

Do Riemann-Roch theorem and Atiyah-Singer index theorem have applications in TGD?

M. Pitkänen

Email: matpitka6@gmail.com.

<http://tgdtheory.com/>.

September 20, 2017

Abstract

Riemann-Roch theorem (RR) is a central piece of algebraic geometry. Atiyah-Singer index theorem is one of its generalizations relating the solution spectrum of partial differential equations and topological data. For instance, characteristic classes classifying bundles associated with Yang-Mills theories have applications in gauge theories and string models.

The advent of octonionic approach to the dynamics of space-time surfaces inspired by $M^8 - H$ duality gives hopes that dynamics at the level of complexified octonionic M^8 could reduce to algebraic equations plus criticality conditions guaranteeing associativity for space-time surfaces representing external particles, in interaction region commutativity and associativity would be broken. The complexification of octonionic M^8 replacing norm in flat space metric with its complexification would unify various signatures for flat space metric and allow to overcome the problems due to Minkowskian signature. Wick rotation would not be a mere calculational trick.

For these reasons time might be ripe for applications of possibly existing generalization of RR to TGD framework. In the following I summarize my admittedly unprofessional understanding of RR discussing the generalization of RR for complex algebraic surfaces having real dimension 4: this is obviously interesting from TGD point of view.

I will also consider the possible interpretation of RR in TGD framework. One interesting idea is possible identification of light-like 3-surfaces and curves (string boundaries) as generalized poles and zeros with topological (but not metric) dimension one unit higher than in Euclidian signature. The discussion of RR as also the notion of infinite primes and infinite rationals as counterparts of zero energy states suggests that rational functions $R = P_1/P_2$ could be more appropriate than mere polynomials. The construction of space-time varieties would not be modified in essential manner: one would have zero loci of $IM(P_i)$ identifiable as space-time sheets and zero- and ∞ -loci of $RE(P_1/P_2)$ naturally identifiable as wormhole contacts connecting the space-time sheets.

Atiyah-Singer index theorem (AS) is one of the generalizations of RR and has shown its power in gauge field theories and string models as a method to deduce the dimensions of various moduli spaces for the solutions of field equations. A natural question is whether AS could be useful in TGD and whether the predictions of AS at H side could be consistent with $M^8 - H$ duality suggesting very simple counting for the numbers of solutions at M^8 side as coefficient combinations of polynomials in given extension of rationals satisfying criticality conditions. One can also ask whether the hierarchy of degrees n for octonion polynomials could correspond to the fractal hierarchy of generalized conformal sub-algebras with conformal weights coming as n -multiples for those for the entire algebras.

Contents

1	Introduction	2
2	Riemann-Roch theorem	2
2.1	Basic notions	3
2.2	Formulation of RR theorem	3
2.3	The dimension of the space of meromorphic functions corresponding to given divisor	4

2.4	RR for algebraic varieties and bundles	6
3	Could a generalization of Riemann-Roch theorem be useful in TGD framework?	6
3.1	What could be the analogs of zeros and poles of meromorphic function?	7
3.2	Could one generalize RR to octonionic algebraic varieties?	8
4	Could the TGD variant of Atiyah-Singer index theorem be useful in TGD?	10
4.1	AS very briefly	10
4.2	AS and TGD	12

1 Introduction

Riemann-Roch theorem (RR) (see <http://tinyurl.com/mdmbcx6>) is a central piece of algebraic geometry. Atiyah-Singer index theorem is one of its generalizations relating the solution spectrum of partial differential equations and topological data. For instance, characteristic classes classifying bundles associated with Yang-Mills theories (see <http://tinyurl.com/y9xvkhyy>) have applications in gauge theories and string models.

The advent of octonionic approach to the dynamics of space-time surfaces inspired by $M^8 - H$ duality [K6] [L1, L2] gives hopes that dynamics at the level of complexified octonionic M^8 could reduce to algebraic equations plus criticality conditions guaranteeing associativity for space-time surfaces representing external particles, in interaction region commutativity and associativity would be broken. The complexification of octonionic M^8 replacing norm in flat space metric with its complexification would unify various signatures for flat space metric and allow to overcome the problems due to Minkowskian signature. Wick rotation would not be a mere calculational trick.

For these reasons time might be ripe for applications of possibly existing generalization of RR to TGD framework. In the following I summarize my admittedly unprofessional understanding of RR discussing the generalization of RR for complex algebraic surfaces having real dimension 4: this is obviously interesting from TGD point of view.

I will also consider the possible interpretation of RR in TGD framework. One interesting idea is possible identification of light-like 3-surfaces and curves (string boundaries) as generalized poles and zeros with topological (but not metric) dimension one unit higher than in Euclidian signature. The discussion of RR as also the notion of infinite primes and infinite rationals as counterparts of zero energy states suggests that rational functions $R = P_1/P_2$ could be more appropriate than mere polynomials. The construction of space-time varieties would not be modified in essential manner: one would have zero loci of $IM(P_i)$ identifiable as space-time sheets and zero- and ∞ -loci of $RE(P_1/P_2)$ naturally identifiable as wormhole contacts connecting the space-time sheets.

Atiyah-Singer index theorem (AS) is one of the generalizations of RR and has shown its power in gauge field theories and string models as a method to deduce the dimensions of various moduli spaces for the solutions of field equations. A natural question is whether AS could be useful in TGD and whether the predictions of AS at H side could be consistent with $M^8 - H$ duality suggesting very simple counting for the numbers of solutions at M^8 side as coefficient combinations of polynomials in given extension of rationals satisfying criticality conditions. One can also ask whether the hierarchy of degrees n for octonion polynomials could correspond to the fractal hierarchy of generalized conformal sub-algebras with conformal weights coming as n -multiples for those for the entire algebras.

2 Riemann-Roch theorem

Riemann-Roch theorem (RR) is also part of enumerative geometry albeit more abstract. Instead of counting of numbers of points, one counts dimensions of various function spaces associated with Riemann surfaces. RR provides information about the dimensions for the spaces of meromorphic functions and 1-forms with prescribed zeros and poles.

2.1 Basic notions

Riemann surface is the basic notion. Riemann surface is orientable is characterized by its genus g and number of holes/punctures in it. Riemann surface can also possess marked points, which seem to be equivalent with punctures. The moduli space of these complex curves is parameterized by a moduli space $\overline{M}_{g,n}$ of curves of genus g with n marked points (see <http://tinyurl.com/yaq8n6dp>) (see <http://tinyurl.com/yaq8n6dp>).

Dolbeault cohomology (see <http://tinyurl.com/y7cvs5sx>) generalizes the notion of differential form so that it has well-defined degrees with respect to complex coordinates and their conjugates: one can write in general complex manifold this kind of form as

$$\omega = \omega_{i_1 i_2 \dots i_n, j_1 j_2 \dots j_n} dz^{i_1} \wedge dz^{i_2} \dots dz^{i_n} d\bar{z}^{j_1} \wedge d\bar{z}^{j_2} \dots d\bar{z}^{j_n} .$$

The ordinary exterior derivative d is replaced with its holomorphic counterpart ∂ and its conjugate. One can construct the counterparts of cohomology groups (Hodge theory) $H^{p,q} = H^{q,p}$. Betti numbers as numbers $h_{i,j}$ defining the dimensions of the cohomology groups forms of degrees i and j with respect to dz^i and $d\bar{z}^j$. One can define the holomorphic Euler's characteristic as $\chi_C = h_{0,0} - h_{0,1} = 1 - g$ whereas ordinary Euler characteristic is $\chi_R = h_{0,0} - (h_{0,1} + h_{1,0}) + h_{1,1} = 2(1 - g)$.

One considers meromorphic functions having poles and zeros as the only singularities (points at which the map does not preserve angles): rational functions provide the basic example. Riemann zeta provides example of meromorphic function not reducing to rational function. Holomorphic functions have only zeros and entire functions have neither zeros nor poles. If analytic functions on Riemann surfaces can be interpreted as maps of compact Riemann surface to itself rather than to complex plane, meromorphy reduces to holomorphy since the point ∞ belongs to the Riemann surface.

The elements of free group of divisors are defined as formal sums of integers associated with the points P of Riemann surface. Divisors $D = \sum_P n(P)$, where (P) is integer, are analogous to integer valued "wave functions" on Riemann surface. The number of points with $n(P) \neq 0$ is countable. The degree of divisor is obtained as the ordinary sum $deg(D)$ of the integers defining the divisor.

Although divisors can be seen as purely formal objects, they are in practice associated to both meromorphic functions and 1-forms. The divisor of a meromorphic function is known as principal divisor. Meromorphic functions and 1-forms differing by a multiplication with meromorphic function are regarded as linearly equivalent - in other words, one can add to a given divisor a divisor of a meromorphic function without changing its equivalence class. It can be shown that all divisors associated with meromorphic 1-forms linearly equivalent and one can talk about canonical divisor. Note that $deg(D)$ is linear invariant since the degree of globally meromorphic function is zero.

The motivation for the divisors is following. Consider the space of meromorphic functions h with the property that the degrees of poles associated with the poles of these functions are not higher than given integers $n(P)$. In other words, one has $\langle h(P) \rangle + D(P) \geq 0$ for all points P ($\langle h \rangle$ is the divisor of h). For $D(P) > 0$ the pole has degree not higher than $D(P)$. For non-positive $D(P)$ the function has zero of order $D(P)$ at least.

2.2 Formulation of RR theorem

With these prerequisites it is possibly to formulate RR (for Wikipedia article see <http://tinyurl.com/mdmbcx6>). The Wikipedia article is somewhat confusing and a more precise description of RR can be found in the article "Riemann-Roch theorem" by Vera Talovikova [A2] (see <http://tinyurl.com/ktww7ks>).

Let $l(D)$ be the dimension of the space of meromorphic functions with principal divisor D or 1-forms linearly equivalent with canonical divisor K . Then the equality

$$l(D) - l(K - D) = deg(D) - g + 1 \tag{2.1}$$

is true for both meromorphic functions and canonical divisors. For $D = K$ one obtains using $l(0) = 1$

$$l(K) = \text{deg}(K) - g + 2 \tag{2.2}$$

giving the dimension of the space of canonical divisors. $l(K) > 0$ in general is not in conflict with the fact that canonical divisors are linearly equivalent. $\text{deg}(K) = 2g - 2$ in the above formula gives $l(K) = g$.

$l(K) = 0$ for $g = 0$ case looks strange: one should actually make notational distinction between dimensions of spaces of meromorphic functions and one-forms (this is done in the article of Talivakova). The explanation is that $l(K)$ here is not the dimension of the space of canonical 1-forms but that of the holomorphic functions with the divisor of K . The canonical form K for the sphere has second order pole at ∞ so that one cannot have meromorphic forms holomorphic outside P .

Riemann's inequality

$$l(D) \geq \text{deg}(D) - g + 1 \tag{2.3}$$

follows from $l(K - D) \geq 0$, which can be seen as a correction term to the formula

$$l(D) = \text{deg}(D) - g + 1 \tag{2.4}$$

In what sense this is true, becomes clear from what follows.

2.3 The dimension of the space of meromorphic functions corresponding to given divisor

The simplest divisor associated with meromorphic function involves only one point. Multiplying a function, which is non-vanishing and finite at P by $(z - z(P))^{-n}$ gives a pole of order n (zero has negative order in this sense). One is interested on the dimension $l(nP)$ of the space nP of meromorphic functions and RR allows to deduce information about $l(nP)$. One is interested on the behavior of $l(nP)$ as function of genus g of Riemann surface (more general situation would allow also punctures). For $n = 0$ one has entire function without poles and zeros. Only constant function is possible: $l(0) = 1$.

First some general observations. K has degree $\text{deg}(K) = 2g - 2$, which gives $l(K) = g$. For $n = \text{deg}(D) > \text{deg}(K) = 2g - 2$ the correction term vanishes since $\text{deg}(K - D)$ becomes negative, and one has $l(D) = \text{deg}(D) - g + 1$. This gives $l(n = 2g - 1) = g$. Therefore $n \in \{2g - 1, 2g, \dots\}$ corresponds to $l(nP) \in \{g, g + 1, \dots\}$. $n < 2g - 2$ corresponds to $l(nP) = 1$. What about the range $n \in \{2, \dots, 2g - 2\}$? Note that $2g - 2$ is the negative of the Euler character of Riemann surface.

1. $g = 0$ case. K on sphere. dz canonical 1-form on Riemann sphere covered by two complex coordinate patches. $z \rightarrow w = 1/z$ relates the coordinates. There is second order pole at infinity ($dw = -dz/z^2$). One has therefore $\text{deg}(K) = -2$ for sphere in accordance with the general formula $\text{deg}(K) = 2g - 2$. The formula $l(nP) = \text{deg}(D) + 1$ holds for all n and there is no correction term now. One as $l(nP) = n + 1$.

2. $g = 1$ case.

One has $\text{deg}(K) = 2g - 2 = 0$ for torus reflecting the fact that the canonical form $\omega = dz$ has no poles or zeros (torus is obtained by identifying the cells of a periodic lattice in complex plane). Correction term vanishes since it would have negative degree for all n and one has $l(nP) \in \{1, 1, 2, 3, \dots\}$.

3. $g = 2$ case.

For $n = \text{deg}(D) \geq 2 \times 2 - 1 = 3$ gives $l(D) = n - 1$ giving for $n \geq 3$ $l(nP) \in \{2, 3, \dots\}$. What about $n = g = 2$? For generic points one has $l(2) = 1$. There are 6 points at which one has $l(D) = 2$ so that there is additional meromorphic function having pole of order 2 at this kind of point. These points are fixed points under Z_2 defining hyper-ellipticity. Note that $g \leq 2$ Riemann surfaces are always hyper-elliptic in the sense that it allows Z_2 as conformal symmetry (see <http://tinyurl.com/y9sdu4o3>).

One has therefore $l(nP) \in \{1, 1, 1, 2, \dots\}$ for a generic point and $l(nP) \in \{1, 1, 2, 2, \dots\}$ for 6 points fixed under Z_2 . An interesting question is whether this phenomenon could have physical interpretation in TGD framework.

4. $g > 2$ case.

For $g > 2$. $l(nP)$ in the range $\{2, 2g - 2\}$ can depend on point and even on the conformal moduli. There are more special points in which correction term differs from that in the generic case. $g = 3$ illustrates the situation. $n \in \{1, 1, 1, 1, 1, 2, \dots\}$ is obtained for a generic point. At special points and for $n < 3$ there are also other options for $l(nP)$. Also the dependence of $l(nP)$ on moduli emerges for $g \geq 3$. The natural guess layman is that these points are fixed points of conformal symmetries. Also now hyper-elliptic surfaces allowing projective Z_2 covering are special. In the general case hyper-ellipticity is not possible.

In TGD framework Weierstrass points(see <http://tinyurl.com/y9wehsm1>) are of special interest physically.

1. My layman guess is that special points known as Weierstrass points (see <http://tinyurl.com/y9wehsm1>) correspond to singularities for projective coverings for which conformal symmetries permute the sheets of the covering. Several points coincide for the covering since a sub-group of conformal symmetries would act trivially on the Weierstrass point.

Note that for $g > 2$ Z_2 covering is not possible except for hyper-elliptic surfaces, and one can wonder whether this relates to the experimental absence of $g > 2$ fermion families [K1]. Second interesting point is that elementary particles indeed correspond to double sheeted structures from the condition that monopole fluxes flow along closed flux tubes (there are no free magnetic monopoles).

2. There is an obvious analogy with the coverings associated with the cognitive representation defined by the points of space-time surface with coordinates in an extension of rationals [K7, K6] [L3]. Fixed points for a sub-group of Galois group generate singularities at which sheets touch each other. These singular points are physically the most interesting and could carry sparticles. The action of discrete conformal groups restricted to cognitive representation could be represented as the action of Galois group on points of cognitive representation. Cognitive representation would indeed represent!

Remarkably, if the tangent spaces are not parallel for the touching sheets, these points are mapped to several points in H in $M^8 - H$ correspondence. If this picture is correct, the hyper-elliptic symmetry Z_2 of genera $g \leq 2$ could give rise to this kind of exceptional singularities for $g \geq 2$.

What is worrying that there are two views about twistorial amplitudes. One view relying on the notion of octonionic super-space M^8 [K6] is analogous to that of SUSYs: sparticles can be seen as completely local composites of fermions. Second view relies on imbedding space $M^4 \times CP_2$ [K5] and on the identification sparticles as non-local many-fermion states at partonic 2-surfaces. These two views could be actually equivalent by $M^8 - H$ duality.

3. The actual construction solutions however strongly suggests that $RE(P) = 0$ and $IM(P) = 0$ conditions can have genuine super counterparts only if the zero locus is 1-D and the super-octonions reduce to those having interpretation as single fermion states. These string like objects could form 1-sub-varieties of 4-D space-time varieties. If this picture makes sense then it is not possible to have 3-vertices for sparticles and H picture would be true. Sparticle lines might however emerge as local degeneration of several branches of space-time variety to single one. Sparticles would be critical and unstable and possess higher symmetry. Their instability would cause breaking of octonionic SUSY and could perhaps even explain why sparticles are not observed at LHC.
4. When these singular points are present at partonic 2-surfaces at boundaries of CD and at vertices, the topology of partonic 2-surface is in well-defined sense between g and $g+1$ external particles: one has criticality. The polynomials representing external particles indeed satisfy criticality conditions guaranteeing associativity or co-associativity (quantum criticality of

TGD Universe is the basic postulate of quantum TGD). At partonic orbits the touching pieces of partonic 2-surface could separate (g) or fuse ($g + 1$). Could this topological mixing give rise to CKM mixing of fermions [K1, K2, K3]?

2.4 RR for algebraic varieties and bundles

RR can be generalized to algebraic varieties (see <http://tinyurl.com/y9asz4qg>). In complex case the real dimension is four so that this generalization is interesting from TGD point of view and will be considered later. The generalization involves rather advanced mathematics such as the notion of sheaf (see <http://tinyurl.com/nudhxo6>). Zeros and poles appearing in the divisor are for complex surfaces replaced with 2-D varieties so that the generalization is far from trivial.

The following is brief summary based on Wikipedia article.

1. Genus g is replaced with algebraic genus and $\deg(D)$ plus correction term is replaced with the intersection number (see <http://tinyurl.com/y7dcffb6>) for D and $D - K$, where K is the canonical divisor associated with 2-forms, which is also unique apart from linear equivalence. Points of divisor are replaced with 2-varieties.
2. The generalization to complex surfaces (with real dimension equal to 4) reads as

$$\chi(D) = \chi(0) + \frac{1}{2}D \cdot (D - K) . \quad (2.5)$$

$\chi(D)$ is holomorphic Euler characteristic associated with the divisor. $\chi(0)$ is defined as $\chi(0) = h_{0,0} - h_{0,1} + h_{0,2}$, where $h_{i,j}$ are Betti numbers for holomorphic forms. \cdot denotes intersection product in cohomology made possibly by Poincare duality. K is canonical two-form which is a section of determinant bundle having unique divisor (there is linear equivalence due to the possibility to multiply with meromorphic function).

One has $\chi(0) = 1 + p_a$, where p_a is arithmetic genus. Noether's formula gives

$$\chi(0) = \frac{c_1^2 + c_2}{12} = \frac{K \cdot K + e}{12} . \quad (2.6)$$

c_1^2 is Chern number and $e = c_2$ is topological Euler characteristic.

Clearly the information given by $\chi(D)$ is about Dolbeault homology. For comparison note that RR for curves states $l(D) - l(K - D) = \chi(D) = \chi(0) + \deg(D)$.

RR can be also generalized so that it applies to vector bundles. Ordinary RR can be interpreted as applying to a bundle for which the fiber is point. This requires the notion of the inverse bundle defined as a bundle with the property that its direct sum (Whitney sum) with the bundle itself is trivial bundle. One ends up with various characteristic classes, which represent homologically non-trivial forms in the base spaces characterizing the bundle. For instance, the generalizations of RR give information about the dimensions of the spaces of sections of the vector bundle.

Atiyah-Singer index theorem (see <http://tinyurl.com/k6daqco>) deals with so called elliptic operators in compact manifolds and represents a generalization important in recent theoretical physics, in particular gauge theories and string models. The theorem relates analytical index - typically characterizing the dimension for the spectrum of solutions of elliptic operator to a topological index. Elliptic operator is assigned with small perturbations for a given solution of field equations. Perturbations of a given solution of say Yang-Mills equations is a representative example.

3 Could a generalization of Riemann-Roch theorem be useful in TGD framework?

The generalization of RR for algebraic varieties, in particular for complex surfaces (real dimension equal to 4) exists. In M^8 picture the complexified metric Minkowskian signature need not cause

any problems since the situation can be reduced to Euclidian sector. Clearly, this picture would provide a realization of Wick rotation as more than a trick to calculate scattering amplitudes.

Consider first the motivations for the desire of having analog of Riemann-Roch theorem (RR) at the level of space-time surfaces in M^8 .

1. It would be very nice if partonic 2-surfaces would have interpretation as analogs of zeros or poles of a meromorphic function. RR applies to the divisors characterizing meromorphic functions and 2-forms, and one could hope of obtaining information about the dimensions of these function spaces giving rise to octonionic space-time varieties. Note however that the reduction to real polynomials or even rational functions might be already enough to give the needed information. Rational functions are required by the simplest generalization whereas the earlier approach assumed only polynomials. This generalization does not however change the construction of space-time varieties as zero loci of polynomials in an essential manner as will be found.
2. One would like to count the degeneracies for the intersections of 2-surfaces of space-time surface and here RR might help since its generalization to complex surfaces involves intersection form as was found in the brief summary of RR for complex surfaces with real dimension 4 (see Eq. 2.5).

In particular, one would like to know about the intersections of partonic 2-surfaces and string world sheets defining the points at which fermions reside. The intersection form reduces the problem via Poincare duality to 2-cohomology of space-time surfaces. More generally, it is known that the intersection form for 2-surfaces tells a lot about the topology of 4-D manifolds (see <http://tinyurl.com/y8tmqtef>). This conforms with SH. Gromow-Witten invariants [A1] (see <http://tinyurl.com/ybobccub>) are more advanced rational valued invariants but might reduce to integer valued invariants in TGD framework [L2].

There are also other challenges to which RR might relate.

1. One would like to know whether the intersection points for string world sheets and partonic 2-surfaces can belong in an extension of rationals used for adèle. If the points belong to cognitive representations and subgroup of Galois group acts trivially then the number of points is reduces as the points at its orbit fuse together. The sheets of the Galois covering would intersect at point. The images of the fused points in H could be disjoint points since tangent spaces need not be parallel.
2. One would also like to have idea about what makes partonic 2-surfaces and string world sheets so special. In 2-D space-time one would have points instead of 2-surfaces. The obvious idea is that at the level of M^8 these 2-surfaces are in some sense analogous to poles and zeros of meromorphic functions. At the level of H the non-local character of $M^8 - H$ would imply that preferred extremals are solutions of an action principle giving partial differential equations.

3.1 What could be the analogs of zeros and poles of meromorphic function?

The basic challenge is to define what notions like pole, zero, meromorphic function, and divisor could mean in TGD context. The most natural approach based on a simple observation that rational functions need not define map of space-time surface to itself. Even though rational function can have pole inside CD, the point ∞ need not belong to the space-time variety defined the rational functions. Hence one can try the modification of the original hypothesis by replacing the octonionic polynomials with rational functions. One cannot exclude the possibility that although the interior of CD contains only finite points, the external particles outside CD could extend to infinity.

1. For octonionic analytic polynomials the notion of zero is well-defined. The notion of pole is well-defined only if one allows rational functions $R = P_1(o)/P_2(o)$ so that poles would correspond to zeros for the denominator of rational function. 0 and ∞ are both unaffected by multiplication and ∞ also by addition so that they are algebraically special. There are several variants of this picture. The most general option is that for a given variety zeros of both P_i are allowed.

2. The zeros of $IM(P_1) = 0$ and $IM(P_2) = 0$ would give solutions as unions of surfaces associated with P_i . This is because $IM(o_1 o_2) = IM(o_1)RE(o_2) + IM(o_2)RE(o_1)$. There is no need to emphasize how important this property of IM for product is. One might say that one has two surfaces which behave like free non-interacting particles.
3. These surfaces should however interact somehow. The intuitive expectation is that the two solutions are glued by wormhole contacts connecting partonic 2-surfaces corresponding to $IM(P_1) = 0$ and $IM(P_2) = 0 = \infty$. For $RE(P_i) = 0$ and $RE(P_i) = \infty$ the solutions do not reduce to separate solutions $RE(P_1) = 0$ and $RE(P_2) = 0$. The reason is that the real part of $o_1 o_2$ satisfies $Re(o_1 o_2) = Re(o_1)Re(o_2) - Im(o_1)Im(o_2)$. There is a genuine interaction, which should generate the wormhole contact. Only at points for which $P_1 = 0$ and $P_2 = 0$ holds true, $RE(P_1) = 0$ and $RE(P_2) = 0$ are satisfied simultaneously. This happens in the discrete intersection of partonic 2-surfaces.
4. Elementary particles correspond even for $h_{eff} = h$ to two-sheeted structures with partonic surfaces defining wormhole throats. The model for elementary particles requires that particles are minimally 2-sheeted structures since otherwise the conservation of monopole Kähler magnetic flux cannot be satisfied: the flux is transferred between space-time sheets through wormhole contacts with Euclidian signature of induced metric and one obtains closed flux loop. Euclidian wormhole contact would connect the two Minkowskian sheets. Could the Minkowskian sheets corresponds to zeros $IM(P_i)$ for P_1 and P_2 and could wormhole contacts emerge as zeros of $RE(P_1/P_2)$?

One can however wonder whether this picture could allow more detailed specification. The simplest possibility would be following. The basic condition is that CD emerges automatically from this picture.

1. The simplest possibility is that one has $P_1(o)$ and $P_2(T - o)$ with the origin of octonions at the “lower” tip of CD. One would have $P_1(0) = 0$ and $P_2(0) = 0$. $P_1(o)$ would give rise to the “lower” boundary of CD and $P_2(T - o)$ to the “upper” boundary of CD.

ZEO combined with the ideas inspired by infinite rationals as counterparts of space-time surfaces connecting 3-surfaces at opposite boundaries of CD [K4] would suggest that the opposite boundaries of CD could correspond zeros and poles respectively and the ratio $P_1(o)/P_2(T - o)$ and to zeros of P_1 resp. P_2 assignable to different boundaries of CD. Both light-like parton orbits and string world sheets would interpolate between the two boundaries of CD at which partonic 2-surface would correspond to zeros and poles.

The notion divisor would be a straightforward generalization of this notion in the case of complex plane. What would matter would be the rational function $P_1(t)/P_2(T - t)$ extended from the real (time) axis of octonions to the entire space of complexified octonions. Positive degree of divisor would multiply $P_1(t)$ with $(t - t_1)^m$ inducing a new zero at or increasing the order of existing zero at t_1 . Negative orders n would multiply the denominator by $(t - t_1)^n$.

2. One can also consider the possibility that both boundaries of CD emerge for both P_1 and P_2 and without assigning either boundary of CD with P_i . In this case P_i would be sum over terms $P_{ik} = P_{ia_k}(o)P_{ib_k}(T - o)$ of this kind of products satisfying $P_{ia_k}(0) = 0$ and $P_{ib_k}(0) = 0$.

One can imagine also an alternative approach in which 0 and ∞ correspond to opposite tips of CD and have geometric meaning. Now zeros and poles would correspond to 2-surfaces, which need not be partonic. Note that in the case of Riemann surfaces ∞ can represent any point. This approach does not however look attractive.

3.2 Could one generalize RR to octonionic algebraic varieties?

RR is associated with complex structure, which in TGD framework seems to make sense independent of signature thanks to complexification of octonions. Divisors are the key notion and characterize what might be called local winding numbers. De-Rham cohomology is replaced with

much richer Dolbeault cohomology (see <http://tinyurl.com/y7cvs5sx>) since the notion of continuity is replaced with that of meromorphy. Symplectic approach about which G-W invariants for symplectic manifolds provide an example define a different approach and now one has ordinary cohomology.

An interesting question is whether $M^8 - H$ -duality corresponds to the mirror symmetry of string models (see <http://tinyurl.com/yc2m2e5m>) relating complex structures and symplectic structures. If this were the case, M^8 would correspond to complex structure and H to symplectic structure.

RR for curves gives information about dimensions for the spaces of meromorphic functions having poles with order not higher than specified by divisor. This kind of interpretation would be very attractive now since the poles and zeros represented as partonic 2-surfaces would have direct physical interpretation in terms of external particles and interaction vertices. RR for curves involves poles with orders not higher than specified by the divisor and gives a formula for the dimension of the space of meromorphic functions for a given divisor. As a special case give the dimension $l(nD)$ for a given divisor.

Could something similar be true in TGD framework?

1. Arithmetic genus makes sense for polynomials $P(t)$ since t can be naturally complexified giving a complex curve with well-defined arithmetic genus. What could correspond to the intersection form for 2-surfaces representing D and $K - D$? The most straightforward possibility is that partonic 2-surfaces correspond to poles and zeros.

Divisor $-D$ would correspond to the inverse of P_2/P_1 representing it. $D - K$ would also have a well-defined meaning provided the canonical divisor associated with holomorphic 2-form has well-defined meaning in the Dolbeault cohomology of the space-time surface with complex structure. RR would give direct information about the space of space-time varieties defined by $RE(P) = 0$ or $IM(P) = 0$ condition.

One could hope of obtaining information about intersection form for string world sheets and partonic 2-surfaces. Whether the divisor $D - K$ has anything to do with string world sheets, is of course far from clear.

2. Complexification means that field property fails in the sense that complexified Euclidian norm vanishes and the inverse of complexified octonion/quaternion/complex number is infinite formally. For Euclidian sector with real coordinates this does not happen but does take place when some coordinates are real and some imaginary so that signature is effectively Minkowskian signature.

At 7-D light-cone of M^8 the condition $P(o) = 0$ reduces to a condition for real polynomial $P(t) = 0$ giving roots t_n . Partonic 2-varieties are intersections of 4-D space-time varieties with 6-spheres with radii t_n . There are good reasons to expect that the 3-D light-like orbits of partonic 3-surfaces are intersections of space-time variety with 7-D light-cone boundary and their H counterparts are obtained as images under $M^8 - H$ duality.

For light-like complexified octonionic points the inverse of octonion does not exist since the complexified norm vanishes. Could the light-like 3-surfaces as partonic orbits correspond to images under $M^8 - H$ duality for zeros and/or poles as 3-D light-like surfaces? Could also the light-like boundaries of strings correspond to this kind of generalized poles or zeros? This could give a dynamical realization for the notions of zero and pole and increase the topological dimension of pole and zero for both 2-varieties and 4-varieties by one unit. The metric dimension would be unaffected and this implies huge extension of conformal symmetries central in TGD since the light-like coordinate appears as additional parameter in the infinitesimal generators of symmetries.

Could one formulate the counterpart of RR at the level of H ? The interpretation of $M^8 - H$ duality as analog of mirror symmetry (see <http://tinyurl.com/yc2m2e5m>) suggests this. In this case the first guess for the identification of the counterpart of canonical divisor could be as Kähler form of CP_2 . This description would provide symplectic dual for the description based on divisors at the level of M^8 . G-W invariants and their possible generalization are natural candidates in this respect.

4 Could the TGD variant of Atiyah-Singer index theorem be useful in TGD?

Atiyah-Singer index theorem (AS) and other generalizations of RR involve extremely abstract concepts. The best manner to get some idea about AS is to learn the motivations for it. The article <http://tinyurl.com/yc4911jp> gives a very nice general view about the motivations of Atiyah-Singer index theorem and also avoids killing the reader with details.

Solving problems of algebraic geometry is very demanding. The spectrum of solutions can be discrete (say number of points of space-time surface having linear M^8 coordinates in an extension of rationals) or continuous such as the space of roots for n :th order polynomials with real coefficients.

An even more difficult challenge is solving of partial differential equations in some space, call it X , of say Yang-Mills gauge field coupled to matter fields. In this case the set of solutions is typically continuous moduli space.

One can however pose easier questions. What is the number of solutions in counting problem? What is the dimension of the moduli space of solutions? Atiyah-Singer index theorem relates this number - analytic index - to topological index expressible in terms of topological invariants assignable to complexified tangent bundle of X and to the bundle structure - call it field bundle - accompanying the fields for which field equations are formulated.

4.1 AS very briefly

Consider first the assumptions of AS.

1. The idea is to study perturbations of a given solution and linearize the equations in some manifold X often assumed to be compact. This leads to a linear partial differential equations defined by linear operator P . One can deduce the dimension of the solution space of P . This number defines the dimension of the tangent space of solution space of full partial differential equations, call it moduli space.
2. The idea is to assign to the partial differential operator P its symbol $\sigma(P)$ obtained by replacing derivatives with what might be called momentum components. The reversal of this operation is familiar from elementary wave mechanics: $p_i \rightarrow id/dx^i$. This operation can be formulated in terms of co-tangent bundle. The resulting object is purely algebraic. If this matrix is reversible for all momentum values and points of X , one says that the operator is elliptic.

Note that for field equations in Minkowski space M^4 the invertibility constraint is not satisfied and this produces problems. For instance, for massive M^4 d'Alembertian for scalar field the symbol is four-momentum squared, which vanishes, when on-mass shell condition is satisfied. Wick rotation is somewhat questionable manner to escape this problem. One replaces Minkowski space with its Euclidian counterpart or by 4-sphere. If all goes well the dimension of the solution space does not depend on the signature of the metric.

3. In the general case one studies linear equation of form $DP = f$, where f is homogeneity term representing external perturbation. f can also vanish. Quite generally, one can write the dimension of the solution space as

$$Ind_{anal}(P) = dim(ker(P)) - dim(coker(P)) . \quad (4.1)$$

$ker(P)$ denotes the solution space for $DP = 0$ without taking into account the possible restrictions coming from the fact that f can involve part f_0 satisfying $Df_0 = 0$ (for instance, f_0 corresponds to resonance frequency of oscillator system) nor boundary conditions guaranteeing hermiticity. Indeed, the hermitian conjugate D^\dagger of D is not automatically identical with D . D^\dagger is defined in terms of the inner product for small perturbations as

$$\langle D^\dagger P_1^* | DP_2 \rangle = \langle P_1 | DP_2 \rangle . \quad (4.2)$$

The inner product involves integration over X and partial integrations transfer the action of partial derivatives from P_2 to P_1^* . This however gives boundary terms given by surface integral and hermiticity requires that they vanish. This poses additional conditions on P and contributes to $\dim(\text{coker}(P))$.

The challenge is to calculate $Ind_{anal}(P)$ and here AS is of enormous help. AS relates analytical index $Ind_{anal}(P)$ for P to topological index $Ind_{top}(\sigma(P))$ for its symbol $\sigma(P)$.

1. $Ind_{top}(\sigma(P))$ involves only data associated with the topology X and with the bundles associated with field variables. In the case of Yang-Mills fields coupled to matter the bundle is the bundle associated with the matter fields with a connection determined by Yang-Mills gauge potentials. So called Todd class $Td(X)$ brings in information about the topology of complexified tangent bundle.
2. $Ind_{top}(\sigma(P))$ is not at all easy to define but is rather easily calculable as integrals of various invariants assignable to the bundle structure involved. Say instanton density for YM fields and various topological invariants expressing the topological invariants associated with the metric of the space. What is so nice and so non-trivial is that the dimension of the moduli space for non-linear partial differential equations is determined by topological invariants. Much of the dynamics reduces to topology.

The expression for $Ind_{top}(\sigma(P))$ involves besides σ_P topological data related to the field bundle and to the complexified tangent bundle. The expression Ind_{top} as a function of the symbol $\sigma(P)$ is given by

$$Ind_{top}(\sigma(P)) = (-1)^n \langle ch(\sigma(P)) \cdot Td(T_C(X), [X]) \rangle . \quad (4.3)$$

The expression involves various topological data.

1. Dimension of X .
2. The quantity $\langle x.y \rangle$ involving cup product $x.y$ of cohomology classes, which contains a contribution in the highest homology group $H^n(X)$ of X corresponding to the dimension of X and is contracted with this fundamental class $[X]$. $\langle x.y \rangle$ denotes matrix trace for the operator $ch(\sigma(P))$ formed as polynomial of $\sigma(P)$. $[X]$ denotes so called fundamental class for X belonging to H^n and defines the orientation of X .
3. Chern character $ch_E(t)$ (see <http://tinyurl.com/ybavu66h>). I must admit that I ended up to a garden of branching paths while trying to understand the definition of ch_E is. In any case, $ch_E(t)$ characterizes complex vector bundle E expressible in terms of Chern classes (see <http://tinyurl.com/y8j1aznc>) of E . E is the bundle assignable to field variables, say Yang Mills fields and various matter fields.

Both direct sums and tensor products of fiber spaces of bundles are possible and the nice feature of Chern class is that it is additive under tensor product and multiplicative under direct sum. The fiber space of the entire bundle is now direct sum of the tangent space of X and field space, which suggests that $Ind(top)$ is actually the analog of Chern character for the entire bundle.

$t = \sigma P$ has interpretation as an argument appearing in the definition of Chern class generalized to Chern character. $t = \sigma(P)$ would naturally correspond to a matrix valued argument of the polynomial defining Chern class as cohomology element. $ch(\sigma(P))$ is a polynomial of the linear operator defined by symbol $\sigma(P)$. ch_E for given complex vector bundle is a polynomial, whose coefficients are relatively easily calculable as topological invariants assignable to bundle E . E must be the field bundle now.

4. Todd class $Td(T_C(X))$ for the complexified tangent bundle (see <http://tinyurl.com/yckv4w84>) appears also in the expression. Note that also now the complexification occurs. The cup product gives element in $H^n(X)$, which is contracted with fundamental class $[X]$ and integrated over X .

4.2 AS and TGD

The dynamics of TGD involves two levels: the level of complexified M^8 (or equivalently E^8) and the level of H related to $M^8 - H$ correspondence.

1. At the level of M^8 one has algebraic equations rather than partial differential equations and the situation is extremely simple as compared to the situation for a general action principle. At the level of H one has action principle and partial differential equations plus infinite number of gauge conditions selecting preferred extremals and making dynamics for partial differential equations dual to the dynamics determined by purely number theoretic conditions.

The space-time varieties representing external particles outside CDs in M^8 satisfy associativity conditions for tangent space or normal space and reducing to criticality conditions for the real coefficients of the polynomials defining the space-time variety. In the interior of CDs associativity conditions are not satisfied but the boundary conditions fix the values of the coefficients to be those determined by criticality conditions guaranteeing associativity outside the CD.

In the interiors space-time surfaces of CDs M^8 -duality does not apply but associativity of tangent spaces or normal spaces at the boundary of CD fixes boundary values and minimal surface dynamics and strong form of holography (SH) fixes the space-time surfaces in the interior of CD.

2. For the H -images of space-time varieties in H under $M^8 - H$ duality the dynamics is universal coupling constant independent critical dynamics of minimal surfaces reducing to holomorphy in appropriate sense. For minimal surfaces the 4-D Kähler current density vanishes so that the solutions are 4-D analogs of geodesic lines outside CD. Inside CD interactions are coupled on and this current is non-vanishing. Infinite number of gauge conditions for various half conformal algebras in generalized sense code at H side for the number theoretical critical conditions at M^8 side. The sub-algebra with conformal weights coming as n -ples of the entire algebra and its commutator with entire algebra gives rise to vanishing classical Noether charges. An attractive assumption is that the value of n at H side corresponds to the order n of the polynomials at M^8 side.
3. The coefficients of polynomials $P(o)$ determining space-time varieties are real numbers (also complexified reals can be considered without losing associativity) restricted to be numbers in extension of rationals. This makes it possible to speak about p-adic variants of the space-time surfaces at the level of M^8 at least.

Could Atiyah-Singer theorem have relevance for TGD?

1. For real polynomials it is easy to calculate the dimension of the moduli space by counting the number of independent real (in octonionic sense) coefficients of the polynomials of real variable (one cannot exclude that the coefficients are in complex extension of rationals). Criticality conditions reduce this number and the condition that coefficients are in extension of rationals reduces it further. One has quite nice overall view about the number of solutions and one can see them as subset of continuous moduli space. If $M^8 - H$ duality really works then this gives also the number of preferred extremals at H side.
2. This picture is not quite complete. It assumes fixing of 8-D CD in M^8 as well as fixing of the decomposition $M^2 \subset M^4 \subset M^4 \times E^4$. This brings in moduli space for different choices of octonion structures (8-D Lorentz group is involved). Also moduli spaces for partonic 2-surfaces are involved. Number theoretical universality seems to require that also these moduli spaces have only points with coordinates in extension of rationals involved.
3. In principle one can try to formulate the counterpart of AS at H side for the linearization of minimal surface equations, which are nothing but the counterpart of massless field equations in a fixed background metric. Note that additional conditions come from the requirement that the term from Kähler action reduces to minimal surface term.

Discrete sets of solutions for the extensions of rationals should correspond to each other at the two sides. One can also ask whether the dimensions for the effective continuous moduli

spaces labelled by n characterizing the sub-algebras of various conformal algebras isomorphic to the entire algebra and those for the polynomials of order n satisfying criticality conditions. One would have a number theoretic analog for a particle in box leading to the quantization of momenta.

All this is of course very speculative and motivated only by the general physical vision. If the speculations were true, they would mean huge amount of new mathematics.

REFERENCES

Mathematics

- [A1] Vakil R. The moduli space of curves and Gromov-Witten theory. Available at: <https://arxiv.org/pdf/math/0602347.pdf>, 2006.
- [A2] Talovikova V. Riemann-Roch theorem. Available at: <http://www.math.uchicago.edu/~may/VIGRE/VIGRE2009/REUPapers/Talovikova.pdf>, 2009.

Books related to TGD

- [K1] Pitkänen M. Construction of elementary particle vacuum functionals. In *p-Adic Physics*. Online book. Available at: http://tgdtheory.fi/public_html/padphys/padphys.html#elvafu, 2006.
- [K2] Pitkänen M. Massless states and particle massivation. In *p-Adic Physics*. Online book. Available at: http://tgdtheory.fi/public_html/padphys/padphys.html#mless, 2006.
- [K3] Pitkänen M. p-Adic Particle Massivation: Hadron Masses. In *p-Adic Length Scale Hypothesis and Dark Matter Hierarchy*. Online book. Available at: http://tgdtheory.fi/public_html/padphys/padphys.html#mass3, 2006.
- [K4] Pitkänen M. TGD as a Generalized Number Theory: Infinite Primes. In *TGD as a Generalized Number Theory*. Online book. Available at: http://tgdtheory.fi/public_html/tgdnumber/tgdnumber.html#visionc, 2006.
- [K5] Pitkänen M. Questions related to the twistor lift of TGD. In *Towards M-Matrix*. Online book. Available at: http://tgdtheory.fi/public_html/tgdquantum/tgdquantum.html#twistquestions, 2016.
- [K6] Pitkänen M. Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry? In *TGD as a Generalized Number Theory*. Online book. Available at: http://tgdtheory.fi/public_html/tgdnumber/tgdnumber.html#ratpoints, 2017.
- [K7] Pitkänen M. Philosophy of Adelic Physics. In *TGD as a Generalized Number Theory*. Online book. Available at: http://tgdtheory.fi/public_html/tgdnumber/tgdnumber.html#adelephysics, 2017.

Articles about TGD

- [L1] Pitkänen M. Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry?: part I. Available at: http://tgdtheory.fi/public_html/articles/ratpoints1.pdf, 2017.
- [L2] Pitkänen M. Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry?: part II. Available at: http://tgdtheory.fi/public_html/articles/ratpoints2.pdf, 2017.
- [L3] Pitkänen M. p-Adicization and adelic physics. Available at: http://tgdtheory.fi/public_html/articles/adelicphysics.pdf, 2017.