About enumerative algebraic geometry in TGD framework

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Abstract

String models and M-theory involve both algebraic and symplectic enumerative geometry. Also in adelic TGD enumerative algebraic geometry emerges. This article gives a brief summary about the basic ideas involved and suggests some applications to TGD.

1. One might want to know the number of points of sub-variety belonging to the number field defining the coefficients of the polynomials. This problem is very relevant in M^8 formulation of TGD, where these points are carriers of sparticles. In TGD based vision about cognition they define cognitive representations as points of space-time surface, whose M^8 coordinates can be thought of as belonging to both real number field and to extensions of various p-adic number fields induced by the extension of rationals. If these cognitive representations define the vertices of analogs of twistor Grassmann diagrams in which sparticle lines meet, one would have number theoretically universal adelic formulation of scattering amplitudes and a deep connection between fundamental physics and cognition.
2. Second kind of problem involves a set algebraic surfaces represented as zero loci for polynomials - lines and circles in the simplest situations. One must find the number of algebraic surfaces intersecting or touching the surfaces in this set. Here the notion of incidence is central. Point can be incident on line or two lines (being their intersection), line on plane, etc.. This leads to the notion of Grassmannians and flag-manifolds. In twistor Grassmannian approach algebraic geometry of Grassmannians play key role. Also in twistor Grassmannian approach to TGD algebraic geometry of Grassmannians play a key role and some aspects of this approach are discussed.
3. In string models the notion of brane leads to what might be called quantum variant of algebraic geometry in which the usual rules of algebraic geometry do not apply as such. Gromow-Witten invariants provide an example of quantum invariants allowing sharper classification of algebraic and symplectic geometries. In TGD framework M^8-H duality suggests that the construction of scattering amplitudes at level of M^8 reduces to a super-space analog of algebraic geometry for complexified octonions. Candidates for TGD analogs of branes emerge naturally and G-W invariants could have applications also in TGD.

In the sequel I will summarize the understanding of novice about enumerative algebraic geometry and discuss possible TGD applications. This material can be also found in earlier articles but it seemed appropriate to collect the material about enumerative algebraic geometry to a separate article.

Contents

1 Introduction

Algebraic geometry is something very different from Riemann geometry, Kähler geometry, or sub-manifold geometry based on local notions. Sub-manifolds are replaced with sub-varieties defined as zero loci for polynomials with coefficients in the field of rationals or extension of rationals. Partial differential equations are replaced with algebraic ones. One can generalize algebraic geometry to any number field.

String theorists have worked with algebraic geometry with motivation coming from various moduli spaces emerging in string theory. The moduli spaces for closed and open strings possibly in presence of branes are involved. Also Calabi-Yau compacticication leads to algebraic geometry, and topological string theories of type A and B involve also moduli spaces and enumerative algebraic geometry.

In TGD the motivation for enumerative algebraic geometry comes from several sources.

1. Twistor lift of TGD lifts space-time surfaces to their 6-D twistor spaces representable as surfaces in the product of 6-D twistor spaces of M^4 and CP_2 and possessing Kähler structure - this makes these spaces completely unique and strongly suggests the role of algebraic geometry, in particular in the generalization of twistor Grassmannian approach [K4] [L1].

2. There are three threads in number theoretic vision: p-adic numbers and adelics, classical number fields, and infinite primes. Adelic physics [L3] as physics of sensory experience and cognition unifies real physics and various p-adic physics in the adele characterized by an extension of rationals inducing those of p-adic number fields. This leads to algebraic geometry and counting of points with imbedding space coordinates in the extension of rationals and defining a discrete cognitive representation. The core of the scattering amplitude would be defined by this cognitive representation identifiable in terms of points appearing as arguments of n-point function in QFT picture [K4].

3. $M^8 - M^4 \times CP_2$ duality is the analog of the rather adhoc spontaneous compactification in string models but would be non-dynamical and thus allow to avoid landscape catastrophe. Classical physics would reduce to octonionic algebraic geometry at the level of complexified octonions with several special features due to non-commutativity and non-associativity: space-time could be seen as 4-surface in the complexification of of octonions. The commuting imaginary unit would make possible the realization of algebraic extensions of rationals.

   The moduli space for the varieties is discrete if the coefficients of the polynomials are in the extension of rationals. If one poses additional conditions such as associativity of 4-surfaces, the moduli space is further reduced by the resulting criticality conditions realizing quantum criticality at the fundamental level raising hopes about extremely simple formulation of scattering amplitudes at the level of M^8 [K4] [L1].

   Also complex and co-complex sub-manifolds of associative space-time surface are important and would realize strong form of holography (SH). For non-associative regions of space-time surface it might not be possible to define complex and co-complex surfaces in unique manner since the basic $M^2 \subset M^4$ local flag structure is missing. String world sheets and partonic 2-surfaces and their moduli spaces are indeed in key role and the topology of partonic surfaces plays a key role in understanding of family replication phenomenon in TGD [K4].

In this framework one cannot avoid enumerative algebraic geometry.

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2. Second kind of problem involves a set algebraic surfaces represented as zero loci for polynomials - lines and circles in the simplest situations. One must find the number of algebraic surfaces intersecting or touching the surfaces in this set. Here the notion of incidence is central. Point can be incident on line or two lines (being their intersection), line on plane, etc.. This leads to the notion of Grassmannians and flag-manifolds. In twistor Grassmannian approach algebraic geometry of Grassmannians play key role. Some aspects of twistor Grassmannian approach in TGD discussed already in [K4] [L1] will be discussed in this article.

3. In string models the notion of brane leads to what might be called quantum variant of algebraic geometry in which the usual rules of algebraic geometry do not apply as such. Gromow-Witten invariants provide an example of quantum invariants allowing sharper classification of algebraic and symplectic geometries. In TGD framework M^8-H duality suggests that the construction of scattering amplitudes at level of M^8 reduces to a super-space analog of algebraic geometry for complexified octonions. Candidates for TGD analogs of branes emerge naturally and G-W invariants could have applications also in TGD.

Moduli spaces parameterizing sub-varieties of given kind - lines, circles, algebraic curves of given degree, are central for the more advanced formulation of algebraic geometry. These moduli spaces emerge also in the formulation of TGD. The moduli space of conformal equivalence classes of partonic 2-surfaces is one example involved with the explanation of family replication phenomenon [K1]. One can assign moduli spaces also to octonion and quaternion structures in M^8 (or equivalently with the complexification of E^8). One can identify CP_2 as a moduli space of quaternionic sub-spaces of octonions containing preferred complex sub-space.

One cannot avoid these moduli spaces in the formulation of the scattering amplitudes and this leads to $M^8 - H$ duality. The hard core of the calculation should however reduce to the understanding of the algebraic geometry of 4-surfaces in octonionic space. Clearly, M^8 picture seems to provide the simplest formulation of the number theoretic vision [K4] [L1].

In the sequel I will summarize the understanding of novice about enumerative algebraic geometry and discuss possible TGD applications. This material can be also found in [K4] [L2, L1] but it seemed appropriate to collect the material about enumerative algebraic geometry to a separate article.

2 Basics of enumerative algebraic geometry

2.1 Some examples

Some examples give an idea about what enumerative algebraic geometry (see `http://tinyurl.com/y7yzt67b`) is.

1. Consider 4 lines in 3-D space. What is the number of lines intersecting these 4 lines [A3] (see `http://tinyurl.com/ycrbr5aj`). One could deduce the number of lines and lines by writing the explicit equations for the lines with each line characterized by 2+3=5 parameters specifying direction t vector and arbitarily chosen point x_0 on the line. 2+3=5 parameters characterize each sought-for line.

   For intersection points x_i of sought for line with i:th one has $x_i = x_0 + k_i t_0$, $i = 1, ..., 4$ for the sought for line with direction t_0. At the intersection points at the 4 lines one has

$x_i = x_{0i} + K_i t_i$ with fixed directions t_i. Combining the two equations for each line one has $4 \times 3 = 12$ equations and 3+4+2 parameters for the sought for line plus 4 parameters K_i for the four lines. This gives 13 unknown parameters corresponding to x_0, k_i, K_i. One would have one parameter set of solutions: something goes wrong.

One has however projective invariance: one can shift x_0 along the line by $x_0 \to x_0 - at$, $k_i \to k_i + a$ and using this freedom assume for instance $k_1 = 0$. This reduces the number of parameters to 12 and one has finite number of solutions in the generic case. Actually the number is 2 in the generic case but can be infinite in some special cases. The challenge is to deduce the number of the solutions by geometric arguments.Below Schubert's argument proving that the number of solutions is 2 will be discussed.

The idea of enumerative geometry is to do this using general geometric arguments allowing to deform the problem topologically to a simpler one in which case the number of solutions is obvious which in the most abstract formulation become topological.

2. Apollonius can be seen as founder of enumerative algebraic geometry. Apollonian circles (see `http://tinyurl.com/ycvxe688`) represent second example. One has 3 circles in plane. What is the number of circles tangential to all these 3 circles. Wikipedia link represents the geometric solution of the problem. The number of circles is 8 in the generic case but there are exceptional cases.

3. In Steiner's conic problem (see `http://tinyurl.com/yahshsjo`) one have 5 conical sections (circles, cones, ellipsoids, hyperbole) in plane. How many different conics tangential to the conics there exist? This problem is rather difficult and the thumb rules of enumerative geometry (dimension counting, Bezout's rule, Schubert calculus) fail. This is a problem in projective geometry where one is forced to introduce moduli space for conics tangential to given conic. This space is algebraic sub-variety of all conics in plane which is 5-D projective space. One must be able to deduce the number of points in the intersection of these sub-varieties so that the original problem in 2-D plane is replaced with a problem in moduli space.

2.2 About methods of algebraic enumerative geometry

A brief summary about methods of algebraic geometry is in order to give some idea about what is involved (see `http://tinyurl.com/y7yzt67b`).

1. Dimension counting is the simplest method. If two geometric objects of n-D space have dimensions k and l, there intersection is $n-k-l$-dimensional for $n-k-l \geq 0$ or empty in the generic case. For $k+l=n$ one obtains discrete set of intersection points.

2. Bezouts theorem is a more advanced method. Consider for instance, curves in plane defined by the curves polynomials $x = P^m(y)$ and $x = P^n(y)$ of degrees $k = m$ and $k = n$. The number N of intersection points in the generic case is bounded above by $N = m \times n$ (in this case all roots are real). One can understand this by noticing that one has m roots y_k or given x giving rise to a m-branched graph of function $y = f(x)$. The number of intersections for the graphs of the two polynomials is at most $m \times n$. If one has curve in plane represented by polynomial equation $P^{m,n}(x,y) = 0$, one can also estimate immediately the minimal multi-degree (m,n) for this polynomials.

3. Schubert calculus `http://tinyurl.com/y766ddw2`) is a more advanced but not completely rigorous method of enumerative geometry [A3] (see `http://tinyurl.com/ycrbr5aj`).

   Schubert's vision was that the number of intersection points is stable against deformations in the generic case. This is not quite true always but in exceptional cases one can say that two separate solutions degenerate to single one, just like roots of polynomial can do for suitable values of coefficients.

   For instance, Schubert's solution to the already mentioned problem of finding a line intersecting 4 lines in generic position relies on this assumption. The idea is to deform the situation so that one has two intersecting pairs of lines. One solution to the problem is a line going through the intersection points for line pairs. Second solution is obtained as intersection of the planes. It can happen that planes are parallel in which case this does not work.

Schubert calculus it applies to linear sub-varieties but can be generalized also to non-linear varieties. The notion of incidence allowing a general formulation for intersection and tangentiality (touching) is central. This leads to the notions of flag, flag manifold, and Schubert variety as sub-variety of Grassmannian.

Flag is a hierarchy of incident subspaces $A_0 \subset A_1 \subset A_2 ... \subset A_n$ with the property that the the dimension $d_i \leq n$ of A_i satisfies $d_i \geq i$. As a special case this notion leads to the notion of Grassmannian $G(k,n)$ consisting of k-planes in n-dimensional space: in this case A_0 corresponds to k-planes and A_2 to space A_n. More general flag manifolds are moduli spaces and sub-varieties of Grassmannian providing a solution to some conditions. Flag varieties as sub-varieties of Grassmannians are Schubert varieties (see `http://tinyurl.com/y7ehcrzg`). They are also examples of singular varieties. More general Grassmannians are obtained as coset spaces of G/P, where G is algebraic group and P is parabolic sub-group of G.

Remark: CP_2 corresponds to the space of complex lines in C^3. CP_2 can be also understood as the space of quaternionic planes in octonionic 8-space containing fixed 2-plane so that also now one has flag. String world sheets inside space-time surfaces define curved flags with 2-D and 4-D tangent spaces defining an integrable distribution of local flags.

4. Cohomology combined with Poincare duality allows a rigorous formulation of Schubert calculus. Schubert's idea about possibility to deform the generic position corresponds to homotopy invariance, when the degeneracies of the solutions are taken into account. Homology and cohomology become basic tools and the so called cup product for cohomology together with Poincare duality and Künneth formula for the cohomology of Cartesian product in terms of cohomologies of factors allows to deduce intersection numbers algebraically. Schubert cells define a basis for the homology of Grassmannian containing only even-dimensional generators.

   Grassmannians play a key role in twistor Grassmannian approach as auxiliary manifolds. In particular, the singularities of the integrand of the scattering amplitude defined as a multiple residue integral over $G(k,n)$ define a hierarchy of Schubert cells. The so called positive Grassmannian [B1] defines a subset of singularities appearing in the scattering amplitudes of $\mathcal{N}=4$ SUSY. This hierarchy and its CP_2 counterpart are expected also in TGD framework.

   Remark: Schubert's vision might be relevant for the notion of conscious intelligence. Could problem solving involve the transformation of a problem to a simple critical problem, which is easy but for which some solutions can become degenerate? The transformation of general position for 4 lines to a pair of intersecting lines would be example of this. One can wonder whether quantum criticality could help problem solving by finding critical cases.

5. Moduli spaces of curves and varieties provide the most refined methods. Flag manifolds define basic examples of moduli spaces. Quantum cohomology represents even more refined conceptualization: the varieties (branes in M-theory terminology) are said to be connected or intersect if each of them has a common point with the same pseudo-holomorphic variety ("string world sheet"). Pseudo-holomorphy - which could have minimal surface property as counterpart - implies that the connecting 2-surface is not arbitrary.

   Quantum intersection for the "string world sheet" and "brane" is possible also when it is not stable classically (the co-dimension of brane is smaller than 2). Even in the case that it possible classically quantum intersection makes possible kind of "telepathic" quantum contact mediated by the "string world sheet" naturally involved with the description of quantum entanglement in TGD framework.

3 Gromow-Witten invariants

Gromow-Witten invariants play a central role in superstring theories and M-theory and are closely related to branes. For instance, partition functions can be expressed in terms of these invariants giving additional invariants of symplectic and algebraic geometries. Hence it is interesting to look whether they could be important also in TGD framework.

3.1 Formal definition

Consider first the definition of Gromow-Witten (G-W) invariants (see `http://tinyurl.com/y9b5vbcw`). G-W invariants are rational number valued topological invariants useful in algebraic and symplectic geometry. These quantum invariants give information about these geometries not provided by classical invariants. Despite being rational numbers in the general case G-W invariants in some sense give the number of string world sheets connecting given branes.

1. One considers collection of n surfaces ("branes") with even dimensions in some symplectic manifold X of dimension $D = 2k$ (say Kähler manifold) and pseudo-holomorphic curves ("string world sheets") X^2, which have the property that they connect these n surfaces in the sense that they intersect the "branes" in the marked points x_i, $i = 1, .., n$.

   "Connect" does not reduce to intersection in topologically stable sense since connecting is possible also for branes with dimension smaller than $D - 2$. One allows all surfaces that X^2 that intersects the n surfaces at marked points if they are pseudo-holomorphic even if the basic dimension rule is not satisfied. In 4-dimensional case this does not seem to have implications since partonic 2-surfaces satisfy automatically the dimension rule. The n branes intersect or touch in quantum sense: there is no concrete intersection but intersection with the mediation of "string world sheet".

2. Pseudo-holomorphy means that the Jacobian df of the imbedding map $f : X^2 \to X$ commutes with the symplectic structures j *resp.* J of X^2 *resp.* X: i.e. one has $df(jT) = Jdf(T)$ for any tangent vector T at given point of X^2. For $X^2 = X = C$ this gives Cauchy-Riemann conditions.

   In the symplectic case X^2 is characterized topologically by its genus g and homology class A as surface of X. In algebraic geometry context the degree d of the polynomial defining X^2 replaces A. In TGD X^2 corresponds to string world sheet having also boundary. X^2 has also n marked points $x_1, ..., x_n$ corresponding to intersections with the n surfaces.

3. G-W invariant $GW^{X,A}_{g,n}$ gives the number of pseudo-holomorphic 2-surfaces X^2 connecting n given surfaces in X - each at single marked point. In TGD these surfaces would be partonic 2-surfaces and marked points would be carriers of sparticles.

The explicit definition of G-W invariant is rather hard to understand by a layman like me. I however try to express the basic idea on basis of Wikipedia definition (see `http://tinyurl.com/y9b5vbcw`). I apologize for my primitive understanding of higher algebraic geometry. The article of Vakil [A1] (see `http://tinyurl.com/ybobccub`) discusses the notion of G-W invariant in detail.

1. The situation is conformally invariant meaning that one considers only the conformal equivalence classes for the marked pseudo-holomorphic curves X^2 parameterized by the points of so called Deligne-Mumford moduli space $\overline{M}_{g,n}$ of curves of genus g with n marked points: note that these curves are just abstract objects without no imbedding as surface to X assumed. $\overline{M}_{g,n}$ has *complex* dimension

   $$d_0 = 3(g-1) + n \ .$$

   n corresponds complex dimensions assignable to the marked points and $3(g-1)$ correspond to the complex moduli in absence of marked points. This space appears in TGD framework in the construction of elementary particle vacuum functionals [K1].

2. Since these curves must be represented as surfaces in X one must introduces the moduli space $\overline{M}_{g,n}(X, A)$ of their maps f to X with given homology equivalence class. The elements in this space are of form $(C, x_1, .., x_n, f)$ where C is one particular representative of A.

3. The complex dimension d of $\overline{M}_{g,n}(X, A)$ can be calculated. One has

   $$d = d_0 + c_1^X(A) + (g-1)k \ .$$

Here $c_1^X(A)$ is the first Chern class defining element of second cohomology of X evaluated for A. For Calabi-Yau manifolds one has $c_1 = 0$. The contribution $(g-1)k$ to the dimension vanishing for torus topology should have some simple explanation.

4. One defines so called evaluation map ev from $\overline{M}_{g,n}(X, A) \rightarrow Y$, $Y = \overline{M}_{g,n} \times X^n$ in terms of stabilization $st(C, x_1, ..., x_n) \in \overline{M}_{g,n}(X, A)$ of C (I understand that stabilization means that the automophism group of the stabilized surface defined by f is finite [A2] (see `http://tinyurl.com/y8r44uhl`). I am not quite sure what the finiteness of the automorphism group means. One might however think that conformal transformations must be in question. One has

$$ev(C, x_1, .., x_n, f) = (st(C, x_1, .., x_n), f(x_1), ...f(x_n)) \ .$$

   Evaluation map assigns to the concrete realization of string world sheet with marked points the abstract curve $st(C, x_1, .., x_n)$ and points $(f(x_i), ..., f(x_n)) \in X^n$ possibly interpretable as positions $f(x_i)$ of n particles. One could say that one has many particle system with particles represented by surfaces of X_i of X connected by X^2 - string world sheet - mediating interaction between X_i via the intersection points.

5. Evaluation map takes the fundamental class of $\overline{M}_{g,n}(X, A)$ in $H_d(\overline{M}_{g,n}(X, A))$ to an element of homology group $H_d(Y)$. This homology equivalence class defines G-W invariant, which is rational valued in the general case.

6. One can make this more concrete by considering homology equivalence class β in $\overline{M}_{g,n}$ and homology equivalence classes α_i, $i = 1, ..., n$ represented by the surfaces X_i. The co-dimensions of these $n+1$ homology equivalence classes must sum up to d. The homologies of $\overline{M}_{g,n}$ and $Y = \overline{M}_{g,n} \times X^n$ induce homology of Y by Künneth formula (see `http://tinyurl.com/yd9ttlfr`) implying that Y has class of $H_d(Y)$ given by the product $\beta \cdot \alpha_1 ... \cdot \alpha_n$.

   One can identify the value of $GW_{g,n}^{X,A}$ for a given class $\beta \cdot \alpha_1 ... \cdot \alpha_n$ as the coefficients in its expansion as sum of all elements in $H_d(Y)$. This coefficient is the value of its intersection product of $GW_{g,n}^{X,A}$ with the product $\beta \cdot \alpha 1 ... \cdot \alpha_n$ and gives element of $H_0(Q)$, which is rational number.

7. There are two non-classical features. Classically intersection must be topologically stable. This would require α_i to have codimension 2 but all even co-dimensions are allowed. That the value for the number of connecting string world sheets is rational number does not conform with the classical geometric intuition. The Wikipedia explanation is that the orbifold singularities for the space $\overline{M}_{g,n}(X, A)$ of stable maps are responsible for rational number.

3.2 Application to string theory

Topological string theories give a physical realization of this picture. Here the review article *Instantons, Topological Strings, and Enumerative Geometry* of Szabo [A2] (see `http://tinyurl.com/y8r44uhl`) is very helpful.

1. In M-theory framework and for topological string models of type A and B the physical interpretation for the varieties associated with α_i would be as branes of various dimensions needed to satisfy Dirichlet boundary conditions for strings.

2. In topological string theories one considers sigma model with target space X, which can be rather general. The symplectic or complex structure of X is however essential. X is forced to be 3-D (in complex sense) Calabi-Yau manifold by consistency of quantum theory. Interestingly, the super twistor space $CP(3|4)$ is super Calabi-Yau manifold although CP_3 is not and must therefore have trivial first Chern class c_1 appearing in the formula for the dimension d above. I must admit that I do not understand why this is the case.

   Closed topological strings have no marked points and one has $n = 0$. Open topological strings world sheets meet n branes at points x_i, where they satisfy Dirichlet boundary conditions. Branes an be identified as even-dimensional Lagrangian sub-manifolds with vanishing induced symplectic form.

3. For topological closed string theories of type A one considers holomorphically imbedded curves in X characterized by genus g and homology class A: one speaks of world sheet instantons. $A = \sum n_i S_i$ is sum over the generating classes S_i with integer coefficients. For given g and A one has analog of product of non-interacting systems at temperatures $1/t_i$ assignable to the homology classes S_i with energies identifiable as n_i. One can assign Boltzmann weight labelled by (g, A) as $Q^\beta = \prod_i Q_i^{n_i}$, $Q_i = exp(-t_i)$.

   One can construct partition function for the entire system as sum over Boltzmann weights with degeneracy factors telling the number of world sheet instantons with given (g, A). One can calculate free energy as sum $\sum N_{g,\beta} Q^\beta$ over contributions labelled by (g, A). The coefficients $N_{g,\beta}$ count the rational valued degeneracies of the world sheet instantons of given type and reduce to G-W invariants $GW_{g,0}^{X,A}$.

   Remark: If one allows powers of a root $e^{-1/n}$, $t = n$, in the extension of rationals or replace e^{-t} with p^n, partition functions make sense also in the p-adic context.

4. For topological open string theories of type A one has also branes. Homology equivalence classes are relative to the brane configuration. The coefficients $N_{g,\beta}$ are given by $GW_{g,n}^{X,A}$ for a given configuration of branes: the above described general formulas correspond to these.

5. For topological string theories of type B, string world sheets reduce to single point and thus correspond to constant solutions to the field equations of sigma model. Quantum intersection reduces to ordinary intersection and one has $x_1 = x_2 ... = x_n$. G-W invariants involve only classical cohomology and give for $n = 2$ the number of common points for two surfaces in X with dimension d_1 and $d_2 = n - d$. The duality between topological string theories of type A and B related to the mirror symmetry supports the idea that one could generalize the calculation of these invariants in theories B to theories A. It is not clear whether this option as any analog in TGD.

The so called Witten conjecture (see `http://tinyurl.com/yccahv3q`) proved by Kontsevich states that the partition function in one formulation of stringy quantum gravity and having as coefficients of free energy G-W invariants of the target space is same as the partition function in second formulation and expressible in terms of so called tau function associated with KdV hierarchy. This leads to non-trivial identities. Witten conjecture actually follows from the invariance of partition function with respect to half Virasoro algebra and Virasoro conjecture (see `http://tinyurl.com/y7xcc9hm`) stating just this generalizes Witten's conjecture.

3.3 About the analogs of branes and G-W invariants in TGD

A couple of comments from TGD perspective are in order.

1. As such the definition of G-W invariants given above do not make sense in TGD framework. For instance, space-time surface is not a closed symplectic manifold whereas M^8 and H are analogs of symplectic spaces. Minkowskian regions of space-time surface have Hamilton-Jacobi structure at the level of both M^8 and H and this might replace the symplectic structure. Space-time surfaces are not closed manifolds.

   Physical intuition however suggests that the generalization exists. The fact that Minkowskian metric and Euclidian metric for complexified octonions are obtained in various sectors for which complex valued length squared is real suggests that signature is not a problem. Kähler form for complexified z gives as special case analog of Kähler form for E^4 and M^4.

2. The quantum intersection defines a description of interactions in terms of string world sheets. If I have understood G-W invariant correctly, one could have for $D > 4$-dimensional symplectic spaces besides partonic $2k-2$-D surfaces also surfaces with smaller but even dimension identifiable as branes of various dimensions. Branes would correspond to a generalization of relative cohomology. In TGD framework one has $2k = 4$ and the partonic 2-surfaces have dimension 2 so that classical intersections consisting of discrete points are possible and stable for string world sheets and partonic 2-surfaces. This is a unique feature of 4-D space-time.

One might think a generalization of G-W invariant allowing to see string world sheets as connecting the spaced-like 3-surfaces at the boundaries of CDs and light-like orbits of partonic 2-surfaces. The intersection is not discrete now and marked points would naturally correspond to the ends points of strings at partonic 2-surfaces associated with the boundaries of CD and with the vertices of topological scattering diagrams.

3. The idea about 2-D string world sheet as interaction region could generalize in TGD to space-time surface inside CD defining 4-D interaction region. In [K4] [L1] one indeed ends up with amazingly similar description of interactions for n external particles entering CD and represented as zero loci for quaternion valued "real" part $RE(P)$ or "imaginary" part $IM(P)$ for the complexified octonionic polynomial.

   Associativity forces quantum criticality posing conditions on the coefficients of the polynomials. Polynomials with the origin of octonion coordinate along the same real axis commute and associate. Since the origins are different for external particles in the general case, the polynomials representing particles neither commute nor associate inside the interaction region defined by CD but one can also now define zero loci for both $RE(\prod P_i)$ and $IM(\prod P_i)$ giving $P_i = 0$ for some i. Now different permutations and different associations give rise to different interaction regions and amplitude must be sum over all these.

   3-vertices would correspond to conditions $P_i = 0$ for 3 indices i simultaneously. The strongest condition is that 3 partonic 2-surfaces X_i^2 co-incide: this condition does not satisfy classical dimension rule and should be posed as essentially 4-D boundary condition. Two partonic 2-surfaces $X_i^2(t_i(n))$ intersect at discrete set of points: could one assume that the sparticle lines intersect and there fusion is forced by boundary condition? Or could one imagine that partonic 2-surfaces turns back in time and second partonic 2-surface intersects it at the turning point?

4. In 4-D context string world sheets are associated with magnetic flux tubes connecting partonic orbits and together with strings serve as correlates for negentropic entanglement assignable to the p-adic sectors of the adele considered, to attention in consciousness theory, and to remote mental interactions in general and occurring routinely between magnetic body and biological body also in ordinary biology. This raises the question whether "quantum touch" generalizes from 2-D string world sheets to 4-D space-time surface (magnetic flux tubes) connecting 3-surfaces at the orbits and partonic orbits.

5. The above formulation applies to closed symplectic manifolds X. One can however generalize the formulation to algebraic geometry. Now the algebraic curve X^2 is characterized by genus g and order of polynomial n defining it. This formulation looks very natural in M^8 picture.

An interesting question is whether the notion of brane makes sense in TGD framework.

1. In TGD branes inside space-time variety are replaced by partonic 2-surfaces and possibly by their light-like orbits at which the induced metric changes signature. These surfaces are metrically 2-D. String world sheets inside space-time surfaces have discrete intersection with the partonic 2-surfaces. The intersection of strings as space-like *resp.* light-like boundaries of string world sheet with partonic orbit sheet *resp.* space-like 3-D ends of space-time surface at boundaries of CD is also discrete classically.

2. An interesting question concerns the role of 6-spheres $S^6(t_n)$ appearing as special solutions to the octonionic zero locus conditions solving both $RE(P_n) = 0$ and $IM(P_n) = 0$ requiring $P_n(o) = 0$. This can be true at 7-D light cone $o = et$, e light-like vector and t a real parameter. The roots t_n of $P(t) = 0$ give 6-spheres $S^6(t_n)$ with radius t_n as solutions to the singularity condition. As found, one can assign to each factor P_i in the product of polynomials defining many-particle state in interaction region its own partonic 2-surfaces $X^2(t_n)$ related to the solution of $P_i(t) = 0$

   Could one interpret 6-spheres as brane like objects, which can be connected by 2-D "free" string world sheets as 2-varieties in M^8 and having discrete intersection with them implied by the classical dimension condition for the intersection. Free string world sheets would be something new and could be seen as trivially associative surfaces whereas 6-spheres would represent trivially co-associative surfaces in M^8.

The 2-D intersections of $S^6(t_n)$ with space-time surfaces define partonic 2-surfaces X^2 appearing at then ends of space-time and as vertices of topological diagrams. Light-like sparticle lines along parton orbits would fuse at the partonic 2-surfaces and give rise to the analog of 3-vertex in $\mathcal{N} = 4$ SUSY.

Some further TGD inspired remarks are in order.

1. Virasoro conjecture generalizing Witten conjecture involves half Virasoro algebra. Super-Virasoro algebra algebra and its super-symplectic counterpart (SSA) play a key role in the formulation of TGD at level of H. Also these algebras are half algebras. The analogs of super-conformal conformal gauge conditions state that sub-algebra of SSA with conformal weights coming as n-ples of those for entire algebra and its commutator with entire SSA give rise to vanishing Noether charges and annihilate physical states.

   These conditions are conjecture to fix the preferred extremals and serve as boundary conditions allowing the formulation of $M^8 - H$ correspondence inside space-time regions (interaction regions), where the associativity conditions fail to be true and direct $M^8 - H$ correspondence does not make sense. Non-trivial solutions to these conditions are possible only if one assumes half super-conformal and half super-symplectic algebras. Otherwise the generators of the entire SSA annihilate the physical states and all SSA Noether charges vanish. The invariance of partition function for string world sheets in this sense could be interpreted in terms of emergent dynamical symmetries.

2. Just for fun one can consider the conjecture that the reduction of quantum intersections to classical intersections mediated by string world sheets implies that the numbers of string world sheets as given by the analog of G-W invariants are integers.

4 Twistor Grassmannians and algebraic geometry

Twistor Grassmannians provide an application of algebraic geometry involving the above described notions [B1] (see `http://tinyurl.com/yd9tf2ya`). This approach allows extremely elegant expressions for planar amplitudes of $\mathcal{N} = 4$ SYM theory in terms of amplitudes formulated in Grassmannians $G(k, n)$.

It seems that this approach generalizes to TGD in such a manner that CP_2 degrees of freedom give rise to additional factors in the amplitudes having form very similar to the M^4 part of amplitudes and involving also $G(k, n)$ with ordinary twistor space CP_3 being replaced with the flag manifold $SU(3)/U(1) \times U(1)$: k would now correspond to the number sparticles with negative weak isospin. Therefore the understanding of the algebraic geometry of twistor amplitudes could be helpful also in TGD framework.

4.1 Twistor Grassmannian approach very concisely

I try to compress my non-professional understanding of twistor Grassmann approach to some key points.

1. Twistor Grassmannian approach constructs the scattering amplitudes by fusing 3-vertices (+,-,-) (one positive helicity) and (-,+,+) (one negative helicity) to a more complex diagrams. All particles are on mass shell and massless but complex. If only real massless momenta are allowed the scattering amplitudes would allow only collinear gluons. Incoming particles have real momenta.

   Remark: Remarkably, $M^4 \times CP_2$ twistor lift of TGD predicts also complex Noether charges, in particular momenta, already at classical level. Also $M^8 - H$ duality requires a complexification of octonions by adding commuting imaginary unit and allows to circumvent problems related to the Minkowski signature since the metric tensor can be regarded as Euclidian metric tensor defining complex value norm as bilinear $m^k m_{kl} m^l$ in complexified M^8 so that real metric is obtained only in sub-spaces with real or purely imaginary coordinates. The additional imaginary unit allows also to define what complex algebraic numbers mean.

The unique property of 3-vertex is that the twistorial formulation for the conservation of four-momentum implies that in the vertex one has either $\lambda_1 \propto \lambda_2 \propto \lambda_3$ or $\overline{\lambda}_1 \propto \overline{\lambda}_2 \propto \overline{\lambda}_3$. These cases correspond to the 2 3-vertices distinguished notationally by the color of the vertex taken to be white or black [B1].

Remark: One must allow octonionic super-space in M^8 formulation so that octonionic SUSY broken by CP_2 geometry reducing to the quaternionicity of 8-momenta in given scattering diagram is obtained.

2. The conservation condition for the total four-momentum is quadratic in twistor variables for incoming particles. One can linearize this condition by introducing auxiliary Grassmannian $G(k,n)$ over which the tree amplitude can be expressed as a residue integral. The number theoretical beauty of the multiple residue integral is that it can make sense also p-adically unlike ordinary integral.

   The outcome of residue integral is a sum of residues at discrete set of points. One can construct general planar diagrams containing loops from tree diagrams with loops by BCFW recursion. I have considered the possibility that BCFW recursion is trivial in TGD since coupling constants should be invariant under the addition of loops: the proposed scattering diagrammatics however assumed that scattering vertices reduce to scattering vertices for 2 fermions. The justification for renormalization group invariance would be number theoretical: there is no guarantee that infinite sum of diagrams gives simple function defined in all number fields with parameters in extension of rationals (say rational function).

3. The general form of the Grassmannian integrand in $G(k,n)$ can be deduced and follows from Yangian invariance meaning that one has conformal symmetries and their duals which expand to full infinite-dimensional Yangian symmetry. The denominator of the integrand of planar tree diagram is the product of determinants of $k \times k$ minors for the $k \times n$ matrix providing representation of a point of $G(k,n)$ unique apart from $SL(k,k)$ transformations. Only minors consisting of k consecutive columns are assumed in the product. The residue integral is determined by the poles of the denominator. There are also dynamical singularities allowing the amplitude to be non-vanishing only for some special configurations of the external momenta.

4. On mass-shell diagrams obtained by fusing 3-vertices are highly redundant. One can describe the general diagram by using a disk such that its boundary contains the external particles with positive or negative helicity. The diagram has certain number n_F of faces. There are moves, which do not affect the amplitude and it is possible to reduce the number of faces to minimal one: this gives what is called reduced diagram. Reduced diagrams with n_F faces define a unique n_F-1-dimensional sub-manifold of $G(k,n)$ over which the residue integral can be defined. Since the dimension of $G(k,n)$ is finite, also n_F is finite so that the number of diagrams is finite.

5. On mass shell diagrams can be labelled by the permutations of the external lines. This gives a connection with 1+1-dimensional QFTs and with braid group. In 1+1-D integral QFTs however scattering matrix induces only particle exchanges.

   The permutation has simple geometric description: one starts from the boundary point of the diagram and moves always from left or right depending on the color of the point from which one started. One arrives some other point at the boundary and the final points are different for different starting points so that the process assigns a unique perturbation for a given diagram. Diagrams which are obtained by moves from each other define the same permutation. BFCW bridge which is a manner to obtain new Yangian invariant corresponds to a permutation of consecutive external particles in the diagram.

6. The poles of the denominator determine the value of the multiple residue integrals. If one allowed all minors, one would have extremely complex structure of singularities. The allowance only cyclically taken minors simplifies the situation dramatically. Singularities correspond to n subgroups of more than 2 collinear k-vectors implying vanishing of some of the minors.

7. Algebraic geometry comes in rescue in the understanding of singularities. Since residue integral is in question, the choice is rather free and only the homology equivalence class of the cell decomposition matters. The poles for a hierarchy with poles inside poles since given singularity contains sub-singularities. This hierarchy gives rise to a what is known as cell composition - stratification - of Grassmannian consisting of varieties with various dimensions. These sub-varieties define representatives for the homology group of Grassmannian. Schubert cells already mentioned define this kind of stratification.

   Remark: The stratification has very strong analogy of the decomposition of catastrophe in Thom's catastrophe theory to pieces of various dimensions. The smaller the dimension, the higher the criticality involved. A connection with quantum criticality of TGD is therefore highly suggestive.

   Cyclicity implies a reduction of the stratification to that for positive Grassmannians for which the points are representable as $k \times n$ matrices with non-negative $k \times k$ determinants. This simplifies the situation even further.

   Yangian symmetries have a geometric interpretation as symmetries of the stratification: level 1 Yangian symmetries are diffeomorphisms preserving the cell decomposition.

4.2 Problems of twistor approach

Twistor approach is extremely beautiful and elegant but has some problems.

1. The notion of twistor structure is problematic in curved space-times. In TGD framework the twistor structures of M^4 and CP_2 (E^4) induce twistor structure of space-time surface and the problem disappears just like the problems related to classical conservation laws are circumvented. Complexification of octonions allows to solve the problems related to the metric signature in twistorialization.

2. The description of massive particles is a problem. In TGD framework M^8 approach allows to replace massive particles with particles with octonionic momenta light-like in 8-D sense belonging to quaternionic subspace for a given diagram. The situation reduces to that for ordinary twistors in this quaternionic sub-space but since quaternionic sub-space can vary, additional degrees of freedom bringing in CP_2 emerge and manifest themselves as transversal 8-D mass giving real mass in 4-D sense.

3. Non-planar diagrams are also a problem. In TGD framework a natural guess is that they correspond to various permutations of free particle octonionic polynomials. Their product defines interaction region in the interior of CD to which free particles satisfying associativity conditions (quantum criticality) arrive. If the origins of polynomials are not along same time axis, the polynomials do not commute nor associate. One must sum over their permutations and for each permutation over its associations.

4.3 Trying to understand the fundamental 3-vertex in twistor approach

As far as the realization of four-momentum conservation is considered 3-vertex unique twistorial properties. Therefore it is fundamental in the construction of scattering diagrams in twistor Grassmannian approach to $\mathcal{N} = 4$ SYM [B1] (see `http://tinyurl.com/yd9tf2ya`). Twistor Grassmann approach suggests that 3-vertex with complexified light-like 8-momenta represents the basic building brick representing from which more complex diagrams can be constructed using the BCFW recursion formula [B1]. In TGD 3-vertex generalized to 8-D light-like quaternionic momenta should be highly analogous to the 4-D 3-vertex and in a well-defined sense reduce to it if all momenta of the diagram belong to the same quaternionic sub-space M_0^4 [K4] [L1]. It is however not completely clear how 3-vertex emerges in TGD framework.

1. A possible identification of the 3-vertex at the level of M^8 would be as a vertex at which 3 sparticle lines with light-like complexified quaternionic 8-momenta meet. This vertex would be associated with the partonic vertex $X^2(t_n) = X^4 \cap S^6(t_n)$. Incoming sparticle lines at the light-like partonic orbits identified as boundaries of string world sheets (for entangled states at least) would be light-like.

Does the fusion of two sparticle lines to third one require that either or both fusing lines become space-like - say pieces of geodesic line inside the Euclidian space-time region- bounded by the partonic orbit? The identification of the lines of twistor diagrams as carriers of light-like complexified quaternionic momenta in 8-D sense does not encourage this interpretation (also classical momenta are complex). Should one pose the fusion of the light-like lines as a boundary condition?

2. As found in [L1], one can challenge the assumption about the existence of string world sheets as commutative regions in the non-associative interaction region. Could one have just fermion lines as light-like curves at partonic orbits inside CD? Or cannot one have even them?

   Even if the polynomial $\prod_i P_i$ defining the interaction region is product of polynomials with origins of octonionic coordinates not along the same real line, the 7-D light-cones of M^8 associated with the particles still make sense in the sense that $P_i(o_i) = 0$ reduces at it to $P_i(t_i) = 0$, t_i real number, giving spheres $S^6(t_i(n))$ and partonic 2-surfaces and vertices $X_2(t_i(n))$. The light-like curves as geodesics the boundary of 7-D light-cones mapped to light-like curves along partonic orbits in H would not be lost inside interaction regions. Hence it seems that light-like curves are there. Interactions can be said to effectively localize sparticles to 1-D lightlike curves.

3. At the level of H this relates to a long standing interpretational problem related to the notion of induced spinor fields. SH suggests strongly the localization of the induced spinor fields at string world sheets and even at sparton lines in absence of entanglement. On the other hand, various super-conformal symmetries require that induced spinor fields are 4-D and thus seems to favor de-localization. The information theoretic interpretation is that the induced spinor fields at string world sheets or even at sparton lines contain all information needed to construct the scattering amplitudes. One can also say that string world sheets and sparton lines correspond to a description in terms of an effective action.

4.4 Could the M^8 view about twistorial scattering amplitudes be consistent with the earlier H picture?

The proposed picture involving super coordinates of M^8 and super-twistors [K4] [L1] does not seem to conform with the earlier proposal for the construction of scattering amplitudes at the level of H [K3]. Is this really true?

In H picture the introduction of super-space does not look natural, and one can say that fundamental fermions are the only fundamental particles [K2, K3]. The H view about supersymmetry is as broken supersymmetry in which many fermion states at partonic 2-surfaces give rise to supermultiplets such that fermions are at different points. Fermion 4-vertex would be the fundamental vertex and involve classical scattering without fusion of fermion lines. Only a redistribution of fermion and anti-fermion lines among the orbits of partonic 2-surfaces would take place in scattering and one would have kind of OZI rule.

Could the earlier H view conform with the recent M^8 view, which is much closer to the SUSY picture.

1. The intuitive idea without a rigorous justification has been that the fermion lines at partonic 2-surfaces correspond to singularities of many-sheeted space-time surface at which some sheets co-incide. M^8 sparticle consists effectively of n fermions at the same point in M^8. Could it be mapped by M^8-H duality to n fermions at distinct locations of partonic 2-surface in H?

2. $M^8 - H$ correspondence maps the points of $M^4 \subset M^4 \times E^4$ to points of $M^4 \subset M^4 \times CP_2$. The tangent plane of space-time surface containing a preferred M^2 is mapped to a point of CP_2. If the effective n-fermion state M^8 is at point at which n sheets of space-time surface co-incide and if the tangent spaces of different sheets are not identical, which is quite possible and even plausible, the point is indeed mapped to n points of H with same M^4 coordinates but different CP_2 coordinates and sparticle would be mapped to a genuine many-fermion state.

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