Generalization of Fermat's last theorem and TGD

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Abstract

The inspiration for writing this article came from a popular article telling about number theoretic developments related to Fermat's last theorem, its connection of Diophantine equations and spectra of automorphic forms and Langlands correspondence. The article describes recent generalizations of Fermat's last theorem and the message is that there is a bridge between two very different mathematics: Diophantine equations at number theory side and automorphic functions associated with the representations of non-compact algebraic groups in highly symmetric spaces at geometry side. I am not a mathematician in the technical sense of word and my humble goal is to understand the relevance to TGD, where number theoretic and geometric visions are complementary to each other as they are in Langlands correspondence.

In TGD an obvious candidate for a space at automorphic side would be the product of $H^3 \times CP_2$ carrying the representations of $SO(1,3) \times SU(3)$. H^3 is 3-D hyperboloid H^3 of M^4 having $SO(1,3)$ as group of isometries. The infinite discrete subgroups of $SO(1,3)$ define tesselations of H^3 analogous lattices in E^3, and one can assign to these automorphic functions as analogs of Bloch waves. They would be associated with separable solutions of spinor d'Alembertian in future light-cone, which corresponds to empty Robertson-Walker cosmology. This is however not the only option: automorphic functions appear also in the description of family replication phenomenon and give rise to modular invariant elementary particle functions in the spaces of conformal moduli for partonic 2-surfaces.

$M^8 - H$ duality states that space-time can be regarded as a 4-surface in either complexified 8-D Minkowski space having interpretation as complexified octonions or $H = M^4 \times CP_2$. At the level M^8 space-time surfaces are algebraic surfaces assignable to an algebraic continuation of a polynomial with rational (or even algebraic) coefficients to M^8. In H one has minimal surfaces with 2-D algebraic singularities - string world sheets and partonic 2-surfaces. Each polynomial defines extension of rationals and the Galois group of extension acts as a symmetry group for the cognitive representations identified as the set of points of space-time surface with coordinate values in the extension of rationals considered. This is central for adelic physics fusing real physics and physics for extensions of p-adic numbers induced by that for rationals. Cognitive representations would define the number theoretic side and Langlands correspondence and generalization of Fermat's theorem would mean that there is many-to-one correspondence from the automorphic side (imbedding space level) to the number theoretic side (cognitive representations). In particular, Galois group of extension would have action as a discrete finite subgroup of $SO(3) \subset SO(1,3)$.

In this article I try to relate this picture to the extension of Fermat's theorem and to Langlands correspondence.

1 Introduction

I received a link to a popular article published in Quanta Magazine (http://tinyurl.com/t44qv8o) with title *Amazing Math Bridge Extended Beyond Fermats Last Theorem* suggesting that Fermat's last theorem could generalize and provide a bridge between two very different pieces of mathematics suggested also by Langlands correspondence [?, A3, A2, A4].

I would be happy to have the technical skills of real number theorist but I must proceed using physical analogies. What the theorem states is that one has two quite different mathematical systems, which have a deep relationship between each other.

1. Diophantine equations give solutions as roots of a polynomial $P_y(x)$ containing second variable y as parameter. The coefficients of $P_y(x)$ and y are integers but one can consider a generalization allowing them to be in extension of rationals.

   The general solution of $P_y(x) = 0$ for given value of n is in extension of rationals, whose dimension is determined by the degree n of $P_y(x)$. One is however interested only on the roots (x, y) of $P_y(x) = 0$ coming pairs of integers.

   Diophantine equations can be solved also in p-adic number fields labelled by primes p and in the adelic physics of TGD they are present. Also are present the extensions of p-adic number fields induced by the extensions of rational numbers. There is infinite hierarchy of them. The dimension n of extension serves as a measure for algebraic complexity and kind of "IQ" and $n = h_{eff}/h_0$ gives to effective Planck constant: the larger the value of n, the longer the scale of quantum coherence. This gives a direct connection to quantum biology.

   In p-adic number fields the p-adic integer solutions of the Diophantine equation can be infinite as real numbers. The solutions which are finite as real integers for all primes p define real solutions as finite integers. The sequence of these solutions modulo prime p - that is in finite field - characterizes Diophantine equations. For large p these solutions would stabilize and start to repeat themselves for finite integer solutions. This picture can be generalized from simple low degree polynomials to higher degree polynomials with rational coefficients and even with coefficients in extension of rationals.

2. Second system consists of automorphic functions in lattice like systems, tesselations. They are encountered in Langlands conjecture [?, A3, A2, A4], whose possible physical meaning I still fail to really understand physically so well that I could immediately explain what it is.

   The hyperboloid L (L for Lobatchevski space) defined as $t^2 - x^2 - y^2 - z^2 = constant$ surface of Minkowski space (particle physicist talks about mass shell) is good example about this kind of system in TGD framework. One can define in this kind of tesselation automorphic functions, which are quasi-periodic in sense that the values of function are fixed once one knows them for single cell of the lattice. Bloch waves serve as condensed matter analog.

   One can assign to automorphic function what the article calls its "energy spectrum". In the case of hyperboloid it could correspond to the spectrum of d'Alembertian - this is physicist's natural guess. Automorphic function could be analogous to a partition function build from basic building bricks invariant under the sub-group of Lorentz group leaving the fundamental cell invariant. Zeta function assignable to extension of rationals as generalization of Riemann zeta is one example [L7].

What the discovery could be? I can make only humble guesses. The popular article tells that the "clock solutions" of given Diophantine equation in various finite fields F_p are in correspondence with the "energy" spectra of some automorphic form defined in some space.

The problem of finding the automorphic forms is difficult and the message is that here a great progress has occurred. So called torsion coefficients for the modular form would correspond the integer value roots of Diophantine equations for various finite fields F_p. What could this statement mean?

1. What does automorphic form mean? One has a non-compact group G and functions from G to some vector space V. For instance, spinor modes could be considered. Automorphic forms are eigenfunctions of Casimir operators of G, d'Alembert type operator is one such operator and in TGD framework $G = SO(1, 3)$ is the interesting group to consider. There is also discrete infinite subgroup $\Gamma \subset G$ under which the eigenfunctions are not left invariant but transform by factor $j(\gamma)$ of automorphy acting as matrix in V - one speaks of twisted representation.

   Basic space of this kind of is upper half plane of complex plane in which $G = SL(2, C)$ acts as also does $\gamma = SL(2, Z)$ and various other discrete subgroups of $SL(2, C)$ and defines analog of lattice consisting of fundamental domains $\gamma \backslash G$ as analogs of lattice cells. 3-D hyperboloid of M^4 allows similar structures and is especially relevant from TGD point of view. When $j(\gamma)$ is non-trivial one has analogy of Bloch waves.

Modular invariant functions is second example. They are defined in the finite-D moduli space for the conformal structures of 2-D surfaces with given genus. Automorphic forms transform by a factor $j(\gamma)$ under modular transformations which do not affect the conformal equivalence class. Modular invariants formed from the modular forms can be constructed from these and the TGD based proposal for family replication phenomenon involves this kind invariants as elementary particle vacuum functions in the space of conformal equivalence classes of partonic 2-surfaces [K1].

One can also pose invariance under a compact group K acting on G from right so that one has automorphic forms in G/K. In the case of $SO(3,1)$ this would give automorphic forms on hyperboloid H^3 ("mass shell") and this is of special interest in TGD. One could also require invariance under discrete finite subgroup acting from the left so that $j(\gamma) = 1$ would be true for these transformations. Here especially interesting is the possibility that Galois group of extension of rationals is represented as this group. The correct prediction of Newton's constant from TGD indeed assumes this [L9].

2. What does the spectrum (`http://tinyurl.com/vakzxye`) mean? Spectrum would be defined by the eigenvalues of Casimir operators of G: simplest of them is analog of d'Alembertian for say $SO(3,1)$. The number of these operators equals to the dimension of Cartan sub-algebra of G. Additional condition is posed by the transformation properties under Γ characterized by $j(\gamma)$.

One can assign to automorphic forms so called torsion coefficients in various finite fields F_p and to the eigen functions of d'Alembertian and other Casimir operators in coset space G/K. Consider discrete but infinite subgroup Γ such that solutions are apart from the factor $j(\gamma)$ of automorphy left invariant under Γ. For trivial $j(\gamma)$ they would be defined in double coset space $\Gamma\backslash G/K$. Besides this Galois group represented as finite discrete subgroup of $SU(2)$ would leave the eigenfunctions invariant.

1. Torsion group T is for the first homotopy group Π_1 (fundamental group) a finite Abelian subgroup decomposing Z_n to direct summands Z_p, p prime. The fundamental group in the recent case would be naturally that of double coset space $\Gamma\backslash G/K$.

2. What could torsion coefficients be (`http://tinyurl.com/u3jv86t`)? Π_1 is Abelian an representable as a product $T \times Z^s$. Z_s is the dimension of Π_1 - rank - as a linear space over Z and $T = Z_{m_1} \times Z_{m_2} \times ....Z_{m_n}$ is the torsion subgroup. The torsion coefficients m_i satisfy the conditions $m_1 \perp m_2 \perp ... \perp m_n$. The torsion coefficients in F_p would be naturally $m_i \ mod \ p$.

   The torsion coefficients characterize also the automorphic functions since they characterize the first homotopy group of $\Gamma\backslash G/K$. If I have understood correctly, torsion coefficients m_i for various finite fields F_p for given automorphic form correspond to a sequence of solutions of Diophantine equation in F_p. This is the bridge.

3. How are the Galois groups related to this (`http://tinyurl.com/tje4hvc`)? Representations of Galois group $Gal(F)$ for finite-D extension F of rationals could act as a discrete finite subgroup of $SO(3) \subset SO(1,3)$ and would leave eigenfunctions invariant: these ADE groups form appear in McKay correspondence and in inclusion hierarchy of hyper-finite factors of type II_1 [K7, K3].

   The invariance under $Gal(F)$ would correspond to a special case of what I call Galois confinement, a notion that I have considered in [L10, L1] with physical motivations coming partially from the TGD based model of genetic code based on dark photon triplets.

   The problem is to understand how dark photon triplets occur as asymptotic states - one would expect many-photon states with single photon as a basic unit. The explanation would be completely analogous to that for the appearance of 3-quark states as asymptotic states in hadron physics - the analog of color confinement. Dark photons would form Z_3 triplets under Z_3 subgroup of Galois group associated with corresponding space-time surface, and only Z_3 singlets realized as 3-photon states would be possible.

Mathematicians talk also about the Galois group $Gal(\overline{Q})$ of algebraic numbers regarded as an extension of finite extension F of rationals such that the Galois group $Gal(F)$ would leave eigenfunctions invariant - this would correspond to what I have called Galois confinement.

4. There is also the idea that the torsion group could have representation as a sub-group of Galois group. In TGD the correspondence between physics as geometry and cognitive physics as number theory supports this idea: in adelic physics [L5] cognition would represent number theoretically.

What could be the general vision concerning the connection between Diophantine equations and automorphic forms in TGD framework?

1. In TGD framework an obvious candidate for a space at automorphic side would be the product of $H^3 \times CP_2$ carrying the representations of $SO(1,3) \times SU(3)$. H^3 is 3-D hyperboloid H^3 of M^4 having $SO(1,3)$ as group of isometries. The infinite discrete subgroups of $SO(1,3)$ define tesselations of H^3 analogous lattices in E^3, and one can assign to these automorphic functions as analogs of Bloch waves. They would be associated with separable solutions of spinor d'Alembertian in future light-cone, which corresponds to empty Robertson-Walker cosmology. This is however not the only option: automorphic functions appear also in the description of family replication phenomenon and give rise to modular invariant elementary particle functions in the spaces of conformal moduli for partonic 2-surfaces [K1].

   M^8-H duality states that space-time can be regarded as a 4-surface in either complexified 8-D Minkowski space having interpretation as complexified octonions or $H = M^4 \times CP_2$. At the level M^8 space-time surfaces are algebraic surfaces assignable to an algebraic continuation of a polynomial with rational (or even algebraic) coefficients to M^8. In H one has minimal surfaces with 2-D algebraic singularities - string world sheets and partonic 2-surfaces. Each polynomial defines extension of rationals and the Galois group of extension acts as a symmetry group for the cognitive representations identified as the set of points of space-time surface with coordinate values in the extension of rationals considered. This is central for adelic physics fusing real physics and physics for extensions of p-adic numbers induced by that for rationals. Cognitive representations would define the number theoretic side and Langlands correspondence and generalization of Fermat's theorem would mean that there is many-to-one correspondence from the automorphic side (imbedding space level) to the number theoretic side (cognitive representations). In particular, Galois group of extension would have action as a discrete finite subgroup of $SO(3) \subset SO(1,3)$.

2. In TGD framework Galois group $Gal(F)$ has natural action on the cognitive representation identified as a set of points of space-time surface for which preferred imbedding space coordinates belong to given extension of rationals [L2, L3, L4, L8]. In general case the action of Galois group gives a cognitive representation related to a new space-time surface, and one can construct representations of Galois group as superpositions of space-time surfaces and they are effectively wave functions in the group algebra of $Gal(F)$. Also the action of discrete subgroup of $SO(3) \subset SO(1,3)$ gives a new space-time surface.

   There would be two actions of $Gal(F)$: one at the level of imbedding spaces at H^3 and second at the level of cognitive representations. Possible applications of Langlands correspondence and generalization of Fermat's last theorem in TGD framework should relate to these two representations. Could the action of Galois group on cognitive representation be equivalent with its action as a discrete subgroup of $SO(3) \subset SO(1,3)$? This would mean concrete geometric constraint on the preferred extremals.

In this article I try to make this picture more concrete.

2 Trying to interpret the discovery in TGD framework

What could this discovery have to do with TGD?

2.1 The analog for Diophantine equations in TGD

Diophantine equations have analogy in TGD framework.

1. In adelic physics [L5, L6] $M^8 - H$ duality is in key role. Space-time surfaces can be regarded either as algebraic 4-surfaces in complexified M^8 determined as roots of polynomial equations. Second representation is as minimal surfaces with 2-D singularities identified as preferred extremals of action principle: analogs of Bohr orbits are in question.

2. The Diophantine equations generalize in TGD framework. One considers the roots of polynomials with rational coefficients and extends them to 4-D space-time surfaces defined as roots of their continuations to octonion polynomials in the space of complexified octonions [L8, L2, L3, L4]. Associativity is the basic dynamical principle: the tangent space of these surfaces is quaternionic, and therefore associative. Each irreducible polynomial defines extension of rationals via its roots and one obtains a hierarchy of them having physical interpretation as evolutionary hierarchy. These surface can be mapped to surface in $H = M^4 \times CP_2$ by $M^8 - H$ duality.

3. So called cognitive representations for given space-time surface are identified as set of points for which points have coordinate in extension of rationals. They realize the notion of finite measurement resolution and scattering ampludes can be expressed using the data provided by cognitive representations: this is extremely strong form of holography.

4. Cognitive representation generalizes the solutions of Diophantine equation: instead of integers one allows points in given extension of rationals. These cognitive representations determine the information that conscious entity can have about space-time surface. As the extensions approaches algebraic numbers, the information is maximal since cognitive representation defines a dense set of space-time surface.

2.2 The analog for automorphic forms in TGD

One can image also analogy for automorphic forms in TGD.

1. The above mentioned hyperboloids H^3 of M^4 are central in zero energy ontology (ZEO) of TGD: in TGD based cosmology they correspond to cosmological time constant surfaces. Also the tesselations of hyperboloids are expected to have a deep physical meaning - quantum coherence even in cosmological scales is possible [K6, K5] and there are pieces of evidence about the lattice like structures in cosmological scales.

2. Also the finite lattices defined by finite discrete subgroups of $SU(3)$ in CP_2 analogous to Platonic solids and and regular polygons for rotation group are expected to be important. For what this could mean in number theoretic vision about TGD see for the correct prediction of the Newton's constant in terms of CP_2 radius [L9] (http://tgdtheory.fi/public_html/articles/Gagain.pdf).

3. One can imagine analogs of automorphic forms for these tesselations. The spectrum would correspond to that for massless spinor d'Alembertian of $L \times CP_2$, where L denotes the hyperboloid, satisfying the boundary conditions given by tesselation. The mass eigenvalues would be determined by the CP_2 spinor Laplacian. In condensed matter physics solutions of Schrödinger equation consistent with lattice symmetries would be in question as quasi-periodic Bloch waves. The spectrum would correspond to mass squared eigenvalues and to the spectra for observables assignable to the discrete subgroup of Lorentz group defining the tesselation.

4. The theorem described in the article suggests a generalization in TGD framework based on physical motivations. The "energy" spectrum of these automorphic forms identified as mass squared eigenvalues and other quantum numbers characterized by the subgroup of Lorentz group are at the other side of the bridge.

At the other side of bridge could be the spectrum of the roots of polynomials defining space-time surfaces: the roots indeed fix the polynomial of one argument and therefore entire space-time surface as a "root" of the octonionic counterpart of the polynomial. A more general conjecture would be that the discrete cognitive representations for space-time surfaces as "roots" of octonionic polynomial are at the other side of bridge. These two would correspond to each other.

Cognitive representations at space-time level would code for the spectrum of d'Alembertian like operator at the level of imbedding space. This could be seen as example of quantum classical correspondence (QCC) , which is basic principle of TGD.

2.3 What is the relation to Langlands conjecture (LC)?

I understand very little about LC [A1, A3, A2, A4] at technical level but I can try to relate it to TGD via physical analogies. I have done this actually two times already earlier [A1, K4].

1. LC relates two kinds of groups.

   (a) Algebraic groups satisfying certain very general additional conditions (complex nxn matrices satisfying algebraic conditions is one example). Matrix groups such as Lorentz group are a good example.

   The Cartesian product of future light-cone and CP_2 would be the basic space. d'Alembertian inside future light-cone in the variables defined by Robertson- Walker coordinates. The separation of variables a as light-cone proper time and coordinates of H^3 for given value of a assuming eigenfunction of H^3 d'Alembertian satisfying additional symmetry conditions would be in question. The dependence on a is fixed by the separability and by the eigenvalue value of CP_2 spinor Laplacian.

   (b) So called L-groups assigned with extensions of rationals and function fields defined by algebraic surfaces as as those defined by roots of polynomials. This brings in adelic physics in TGD.

2. The physical meaning in TGD could be that the discrete the representations provided by the extensions of rationals and function fields on algebraic surfaces (space-time surfaces in TGD) determined by them. Function fields might be assigned to the modes of induce spinor fields.

   The physics at the level of imbedding space (M^8 or $H = M^4 \times CP_2$) described in terms of real and complex numbers - the physics as we usually understand it - would by LC corresponds to the physics provided by discretizations of space-time surfaces as algebraic surfaces. This correspondence would not be 1-1 but many-to-one. The discretizations provided by cognitive representations would provide hierarchy of unique approximations. Langlands conjecture (or rather, its proof!) would justify this vision.

3. Galois groups of extensions are excellent examples of L-groups an indeed play central role in TGD. The proposal is that Galois groups provide a representation for the isometries of the imbedding space and also for the hierarchy of dynamically generated symmetries. This is just what the Langlands conjecture motivates to say.

   Amusingly, just last week I wrote an article deducing the value of Newton's constant using the conjecture that discrete subgroup of isometries common to M^8 and $M^4 \times CP_2$ consisting of a product of icosahedral group with 3 copies of its covering corresponds to Galois group for extension of rationals. The prediction is correct. The possible connection with Langlands conjecture came into my mind while writing these comments.

To sum up, Langlands correspondence would relate two descriptions. Discrete description for cognitive representations at space-time level and continuum description at imbedding space level in terms of eigenfunctions of spinor d'Alembertian.

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