

# Could McKay correspondence generalize in TGD framework?

M. Pitkänen

Email: matpitka6@gmail.com.

<http://tgdtheory.com/>.

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## Abstract

McKay correspondence states that the McKay graphs for the irreducible representations (irreps) of finite subgroups of  $G \subset SU(2)$  characterizing their fusion algebra is given by extended Dynkin diagram of ADE type Lie group. Minimal conformal models with  $SU(2)$  Kac-Moody algebra (KMA) allow a classification by the same diagrams as fusion algebras of primary fields. The resolution of the singularities of complex algebraic surfaces in  $C^3$  by blowing implies the emergence of complex lines  $CP_1$ . The intersection matrix for the  $CP_1$ s is Dynkin diagram of ADE type Lie group. These results are highly inspiring concerning adelic TGD.

1. The appearance of Dynkin diagrams in the classification of minimal conformal field theories (CFTs) inspires the conjecture that in adelic physics Galois groups  $Gal$  or semidirect products of  $Gal$  with a discrete subgroup  $G$  of automorphism group  $SO(3)$  (having  $SU(2)$  as double covering!) classifies TGD generalizations of minimal CFTs. Also discrete subgroups of octonionic automorphism group can be considered. The fusion algebra of irreps of  $Gal$  would define also the fusion algebra for KMA for the counterparts of minimal fields. This would provide deep insights to the general structure of adelic physics.
2. One cannot avoid the question whether the extended ADE diagram could code for a dynamical symmetry of a minimal CFT or its modification? If the  $Gal$  singlets formed from the primary fields of minimal model define primary fields in Cartan algebra of ADE type KMA, then standard free field construction would give the charged KMA generators. In TGD framework this conjecture generalizes.
3. A further conjecture is that the singularities of space-time surface imbedded as 4-surface in its 6-D twistor bundle with twistor sphere as fiber could be classified by McKay graph of  $Gal$ . The singular intersection of the Euclidian and Minkowskian regions of space-time surface is especially interesting: the twistor spheres at the common points defining light-like partonic orbits need not be same but have intersections with intersection matrix given by McKay graph for  $Gal$ . The basic information about adelic CFT would be coded by the general character of singularities for the twistor bundle.
4. In TGD also singularities in which the group  $Gal$  is reduced to its subgroup  $Gal/H$ , where  $H$  is normal group are possible and would correspond to phase transition reducing the value of Planck constant. What happens in these phase transitions to single particle states would be dictated by the decomposition of representations of  $Gal$  to those of  $Gal/H$  and transition matrix elements could be evaluated.

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## 1 Introduction

McKay correspondence is rather mysterious looking correspondence appearing in several fields. This correspondence is extremely interesting from point of view of adelic TGD [?]delephysics [L2].

1. McKay graphs code for the fusion algebra of irreducible representations (irreps) of finite groups (see <http://tinyurl.com/z48d92t>). For finite subgroups of  $G \subset SU(2)$  McKay graphs are extended Dynkin diagrams for affine (Kac-Moody) algebras of ADE type coding the structure of the root diagram for these algebras. The correspondence looks mysterious since Dynkin diagrams have quite different geometric interpretation.

2. McKay graphs for finite subgroups of  $G \subset SU(2)$  characterize also the fusion rules of minimal conformal field theories (CFTs) having Kac-Moody algebra (KMA) of  $SU(2)$  as symmetries (see <http://tinyurl.com/y7dof7pe>). Fusion rules characterize the decomposition of the tensor products of primary fields in CFT. For minimal CFTs the primary fields belonging to the irreps of  $SU(2)$  are in 1-1 correspondence with irreps of  $G$ , and the fusion rules for primary fields are same as for the irreps of  $G$ . The irreps of  $SU(2)$  are also irreps of  $G$ .

Could the ADE type affine algebra appear as dynamical symmetry algebra too? Could the primary fields for ADE defining extended ADE Cartan algebra be constructed as  $G$ -invariants formed from the irreps of  $G$  and be exponentiated using the standard free field construction using the roots of the ADE KMA a give ADE KMA acting as dynamical symmetries?

3. McKay graphs for  $G \subset SU(2)$  characterize also the double point singularities of algebraic surfaces of real dimension 4 in  $C^3$  (or  $CP^3$ , one variant of twistor space!) with real dimension 6 (see <http://tinyurl.com/ydz93hle>). The subgroup  $G \subset SU(2)$  has a natural action in  $C^2$  and it appears in the canonical representation of the singularity as orbifold  $C^2/G$ . This partially explains the appearance of the McKay graph of  $G$ . The resolved singularities are characterized by a set of projective lines  $CP_1$  with intersection matrix in  $CP_2$  characterized by McKay graph of  $G$ . Why the number of spheres is the number of irreps for  $G$  is not obvious to me.

The double point singularities of  $C^2 \subset C^3$  allow thus ADE classification. The number of added points corresponds to the dimension of Cartan algebra for ADE type affine algebra, whose Dynkin diagram codes for the finite subgroup  $G \subset SU(2)$  leaving the algebraic surface looking locally like  $C^2$  invariant and acting as isotropy group of the singularity.

These results are highly inspiring concerning adelic TGD.

1. The appearance of Dynkin diagrams in the classification of minimal CFTs inspires the conjecture that in adelic physics Galois groups  $Gal$  or semi-direct products  $G \triangleleft Gal$  of  $Gal$  with a discrete subgroup  $G$  of automorphism group  $SO(3)$  (having  $SU(2)$  as double covering!) classifies TGD generalizations of minimal CFTs. Also discrete subgroups of octonionic automorphism group can be considered. The fusion algebra of irreps of  $Gal$  would define also

the fusion algebra for KMA for the counterparts of minimal fields. This would provide deep insights to the general structure of adelic physics.

2. One cannot avoid the question whether the extended ADE diagram could code for a dynamical symmetry of a minimal CFT or its modification? If the  $Gal$  singlets formed from the primary fields of minimal model define primary fields in Cartan algebra of ADE type KMA, then standard free field construction would give the charged KMA generators. In TGD framework this conjecture generalizes.
3. A further conjecture is that the singularities of space-time surface imbedded as 4-surface in its 6-D twistor bundle with twistor sphere as fiber could be classified by McKay graph of  $Gal$ . The singular intersection of the Euclidian and Minkowskian regions of space-time surface is especially interesting: the twistor spheres at the common points defining light-like partonic orbits need not be same but have intersections with intersection matrix given by McKay graph for  $Gal$ . The basic information about adelic CFT would be coded by the general character of singularities for the twistor bundle.
4. In TGD also singularities in which the group  $Gal$  is reduced to its subgroup  $Gal/H$ , where  $H$  is normal group are possible and would correspond to phase transition reducing the value of Planck constant. What happens in these phase transitions to single particle states would be dictated by the decomposition of representations of  $Gal$  to those of  $Gal/H$  and transition matrix elements could be evaluated.

One can find from web excellent articles about the topics to be discussed in this article.

1. The article "*Cartan matrices, finite groups of quaternions, and Kleinian singularities*" of John McKay [A2] (see <http://tinyurl.com/ydygjgge>) summarizes McKay correspondence.
2. Miles Reid has written an article "*The Du Val singularities  $A_n, D_n, E_6, E_7, E_8$* " [A3] (see <http://tinyurl.com/ydz93hle>). Also the article "*Chapters on algebraic surfaces*" [A4] (see <https://arxiv.org/abs/alg-geom/9602006>) of Reid should be helpful. There is also an article "*Resolution of Singularities in Algebraic Varieties*" [A1] (see <http://tinyurl.com/yb7cukf>) of Emma Whitten about resolution of singularities.
3. Andrea Cappelli and Jean-Benoit Zuber have written an article "*A-D-E Classification of Conformal Field Theories*" [B1] about ADE classification of minimal CFT models (see <http://tinyurl.com/y7dofpfe>).
4. McKay correspondence appears also in M-theory, and the thesis "*On Algebraic Singularities, Finite Graphs and D-Brane Gauge Theories: A String Theoretic Perspective*" [B2] (see <http://tinyurl.com/ycmyjukn>) of Yang-Hui He might be helpful for the reader. In this work the possible generalization of McKay correspondence so that it would apply form finite subgroups of  $SU(n)$  is discussed.  $SU(3)$  acting as subgroup of automorphism group  $G_2$  of octonions is especially interesting in this respect. The idea is rather obvious: the fusion diagram for the theory in question would be the McKay graph for the finite group in question.

## 2 McKay graphs in mathematics and physics

McKay graphs for subgroups of  $SU(2)$  reducing to Dynkin diagrams for affine Lie algebras of ADE type appear in several manners in mathematics and physics.

### 2.1 McKay graphs

McKay graphs [A2] (see <http://tinyurl.com/ydygjgge>) code for the fusion algebra of irreps of finite groups  $G$  (for Wikipedia article see <http://tinyurl.com/z48d92t>). One considers the tensor products of irreps with the canonical representation (doublet representation for the finite sub-groups of  $SU(2)$ ), call it  $V$ . The irreps  $V_i$  correspond to nodes and their number is equal to the number of irreps  $G$ .

Two nodes  $i$  and  $j$  are no connected if the decomposition of  $V \otimes V_i$  to irreps does not contain  $V_j$ . There is arrow pointing from  $i \rightarrow j$  in this case. The number  $n_{ij} > 0$  or number of arrows tells how many times  $j$  is contained in  $V \otimes V_i$ . For  $n_{ij} = n_{ji}$  there is no arrow.

One can characterize the fusion rules by matrix  $A = d\delta_{ij} - n_{ij}$ , where  $d$  is the dimension of the canonical representation. The eigenvalues of this matrix turn out to be given by  $d - \xi_V(g)$ , where  $\xi_V(g)$  is the character of the canonical representation, which depends on the conjugacy class of  $g$  only. The number of eigenvalues is therefore equal to the number  $n(class, G)$  of conjugacy classes. The components of eigenvectors in turn are given by the values  $\chi_i(g)$  of characters of irreps.

## 2.2 MacKay graphs and Dynkin diagrams

The nodes of the Dynkin diagram (see <http://tinyurl.com/hpm5y9s>) are positive simple root vectors identified as vectors formed by the eigenvalues of the Cartan sub-algebra generators under adjoint action on Lie algebra. In the case of affine Lie algebra the Cartan algebra contains besides the Cartan algebra of the Lie group also scaling generator  $L_0 = td/dt$  and the number of nodes increases by one.

The number of positive simple roots equals to the dimension of the root space. The number  $n_{ij}$  codes now for the angle between positive simple roots. The number of edges connecting root vectors is  $n = 0, 1, 2, 3$  depending on whether the the angle between root vectors is  $\pi/2, 2\pi/3, 3\pi/4$ , or  $5\pi/6$ . The ratios of lengths of connected roots can have values  $\sqrt{n}$ ,  $n \in \{1, 2, 3\}$ , and the number  $n$  of edges corresponds to this ratio. The arrow is directed to the shorter root if present. For simply laced Lie groups (ADE groups) the roots have unit length so that only single undirected edge can connect the roots. Weyl group acts as symmetries of the root diagram as reflections in hyperplanes orthogonal to the roots.

The Dynkin diagrams of affine algebras are obtained by adding to the Cartan algebra a generator which corresponds to the scaling generator  $L_0 = td/dt$  of affine algebra assumed to act via adjoint action to the Lie algebra. Depending on the position of the added node one obtains also twisted versions of the KMA.

For the finite subgroups of  $SU(2)$  the McKay graphs reduce to Dynkin diagrams of affine Lie algebras of ADE type [A2] (see <http://tinyurl.com/ydygjjgge>) so that one has either  $n_{ij} = 0$  or  $n_{ij} = 1$  for  $i \neq j$ . There are no self-loops ( $n_{ii} \neq 0$ ). The result looks mysterious since the two diagrams describe quite different things. One can also raise the question whether ADE type affine algebra might somehow emerge in minimal CFT involving  $SU(2)$  KMA for which ADE classification emerges.

In TGD framework the interpretation of finite groups  $G \subset SU(2)$  in terms of quaternions is an attractive possibility since rotation group  $SO(3)$  acts as automorphisms of quaternions and has  $SU(2)$  as its covering group.

## 2.3 ADE diagrams and subfactors

ADE classification emerges also naturally for the inclusions of hyper-finite factors of type  $II_1$  [K3, K1]. Subfactors with index smaller than four have so called principal graphs characterizing the sequence of inclusions equal to one of the A, D or E Coxeter-Dynkin diagrams: see the article “*In and around the origin of quantum groups*” of Vaughan Jones [A5] (see <http://tinyurl.com/ybbbbvpq>). As a matter of fact, only the  $D_{2n}$  and  $E_6$  and  $E_8$  do occur. It is also possible to construct  $M : N = 4$  sub-factor such that the principle graph is that for any subgroup  $G \subset SU(2)$ . This suggests that the subfactors  $M : N = 4\cos^2(\pi/n) < 4$  correspond to quantum groups. The basic objects can be seen as quantum spinors so that again the appearance of subgroups of  $SU(2)$  looks natural. One can still wonder whether ADE KMAs might be involved.

## 2.4 ADE classification for minimal CFTs

CFTs on torus [B1] are characterized by modular invariant partition functions, which can be expressed in terms of characters of the scaling generator  $L_0$  of Virasoro algebra (VA) given by

$$Z(\tau) = Tr(X) \quad , \quad X = exp\{i2\pi [\tau(L_0 - c/24) - \bar{\tau}(\bar{L}_0 - c/24)]\} \quad . \quad (2.1)$$

Modular invariance requires that  $Z(\tau)$  is invariant under modular transformations leaving the conformal equivalence class of torus invariant. Modular group equals to  $SL(2, Z)$  has as generators the transformations  $T : \tau \rightarrow \tau + 1$  and  $S : \tau \rightarrow -1/\tau$ . The partition function can be expressed as

$$Z(\tau) = \sum N_{j\bar{j}} \chi_j(q) \chi_{\bar{j}}(\bar{q}) \quad , \quad q = \exp(i2\pi\tau) \quad , \quad \bar{q} = \exp(-i2\pi\bar{\tau}) \quad . \quad (2.2)$$

Here  $\chi_j$  corresponds to the trace of  $L_0 - c/24$  for a representation of KMA inducing the VA representation. Modular invariance of the partition function requires  $SNS^\dagger = N$  and  $TNT^\dagger = N$ .

The ADE classification for minimal conformal models summarized in [B1] (see <http://tinyurl.com/y7dof7pe>) involves  $SU(2)$  affine algebra with central extension parameter  $k$ . The central extension parameter for the VA is  $c < 1$ . The fusion algebra for primary fields in representations of  $SU(2)$  KMA characterizes the CFT to a high degree.

The fusion rules characterized the decomposition of the tensor product of representation  $D_i$  with representation  $D_j$  as  $i \otimes j = N_{ij}^k D_k$ . Due to the properties of the tensor product the matrices  $\mathcal{N}_i = N_{ij}^k$  form and associative and commutative algebra and one can diagonalize these matrices simultaneously. This algebra is known as Verlinde algebra and its elements can be expressed in terms of unitary modular matrix  $S_{ij}$  representing the transformation of characters in the modular transformation  $\tau \rightarrow -1/\tau$ .

The generator of the Verlinde algebra is fusion algebra for the 2-D representation of  $SU(2)$  generating the fusion algebra (this corresponds to the fact that tensor powers of this representations give rise to all representations of  $SU(2)$ ). It turns out that for minimal models with a finite number of primary fields (KMA representations) the fusion algebra of KMA reduces to that for a finite subgroup of  $SU(2)$  and thus corresponds to ADE KMA. The natural interpretation is that the condition that the number of primary fields is finite is realized if the primary fields correspond also to the irreps of finite subgroup of  $SU(2)$ .

Could the ADE type KMA actually correspond to a genuine dynamical symmetry of minimal CFT? For this conjecture makes sense, the roots of ADE type KMA should be in 1-1 correspondence with the irreps of  $G \subset SU(2)$  assignable to primary fields. How could this be possible? In the free field construction of ADE type KMA generators one constructs charged KMA generators from free fields in Cartan algebra by exponentiating the quantities  $\alpha \cdot \phi$ , where  $\alpha$  is the root and  $\phi$  is a primary field corresponding to the element of Cartan algebra of KMA. Could  $SU(2)$  invariants formed from the primary fields defined by each  $G$ - (equivalently  $SU(2)$ -) multiplet give rise to  $SU(2)$  neutral multiplet of primary fields of ADE type Cartan algebra and could their exponentiation give rise to ADE type KMA acting as dynamical symmetries of a minimal CFT?

## 2.5 The resolution of singularities of algebraic surfaces and extended Dynkin diagrams of ADE type

The classification of singularities of algebraic surfaces leads also to extended Dynkin diagrams of ADE type.

### 2.5.1 Classification of singularities

In algebraic geometry the classification of singularities of algebraic varieties [A1] is a central task. The singularities of curves in plane represent simplest singularities (see <http://tinyurl.com/y8ub2c4s>). The resolution of singularities of complex curves in  $C^3$  is less trivial task.

The resolution of singularity (<http://tinyurl.com/y8veht3p>) is a central concept and means elimination of singularity by modifying it locally. There is extremely general theorem by Hiroka stating that the resolution of singularities of algebraic varieties is always possible for fields with characteristic zero (reals and p-adic number fields included) using a sequence of birational transformations. For finite groups the situation is unclear for dimensions  $d > 3$ .

The articles of Reid [A3] and Whitten [A1] describe the resolution for algebraic surfaces (2-D surfaces with real dimension equal to four). The article of Reid describes how the resolutions of double-point singularities of  $m = d_c = 2$ -D surfaces in  $n = d_c = 3$ -D  $C^3$  or  $CP_3$  ( $d_c$  refers to complex dimension) are classified by ADE type extended Dynkin diagrams. Subgroups  $G \subset SU(2)$  appear naturally because the surface has dimension  $d_c = 2$ . This is the simplest non-trivial

situation since for Riemann surface with  $(m, n) = (1, 2)$  the group would be discrete subgroup of  $U(1)$ .

### 2.5.2 Singularity and Jacobians

What does one mean with singularity and its resolution? Reid [A3] (see <http://tinyurl.com/ydz93h1e>) discusses several examples. The first example is the singularity of the surface  $P(x_1, x_2, x_3) = x_1^2 - x_2x_3 = 0$ .

1. One can look the situation from the point of view of imbedding of the 2-surface to  $C^3$ : one considers map from tangent space of the surface to the imbedding space  $C^3$ . The Jacobian of the imbedding map  $(x_2, x_3) \rightarrow (x_1, x_2, x_3) = (\pm\sqrt{x_2x_3}, x_2, x_3)$  becomes ill-defined at origin since the partial derivatives  $\partial x_1/\partial x_2 = (\sqrt{x_3/x_2})/2$  and  $\partial x_1/\partial x_3 = (\sqrt{x_2/x_3})/2$  have all possible limiting values at singularity. The resolution of singularity must as a coordinate transformation singular at the origin should make the Jacobian well-defined. Obviously this must mean addition of points corresponding to the directions of various lines of the surface through origin.
2. A more elegant dual approach replaces parametric representation with representation in terms of conditions requiring function to be constant on the surface. Now the Jacobian of a map from  $C^3$  to the 1-D normal space of the singularity having polynomial  $P(x_1, x_2, x_3)$  as coordinate is considered. Singularity corresponds to the situation when the rank of the Jacobian defined by partial derivatives is less than maximal so that one has  $\partial P/\partial x_i = 0$ . The resolution of singularity means that the rank becomes maximal. Quite generally, for co-dimension  $m$  algebraic surface the vanishing of polynomials  $P_i, i = 1, \dots, m$  defines the surface. At the singularity the reduction of the rank for the matrix  $\partial P_i/\partial x_n$  from its maximal value takes place.

### 2.5.3 Blowing up of singularity

Codimension one algebraic surface is defined by the condition  $P(x_1, x_2, \dots, x_n) = 0$ , where  $P(x_1, \dots, x_n)$  is polynomial. For higher codimensions one needs more polynomials and the situation is not so neat anymore since so called complete intersection property need not hold anymore. Reid [A3] gives an easy-to-understand introduction to the blowing up of double-point singularities. Also the article “*Resolution of Singularities in Algebraic Varieties*” of Emma Whitten [A1] (see <http://tinyurl.com/yb7cuwkf>) is very helpful.

1. Coordinates are chosen such that the singularity is at the origin  $(x, y, z) = (0, 0, 0)$  of complex coordinates. The polynomial has vanishing linear terms at singularity and the first non-vanishing term is second power of some coordinate, say  $x_1$ , so that one has  $x_1 = \pm\sqrt{P_1(x_1, x_2, x_3)}$ , where  $x_1$  in  $P_1$  appears in powers higher than 2. At the singularity the two roots co-incide. One can of course have also more complex singularities such as triple-points.
2. The simplest example  $P(x_1, x_2, x_3) = x_1^2 - x_2x_3 = 0$  has been already mentioned. This singularity has the structure of double cone since one as  $x_1 = \pm\sqrt{x_2x_3}$ . At  $(0, 0, 0)$  the vertices of the two cones meet.
3. One can look this particular situation from the perspective of projective geometry. Homogeneous polynomials define a surface invariant under scalings of coordinates so that modulo scalings the surface can be regarded also as complex curve in  $CP_2$ . The conical surface can be indeed seen as a union of lines  $(x_1 = k^2x_3, x_2 = kx_3)$ , where  $k$  is complex number. The ratio  $x_1 : x_2 : x_3$  for the coordinates at given line is determined by  $x_1 : x_2 = k$  and  $x_2 : x_3 = k$  so that the surface can be parameterized by  $k$  and the coordinate along given line.

In this perspective the singularity decomposes to the directions of the lines going through it and the situation becomes non-singular. The replacement of the original view with this gives a geometric view idea about the resolution of singularity: the 2-surface is replaced by a bundle lines of surfaces going through the singularity and singularity is replaced with a union of directions for these lines.

Quite generally, in the resolution of singularity, origin is replaced by a set of points  $(x_1, x_2, x_3)$  with a well-defined ratio  $(x_1 : x_2 : x_3)$ . This interpretation applies also to more general singularities. One can say that origin is replaced with a projective sub-manifold of 2-D projective space  $CP_2$  (very familiar to me)! This procedure is known as blowing up. Strictly speaking, one only replaces origin with the directions of lines in  $C^3$ .

**Remark:** In TGD the wormhole contacts connecting space-time sheets of many-sheeted space-time could be seen as outcomes of blowing up procedure.

Blowing up replaces the singular point with projective space  $CP_1$  for which points with same value of  $(x_1 : x_2 : x_3)$  are identified. Blowing up can be also seen as a process analogous to seeing the singularity such as self-intersection of curve as an illusion: the curve is actually a projection of a curve in higher dimensional space to which it is lifted so that the intersection disappears [A1] (see <http://tinyurl.com/yb7cuwkf>). Physicist can of course protest by saying that in space-time physics is not allowed to introduce additional dimensions in this manner!

There is an analytic description for what happens at the singular point in blowing up process [A1] (see <http://tinyurl.com/yb7cuwkf>).

1. In blowing up one lifts the surface in higher-dimensional space  $C^3 \times CP_2$  ( $C^3$  can be replaced by any affine space). The blowing up of the singularity would be the set of lines  $\bar{q}$  of the surface  $S$  going through the singularity that is the set  $B = \{(q, \bar{q}) | q \in S\}$ . This set can be seen as a subset of  $C^3 \times CP_2$  and one can represent it explicitly by using projective coordinates  $(y_1, y_2, y_3)$  for  $CP_2$ . Consider points of  $C^3$  and  $CP_2$  with coordinates  $z = (x_1, x_2, x_3)$  and  $y = (y_1, y_2, y_3)$ . The coordinate vectors must be parallel  $x$  is to be at line  $y$ . This requires that all  $2 \times 2$  sub-determinants of the matrix

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \quad (2.3)$$

vanish: that is  $x_i y_j - x_j y_i = 0$  for all pairs  $i < j$ . This description generalizes to the higher-dimensional case. The added  $CP_1$ s defined what is called exceptional divisor in the blown up surface. Recall that divisors (see <http://tinyurl.com/yc7x3ohx>) are by definition formal combinations of points of algebraic surface with integer coefficients. The principal divisors defined by functions are sums over their zeros and poles with integer weight equal to the order of zero (negative for pole).

The above example considers a surface  $x_1^2 - x_2 x_3 = 0$  which allows interpretation as a projective surface. The method however works also for more general case since the idea about replacing point with directions is applied only at origin.

2. One can consider a more practical resolution of singularity by performing a bi-rational coordinate transformation becoming singular at the singular point. This can improve the singularity by blowing it up or make it worse by inducing blowing down. The idea is to perform a sequence of this kind of coordinate changes inducing blowing ups so that final outcome is free of singularities.

Since one considers polynomial equations both blowing up and its reversal must map polynomials to polynomials. Hence a bi-rational transformation  $b$  acting as a surjection from the modified surface to the original one must be in question (for bi-rational geometry see <http://tinyurl.com/yadoo3ot>). At the singularity  $b$  is many-to-one so that at this point inverse image is multivalued and gives rise to the blowing up.

The equation  $P(x_1, x_2, x_3) = 0$  combined with the equations  $x_i y_j - x_j y_i = 0$  by putting  $y_3 = 1$  (the coordinates are projective) leads to a parametric representation of  $S$  using  $y_1$  and  $y_2$  as coordinates instead of  $x_1$  and  $x_2$ . Origin is replaced with  $CP_1$ . This representation is actually much more general. Whitten [A1] gives a systematic description of resolution of singularities using this representation. For instance, cusp singularity  $P(x_1, x_2) = x_1^2 - x_2^3 = 0$  is discussed as a special case.

3. Topologically the blow up process corresponds to the gluing of  $CP_2$  to the algebraic surface  $A : A \rightarrow A \# CP_2$  and clearly makes it more complex. One can say that gluing occurs along

sphere  $CP_1$  and since the process involves several steps several spheres are involved with the resolution of singularities.

#### 2.5.4 ADE classification for resolutions of double point singularities of algebraic surfaces

ADE classification emerges for co-dimension one double point singularities of complex surfaces in  $C^3$  known as Du Val singularities. The surface itself can be seen locally as  $C^2$ . These surfaces are 4-D in real sense can have self-intersections with real dimension 2. In the singular point the dimension of the intersection is reduced and the dimension of tangent space is reduced (the rank of Jacobian is not maximal). The vertices of cone and cusp are good examples of singularities.

The subgroup  $G \subset SU(2)$  has a natural action in  $C^2$  and it appears in the canonical representation of the singularity as orbifold  $C^2/G$ . This helps to understand the appearance of the McKay graph of  $G$ . The resolved singularities are characterized by a set of projective lines  $CP_1$  with intersection matrix in  $CP_2$  characterized by McKay graph of  $G$ . Why the number of projective lines equals to the number of irreps of  $G$  appearing as nodes in McKay graph looks to me rather mysterious. Reid's article [A3] gives the characterization of groups  $G$  and canonical forms of the polynomials defining the singular surfaces.

The reason why Du Val singularities are so interesting from TGD point of view is that complex surfaces in Du Val theory have real dimension 4 and are surfaces in space of real dimension 6. The intersections of the branches of the 4-surfaces have real dimension  $D = 2$  in the generic case. In TGD space-time surfaces as preferred extremals have real dimension 4 and assumed possess complex structure or its Minkowskian generalization that I have called Hamilton-Jacobi structure [K2].

### 3 Do McKay graphs of Galois groups give overall view about classical and quantum dynamics of quantum TGD?

McKay graphs for Galois groups are interesting from TGD view point for several reasons. Galois groups are conjectured to be the number theoretical symmetries for the hierarchy of extensions of rationals defining hierarchy of adelic physics [K6] [L2] and the notion of CFT is expected to generalize in TGD framework so that ADE classification for minimal CFTs might generalize to a classification of minimal number theoretic CFTs by Galois groups.

#### 3.1 Vision

The arguments leading to the vision are roughly following.

1. Adelic physics postulates a hierarchy of quantum physics with adeles at given level associated with extension of rationals characterized partially by Galois group and ramified primes of extension. The dimension of the extension is excellent candidate for defining the value of Planck constant  $h_{eff}/h = n$  and ramified primes could correspond to preferred p-adic primes. The discrete sets of points of space-time surface for which imbedding space coordinates are in the extension define what I have interpreted as cognitive representations and can be said to be in the intersection of all number fields involved forming kind of book like structure with pages intersecting at the points with coordinates in extension.

Galois groups would define a hierarchy of theories and the natural first guess is that Galois groups take the role of subgroups of  $SU(2)$  in CFTs with  $SU(2)$  KMA as symmetry. Could the MacKay graphs defining the fusion algebra of Galois group define the fusion algebra of corresponding minimal number theoretic QFTs in analogy with minimal conformal models? This would fix the primary fields of theories assignable to given level of adele hierarchy to be minimal representations of *Gal* perhaps having also interpretation as representations of KMAs or their generalization to TGD framework.

2. The analogies between TGD and the theory of Du Val singularities is intriguing. Complex surfaces in Du Val theory have real dimension 4 and are surfaces in space of real dimension 6.



The intersections of the branches of the 4-surfaces have real dimension  $D = 2$  in the generic case. In TGD space-time surfaces have real dimension 4 and possess complex structure or its Minkowskian generalization that I have called Hamilton-Jacobi structure.

The twistor bundle of space-time surface has 2-sphere  $CP_1$  as a fiber and space-time surface as base [K4, K7]. Space-time surfaces can be realized as sections in their own 6-D twistor bundle obtained by inducing twistor structure from the product  $T(M^4) \times T(CP_2)$  of twistor bundles of  $M^4$  and  $CP_2$ . Section is fixed only modulo gauge choice, which could correspond to the choice of the Kähler form defining twistor structure from quaternionic units represented as points of  $S^2$ . Even if this choice is made,  $U(1)$  gauge transformations remain and could correspond to gauge transformations of WCW changing its Kähler gauge potential by gradient and adding to Kähler function a real part of holomorphic function of WCW coordinates.

If the imbedding of 4-D space-time surface as section can become singular in given gauge, it will have self-intersections with dimension 2 possibly assignable to partonic 2-surfaces and maybe also string world sheets playing a key role in strong form of holography (SH). Could SH mean that information about classical and quantum theory is coded by singularities of the imbedding of space-time surface to twistor bundle. This would be highly analogous to what happens in the case of complex functions and also in twistor Grassmann theory whether the amplitudes are determined by the data at singularities.

3. Where would the intersections take place? Space-time regions with Minkowskian and Euclidian signature of metric have light-like orbits of partonic 2-surfaces as intersections. These surfaces are singular in the sense that the metric determinant vanishes and tangent space of space-time surface becomes effectively 3-D: this would correspond to the reduction of tangent space dimension of algebraic surface at singularity. It is attractive to think that the lifts of Minkowskian and Euclidian space-time sheets have twistor spheres, which only intersect and have intersection matrix represented by McKay graph of *Gal*.

What about string world sheets? Does it make sense to regard them as intersections of 4-D surfaces? This does not look plausible idea but there are also other characterizations of string world sheets. One can also ask about the interpretation of the boundaries of string world sheets, in particular the points at the partonic 2-surfaces. How could they relate to singularities? The points of cognitive representation at partonic 2-surfaces carrying fermion number should belong to cognitive representation with imbedding space coordinates belonging to an extension of rationals.

4. In Du Val theory the resolution of singularity means that one adds additional points to a double singularity: the added points form projective sphere  $CP_1$ . The blowing up process is like lifting self-intersecting curve to a non-singular curve by imbedding it into 3-D space so that the original curve is its projection. Could singularity disappear as one looks at 6-D objects instead of 4-D object? Could the blowing up correspond in TGD to a transition to a new gauge in which the self intersection disappears or is shifted on new place? The intersections of 4-surfaces in 6-space analogous to roots of polynomial are topologically stable suggesting that they can be only shifted by a new choice of gauge.

Self-intersection be a genuine singularity if the spheres  $CP_1$  defining the fibers of the twistor bundles of branches of the space-time surface do not co-incide in the 2-D intersection. In the generic case they would only intersect in the intersection. Could the McKay diagram of Galois group characterize the intersection matrix?

5. The big vision could be following. Galois groups characterize the singularities at given level of the adelic hierarchy and code for the multiplets of primary fields and for the analogs of their fusion rules for TGD counterparts of minimal CFTs. Note that singularities themselves identified as partonic 2-surfaces and possibly also light partonic orbits and possibly even string world sheets are not restricted in any manner.

This idea need not be so far-fetched as it might look at first.

1. One considers twistor lift and self-intersections indeed occur also in twistor theory. When the  $M^4$  projections of two spheres of twistor space  $CP_3$  (to which the geometric twistor space

$T(M^4) = M^4 \times S^2$  has a projection) have light-like separation, they intersect. In twistor diagrams the intersection corresponds to an emission of massless particle.

2. The physical expectation is that this kind of intersections could occur also for the twistor bundle associated with the space-time surface. Most naturally, they could occur along the light-like boundary of causal diamond (CD) for points with light-like separation. They could also occur along the partonic orbits which are light-like 3-surfaces defining the boundaries between Minkowskian and Euclidian space-time regions. The twistor spheres at the ends of light-like curve could intersect.

Why the number of intersecting twistor spheres should reduce to the number  $n(irred, Gal)$  of irreducible representations (irreps) of  $Gal$ , which equals to  $n(Gal)$  in Abelian case but is otherwise smaller? This question could be seen as a serious objection.

1. Does it make sense to think that although there are  $n(Gal)$  in the local fiber of twistor bundle, the part of Galois fiber associated with the twistor fiber  $CP_1$  has only  $n(irrep, Gal)$   $CP_1$ :s and even that the spheres could correspond to irreps of  $Gal$ . I cannot invent any obvious objection against this. What would happen that Could this mean realization of quantum classical correspondence at space-time level.
2. There are  $n(irrep, G)$  irreps and  $\sum_i n_i^2 = n(G)$ .  $n_i^2$  points at corresponding sheet labelled by irrep. The number of twistor spheres collapsing to single one would be  $n_i$  for  $n_i$ -D irrep so that instead of states of representations the twistor spheres would correspond to irrep. One would have analogy with the fractionization of quantum numbers. The points assignable to  $n_i$ -D representations would become effectively  $1/n_i$ -fractionized. At the level of base space this would not happen.

### 3.2 Phase transitions reducing $h_{eff}/h$

In TGD framework one can imagine also other kinds of singularities. The reduction of  $Gal$  to its subgroup  $Gal/H$ , where  $H$  is normal subgroup defining Galois group for the  $Gal$  as extension of  $Gal/H$  is one such singularity meaning that the the  $H$  orbits of space-time sheets become trivial.

1. The action of  $Gal$  could reduce locally to a normal subgroup  $H$  so that  $Gal$  would be replaced with  $Gal/H$ . In TGD framework this would correspond to a phase transition reducing the value of Planck constant  $h_{eff}/h = n(Gal)$  labelling dark matter phases to  $h_{eff}/h = n(Gal/H) = n(Gal)/n(H)$ . The reduction to  $Gal/H$  would occur automatically for the points of cognitive representation belonging to a lower dimensional extension having  $Gal/H$  as Galois group. The singularity would occur for the cognitive points of both space-time surface and twistor sphere and would be analogous to  $n(H)$ -point singularity.
2. A singularity of the discrete bundle defined by Galois group would be in question and is assumed to induce similar singularity of  $n(Gal)$  -sheeted space-time surface and its twistor lift. Although the singularity would occur for the ends of strings it would induce reduction of the extension of rationals to  $Gal/H$ , which should also mean that string world sheets have representation with WCW coordinates in smaller extension of rationals.
3. This would be visible as a reduction in the spectrum of primary fields of number theoretic variant of minimal model. I have considered the possibility that the points at partonic 2-surfaces carrying fermions located at the ends of string world sheets could correspond to singularities of this kind. Could string world sheets could correspond to this kind of bundle singularities? This singularity would not have anything to do with the above described self-interactions of the twistor spheres associated with the Minkowskian and Euclidian regions meeting at light-like orbits of partonic 2-surfaces.
4. This provides a systematic procedure for constructing amplitudes for the phase transitions reducing  $h_{eff}/h = n(Gal)$  to  $h_{eff}/h = n(Gal/H)$ . The representations of  $Gal$  would be simply decomposed to the representations of  $Gal(G/H)$  in the vertex describing the phase transition. In the simplest 2-particle vertex the representation of  $Gal$  remains irreducible as

representation of  $Gal/H$ . Transition amplitudes are given by overlap integrals of representation functions of group algebra representations of  $Gal$  restricted to  $Gal/H$  with those of  $Gal/H$ .

The description of transitions in which particles with different Galois groups arrive in same diagram would look like follows. The Galois groups must form an increasing sequence  $\dots \subset Gal_i = Gal_{i+1}/H_{i+1} \subset \dots$ . The representations of the largest Galois group would be decomposed to the representations of smallest Galois group so that the scattering amplitudes could be constructed using the fusion algebra of the smallest Galois group. The decomposition should be associative and commutative and could be carried in many manners giving the same outcome at the final step.

### 3.3 Also quaternionic and octonionic automorphisms might be important

What about the role of subgroups of  $SU(2)$ ? What roles they could have? Could also they classify singularities in TGD framework?

1.  $SU(2)$  is indeed realize as multiplication of quaternions.  $M^8 - H$  correspondence suggests that space-time surfaces in  $M^8$  can be regarded as associative or co-associative (normal space is associative. Associative translates to quaternionic. Associativity makes sense also at the level of  $H$  although it is not necessary. This would mean that the tangent space of space-time surface has quaternionic structure and the multiplication by quaternions is makes sense.
2. The Galois group of quaternions is  $SO(3)$  and has discrete subgroups having discrete subgroups of  $SU(2)$  as covering groups. Quaternions have action on the spinors from which twistors are formed as pairs of spinors. Could quaternionic automorphisms be lifted to a an  $SU(2)$  action on these spinors by quaternion multiplication? Could one imagine that the representations formed as tensor powers of these representations give finite irreps of discrete subgroups of  $SU(2)$  defining ground states of  $SU(2)$  KMA a representations and define the primary fields of minimal models in this manner?
3. Galois groups for extensions of rationals have automorphic action on  $SO(3)$  and its algebraic subgroups replacing matrix elements with their automorphs: for subgroups represented by rational matrices the action is trivial. One would have analogs of representations of Lorentz group  $SL(2, C)$  induced from spin representations of finite subgroups  $G \subset SU(2)$  by Lorentz transformations realizing the representation in Lobatchevski space. Lorentz group would be replaced by  $Gal$  and the Lobatchevski spaces as orbit with the representation of  $Gal$  in its group algebra. An interesting question is whether the hierarchy of discrete subgroups of  $SU(2)$  in McKay correspondence relates to quaternionicity.

$G_2$  acts as octonionic automorphisms and  $SU(3)$  appears as its subgroup leaving on octonionic imaginary unit invariant. Could these semi-direct products of  $Gal$  with these automorphism groups have some role in adelic physics?

### 3.4 About TGD variant of ADE classification for minimal models

I already considered the ADE classification of minimal models. The first question is whether the finite subgroups  $G \subset SU(2)$  are replaced in TGD context with Galois groups or with their semi-direct products  $G \triangleleft Gal$ . Second question concerns the interpretation of the Dynkin diagram of affine ADE type Lie algebra. Does it correspond to a real dynamical symmetries.

1. Could the MacKay correspondence and ADE classification generalize? Could fusion algebras of minimal models for KMA associated with general compact Lie group  $G$  be classified by the fusion algebras of the finite subgroups of  $G$ . This generalization seems to be discussed in [B2] (see <http://tinyurl.com/ycmyjukn>).
2. Could the fusion algebra of Galois group  $Gal$  give rise to a generalization of the minimal model associated with a KMA of Lie group  $G \supset Gal$ . The fusion algebra of  $Gal$  would be

identical with that for the primary fields of KMA for  $G$ . Galois groups could be also grouped to classes consisting of Galois groups appearing as a subgroup of a given Lie group  $G$ .

3. In TGD one has a fractal hierarchy of isomorphic supersymplectic algebras (SSAs) (the conformal weights of sub-algebra are integer multiples of those of algebra) with gauge conditions stating that given sub-algebra of SSA and its commutator with the entire algebra annihilates the physical states. The remnant of the full SSA symmetry algebra would be naturally KMA. The pair formed by full SSA and sub-SSA would correspond to pair formed by group  $G$  and normal subgroup  $H$  and the dynamical KMA would correspond to the factor group  $G/H$ . This conjecture generalizes: one can replace  $G$  with Galois group and  $SU(2)$  KMA with a KMA continuing  $Gal$  as subgroup. One the other hand, one has also hierarchies of extensions of rationals such that  $i + 1$ :th extension of rationals is extension of  $i$ :th extension.  $G_i$  is a normal subgroup of  $G_{i+1}$  so that the group  $Gal_{i+1,i} = Gal_{i+1}/Gal_i$  acts as the relative Galois group for  $i + 1$ :th extension as extensions of  $i$ :th extension.

This suggest the conjecture that the Galois groups  $Gal_i$  for extension hierarchies correspond to the inclusion hierarchies  $SSA_i \supset SSA_{i+1}$  of fractal sub-algebras of SSA such that the gauge conditions for  $SSA_i$  define a hierarchy  $KMA_i$  of dynamical KMAs acting as dynamical symmetries of the theory. The fusion algebra of  $KMA_i$  theory would be characterized by Galois group  $Gal_i$ .

4. I have considered the possibility that the McKay graphs for finite subgroups  $G \subset SU(2)$  indeed code for root diagrams of ADE type KMAs acting as dynamical symmetries to be distinguished from  $SU(2)$  KMA symmetry and from fundamental KMA symmetries assignable to the isometries and holonomies of  $M^4 \times CP_2$ .

One can of course ask whether also the fundamental symmetries could have a representation in terms of  $Gal$  or its semi-direct product  $G \triangleleft Gal$  with a finite sub-group automorphism group  $SO(3)$  of quaternions lifting to finite subgroup  $G \subset SU(2)$  acting on spinors. This is not necessary since  $Gal$  can form semidirect products with the algebraic subgroups of Lie groups of fundamental symmetries (Langlands program relies on this). In the generic case the algebraic subgroups spanned by given extension of rationals are infinite. When the finite subgroup  $G \subset SU(2)$  is closed under  $Gal$  automorphism, the situation changes, and these extensions are expected to be in a special role physically.

The number theoretic generalization of the idea that affine ADE group acts as symmetries would be roughly like following. The nodes of the McKay graph of  $G \triangleleft Gal$  label its irreps, which should be in 1-1 correspondence with the Cartan algebra of the KMA. The KMA counterparts of the local bilinear  $Gal$  invariants associated with  $Gal$  irreps would give currents of dynamical KMA having unit conformal weight. The convolution of primary fields with respect to conformal weight would be completely analogous to that occurring in the expression of energy momentum tensor as local bilinears of KMA currents.

If the free field construction using the local invariants as Cartan algebra defined by the irreps of  $G \triangleleft Gal$  works, it gives rise to charged primary fields for the dynamical KMA labelled by roots of the corresponding Lie algebra. For trivial  $Gal$  one would have ADE group acting as dynamical symmetries of minimal model associated with  $G \subset SU(2)$ .

5. Number theoretic Langlands conjecture [L1] [K5] generalizes this to the semidirect product  $G_0 \triangleleft Gal$  algebraic subgroup  $G_0$  of the original KMA Lie group (p-adicization allows also powers of roots of  $e$  in extension). One can imagine a hierarchy of KMA type algebras  $KMA_n$  obtained by repeating the procedure for the  $G_1 \triangleleft Gal$ , where  $G_1$  is discrete subgroup of the new KMA Lie group.
6. In CFTs are also other manners to extend VA or SVA (Super-Virasoro algebra) to a larger algebra by discovering new dynamical symmetries. The hope is that symmetries would allow to solve the CFT in question. The general constraint is that the conformal weights of symmetry generators are integer or half-integer valued. For the energy momentum tensor defining VA the conformal weight is  $h = 2$  whereas the conformal weights of primary fields for minimal models are rational numbers.

The simplest extension is SVA involving super generators with  $h = 3/2$ . Extension of (S)VA by (S)KMA so that (S)VA acts by semidirect product on (S)KMA means adding (S)KMA generators with  $h = 1$  (and  $1/2$ ). The generators of  $W_n$ -algebras (see <http://tinyurl.com/y93f6eoo>) have either integer or half integer conformal weights and the algebraic operations are defined as ordered products (an associative operation). These extensions are different from the proposed number theoretic extension for which the restriction to a discrete subgroup of KMA Lie group is essential.

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