Boolean algebras, Stone spaces and TGD

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Abstract

The Facebook discussion with Stephen King about Stone spaces led to a highly interesting development of ideas concerning Boolean, algebras, Stone spaces, and p-adic physics. I have discussed these ideas already earlier but the improved understanding of the notion of Stone space helped to make the ideas more concrete. The basic ideas are briefly summarized.

p-adic integers/numbers correspond to the Stone space assignable to Boolean algebra of natural numbers/rationals with $p = 2$ assignable to Boolean logic. Boolean logic generalizes for n-valued logics with prime values of n in special role. The decomposition of set to n subsets defined by an element of n-Boolean algebra is obtained by iterating Boolean decomposition $n-2$ times. n-valued logics could be interpreted in terms of error correction allowing only bit sequences, which correspond to $n < p < 2^k$ in k-bit Boolean algebra. Adelic physics would correspond to the inclusion of all p-valued logics in single adelic logic.

The Stone spaces of p-adics, reals, etc.. have huge size and a possible identification (in absence of any other!) is in terms of concept of real number assigning to real/p-adic/etc... number a fiber space consisting of all units obtained as ratios of infinite primes. As real numbers they are just units but has complex number theoretic anatomy and would give rise to what I have assigned the terms algebraic holography and number theoretic Brahman = Atman.

1 Introduction

The Facebook discussion with Stephen King about Stone spaces (see http://tinyurl.com/ze2o4o5) led to a highly interesting development of ideas concerning Boolean, algebras, Stone spaces, and p-adic physics. I have discussed these ideas already earlier but the improved understanding of the notion of Stone space helped to make the ideas more concrete. The following piece of text emerged from the attempt to clarify thoughts and to summarize what I think (just now).

1.1 Boolean algebras

The most familiar representation of Boolean algebras (see http://tinyurl.com/cwhw8kd and http://tinyurl.com/jznz7kq) is in terms of set theory. Intersection $\cap$ and union $\cup$ for subsets of given set are the basic commutative and associative set theoretic operations having logical meaning as $\wedge$ (AND) and $\vee$. Negation $\neg$ corresponds to complement of set and is reflection like operation. $\wedge$ ($\cap$) is distributive over $\vee$ ($\cup$) just like product is distributive over sum in arithmetics ($a(b+c) = ab + ac$). $\wedge$ ($\cap$) has unit element 1 (entire set) acting as annihilator for $\vee$ ($\cup$). $\vee$ ($\cup$) has unit element 0 (empty set) acting as annihilator for $\wedge$ ($\cap$). Both $\wedge$ ($\cap$) and $\vee$ ($\cup$) are idempotent and are thus analogous to projection operations. The law of absorbtion states $x \wedge (x \vee y) = x \vee (x \wedge y) = x$. Only distribution law breaks the symmetry between $\wedge$ and $\vee$.

For sets the Boolean algebra B of sets can be realized algebraically as maps from set to Boolean algebra Z_2. Given set is defined as points for which the value of map is 1 and its complement as points for which it is zero: the points of the entire set are colored with black or white, and white points form the subset. Boolean operations correspond to simple operations for these Z_2 valued functions in the set representable as bit sequences with one bit for each element of set. AND as intersection of sets corresponds to bit-wise product

$$f_1 \wedge f_2 = f_1 \times f_2 \ .$$

OR as union of sets to

$$f_1 \vee f_2 = f_1 + f_2 + f_1 \times f_2 \ .$$

Negation corresponds to the addition of bit 1 to each bit:

$$\neg f = f + 1 \ .$$

For finite sets Boolean algebra is identical to its power set consisting of its subsets and having 2^N elements if the set has N elements: each element of set corresponds to a bit telling whether it is present in the subset or not.

For infinite sets situation is not at all so obvious. For instance, for subsets of real line the condition that sets are open is in conflict with the existence of negation. The complement of open set is closed (containing its boundaries).

Stone spaces (see `http://tinyurl.com/ze2o4o5`) could be seen as a formulation of Boolean logic in which one gets rid of the difficulty. One does not try to make the topology of set consistent with Boolean algebra (by assuming that open sets correspond to all elements of Boolean algebra: this would produce discrete topology, which is totally trivial). Instead, one topologizes the Boolean algebra and the outcome is so called Stone space (or pro-finite space) in honour of Marshal Stone who discovered the notion. Stone spaces have compact-open topology meaning that open sets are also compact sets. This means that points of space - if they belong to the Boolean algebra - are open sets. If I have understood correctly the idea of Stone space is to give up the points of continuum as elements of Boolean algebra and replace Boolean algebra with the space of ultrafilters defining Stone space.

What makes Stone spaces so interesting from the point of view of TGD is that also p-adic numbers are Stone spaces. My first misunderstanding was that *all* Stone spaces are associated with Boolean algebras. This is not the case. The obvious guess is however that 2-adic numbers as sequences of possibility infinite bits ordered by their significance correspond to some Boolean algebra. A slight generalization would suggest that p-adic numbers correspond to p-valued logics and "p-Boolean" algebra for some set. Some-one has said that God created the natural numbers and humans did the rest so that the first guess is that this set consists of natural numbers. In the following also these innocent guesses are considered in more detail.

1.2 Stone spaces

First some basic notions about Boolean algebras relevant to the notion of Stone space.

1. The notion of filter is important in the theory of Boolean algebras and Stone spaces (see `http://tinyurl.com/hhvvpe4`). Non-empty subset of F of Boolean algebra B is a filter if

   (a) for any pair x, y elements of F there exists $z \leq x, z \leq y$,

   (b) for any x in F and $x \leq y$, also y belongs to F.

   It is easy to see that filter does not contain mutually inconsistent statements. It is like the set of all theorems of axiomatic system with some basic axioms from which theorems are deduced.

2. Ultrafilter is a filter not contained in any filter. Ultrafilter has an important property that for every element x in Boolean algebra either x or its negation $\neg x$ but not both belongs to ultrafilter.

   Ultrafilters on a Boolean algebra can be related to prime ideals, maximal ideals, and homomorphisms to the 2-element Boolean algebra Z_2. For given homomorphism of this kind the inverse image of "true" is ultrafilter. The inverse image of false is a maximal ideal. Given a maximal ideal, its complement is an ultrafilter and there is unique homomorphism taking the maximal ideal to "false". The dual of this statement holds for given ultrafilter.

Prime ideals of Boolean algebra are maximal and have the property that if $x \wedge y$ belongs to the ideal, then either x or y does so. In finite case maximal ultrafilter the number of elements in maximal ultrafilter is one half of that for the entire Boolean algebra. Maximal ideal and ultrafilter correspond to subset and its complement in Boolean algebra ideal contains empty set and ultrafilter entire set.

3. Stone space (see `http://tinyurl.com/jsapyeq`) for a set S is defined as the set of ultrafilters for the Boolean algebra associated with it. It is contained by the power set of S consisting of its subsets but not equal to it if the set is infinite. Ultrafilters are equivalent with Z_2 homomorphisms from the set. The realization of ultrafilters as inverse images of "true" for Z_2 valued homomorphisms allow to understand Stone space as the set of true statements about fundamental statements defined by the points of the set.

   Homomorphism property tells that these statements about fundamental statements are logically consistent: either given element of Boolean algebra or its negation belongs to the ultrafilter. From Wikipedia (see `http://tinyurl.com/ofysow5`) one learns that for a finite set Boolean algebra equals to its power set. The Boolean algebra of infinite set is a subset of power set. One can intuit that at least points and discrete subsets are excluded.

4. Category theory is an additional aspect. Homomorphisms between Boolean algebras correspond to homeomorphisms between their Stone spaces.

A couple of additional remarks relevant for TGD point of view about Stone spaces are in order.

1. Given Stone space is not necessarily associated with any Boolean algebra as the space of its ultrafilters.

2. What is important is the "statements about statements" structure and ultrafilter as set of true statements about statements. Stone space represents higher level of abstraction hierarchy.

   Around 1990 or so I discussed for the first time a model of genetic code inspired by so called Combinatorial Hierarchy [K2] [L1]. Mersenne prime $M_7 = 2^{127}-1$ corresponds to the number of elements a Boolean algebra of 7 bits with the statement corresponding to physically non-realizable empty set thrown away. One can however choose 64 statements representable using 6 bits and identified in terms of genetic code as mutually consistent statements which are identically true, I talked about axioms but the correct interpretation is perhaps as theorems deducible from axioms. This selection of 64 bit sequences is nothing but selection of an ultrafilter, which I did not realize because I could not go to Wikipedia and check what it says about Boolean algebras.

2 Stone spaces and TGD

p-Adic number fields define Stone space and one expect that 2-adic numbers correspond to Boolean algebra. p-Adic numbers would most naturally correspond to p-valued logic. What could be the interpretation of p-valued logic? The difficult quesetion concerns the Stone spaces associated various classical number fields? Could TGD allow to speculate about them?

2.1 p-Adic numbers and Stone spaces

Some examples might make the notion of Stone space more concrete and clarify the connection to p-adic physics as physics of cognition and therefore also physics of Boolean mind.

1. 2-adic integers define Stone space for natural numbers very naturally. The '1':s in the bit sequence specify the elements of the subset.

2. Also p-adic integers are a Stone space but defined in terms of Z_p valued homomorphisms from natural numbers to Z_p defining p-valued logic and analogs of its ultrafilters. In this case the set is decomposed to p subsets with different colors and generalized union and intersection can be defined for these decompositions using exactly the same algebraic formulas as in the case of Boolean algebra.

What is important is that these operations are not anymore operations for a pair of subsets but for two decompositions of the set to p subsets. Cyclic transformations in Z_p are natural operations analogous to negation. Now however p:th power represents identify transformation. The operation $x \to -x$ is possible since Z_p is finite field but is trivial for $p = 2$.

For natural numbers p-valued logic gives p-adic integers as decomposition of natural numbers to p sub-sets. The homomorphisms generalize also to positive rationals and one expects that Stone space consists of all p-adic numbers. There are good reasons to expect that one can extend this notion also to algebraic extensions of rationals and corresponding integers so that algebraic extensions of p-adic numbers have interpretation as Stone space for corresponding algebraic extension of rationals.

3. Also n-valued logic is possible and correspond to expansions of natural numbers in powers of n. Z_n is not however finite field unless n is a power of prime - for $n = p^k$ one obtains finite field $G(p, k)$ reducing to Z_p for $k = 1$. It also makes sense to speak about n-adic topology but n-adic numbers numbers form only ring rather than number field unless n is prime. For general n the operation $x \to -x$ does not exist by the loss of field property anymore but other operations are well-defined.

4. In TGD framework adelic picture strongly suggests that 2-valued logic is only the lowest one in the hierarchy of p-valued logics. A possible interpretation for $p < 2^k$-valued logics is in terms of error correction and will be discussed below. One selects p statements from k-bit Boolean algebra and error correction routine checks whether the k-bit sequence belongs to this sub-space. The classical analog of error correction in quantum computation. p-Adic evolution would have interpretation in terms of evolving error correction mechanisms.

Could the generation of elements of n-valued logic (n-Boolean algebra) consisting of n subsets of set be reduced to Boolean measurements decomposing set to subset and its complement?

1. A natural operation yielding decomposition of a set to n-subsets is as a sequence of Boolean measurements. Decompose first the set to set x and its complement by Boolean measurement, decompose then x to set y and its complement, etc... n-valued logic would require $n - 2$ Boolean measurements for independent observables. The problem is how one selects the set to be decomposed at given step and there are $n - 2$ choices meaning 2^{n-2} manners to do the operation labelled by elements of $n - 2$-bit Boolean algebra. One possibility is that at each step the next set to be decomposed corresponds to "true" for the previous Boolean measurement. This construction might allow to express elements of n-Boolean algebra as sequences of elements for Boolean algebra and sub-algebras associated with subsets.

2. Physically this process could correspond to a sequence of Boolean measurements. Measure first the Boolean variable P_1 for the elements of set. After than measure whether Boolean variable P_2 is true for the subset for which P_1 is true and false. This gives decomposition of this set to n subsets defining a sequence of truth values (P_1=false, P_1=true and P_2=false,..., P_i=true, $i = 1, .., n - 3$ and P_{n-2}= true/false). The sets of decomposition are ordered with respect to the number of measured properties P_i and thus amount of information. It is not clear how unique this decomposition process is.

One can consider several physical realizations of the n-valued logics. An attractive idea is that all discrete quantum numbers could provide a realization for these logics.

1. 2-valued logic allows a natural fermionic realization. In positive energy ontology superselection rule for fermion numbers makes this realization problematic but in zero energy ontology (ZEO) the problem is avoided. In ZEO one can interpret zero energy states as assigning to a quantum superposition of quantum Boolean statements represented by positive energy state similar similar superposition represented by negative energy state. Physical laws correspond to the conservation laws for various fermionic quantum numbers.

2. Pairs of space-time sheets connected by wormhole contact are fundamental in TGD: for instance, elementary particles correspond to this kind of structures. An interesting question is whether they could give rise to a geometric realization of Boolean logic.

3. n-valued logics could allow alternative realization realization in terms of algebraic extensions of rationals defined by roots of unity. In p-adic context 2-valued logic does not require extensions ($exp(i\pi) = -1$) and this could exclude their realization in this manner.

4. The inclusions of hyperfinite factors are labelled by n:th roots of unity with $n > 2$ and one can assign to this hierarchy Kac-Moody groups defined by simply laced Lie-groups which are excellent candidates for dynamical symmetries in TGD Universe.

   The hierarchy of Planck constants realized in terms of n-sheeted covering spaces could provide a realization of n-valued logic. The internal quantum numbers assignable to the internal dynamical symmetries would emerge as remnants of the huge super-symplectic symmetry [K1]. For preferred extremals a sub-algebra isomorphic to super-symmetric algebra and its commutator with super-symmetric algebra annihilate physical states and what is obtained is presumably Kac-Moody algebra fof simply laced Lie-group. For this algebra also classical Noether charges are non-vanishing. This would give additional spin like degrees of freedom and could allow to realize n-valued logics in terms of quantum states.

2.1.1 p-valued logic and hierarchy of partition algebras

As found, one can formally generalize Boolean logic to a logic in finite field $G(p)$ with p elements. p-Logics have very nice features. For a given set the p-Boolean algebra can be represented as maps having values in finite field $G(p)$. The subsets with a given value $0 \leq k < p$ define subsets of a partition and one indeed obtains p subsets some of which are empty unless the map is surjection.

The basic challenges are following: generalize logical negation and generalize Boolean operations AND and OR. I have considered several options but the one based on category theoretical thinking seems to be the most promising one. One can imbed p_1-Boolean algebras to p-Boolean algebra by considering functions which have values in $G(p_1) \subset G(p)$. One can also project $G(p)$ valued functions to $G(p_1)$ by mod p_1 operation. The operations should respect the logical negation and p-Boolean operations if possible.

1. The basic question is how to define logical negation. Since 2-Boolean algebra is imbeddable to any p-Boolean algebra, it is natural to require that also in p-Boolean case the operation permute 0 and 1. These elements are also preferred elements algebraically since they are neutral elements for sum and product. This condition could be satisfied by simply defining negation as an operation leaving other elements of $G(p)$ un-affected. An alternative definition would be as shift $k \to k-1$. This is an attractive option since it corresponds to a cyclic symmetry.For $G(p)$ also higher powers of this operation would define analogs of negation in accordance with p-valuedness.

   I have considered also the possibility that for $p > 2$ the analog of logical negation could be defined as an additive inverse $k \to p-k$ in $G(p)$ and $k = p-1$ would be mapped to $k = 1$ as one might expect. The non-allowed value $k = 0$ is mapped to $k = p = 0$. $k = 0$ would be its own negation. This would suggest that $k = 0$ corresponds to an ill-defined truth value for $p > 2$. For $p = 2$ $k = 0$ must however correspond to false. This option is not however consistent with category theory inspired thinking.

2. For $G(p)$-valued functions f, one can define the p-analogs of both XOR (excluded or [(A OR B) but not (A AND B)] and AND using local sum and product for the everywhere-non-vanishing $G(p)$-valued functions. One can also define the analog of OR in terms of $f_1 + f_2 - f_1 f_2$ for arbitrary $G(p)$-valued functions. Note that minus sign is essential as one can see by considering $p = 3$ case ($1 + 1 - 1 \times 1 = 1$ and $1 + 1 + 1 \times 1 = 0$). For $p = 2$ this would give ordinary OR and it would be obviously non-vanishing unless both functions are identically zero. For $p > 2$ $AORB$ defined in this manner $f_1 + f_2 - f_1 f_2$ for functions having no zeros can however have zeros. The mod p_1 projection from $G(p) \to G(p_1)$ indeed commutes with these operations.

   Could 3-logic with 0 interpreted as ill-defined logical value serve as a representation of Boolean logic? This is not the case: $1 \times 2 = 2$ would correspond to $1 \times 0 = 0$ but $2 \times 2 = 1$ does not correspond to $0 \times 0 = 0$.

3. It would be nice to have well-defined inverse of Boolean function giving additional algebra structure for the partitions. For non-vanishing values of $f(x)$ one would have $(1/f)(x) = 1/f(x)$. How to define $(1/f)(x)$ for $f(x) = 0$? One can consider three options.

   (a) Option I: If 0 is interpreted as ill-defined value of p-Boolean function, there is a temptation to argue that the value of $1/f$ is also ill defined: $(1/f)(x) = 0$ for $f(x) = 0$. That function values would be replaced with their inverses only at points, where they are no-vanishing would conform with how ill-defined Boolean values are treated in computation. This leads to a well-defined algebra structure but the inverse defined in this manner is only local inverse. One has $f \circ f^{-1)(x)=1}$ only for $f(x) \neq 0$. One has algebra but not a field.

   (b) Option II: One could consider the extension of $G(p)$ by the inverse of 0, call it ∞, satisfying $0 \times \infty = 1$ ("false" AND ∞ = "true"!). Arithmetic intuition would suggest $k \times \infty = \infty$ for $k > 0$ and $k + \infty = \infty$ for all k.

   On the other hand, the interpretation of + as XOR would suggest that $k+\infty$ corresponds to [(k OR ∞) but not (k AND ∞)=∞] suggesting $k + \infty = k$ so that 0 and ∞ would be in completely symmetrical position with respect to product and sum ($k\infty = k$ and $k+0 = k$; $k\times\infty = \infty$ and $k\times 0 = 0$). It would be nice to have a logical interpretation for the inverse and for the element ∞. Especially so in 2-Boolean case. A plausible looking interpretation of ∞ would be as "ill-defined" implying that [$k \sum \infty$] and [k AND ∞] is also "ill-defined". ["false" AND "ill-defined"]="true" sounds however strange.

   For a set with N elements this would give a genuine field with $(p+1)^N$ elements. For the more convincing arithmetic option the outcome is completely analogous to the addition of point ∞ to real or complex numbers.

   (c) Option III: One could also consider functions, which are non-vanishing at all points of the set are allowed. This function space is not however closed under summation.

4. For these three options one would have $K(N) = p^N$, $K(N) = (p+1)^N$ and $K(N) = (p-1)^N$ different maps of this kind having additive and multiplicative inverses. This hierarchy of statements about statements continues ad infinitum with $K(n) = K(K(n-1))$. For Option II this gives $M(n) = (p+1)^{M(n-1)}$ so that one does not obtain finite field $G(p, N)$ with p^N elements but function field.

5. One can also consider maps for which values are in the range $0 < k < p$. This set of maps would be however closed with respect to OR and would not obtain hierarchy of finite fields. In this case the interpretation of 0 would be is un-determined and for $p = 2$ this option would be trivial. For $p = 3$ one would have effectively two well-defined logic values but the algebra would not be equivalent with ordinary Boolean algebra.

The outcome for Option II would be a very nice algebraic structure having also geometric interpretation possibly interesting from the point of view of logic. p-Boolean algebra provides p-partitions with generalizations of XOR, OR, AND, negation, and finite field structure at each level of the hierarchy: kind of calculus for p-partitions.

The lowest level of the algebraic structure generalizes as such also to p-adic-valued functions in discrete or even continuous set. The negation fails to have an obvious generalization and the second level of the hierarchy would require defining functions in the infinite-D space of p-adic-valued functions.

2.1.2 p-Valued logics and error correction

Can one imagine any interpretation for the p-valued - and more generally - n-valued logics?

1. Error correction suggests a possible interpretation of p-valued logic. In quantum computation error correction poses conditions on the quantum states so that sub-space of all possible quantum states is realized. The idea is to check whether the state belongs to this space: if not, error has occurred and must be corrected.

   In the same manner one could perhaps choose a n-element subset in n-bit Boolean algebra having $2^k > p$ elements by some constraints. Error correction algorithm would check whether the bit sequence belongs to this subset. The elements elements of k-bit Boolean algebra are labelled by integers $0, ...2^k - 1$ in a natural manner. Could the map $x \to x \mod n$ project these elements to elements of n-Boolean algebra? The elements $x \geq p$ would be mapped to same elements as x mod n or that only bit sequences $x < p$ are used. This would have a natural interpretation as pinary cutoff in p-adic topology. For some prime values of k dropping just the empty set gives Mersenne prime $M_k = 2^k - 1$ and M_k-valued logic would have a natural realization.

2. It seems that the error correction using n-valued logic does not allow a description in terms of Boolean ultrafilters and ideals for the full set. By studying the illustration of the Wikipedia article (see `http://tinyurl.com/hhvvpe4`) one can indeed get convinced that the number of elements for filters is power of two as one might expect from the logical consistency condition.

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2.3 What about Stone spaces of reals, p-adic numbers, etc.?

Can one speculate anything interesting about the Boolean algebra and Stone spaces of *real line* , *complex numbers*, or *p-adic numbers*? TGD suggests two very interesting structures. Adeles and hierarchy of infinite primes (, integers and rationals). It however seems that adeles provide as coherent description of Stone space for the product of all p-valued logics so that only infinite primes [K3] are left under consideration.

1. Real numbers are in a well-defined sense at the same hierarchy level as p-adic number fields as extensions of rationals. This is suggested also by canonical identification mapping p-adics to reals.

2. In the case of real/p-adic numbers one would have possibily infinite sequences of real/p-adic numbers and one would map each such sequence to Z_p (a map from real line to Z_p). The map cannot be continuous in real topology.

   In the case of p-adic numbers one would have Stone space of Stone space. In the general p-adic case one would have q-valued statements about p-valued statements about natural numbers realized as collections of q subsets of p-adic numbers. A priori it is not necessary to have $q = p$ although internal consistency might demand this. This might help to get some grasp about the complexity involved.

   The set of Z_q valued maps forming q-ultra-filter is extremely large and expected to have naturally q-adic topology. What this monster could be? The "world of classical worlds"

(WCW) and the generalization of the notion of real and p-adic number using the notion of algebraic holography suggested by the hierarchy of infinite primes is what comes in mind in TGD framework [K3].

If it is possible to continue to make statements about statements indefinitely (we would represent rather low level in this hierarchy!), a hierarchical structure should be in question given p_n-Boolean algebras of p_{n-1}-Boolean algebras of.... At given level one has statements about statements of previous level that is Z_{p_n} valued maps from p_{n-1}-Boolean algebra having interpretation as subsets of p_{n-1}-Boolean Stone space/p_{n-1}-Boolean algebra. The first task is to try to identify a hierarchical abstraction structure and TGD Universe is indeed full of them.

3. *Infinite primes* (integers, and rationals) could define this kind of hierarchical structure [K3]. They are obtained by a repeated second quantization of an arithmetic QFT with supersymmetry. The single particle states at the lowest level are labelled by primes and are both bosons and fermions. Infinite primes correspond to both Fock states of free bosons and fermions and to analogs of bound states. These many-particle states define single particle states at the next level of hierarchy. Infinite primes are infinite only with respect to real norm. With respect to p-adic norms they have unit norm.

   By repeated second quantization infinite primes themselves form an infinite hierarchy mappable to polynomial primes at the first level of hierarchy: these irreducible polynomials depend on single variable only. At higher levels of hierarchy one has primes, which correspond to functions of $n > 1$ variables. There is resemblance with the statements about statements hierarchy of Boolean algebras but the correspondence is not so obvious. What is common that new level is constructed using primes of previous level as building bricks.

   The interpretation of finite fermionic part of infinite prime is as a Boolean statement with true assignable to a finite number of primes of the previous level. Besides this infinite primes contain analogs of n-boson Bose Einstein condensates in various modes labelled by primes serving as analogs of momenta. Their interpretation is open.

   The physical correlate for the hierarchy infinite primes could be the hierarchy of space-time sheets and would define a hierarchy of WCWs. At the level of logic one expects also a hierarchy. The attempt to identify somehow the Stone space as the space of infinite primes does not however look a promising idea. Could it be better to try to guess the hierarchy of Stone spaces?

4. Infinite primes lead to what might be called algebraic holography or number theoretic Atman= Brahman identity [K3]. There exists a huge number of infinite integers, whose ratio equals to one as real number and has lower level p-adic norms equal to 1. These pairs of integers have also interpretation as analogs of zero energy states. Conservation of quantum numbers implying the vanishing of total quantum numbers for zero energy states would correspond to the fact that incoming and outgoing infinite integer have unit ratio in real topology although they have different number theoretic anatomies.

   The first thing to come in mind is to proceed using analogy. If p-adic number fields give Stone spaces for p-Boolean algebras of natural numbers then one might expect that the analogs of p-adic number fields for infinite primes - call them P - could give rise to Stone space for reals and p-adics. The pinary expansion of P-adic integer in power of infinite prime P however contains effectively only the lowest term for p-adic integers since already $O(P)$ term has p-adic norm $1/P = 0$. The second problem is how to make sense of the generalization of the condition $0 \leq k < p$ for the coefficients of the powers of p for infinite primes. On the other hand, infinite rationals with finite real norm make sense. This would suggest that infinite-P P-adic numbers are are just infinite-rationals of finite real norm.

   Infinite rationals of unit norm can be interpreted in both real and p-adic senses and would be number theoretically universal. Finiteness condition and ZEO suggests that one could restrict the consideration to those infinite rationals for which the real norm and p-adic norms for lower level primes equals to one. Thus one would have huge space of real units.

   One could replace both reals and p-adics and even adelics with the bundle with fiber formed by the huge infinite-D space of these units. This generalizes also to higher dimensional spaces.

Could these bundles or their fiber spaces of be identifiable as Stone spaces for reals, p-adics, and adeles in a number theoretically universal manner? There would be infinite hierarchy of these spaces.

I have proposed earlier that this extension of imbedding space and its p-adic and adelic variants could allow to realize WCW as fiber bundle with imbedding space as base space. Could this hierarchy correspond to the hierarchy of Stone spaces assignable to reals, p-adics and adeles? The only new thing would be the replacement of space-time points with a space of real units, whose structure would not be visible in real number based space-time geometry and visible only via the number theoretical anatomy and via our ability to think mathematically. Single point of space-time would represent - if not entire WCW - at least some hierarchy levels of WCW. This opens up rather wild vision about what might be behind mathematical consciousness.

5. To make this really complicated, one can of course ask whether also infinite primes could contribute to adeles at higher hierarchy levels! The definition of p-adic number fields for infinite primes is problematic unless it is possible to make the p-adic norm finite.

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