second fundamental form appearing in field equations involves second order partial derivatives $\partial_\alpha \partial_\beta h^k$.

   Field equations are hydrodynamical, in other worlds represent conservation laws for the Noether currents associated with the isometries of $M^4 \times CP_2$. By GCI there are only 4 independent dynamical variables so that the conservation of $m \leq 4$ isometry currents is enough if chosen to be independent. The dimension m of the tangent space spanned by the conserved currents can be smaller than 4. For vacuum extremals one has $m = 0$ and for massless extremals (MEs) $m = 1$! The conservation of these currents can be also interpreted as an existence of $m \leq 4$ closed 3-forms defined by the duals of these currents.

5. The hydrodynamical picture suggests that in some situations it might be possible to assign to the conserved currents flow lines of currents even globally. They would define $m \leq 4$ global coordinates for some subset of conserved currents (4+8 for four-momentum and color quantum numbers). Without additional conditions the individual flow lines are well-defined but do not organize to a coherent hydrodynamic flow but are more like orbits of randomly moving gas particles. To achieve global flow the flow lines must satisfy the condition $d\phi^A/dx^\mu = k^A_B J^B_\mu$ or $d\phi^A = k^A_B J^B$ so that one can special of 3-D family of flow lines parallel to $k^A_B J^B$ at each point - I have considered this kind of possibly in [K1] at detail but the treatment is not so general as in the recent case.

   Frobenius integrability conditions (`https://en.wikipedia.org/wiki/Frobenius_theorem_(differential_topology)`) follow from the condition $d^2\phi^A = 0 = dk^A_B \wedge J^B + k^A_B dJ^B = 0$ and implies that dJ^B is in the ideal of exterior algebra generated by the J^A appearing in $k^A_B J^B$. If Frobenius conditions are satisfied, the field equations can define coordinates for which the coordinate lines are along the basis elements for a sub-space of at most 4-D space defined by conserved currents. Of course, the possibility that for preferred extremals there exists $m \leq 4$ conserved currents satisfying integrability conditions is only a conjecture.

   It is quite possible to have $m < 4$. For instance for vacuum extremals the currents vanish identically For MEs various currents are parallel and light-like so that only single light-like coordinate can be defined globally as flow lines. For cosmic strings (cartesian products of minimal surfaces X^2 in M^4 and geodesic spheres S^2 in CP_2 4 independent currents exist). This is expected to be true also for the deformations of cosmic strings defining magnetic flux tubes.

6. Cauchy-Riemann conditions in 2-D situation represent a special case of Frobenius conditions. Now the gradients of real and imaginary parts of complex function $w = w(z) = u + iv$ define two conserved currents by Laplace equations. In TGD isometry currents would be gradients

apart from scalar function multipliers and one would have generalization of C-R conditions. In citeallbprefextremals,twistorstory I have considered the possibility that the generalization of Cauchy-Riemann-Fuerter conditions [A1, A5] (http://arxiv.org/pdf/hep-th/9306080v2.pdf) could define quaternion analyticity - having many non-equivalent variants - as a defining property of preferred extremals. The integrability conditions for the isometry currents would be the natural physical formulation of CRF conditions. Different variants of CRF conditions would correspond to varying number of independent conserved isometry currents.

7. The problem caused by GCI is that there is infinite number of coordinate choices. How to pick a physically preferred coordinate system? One possible manner to do this is to use coordinates for the projection of space-time surface to some preferred sub-space of imbedding - geodesic manifold is an excellent choice. Only $M^1 \times X^3$ geodesic manifolds are not possible but these correspond to vacuum extremals.

   One could also consider a philosophical principle behind integrability. The variational principle itself could give rise to at least some preferred space-time coordinates in the same manner as TGD based quantum physics would realize finite measurement resolution in terms of inclusions of HFFs in terms of hierarchy of quantum criticalities and fermionic strings connecting partonic 2-surfaces. Frobenius integrability of the isometry currents would define some preferred coordinates. Their number need not be the maximal four however.

   For instance, for massless extremals only light-like coordinate corresponding to the light-like momentum is obtained. To this one can however assign another local light-like coordinate uniquely to obtain integrable distribution of planes M^2. The solution ansatz however defines directly an integrable choice of two pairs of coordinates at imbedding space level usable also as space-time coordinates - light-like local direction defining local plane M^2 and polarization direction defining a local plane E^2. These choices define integrable distributions of orthogonal planes and local hypercomplex and complex coordinates. Pair of analogs of C-R equations is the outcome. I have called these coordinates Hamilton-Jacobi coordinates for M^4.

8. This picture allows to consider a generalization of the notion of solution of field equation to that of integral manifold (https://en.wikipedia.org/wiki/Integrability_conditions_for_differential_systems. If the number of independent isometry currents is smaller than 4 (possibly locally) and the integrability conditions hold true, lower-dimensional submanifolds of space-time surface define integral manifolds as kind of lower-dimensional effective solutions. Genuinely lower-dimensional solutions would of course have vanishing $\sqrt{g_4}$ and vanishing Kähler action.

   String world sheets can be regarded as 2-D integral surfaces. Charged (possibly all) weak boson gauge fields vanish at them since otherwise the electromagnetic charge for spinors would not be well-defined. These conditions force string world sheets to be 2-D in the generic case. In special case 4-D space-time region as a whole can satisfy these conditions. Well-definedness of Kähler-Dirac equation [K4, K5] demands that the isometry currents of Kähler action flow along these string world sheets so that one has integral manifold. The integrability conditions would allow $2 < m \leq n$ integrable flows outside the string world sheets, and at string world sheets one or two isometry currents would vanish so that the flows would give rise 2-D independent sub-flow.

9. The method of characteristics (https://en.wikipedia.org/wiki/Method_of_characteristics) is used to solve hyperbolic partial differential equations by reducing them to ordinary differential equations. The (say 4-D) surface representing the solution in the field space has a foliation using 1-D characteristics. The method is especially simple for linear equations but can work also in the non-linear case. For instance, the expansion of wave front can be described in terms of characteristics representing light rays. It can happen that two characteristics intersect and a singularity results. This gives rise to physical phenomena like caustics and shock waves.

   In TGD framework the flow lines for a given isometry current in the case of an integrable flow would be analogous to characteristics, and one could also have purely geometric counterparts of shockwaves and caustics. The light-like orbits of partonic 2-surface at which the signature of the induced metric changes from Minkowskian to Euclidian might be seen as an example about the analog of wave front in induced geometry. These surfaces serve as carriers of fermion lines in generalized Feynman diagrams. Could one see the particle vertices at which the 4-D space-time surfaces intersect along their ends as analogs of intersections of characteristics - kind of caustics? At these 3-surfaces the isometry currents should be continuous although the space-time surface has "edge".

3.2 Jet bundle formalism

Jet bundle formalism (`https://en.wikipedia.org/wiki/Jet_bundle`) is a modern manner to formulate PDEs in a coordinate independent manner emphasizing the local algebraic character of field equations. In TGD framework GCI of course guarantees this automatically. Beside this integrability conditions formulated in terms of Cartan's contact forms are needed.

1. The basic idea is to take the partial derivatives of imbedding space coordinates as functions of space-time coordinates as independent variables. This increases the number of independent variables. Their number depends on the degree of the jet defined and for partial differential equation of order r, for n dependent variables, and for N independent variables the number of new degrees of freedom is determined by r, n, and N -just by counting the total number of various partial derivatives from $k = 0$ to r. For $r = 1$ (first order PDE) it is $N \times (1 + n)$.

2. Jet at given space-time point is defined as a Taylor polynomial of the imbedding space coordinates as functions of space-time coordinates and is characterized by the partial derivatives at various points treated as independent coordinates analogous to imbedding space coordinate. Jet degree r is characterized by the degree of the Taylor polynomial. One can sum and multiply jets just like Taylor polynomials. Jet bundle assigns to the fiber bundle associated with the solutions of PDE corresponding jet bundle with fiber at each point consisting of jets for the independent variables ($CD \times CP_2$ coordinates) as functions of the dependent variables (space-time coordinates).

3. The field equations from the variation of Kähler action are second order partial differential equations and in terms of jet coefficients they reduce to local algebraic equations plus integrability conditions. Since TGD is very non-linear one obtains polynomial equations at each point - one for each imbedding space coordinate. Their number reduces to four by GCI. The minimum degree of jet bundle is $r = 2$ if one wants algebraic equations since field equations are second order PDEs.

4. The local algebraic conditions are not enough. One must have also conditions stating that the new independent variables associated with partial derivatives of various order reduces to appropriate multiple partial derivatives of imbedding space coordinates. These conditions can be formulated in terms of Cartan's contact forms, whose vanishing states these conditions. For instance, if dh^k is replaced by independent variable u^k, the condition $dh^k - u^k = 0$ is true for the solution surfaces.

5. In TGD framework there are good motivations to break the non-orthodoxy and use 1-jets so that algebraic equations replaced by first order PDEs plus conditions requiring vanishing of contact forms. These equations state the conservation of isometry currents implying that the 3-forms defined by the duals of isometry currents are closed. As found, this formulation reveals in TGD framework the hydrodynamic picture and suggests conditions making the system integrable in Frobenius sense.

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