What does cognitive representability really mean?

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Abstract

The article discusses the notion of cognitive representability. Numbers in the extensions of rationals are assumed to be cognitively representable in terms of points common to real and various p-adic space-time sheets (correlates for sensory and cognitive). One allows extensions of p-adics induced by extension of rationals in question and the hierarchy of adeles defined by them.

One can however argue that algebraic numbers do not allow finite representation as do rational numbers. A weaker condition is that the coding of information about algorithm producing the cognitively representable number contains a finite amount of information although it might take an infinite time to run the algorithm (say containing infinite loops). Furthermore, cognitive representations in TGD sense are also sensory representations allowing to represent algebraic numbers geometrically ($\sqrt{2}$ as the diameter of unit square). Stern-Brocot tree associated with partial fractions indeed allows to identify rationals as finite paths connecting the root of S-B tree to the rational in question. Algebraic numbers can be identified as infinite periodic paths so that finite amount of information specifies the path. Transcendental numbers would correspond to infinite non-periodic paths. A very close analogy with chaos theory suggests itself.

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1 Introduction

The following considerations reflect the ideas inspired by Face Book debate with Santeri Satama (SS) relating to the notion of number and the notion of cognitive representation.

SS wants to accept only those numbers that are constructible, and SS mentioned the notion of demonstrability due to Gödel. According to my impression demonstrability means that number can be constructed by a finite algorithm or at least that the information needed to construct the number can be constructed by a finite algorithm although the construction itself would not be possible as digit sequence in finite time. If the constructibility condition is taken to extreme, one is left only with rationals.

As a physicists, I cannot consider starting to do physics armed only with rationals: for instance, continuous symmetries and the notion of Riemann manifold would be lost. My basic view is that we should identify the limitations of cognitive representability as limitations for what can exist.

I talked about cognitive representability of numbers central in the adelic physics approach to TGD [L1, L2]. Not all real numbers are cognitively representable and need not be so.

Numbers in the extensions of rationals would be cognitively representable as points with coordinates in an extension of rationals. The coordinates themselves are highly unique in the octonionic approach to TGD and different coordinates choices for complexified octonionic M^8 are related by transformations changing the moduli of the octonion structure. Hence one avoids problems with general coordinate invariance). Not only algebraic extensions of rationals are allowed. Neper number e is an exceptional transcendental in that e^p is p-adic number and finite-D extensions of p-adic numbers by powers for root of e are possible.

My own basic interest is to find a deeper intuitive justification for why algebraic numbers shoud be cognitively representable. The naive view about cognitive representability is that the number can be produced in a finite number of steps using an algorithm. This would leave only rationals under consideration and would mean intellectual time travel to ancient Greece.

Situation changes if one requires that only the information about the construction of number can be produced in a finite number of steps using an algorithm. This would replace construction with the recipe for construction and lead to a higher abstraction level. The concrete construction itself need not be possible in a finite time as bit sequence but could be possible physically ($\sqrt{2}$ as a diagonal of unit square, one can of course wonder where to buy ideal unit squares). Both number theory and geometry would be needed.

Stern-Brocot tree associated with partial fractions indeed allows to identify rationals as finite paths connecting the root of S-B tree to the rational in question. Algebraic numbers can be identified as infinite periodic paths so that finite amount of information specifies the path. Transcendental numbers would correspond to infinite non-periodic paths. A very close analogy with chaos theory suggests itself.

2 Demonstrability viz. cognitive representability

SS talked about demonstrable numbers. According to Gödel demonstrable number would be representable by a formula G, which is provable in some axiom system. I understand this that G would give a recipe for constructing that number. In computer programs this can even mean infinite loop, which is easy to write but impossible to realize in practice. Here comes the possibility that demonstrability does not mean constructibility in finite number of steps but only a finite recipe for this.

The requirement that all numbers are demonstrable looks strange to me. I would talk about cognitive representability and reals and p-adic number fields emerge unavoidably as prerequisites for this notion: cognitive representation must be about something in order to be a representation.

About the construction of reals or something bigger - such as surreals - containing them, there are many views and I am not mathematician enough to take strong stance here. Note however that if one accepts surreals as being demonstrable (I do not really understand what this could mean) one also accept reals as such. These delicacies are not very interesting for the formulation of physics as it is now.

The algorithm defining G defines a proof. But what does proof mean? Proof in mathematical sense would reduce in TGD framework be a purely cognitive act and assignable to the p-adic sectors of adele. Mathematicians however tend to forget that for physicist the demonstration is also experimental. Physicist does not believe unless he sees: sensory perception is needed. Experimental proofs are what physicists want. The existence of $\sqrt{2}$ as a diagonal of unit square is experimentally demonstrable in the sense of being cognitively representable but not deducible from the axioms for rational numbers. As a physicist I cannot but accept both sensory and cognitive aspects of existence.

Instead of demonstrable numbers I prefer to talk about cognitively representable numbers.

1. All numbers are cognizable (p-adic) or sensorily perceivable (real). These must form continua if one wants to avoid problems in the construction of physical theories, where continuous symmetries are in a key role.

   Some numbers but not all are also *cognitively representable* that is being in the intersection reals and p-adics - that is in extension of rationals if one allows extensions of p-adics induced by extensions of rationals. This generalizes to intersection of space-time surfaces with real/p-adic coordinates, which are highly unique linear coordinates at octonionic level so that objections relating to a loss of general coordinate invariance are circumvented. General coordinate transformations reduce to automorphisms of octonions.

The relationship to the axiom of choice is interesting. Should axiom of choice be restricted to the points of complexified octonions with coordinates in extensions of rationals? Only points in the extensions could be selected and this selection process would be physical in the sense that fermions providing realization of quantum Boolean algebra would reside at these points. In preferred octonionic coordinates the M^8 coordinates of these points would be in given extension of rationals. At the limit of algebraic numbers these points would form a dense set of reals.

Remark: The spinor structure of "world of classical worlds" (WCW) gives rise to WCW spinors as fermionic Fock states at given 3-surface. In zero energy ontology (ZEO) many-fermion states have interpretation in terms of superpositions of pairs of Boolean statements $A \rightarrow B$ with A and B represented as many-fermion states at the ends of space-time surface located at the opposite light-like boundaries of causal diamond (CD). One could say that quantum Boolean logic emerges as square root of Kähler geometry of WCW.

At partonic 2-surfaces these special points correspond to points at which fermions can be localized so that the representation is physical. Universe itself would come in rescue to make representability possible. One would not anymore try to construct mathematics and physics as distinct independent disciplines.

Even observer as conscious entity is necessarily brought into both mathematics and physics. TGD Universe as a spinor field in WCW is re-created state function reduction by reduction and evolves: evolution for given CD corresponds to the increase of the size of extension of rationals in statistical sense. Hence also mathematics with fixed axioms is replaced with a q dynamical structure adding to itself new axioms discovery by discovery [L2, L1].

2. Rationals as cognitively representable numbers conforms with naive intuition. One can however criticize the assumption that also algebraic numbers are such. Consider $\sqrt{2}$: one can simply define it as length of diagonal of unit square and this gives a meter stick of length $\sqrt{2}$: one can represent any algebraic number of form $m + n\sqrt{2}$ by using meter stricks with length of 1 and $\sqrt{2}$.

   Note that algebraic numbers in n-dimensional extension are points of n-dimensional space and their cognitive representations as points on real axis obtained by using the meter sticks assignable to the algebraic numbers defining base vectors. This should generalize to the roots arbitrary polynomials with rational or even algebraic coefficients. Essentially projection form n-D extension to 1-D real line is in question. This kind of projection might be important in number theoretical dynamics. For instance, quasi-periodic quasi-crystals are obtained from higher-D periodic crystals as projections.

   n-D algebraic extensions of p-adics induced by those of rationals might also related to our ability to imagine higher-dimensional spaces.

3. In TGD Universe cognitive representability would emerge from fundamental physics. Extensions of rationals define a hierarchy of adeles and octonionic surfaces are defined as zero loci for real or imaginary parts (in quaternionic sense) of polynomials of real argument with coefficients in extension continued to octonionic polynomials [K2]. The zeros of real polynomial have a direct physical interpretation and would represent algebraic numbers physically. They would give the temporal positions of partonic 2-surfaces representing particles at light-like boundary of CD.

4. Note that all calculations with algebraic numbers can be done without using approximations for the genuinely algebraic numbers defining the basis for the extension. This actually simplifies enormously the calculation and one avoids accumulating errors. Only at the end one represents the algebraic units concretely and is forced to use rational approximation unless one uses above kind of cognitive representation.

For these reasons I do not feel any need to get rid of algebraics or even transcendentals. Sensory aspects of experience require reals and cognitive aspects of experience require p-adic numbers fields and one ends up with adelic physics. Cognitive representations are in the intersection of reality and various p-adicities, something expressible as formulas and concrete physical realizations or at least finite recipes for them.

3 What the cognitive representability of algebraic numbers could mean?

Algebraic numbers should be in some sense simple in order to be cognitively representable.

1. For rationals representation as partial fractions produces the rational number by using a finite number of steps. One starts from the top of Stern-Brocot (S-B) tree (see `http://tinyurl.com/yb6ldekq`) and moves to right or left at each step and ends up to the rational number appearing only once in S-B tree.

2. Algebraic numbers cannot be produced in a finite number of steps. During the discussion I however realized that one can produce the information needed to construct the algebraic number in a finite number of steps. One steps to a new level of abstraction by replacing the object with the information allowing to construct the object using infinite number of steps but repeating the same sub-algorithm with finite number of steps: infinite loop would be in question.

   Similar abstraction takes place as one makes a step from the level of space-time surface to the level of WCW. Space-time surface with a continuum of points is represented by a finite number of WCW coordinates, in the octonionic representation of space-time surface by the coefficients of polynomial of finite degree belonging to an extension of rationals [K2]. Criticality conditions pose additional conditions on the coefficients. Finite number of algebraic points at space-time surface determines the entire space-time surface under these conditions! Simple names for complex things replacing the complex things is the essence of cognition!

3. The interpretation for expansions of numbers in given base suggests an analog with complexity theory and symbolic dynamics associated with division. For cognitively representable numbers the information about this dynamics should be coded by an algorithm with finite steps. Periodic orbit or fixed point orbit would be the dynamical analog for simplicity. Non-periodic orbit would correspond to complexity and possibly also chaos.

These ideas led to two approaches in attempt to understand the cognitive representability of algebraic numbers.

3.1 Generalized rationals in extensions of rationals as periodic orbits for the dynamics of division

The first approach allows to represent ratios of algebraic integers for given extension using periodic expansion in the base so that a finite amount of information is needed to code the number if one accepts the numbers defining the basis of the algebraic extension as given.

1. Rationals allow periodic expansion with respect to any base. For p-adic numbers the base is naturally prime. Therefore the information about rational is finite. One can see the expansion as a periodic orbit in dynamics determining the expansion by division m/n in given base. Periodicity follows from the fact that the output of the division algorithm for a given digit has only a finite number of outcomes so that the process begins to repeat itself sooner or later.

2. This generalizes to generalized rationals in given extension of rationals defined as ratios of algebraic integers. One can reduce the division to the construction of the expansion of ordinary rational identified as number theoretic norm $|N|$ of the denominator in the extension of rationals considered.

The norm $|N|$ of N is the determinant $|N| = det(N)$ for the linear map of extension induced by multiplication with N. $det(N)$ is ordinary (possibly p-adic) integer. This is achieved by multiplying $1/N$ by $n-1$ conjugates of N both in numerator and denominator so that one obtains product of $n-1$ conjugates in the numerator and $det(N)$ in the denominator. The computation of $1/N$ as series in the base used reduces to that in the case of rationals.

3. One has now periodic orbits in n-dimensional space defined by algebraic extensions which for ordinary rationals reduced to periodic orbits in 1-D space. This supports the interpretation of numbers as orbits of number theoretic dynamics determining the next digit of the generalized rational for given base. This picture also suggests that transcendentals correspond to non-periodic orbits. Some transcendentals could still allow a finite algorithm: in this case the dynamics would be still deterministic. Some transcendentals would be chaotic.

4. Given expansion of algebraic number is same for all extensions of rationals containing the extension in question and the ultimate extension corresponds to algebraic numbers.

The problem of this approach is that the algebraic numbers defining the extension do not have representation and must be accepted as irreducibles.

3.2 Algebraic numbers as infinite periodic orbits in the dynamics of partial fractions

Second approach is based on partial fractions and Stern-Brocot tree (see `http://tinyurl.com/yb6ldekq`, see also `http://tinyurl.com/yc6hhboo`) and indeed allows to see information about algebraic numbers as constructible by using an algorithm with finite number of steps, which is allowed if one accepts abstraction as basic aspect of cognition. I had managed to not become aware of this possibility and am grateful for SS for mentioning the representation of algebraics in terms of S-B tree.

1. The definition S-B tree is simple: if m/n and m'/n' are any neighboring rationals at given level in the tree, one adds $(m+m')/(n+n')$ between them and obtains in this manner the next level in the tree. By starting from $(0/1)$ and $(1/0)$ as representations of zero and ∞ one obtains $(0/1)(1/1)(1/0)$ as the next level. One can continue in this manner ad infinitum. The nodes of S-B tree represent rational points and it can be shown that given rational appears only once in the tree.

   Given rational can be represented as a finite path beginning from $1/1$ at the top of tree consisting of left moves L and right moves R and ending to the rational which appears only once in S-B tree. Rational can be thus constructured by a sequences $R^{a_0}L^{a_1}L^{a_2}....$ characterized by the sequence $a_0; a_1, a_2....$ For instance, $4/11 = 0+1/(2+x)$, $x = 1/(1+1/3)$ corresponds to $R^0L^2R^1L^{3-1}$ labelled by $0; 2, 1, 3$.

2. Algebraic numbers correspond to infinite but periodic paths in S-B tree in the sense that some sequence of L:s and R:s characterized by sequences of non-negative integers starts to repeat itself. Periodicity means that the information needed to construct the number is finite.

   The actual construction as a digit sequence representing algebraic number requires infinite amount of time. In TGD framework octonionic physics would come in rescue and construct algebraic numbers as roots of polynomials having concrete interpretations as coordinate values assignable to fermions at partonic 2-surfaces.

3. Transcendentals would correspond to non-periodic infinite sequences of L:s and R:s. This does not exclude the possibility that these sequences are expressible in terms of some rule involving finite number of steps so that the amount of information would be also now finite. Information about number would be replaced by information about rule.

   This picture conforms with the idea about transition to chaos. Rationals have finite paths. A possible dynamical analog is particle coming at rest due to the dissipation. Algebraic numbers would correspond to periodic orbits possible in presence of dissipation if there is external feed of energy. They would correspond to dynamical self-organization patterns.

Remark: If one interprets the situation in terms of conservative dynamics, rationals would correspond to potential minima and algebraic numbers closed orbits around them.

The assignment of period doubling and p-pling to this dynamics as the dimension of extension increases is an attractive idea. One would expect that the complexity of periodic orbits increases as the degree of the defining irreducible polynomial increases. Algebraic numbers as maximal extension of rationals possibly also containing extension containing all rational roots of e and transcendentals would correspond to chaos.

Transcendentals would correspond to non-periodic orbits. These orbits need not be always chaotic in the sense of being non-predictable. For instance, Neper number e can be said to be p-adically algebraic number (e^p is p-adic integer albeit infinite as real integer). Does the sequence of L:s and R:s allow a formula for the powers of L and R in this case?

4. TGD should be an integrable theory which suggests that scattering amplitudes involve only cognitive representations as number theoretic vision indeed strongly suggets [K2]. Cognitively representable numbers would correspond to the integrable sub-dynamics. Also in chaotic systems both periodic and chaotic orbits are present. Complexity theory for characterization of real numbers exists. The basic idea is that complexity is measured by the length of the shortest program needed to code the bit sequences coding for the number.

4 Surreals and ZEO

The following comment is not directly related to cognitive representability but since it emerged during discussion, I will include it. SS favors surreals (see `http://tinyurl.com/86jatas`) as ultimate number field containing reals as sub-field. I must admit that my knowledge and understanding of surreals is rather fragmentary.

I am agnostic in these issues and see no conflict between TGD view about numbers and surreals. Personally I however like very much infinite primes, integers, and rationals over surreals since they allow infinite numbers to have number theoretical anatomy [K1]. A further reason is that the construction of infinite primes resembles structurally repeated second quantization of the arithmetic number field theory and could have direct space-time correlate at the level of many-sheeted space-time. One ends up also to a generalization of real number. Infinity can be seen as something related to real norm: everything is finite with respect to various p-adic norms.

Infinite rationals with unit real norm and various p-adic norms bring in infinitely complex number theoretic anatomy, which could be even able to represent even the huge WCW and the space of WCW spinor fields. One could speak of number theoretical holography or algebraic Brahman=Atman principle. One would have just complexified octonions with infinitely richly structure points.

Surreals are represented in terms of pairs of sets. One starts the recursive construction from empty set identified as 0. The definition says that the pairs $(.|.)$ of sets defining surreals x and y satisfy $x \leq y$ if the left hand part of x as set is to left from the pair defining y and the right hand part of y is to the right from the pair defining x. This does not imply that one has always $x < y$, $y < x$ or $x = y$ as for reals.

What is interesting that the pair of sets defining surreal x is analogous to a pair of states at boundaries of CD defining zero energy state. Is there a connection with ZEO? One could perhaps say at the level of CD - forgetting everything related to zero energy states - following. The number represented by CD_1 - say represented as the distance between its tip - is smaller than than the number represented by CD_2, if CD_1 is inside CD_2. This conforms with the left and righ rule if left and right correspond to the opposite boundaries of CD. A more detailed definition would presumably say that CD_1 can be moved so that it is inside CD_2.

What makes this also interesting is that CD is the geometric correlate for self, conscious entity, also mathematical mental image about number.

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