From amplituhedron to associahedron

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December 5, 2017

Abstract

Nima Arkani-Hamed et al have published an article generalizing the notion of amplituhedron to associahedron and shown that it emerges also in string theory. Associahedron characterizes the non-associativity for a product of n algebraic objects. Each face of associahedron defined in $n-2$-D space corresponds to one particular association for the product (particular bracketing). Also the proposal is made that color corresponds to something less trivial than Chan-Paton factors.

In TGD non-associativity at the level of arguments of scattering amplitude is induced from that for octonions: one can assign to space-time surfaces octonionic polynomials and induce arithmetic operations for space-time surface from those for polynomials (or even rational or analytic functions). I have already earlier demonstrated that associahedron and construction of scattering amplitudes by summing over different permutations and associations of external particles (space-time surfaces). Therefore the notion of associahedron makes sense also in TGD framework and summation reduces to "integration" over the faces of associahedron. TGD thus provides a concrete interpretation for the associations and permutations at the level of space-time geometry.

In TGD framework the description of color and four-momentum is unified at the level and the notion of twistor generalizes: one has twistors in 8-D space-time instead of twistors in 4-D space-time so Chan-Paton factors are replaced with something non-trivial.

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1 Introduction

Lubos has a nice blog posting (see http://tinyurl.com/y7ywhxew) explaining the proposal represented in the newest article by Nima Arkani-Hamed, Yuntao Bai, Song He, Gongwang Yan [?]see http://tinyurl.com/ya8zstll). Amplituhedron is generalized to a purely combinatorial notion of associahedron and shown to make sense also in string theory context (particular bracketing). The hope is that the generalization of amplituhedron to associahedron allows to compute also the contributions of non-planar diagrams to the scattering amplitudes - at least in $\mathcal{N}=4$ SYM. Also the proposal is made that color corresponds to something less trivial than Chan-Paton factors.

The remaining problem is that 4-D conformal invariance requires massless particles and TGD allows to overcome this problem by using a generalization of the notion of twistor: masslessness is realized in 8-D sense and particles massless in 8-D sense can be massive in 4-D sense.

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2 Associahedrons and scattering amplitudes

The following describes briefly the basic idea between associahedrons.

2.1 Permutations and associations

One starts from a non-commutative and non-associative algebra with product (in TGD framework this algebra is formed by octonionic polynomials with real coefficients defining space-time surfaces as the zero loci of their real or imaginary parts in quaternionic sense. One can indeed multiply space-time surface by multiplying corresponding polynomials! Also sum is possible. If one allows rational functions also division becomes possible.

All permutations of the product of n elements are in principle different. This is due to non-commutativity. All associations for a given ordering obtained by scattering bracket pairs in the product are also different in general. In the simplest case one has either $a(bc)$ or $(ab)c$ and these 2 give different outcomes. These primitive associations are building bricks of general associations: for instance, abc does not have well-defined meaning in non-associative case.

If the product contains n factors, one can proceed recursively to build all associations allowed by it. Decompose the n factors to groups of m and $n-m$ factors. Continue by decomposing these two groups to two groups and repeat until you have have groups consisting of 1 or two elements. You get a large number of associations and you can write a computer code computing recursively the number $N(n)$ of associations for n letters.

Two examples help to understand. For $n = 3$ letters one obviously has $N(n = 3) = 2$. For $n = 4$ one has $N(4) = 5$: decompose first $abcd$ to $(abc)d$, $a(bcd)$ and $(ab)(cd)$ and then the two 3 letter groups to two groups: this gives $N(4) = 2 + 2 + 1 = 5$associations and associahedron in 3-D space has therefore 5 faces.

2.2 Geometric representation of association as face of associahedron

Associations of n letters can be represented geometrically as so called Stasheff polytope (see `http://tinyurl.com/q9ga785`). The idea is that each association of n letters corresponds to a face of polytope in $n-2$-dimensional space with faces represented by the associations.

Associahedron is constructed by using the condition that adjacent faces (now 2-D polygons) intersecting along common face (now 1-D edges). The number of edges of the face codes for the structure particular association. Neighboring faces are obtained by doing minimal change which means replacement of some $(ab)c$ with $a(bc)$ appearing in the association as a building bricks or vice versa. This means that the changes are carried out at the root level.

2.3 How does this relate to particle physics?

In scattering amplitude letters correspond to external particles. Scattering amplitude must be invariant under permutations and associations of the external particles. In particular, this means that one sums over all associations by assigning an amplitude to each association. Geometrically this means that one "integrates" over the boundary of associahedron by assigning to each face an amplitude. This leads to the notion of associahedron generalizing that of amplituhedron.

Personally I find it difficult to believe that the mere combinatorial structure leading to associahedron would fix the theory completely. It is however clear that it poses very strong conditions on the structure of scattering amplitudes. Especially so if the scattering amplitudes are defined in terms of "volumes" of the polyhedrons involved so that the scattering amplitude has singularities at the faces of associahedron.

An important constraint on the scattering amplitudes is the realization of the Yangian generalization of conformal symmetries of Minkowski space. The representation of the scattering amplitudes utilizing moduli spaces (projective spaces of various dimensions) and associahedron indeed allows Yangian symmetries as diffeomorphisms of associahedron respecting the positivity constraint. The hope is that the generalization of amplituhedron to associahedron allows to generalize the construction of scattering amplitudes to include also the contribution of non-planar diagrams of at $\mathcal{N} = 4$ SYM in QFT framework.

3 Associations and permutations in TGD framework

Also in the number theoretical vision about quantum TGD one encounters associativity constraints leading to the notion of associahedron. This is closely related to the generalization of twistor approach to TGD forcing to introduce 8-D analogs of twistors [L1] (see `http://tinyurl.com/yd43o2n2`).

3.1 Non-associativity is induced by octonic non-associativity

As found in [L1], non-associativity at the level of space-time geometry and at the level of scattering amplitudes is induced from octonionic non-associativity in M^8.

1. By $M^8 - H$ duality ($H = M^4 \times CP_2$) the scattering are assignable to complexified 4-surfaces in complexified M^8. Complexified M^8 is obtained by adding imaginary unit i commutating with octonionic units I_k, $k = 1, , .., 7$. Real space-time surfaces are obtained as restrictions to a Minkowskian subspace complexified M^8 in which the complexified metric reduces to real valued 8-D Minkowski metric. This allows to define notions like Kähler structure in Minkowskian signature and the notion of Wick rotations ceases to be ad hoc concept. Without complexification one does not obtain algebraic geometry allowing to reduces the dynamics defined by partial differential equations for preferred extremals in H to purely algebraic conditions in M^8. This means huge simplifications but the simplicity is lost at the QFT-GRT limit when many-sheeted space-time is replaced with slightly curved piece of M^4.

2. The real 4-surface is determined by a vanishing condition for the real or imaginary part of octonionic polynomial with $RE(P)$ and and $IM(P)$ defined by the composition of octonion to two quaternions: $o = RE(o) + I_4 IM(o)$, where I_4 is octonionic unit orthogonal to a quaternionic sub-space and $RE(o)$ and $IM(o)$ are quaternions. The coefficients of the polynomials are assumed to be real. The products of octonionic polynomials are also octonionic polynomials (this holds for also for general power series with real coefficients (no dependence on I_k). The product is not however neither commutative nor associative without additional conditions. Permutations and their associations define different space-time surfaces. The exchange of particles changes space-time surface. Even associations do it. Both non-commutativity and non-associativity have a geometric meaning at the level of space-time geometry!

3. For space-time surfaces representing external particles associativity is assumed to hold true: this in fact guarantees $M^8 - H$ correspondence for them! For interaction regions associativity does not hold true but the field equations and preferred extremal property allow to construct the counterpart of space-time surface in H from the boundary data at the boundaries of CD fixing the ends of space-time surface.

   Associativity poses quantization conditions on the coefficients of the polynomial determining it. The conditions are interpreted in terms of quantum criticality. In the interaction region identified naturally as causal diamond (CD), associativity does not hold true. For instance, if external particles as space-time surfaces correspond to vanishing of $RE(P_i)$ for polynomials representing particles labelled by i, the interaction region (CD) could correspond to the vanishing of $IM(P_i)$ and associativity would fail. At the level of H associativity and criticality corresponds to minimal surface property so that quantum criticality corresponds to universal free particle dynamics having no dependence on coupling constants.

4. Scattering amplitudes must be commutative and associative with respect to their arguments which are now external particles represented by polynomials P_i This requires that scattering amplitude is sum over amplitudes assignable to 4-surfaces obtained by allowing all permutations and all associations of a given permutation. Associations can be described combinatorially by the associahedron!

   Remark:. In quantum theory associative statistics allowing associations to be represented by phase factors can be considered (this would be associative analog of Fermi statistics). Even a generalization of braid statistics can be considered.

Yangian variants of various symmetries are in central piece also in TGD although supersymmetries are realized in different manner and generalized to super-conformal symmetries: these include generalization of super-conformal symmetries by replacing 2-D surfaces with light-like 3-surfaces, supersymplectic symmetries and dynamical Kac-Moody symmetries serving as remnants of these symmetries after supersymplectic gauge conditions characterizing preferred extremals are applied, and Kac-Moody symmetries associated with the isometries of H . The representation of Yangian symmetries as diffeomorphisms of the associahedron respecting positivity constraint encourages to think that associahedron is a useful auxiliary tool also in TGD.

3.2 Is color something more than Chan-Paton factors?

Nima et al talk also about color structure of the scattering amplitudes usually regarded as trivial. It is claimed that this is actually not the case and that there is non-trivial dynamics involved. This is indeed the case in TGD framework. Also color quantum numbers are twistorialized in terms of the twistor space of CP_2, and one performs a twistorialization at the level of M^8 and $M^4 \times CP_2$. At the level of M^8 momenta and color quantum numbers correspond to associative 8-momenta. Massless particles are now massless in 8-D sense but can be massive in 4-D sense. This solves one of the basic difficulty of the ordinary twistor approach. A further bonus is that the choice of the imbedding space H becomes unique: only the twistor spaces of S^4 (and generalized twistor space of M^4 and CP_2 have Kähler structure playing a crucial role in the twistorialization of TGD. To sum up, all roads lead to Rome. Everyone is well-come to Rome!

4 Questions inspired by quantum associations

Associations have (or seem to have) different meaning depending on whether one is talking about cognition or mathematics. In mathematics the associations correspond to different bracketings of mathematical expressions involving symbols denoting mathematical objects and operations between them. The meaning of the expression - in the case that it has meaning - depends on the bracketing of the expression. For instance, one has $a(b+c) \neq (ab)+c$, that is $ab+ac \neq ab+c$). Note that one can change the order of bracket and operation but not that of bracket and object.

For ordinary product and sum of real numbers one has associativity: $a(bc) = (ab)c$ and $a+(b+c) = (a+b)+c$. Most algebraic operations such as group product are associative. Associativity of product holds true for reals, complex numbers, and quaternions but not for octonions and this would be fundamental in both classical and quantum TGD.

The building of different associations means different groupings of n objects. This can be done recursively. Divide first the objects to two groups, divide these tow groups to to two groups each, and continue until you jave division of 3 objects to two groups - that is abc divided into $(ab)c$ or $a(bc)$. Numbers 3 and 2 are clearly the magic numbers.

This inspire several speculative quetions related to the twistorial construction of scattering amplitudes as associative singlets, the general structure of quantum entanglement, quantum measurement cascade as formation of association, the associative structure of many-sheeted space-time as a kind of linguistic structure, spin glass as a strongly associative system, and even the tendency of social structures to form associations leading from a fully democratic paradise to cliques of cliques of ... .

1. In standard twistor approach 3-gluon amplitude is the fundamental building brick of twistor amplitudes constructed from on-shell-amplitudes with complex momenta recursively. Also in TGD proposal this holds true. This would naturally follow from the fact that associations can be reduced recursively to those of 3 objects. 2- and 3-vertex would correspond to a fundamental associations. The association defined 2-particle pairing (both associated particles having either positive or negative helicities for twistor amplitudes) and 3-vertex would have universal structure although the states would be in general decompose to associations.

2. Consider first the space-time picture about scattering [L1]. CD defines interaction region for scattering amplitudes. External particles entering or leaving CD correspond to associative space-time surfaces in the sense that the tangent space or normal space for these space-time surfaces is associative. This gives rise to M^8-H correspondence.

   These surfaces correspond to zero loci for the imaginary parts (in quaternionic sense) for octonionic polynomial with coefficients, which are real in octonionic sense. The product of $\prod_i P_i)$ of polynomials with same octonion structure satisfying $IM(P_i)=0$ has also vanishing imaginary part and space-time surface corresponds to a disjoint union of surfaces associated with factors so that these states can be said to be non-interacting.

   Neither the choice of quaternion structure nor the choice of the direction of time axis assignable to the octonionic real unit need be same for external particles: if it is the particles correspond to same external particle. This requires that one treats the space of external particles (4-surfaces) as a Cartesian product of of single particle 4-surfaces as in ordinary scattering theory.

   Space-time surfaces inside CD are non-associative in the sense that the neither normal nor tangent space is associative: $M^8-M^4\times CP_2$ correspondence fails and space-time surfaces inside CD must be constructed by applying boundary conditions defining preferred extremals. Now the real part of $RE(\prod_i P_i)$ in quaternionic sense vanishes: there is genuine interaction even when the incoming particles correspond to the same octonion structure since one does not have union of surfaces with vanishing $RE(P_i)$. This follows from s rather trivial observation holding true already for complex numbers: imaginary part of zw vanishes if it vanishes for z and w but this does not hold true for the real part. If octonionic structures are different, the interaction is present irrespective of whether one assumes $RE(\prod_i P_i)=0$ or $IM(\prod_i P_i)=0$. $RE(\prod_i P_i)=0$ is favoured since for $IM(\prod_i P_i)=0$ one would obtain solutions for which $IM(P_i)=0$ would vanish for the i:th particle: the scattering dynamics would select i:th particle as non-interacting one.

3. The proposal is that the entire scattering amplitude defined by the zero energy state - is associative, perhaps in the projective sense meaning that the amplitudes related to different associations relate by a phase factor (recall that complexified octonions are considered), which could be even octonionic. This would be achieved by summing over all possible associations.

4. Quantum classical correspondence (QCC) suggests that in ZEO the zero energy states - that is scattering amplitudes determined by the classically non-associative dynamics inside CD - form a representation for the non-associative product of space-time surfaces defined by the condition $RE(\prod_i P_i)=0$. Could the scattering amplitude be constructed from products of octonion valued single particle amplitudes. This kind of condition would pose strong constraints on the theory. Could the scattering amplitudes associated with different associations be octonionic - may be differing by octonion-valued phase factors - and could only their sum be real in octonionic sense (recall that complexified octonions involving imaginary unit i commuting with the octonionic imaginary units are considered)?

One can look the situation also from the point of view of positive and negative energy states defining zero energy states as they pairs.

1. The formation of association as subset is like formation of bound state of bound states of ... . Could each external line of zero energy state have the structure of association? Could also the internal entanglement associated with a given external line be characterized in terms of association.

   Could the so called monogamy theorem stating that only two-particle entanglement can be maximal correspond to the decomposing of $n = 3$ association to one- and two-particle associations? If quantum entanglement is behind associations in cognitive sense, the cognitive meaning of association could reduce to its mathematical meaning.

   An interesting question relates to the notion of identical particle: are the many-particle states of identical particles invariant under associations or do they transform by phase factor under association. Does a generalization of braid statistics make sense?

2. In ZEO based quantum measurement theory the cascade of quantum measurements proceeds from long to short scales and at each step decomposes a given system to two subsystems. The cascade stops when the reduction of entanglement is impossible: this is the case if the entanglement probabilities belong to an extension of extension of rationals characterizing the extension in question. This cascade is nothing but a formation of an association! Since only the state at the second boundary of CD changes, the natural interpretation is that state function reduction mean a selection of association in 3-D sense.

3. The division of n objects to groups has also social meaning: all social groups tend to divide into cliques spoiling the dream about full democracy. Only a group with 2 members - Romeo and Julia or Adam and Eve - can be a full democracy in practice. Already in a group of 3 members 2 members tend to form a clique leaving the third member outside. Jules and Catherine, Jim and Catherine, or maybe Jules and Jim! Only a paradise allows a full democracy in which non-associativity holds true. In ZEO it would be realized only at the quantum critical external lines of scattering diagram and quantum criticality means instability. Quantum superposition of all associations could realize this democracy in 4-D sense.

A further perspective is provided by many-sheeted space-time providing classical correlate for quantum dynamics.

1. Many-sheeted space-time means that physical states have a hierarchical structure - just like associations do. Could the formation of association (AB) correspond basically to a formation of flux tube bond between A and B to give AB and serve as space-time correlate for (negentropic) entanglement. Could ((AB)C) would correspond to (AB) and (C) "topologically condensed" to a larger surface. If so, the hierarchical structure of many-sheeted space-time would represent associations and also the basic structures of language.

2. Spin glass (see `http://tinyurl.com/y9yyq8ga`) is a system characterized by so called frustrations. Spin glass as a thermodynamical system has a very large number of minima of free energy and one has fractal energy landscape with valleys inside valleys. Typically there is a competition between different pairings (associations) of the basic building bricks of the system.

   Could spin glass be describable in terms of associations? The modelling of spin glass leads to the introduction of ultrametric topology characterizing the natural distance function for the free energy landscape. Interestingly, p-adic topologies are ultrametric. In TGD framework I have considered the possibility that space-time is like 4-D spin glass: this idea was originally inspired by the huge vacuum degeneracy of Kähler action. The twistor lift of TGD breaks this degeneracy but 4-D spin glass idea could still be relevant.

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