TGD Inspired Comments about Bognitive Bell inequalities

M. Pitkänen
Dept. of Physics, University of Helsinki, Helsinki, Finland.
Email: matpitka@rock.helsinki.fi.
http://www.helsinki.fi/~matpitka/.

Contents

1 On direct testing of quantum consciousness and DNA as tqc

Quantum entanglement and its reduction in "cognitive" quantum measurement could provide a direct test of quantum consciousness. Andrei Khrennikov [2] has proposed a mathematical formulation of "quantum like" behavior based on his proposal that so called context dependent probabilities could provide alternative description for quantum mechanical interference phenomenon. In quantum theory context would correspond to the choice of quantization axis. Khrennikov has also proposed a modification of Bell inequalities so that they apply on conditional probabilities: this would make it possible to avoid the task of preparing entangled state of brains. The hope is that one could forget completely the microscopic structure of quantum brain and test quantum like behavior by making simple experiments involving just questions to the subject persons and finding whether or not classical rules for conditional probabilities hold true or not. This article was born as a response in email discussions with Elio Conte and I am grateful for these interesting exchanges.

1.1 First experiment

Bistable percepts induced by ambiguous figures are especially attractive from the point of view of experimentation. The question would be "Which of the two possible percepts?" and the outcome would be answer to this question. The first experiment reported in [3] was following.

a) Consider a group S of subject persons. Divide it into two groups U and V containing equally many subject persons. Represent for members of U the question A (bistable percept A). From this one can deduce the probalities $p(A=+)$ and $p(A=-)$. Represent for members of V the question B and and immediately after than the question A (bistable percept A) for those who answered $B=-$. This experiment gives the conditional probabilities $p(A=\pm/B=\pm)$.

b) The quantity

$$cos(\theta_{\pm}) = \frac{p(A=\pm)-p(B=+)p(A=\pm/B=+)-p(B=-)p(A=\pm/B=-)}{2\sqrt{p(B=+)p(B=-)p(A=\pm/B=+)p(A=\pm/B=-)}}$$

measures the failure of the basic rule $p(A=+)-p(B=+)p(A=+/B=+)-p(B=-)p(A=+/B=-)$ for classical conditional probability. Note that in quantum theory similar rules applies to transition amplitudes (conditional probability amplitudes) corresponding to the addition of a complete set of states in the inner product between two states (for instance, repeated application of this gives rise to path integral formulation).

c) One can describe the situation in terms of "quantum like state"

$$\Psi(A=\pm) = \sqrt{p(B=+)p(A=\pm/B=+)}+e^{i\theta_\pm}\sqrt{p(B=-)p(A=\pm/B=-)}$$

satisfying $p(A=\pm) = |\Psi(A=\pm)|^2$. If $cos(\theta_\pm)$ is non-vanishing one can say that that the situation is quantum like.

1.2 Second experiment

Second experimental test is more complex and involves generalization of Bell's inequality so that it involves conditional probabilities [2] Let $A, B, C = \pm$ be arbitrary dichotomous random variables satisfying Kolmogorov axioms characterizing classical probability. Then the following analog of Bell inequality can be shown to hold true:

$$P(A=+,B=+)+P(C=+,B=-) \quad \geq \quad P(A=+,C=+) \ . \qquad (1)$$

In terms of conditional probabilities one has

$$\frac{P(A=+/B=+)}{P(B=+)}+\frac{P(C=+/B=-)}{P(C=+)}) \quad \geq \quad \frac{P(A=+/C=+)}{P(C=+)} \ . \qquad (2)$$

If the random variables are symmetrically distributed so that one has $P(X=\pm) = 1/2$, for $X = A, B, C$ one obtains

$$P(A=+/B=+)+P(C=+/B=-) \quad \geq \quad P(A=+/C=+) \ . \qquad (3)$$

Note that this form of equality is by no means necessary. The symmetric distributions for the random variables would however correspond to maximal entanglement in spin system given best hopes for the violation of the Bell inequality.

d) The test is following. Consider a group S of subject persons divided into subgroups U and V as above. Pose to the members of U question B and immediately after that question A for those who answered $B = +$ and question C for those who answered $B = -1$. For group V represent first the question C and for those who answer $C = +$ represent the question A. The failure of inequality could regarded as a direct proof for quantum like behavior. That failure does not occur does not of course mean that system is classical but only that the quantal effects are not large enough.

e) The analogy with Bell's inequality suggest that the questions are analogous to posing the spins of spin pair in spin singlet state to an external magnetic fields determining the quantization axis. The inequality tend to fail when the directions of the magnetic fields for the two spins differ enough. Thus the failure is expected if the questions, in other words ambiguous figures producing bistable percepts differ enough.

1.3 Criticism and possible improvement of the experiment

In the case of spin pairs the tests of quantum behavior are carried out for the members of spin pair by putting them to magnetic fields having different directions. Now the pair of experiments is made for a single subject person. Hence there is no need to prepare quantum entangled pair of conscious entities.

The use of ensemble is the problematic aspect of experiments. Human beings are extremely complex systems and one can argue that it is impossible to prepare an ensemble of identical systems in cognitive sense. A possible manner to avoid the problem would be the replacement of ensembles with a time series of experiments performed for a single subject person. In both experiments one could perform the two kinds of experiments many times to single subject person and deduce various probabilities and $cos(\theta)$ from the outcome of the experiments.

1.4 Interpretation in terms of zero energy ontology and DNA as tqc

The discussions with Elio Conte led to the realization that in zero energy ontology the experiments differ from the corresponding experiments for two-spin system only in that space-like entanglement is replaced with time like entanglement. The experiment would be essentially a measurement of probabilities defined by the matrix elements of M-matrix defining the generalization of S-matrix. Hence Bell's inequalities and their generalizations should apply in genuine quantum sense. By performing the experiments for a single subject person as time series one might be therefore able study whether quantum consciousness in the sense of TGD makes sense.

Quantum consciousness approach however requires that bistable percepts have genuine microscopic quantum states as their physical correlates. This is not assumed in the approach of Khrennikov.

a) If the vision about DNA as topological quantum computer makes sense, the question to the answer "Which of the two possible percepts?" can be regarded as a qubit which is some function of a large number of qubits and same function irrespective of the ambiguous figure used. This could hold quite generally, at least for a given sensory modality. The qubits appearing as arguments of this function are determined by the sensory input defined by the ambiguous figure. The ambiguous figure would take the role of magnetic field determining the directions of quantization axes for a large collection of qubits appearing as arguments of the Boolean function (one cannot exclude the possibility that neuronal synchrony forces all these axes to have same direction). Qubit could correspond to spin or some spin like variable. The quantization axes could correspond in this case to the direction of external magnetic field acting on 1-gate of tqc.

b) Qubit could be replaced with an n-state system: this would require a generalization of the Bell inequalities. The model of DNA as tqc suggests that qubit might be replaced with qutrit defined by a quark triplet (third quark with vanishing color isospin would correspond to ill-defined truth value). The inability of subject persons to identify the percept always indeed encourages to consider this option. Color group $SU(3)$ ($SO(3) \subset SU(3)$) defines the set of possible quantization axes as points of the flag manifold $F = SU(3)/U(1) \times U(1)$ ($SO(3)/SO(2) = S^2$). Quantization axes would be determined by the direction of color magnetic field in color Lie algebra and sensory input would define a sequence of 1-gates at the lipids ends of the braid strands, and realized as color rotations of the flux tube defining braid strand. This hypothesis would conform with the proposal of Barbara Shipman that honeybee dance that quarks are in some mysterious manner involved with cognition [4].

2 A further comment about Bell inequality and its cognitive counterpart

The attempt to explain Bell's inequality for a non-physicist looks to me a mission impossible. One can deduce the failure of Bell's inequalities easily in the formalism of quantum theory but this formalism is not available for the poor popularizer. Popularizer should explain something completely counter intuitive by applying to the every day intuition of the listener. The generalization of this inequality to cognitive context makes the situation even more difficult since one cannot refer to geometric concepts like spin rotations.

In the following I try to concretize the statement "Quantum behavior need not imply violation of Bell's inequalities" and answer the question "What kind of experimental situation gives rise to a violation of ordinary Bell inequalities in case of spin pair (what the directions of magnetic field must be)" by an argument involving basic notions of group theory of rotation group. The third question is "What kind of cognitive experiment using ambiguous figures might be analogous to the experiment demonstrating violation of Bell's inequalities for spin pair?". TGD view about brain does not give very good hopes of the violation - basically because the cognitive bits are Boolean functions of very many qubits. In the context of Khrennikov's theory the situation might be better.

2.1 Non-violation and violation of Bell inequalities in spin-spin system

2.1.1 Bell inequality in spin-spin system

For simplicity consider a system of two spins $|j,m\rangle = |1/2, \pm 1/2 >$ in maximally entangled state with total spin zero and forget considerations related to fermion statistics. For convenience, let us use the brief hand notation $|\pm 1/2\rangle$ for the states. The state is given by

$$\frac{1}{\sqrt{2}}[|1/2\rangle|-1/2\rangle + |-1/2\rangle|1/2\rangle] \tag{4}$$

and invariant under rotations.

Let a,b,c and a',b',c' denote the measurements of first *resp.* second spin with three directions $n \in \{e_1, e_2, e_3\}$ of measurement axes. $a = \pm 1$ correspond to spin $s_n = \pm 1/2$.

Let us consider Bell's inequality which appears in Andrei Khrennikov's article in the form

$$P(a=-1, b'=+1) + P(c=-1, b'=-1) \geq P(a=-1, c'=+1) \ . \tag{5}$$

An equivalent form is

$$P(s_1=-1/2, s_2=+1/2) + P(s_3=-1/2, s_2=-1/2) \geq P(s_1=-1/2, s_3=+1/2) \ . \tag{6}$$

2.1.2 Expression of probabilities in terms of rotation matrices

One can transform Bell inequality to a purely geometric condition by noticing that the probabilities are defined by matrix elements of matrices rotating quantization axis n_i to n_f to each other. The rotations are not unique since one can perform first an arbitrary rotation around n_i, rotate then then n_i to n_f, and then perform arbitrary rotation around n_f. The initial/final rotation induces only a phase factor to an eigen state of spin with respect to initial/final axes and does not affect probabilities appearing in the Bell inequalities.

a) Quite generally, $P(s_a = \pm 1/2, s_b = \pm 1/2)$ is expressible as

$$P(s_a = \pm 1/2, s_b = \pm 1/2) = |R(n_a, n_b)_{\pm 1/2, \mp 1/2}|^2 \ , \tag{7}$$

where the rotation matrix $R(n_a, n_b)^{\pm 1/2, \pm 1/2}$ represents the action of rotation taking n_a to n_b on spinors. Note that the sign of spin is changed in the second argument on the right hand side: this corresponds to the fact that the measurement of first spin giving spin $s = \pm 1/2$ fixes the value of second spin to be opposite.

This representation follows from the unitary action of rotations on spin eigen states given by

$$|s_a = \pm 1/2\rangle \ = \ R(n_a, n_b)_{\pm 1/2, 1/2} |s_b = 1/2\rangle + R(n_a, n_b)_{\pm 1/2, -1/2} |s_b = -1/2\rangle \ . \tag{8}$$

This action means that the 2 spin 1/2 eigenstates are analogous to the 3 components of position vector in that they transform under rotations to linear combinations of each other.

The above expression for the probability follows using standard rules of quantum theory: that is, as moduli squared of the inner product

$$P(s_a = \pm 1/2, s_b = \pm 1/2) = \langle s_a = \pm 1/2 | s_b = \pm 1/2\rangle|^2 \ , \tag{9}$$

between the spin eigen states associated with quantization axes n_a and n_b using the orthonormality of the state basis.

2.1.3 Expression of Bell's inequality in terms of rotation matrices

a) Using the representation of 9 for probabilities, Bell's inequality reads as

$$|R(n_1,n_2)_{-1/2,-1/2}|^2 + |R(n_2,n_3)_{-1/2,1/2}|^2 \geq |R(n_1,n_3)_{-1/2,-1/2}|^2 \qquad (10)$$

and has obviously purely geometric content.

b) The rotation $n_a \rightarrow n_b$ can be parameterized in terms of three angles:

$$R(n_a,n_b) = R(\alpha,\beta,\gamma) \ , \qquad (11)$$

where α is the angle between n_a and n_b. One can choose always $\beta = \gamma = 0$ without affecting the probabilities since these angle correspond to an arbitrary rotation around the direction of n_b *resp.* n_a inducing only a phase factor. $R(\alpha,0,0)$ has a simple form

$$R(\alpha,0,0) = \left(\begin{array}{cc} cos(\alpha/2) & sin(\alpha/2) \\ -sin(\alpha/2) & cos(\alpha/2) \end{array} \right) \ . \qquad (12)$$

Diagonal elements correspond to $R_{1/2,1/2} = R_{-1/2,-1/2} = cos(\theta/2)$ and non-diagonal ones to $R_{1/2,-1/2} = -R_{-1/2,1/2} = sin(\theta/2)$. This formula follows from the requirement that rotations form a group and can be found in any text book of quantum mechanics.

c) Using this representation of rotation matrices the condition for the *violation*(!) of Bell inequality reads as

$$cos^2(\theta_{12}/2) + sin^2(\theta_{23}/2) < cos^2(\theta_{13}/2) \ . \qquad (13)$$

By using trigonometric identities $cos^2(x/2) = (cos(x)+1)/2$ and $sin^2(x/2) = (1-cos(x))/2$ one can write the inequality as

$$cos(\theta_{23}) - cos(\theta_{21}) > 1 - cos(\theta_{13}) \ . \qquad (14)$$

This is a purely geometric condition for angles between three unit vectors characterizing the quantization axes.

2.1.4 Analysis of the violation Bell's inequalities

By choosing the coordinates suitably one always have e_3 is in z-direction: $e_3 == e_z$, and e_1 in $y-z$ plane.

$$e_3 = e_z \quad , \quad e_1 = cos(\alpha)e_z + sin(\alpha)e_y \quad , \tag{15}$$

e_2 can be always expressed as

$$e_2 = cos(\beta)e_z + sin(\beta)(cos(\phi)e_x + sin(\phi)e_y) \tag{16}$$

The cosines are given in terms of inner products of these vectors: $cos(\theta_{ij}) = n_i \cdot n_j$:

$$\begin{aligned} &cos(\theta_{23}) = cos(\beta) \quad , \\ &cos(\theta_{21}) = cos(\alpha)cos(\beta) + sin(\alpha)sin(\beta)sin(\phi) \quad , \\ &cos(\theta_{13}) = cos(\alpha) \quad . \end{aligned} \tag{17}$$

a) The condition for the violation reads as

$$cos(\beta) - cos(\alpha)cos(\beta) + sin(\alpha)sin(\beta)sin(\phi) > 1 - cos(\alpha) \quad . \tag{18}$$

giving

$$sin(\phi) > \frac{[1 - cos(\alpha)][1 - cos(\beta)]}{sin(\alpha)sin(\beta)} \quad . \tag{19}$$

The condition states that the third quantization axis (e_2) is sufficiently far from the plane defined by the first two (e_3 and e_1).
The region

$$\frac{[1 - cos(\alpha)][1 - cos(\beta)]}{sin(\alpha)sin(\beta)} > 1 \quad . \tag{20}$$

corresponds to pseudo-classical regime allowing for which $sin(\phi) > 1$ would hold true, and the curve

$$\frac{[1 - cos(\alpha)][1 - cos(\beta)]}{sin(\alpha)sin(\beta)} = 1 \quad . \tag{21}$$

separates the pseudoclassical and strictly non-classical regions from each other. Using the variables $x = [1 - cos(\alpha)]/sin(\alpha)$ and $y = [1 - cos(\beta)]/sin(\beta)$ this equation reduces to an equation

$$xy = 1 \ . \tag{22}$$

of a hyperbole. The region between x- and y-axes and hyperbole is strictly non-classical and the region above the hyperbole is pseudoclassical. The allowed variation range of x and y is $(0, \infty)$ corresponding to $(0, \pi)$.

The non-classical region divides into three regions. The two regions between x- (y-) coordinate axis and asymptotic of hyperbole and the middles region with $x \simeq y$.

a) In asymptotic region α is near zero and β near to π or vice versa so that the directions of all three quantization axes are near to x-aces but nearly opposite for α and β. The possible problem is the instability against small variations of α and β. The variation of ϕ is almost entire range $(0, \pi)$ but actually this corresponds to a very small half-circle around z-axis so that there is not so much room than one might think.

b) The middle region of the open square $(0 < x < 1, 0 < y < 1)$ is perhaps the most promising one experimentally. For instance, one can consider large enough equilateral triangles on surface of unit sphere defined by quantization directions with $x = y$ and $sin(\beta)\phi = \alpha = \beta$.

2.1.5 Examples about the violation and non-violation of Bell's inequalities

Consider now examples about violation and non-violation of Bell inequalities.

a) For instance, for α and/or β sufficiently near to zero one can write $1 - cos(\alpha) \simeq \alpha^2/2$ and $sin(\alpha) \simeq \alpha$ (same for β). Suppose that α is small. In this case one has

$$sin(\phi) > \frac{[1 - cos(\beta)]}{sin(\beta)} \times \frac{\alpha}{2} \ . \tag{23}$$

Solutions obviously exist for sufficiently small values of α for given value of $\beta > 0$ (note that $sin(\beta) \geq 0$ is always true). Thus by taking quantization e_3 and e_1 sufficiently near to each other one can always find violation of Bell inequalities.

b) If both α and β are near to zero so that all three quantization axes are near to each other but not identical, the condition for violation reduces in a good approximation to

$$sin(\phi) > \frac{\alpha\beta}{4} \ . \tag{24}$$

and solutions to this condition obviously exist. Obviously this option gives the weakest condition on $sin(\phi)$.

The inequality is satisfied also for $\alpha = 0$ or $\beta = 0$ but this does not actually yield violation of Bell since for $\alpha = 0$ or $\beta = 0$ corresponds to a situation in which two quantization axes are in the same direction (different values of ϕ correspond to the same direction $e_2 = e_z$ for $\beta = 0$). The original naive expectation that the directions of quantization axes should differ very much in order to have violation, is wrong.

c) In the special case $\alpha = \beta > 0$ e_1 and e_2 are on the cone around e_z and violation is obtained for

$$sin(\phi) > \frac{[1 - cos(\alpha)]^2}{sin(\alpha)^2} \ . \tag{25}$$

provided the condition $cos(\alpha) + sin(\alpha) > 1$ guaranteing $sin(\phi) \leq 1$ is satisfied. The condition is true for $0 < \alpha < \pi/2$ so that upper hemisphere with North Pole and equator excluded gives violation. Note that in this case the directions of quantization axes e_1, e_2, e_3 must be far enough from each other unlike in the previous case and this case corresponds to the original intuitive expectations.

d) Bell inequalities are not violated for

$$sin(\phi) \leq \frac{[1 - cos(\alpha)][1 - cos(\beta)]}{sin(\alpha)sin(\beta)} \ . \tag{26}$$

Note that pseudo-classicality implied by too small values of $sin(\phi)$ means that the plane defined by e_2 and e_3 is nearly orthogonal to the plane defined by e_1 and e_3. In particular, for $sin(\phi) = 0$ stating that third quantization axes is in a plane orthogonal to the first two axes.

2.2 Cognitive measurements in TGD framework

Consider first the question what cognitive Bell's inequalities could mean in TGD framework and whether they could be violated.

a) Zero energy ontology reduces the failure of cognitive Bell's inequality to a situation mathematically equivalent with the failure ordinary Bell's inequality for a spin system. The only difference is that space-like entanglement is replaced with time-like entanglement. Two spins located are replaced with the states of cognitive system at times $t = T$ and $t = T + \tau$.

b) The notion of quantization direction for spin is absolutely essential for the failure of the ordinary Bell's inequality and should have a cognitive analog. A concrete representation of Boolean variable as a spin like observable (ordinary spin or color isospin as in TGD based model of DNA as topological quantum computer) gives ore than a mere analog.

c) An ambiguous figure giving rise to a bistable percept serves as the analog for the magnetic field fixing the quantization direction. If figures are close to each other, the quantization axis have almost same direction. If figures are very different, the axis have large relative angle.

d) One might hope the failure of Bell inequality in the following two basic situations involving at least two quantization axes near to each other. Both situations are however somewhat questionable since they are unstable against small changes.

i) All figures are reasonably near to each other (region near origin and below the hyperboloid in (α, β) plane).

ii) Two of figures are near to each other and the third one differs widely from the two (the region below hyperboloid but far along x or y-axis: in his case small variation of β or α and destroy the violation of Bell inequality).

e) The situation in which all figures are analogous to the three quantization axes defining vertices of nearly equilateral and large enough triangle at the surface of sphere defined by the directions of the quantization axes would be perhaps the best one experimentally but it is not easy to quantify what this means in terms of the unambiguous figures. This situation corresponds to the intuitive idea that figures must differ from each other enough.

Unfortunately, there is a serious objection against this optimistic picture is following. The implicit assumption is that single qubit represents directly the percept. One how however expects that the Boolean variable, the answer to question "Which percept", in the experiments considered is a function of very many qubits and in this kind of system it might be very difficult to achieve the failure of Bell's inequality. More concretely, the replacement of ambiguous figure with a modified one means that a large number of spin rotations for a collection of spins is performed and since these rotations are different, the net effect might be that the situation becomes effectively classical and one would obtains no violation of Bell inequality.

2.3 How could one violate cognitive Bell inequality in Khrennikov's approach?

The question is "How to produce experimentally a violation of cognitive Bell inequality?". If one wants to work without a physical realization of Boolean cognition in terms of qubits, the challenge becomes much more difficult. The only hope is to start from the violation of Bell inequality in spin system and try to guess what kind of experimental arrangement could yield the violation.

The idea is to try to translate the above argument for the violation of Bell inequality for spin system to an experimental arrangement in which one uses ambiguous figures yielding bistable percept and the yes/no answer to the question which percept corresponds to $s = \pm 1/2$.

The previous examples about violation suggest that one could choose two of the ambiguous figures to be near to each other so that the quantization axes e_1 and e_3 defined by them are near to each other. The third unambiguous figure could be chosen more freely but in such a manner that the counterpart for the statement "the plane defined by e_1 and e_3 is sufficiently far from a plane orthogonal to that defined by e_2 and e_3". It is not clear what the content of this statement could be. The situation is optimal if all percepts are near to each other but not identical so that the naive expectation that percepts should differ very much is wrong (if the analogy with spin system makes sense).

One can consider following translation to an experimental protocol.

a) Consider the situation in which spin measurements with three quantization directions are replaced with three questions "Which percept?" for three different percepts. Denote by a, b, c these cognitive measurements at $t = T$ and by a', b', c' at $t = T + \tau$.

b) $(s_z = -1/2, s_z = 1/2)$ appearing at the right hand side means that same ambiguous figure $a = c$ is represented at $t = T$ and $t = T + \tau$. What this means that at $t = T$ percept is represented, then it disappears for a short time, and is represented again at $t = T + \tau$. One must check that the percept is not changed in this operation. If not, one cannot use the analogy with the measurement of spin. Obviously this poses some upper bound on τ.

c) As already explained, one might hope the failure of Bell inequality in the following two basic situations involving at least two quantization axes near to each other. Both situations are however somewhat questionable since they are unstable against small changes.

i) All figures are reasonably near to each other (region near origin and below the hyperboloid in (α, β) plane).

ii) Two of figures are near to each other and the third one differs widely from the two (the region below hyperboloid but far along x or y-axis: in his case small variation of β or α and destroy the violation of Bell inequality).

d) The situation in which all figures are analogous to the three quantization axes defining vertices of nearly equilateral and large enough triangle at the surface of sphere defined by the directions of the quantization axes would be perhaps the best one experimentally but it is not easy to quantify what this means in terms of the unambiguous figures. This situation corresponds to the intuitive idea that figures must differ from each other enough.

References

[1] The chapter *DNA as Topological Quantum Computer* of M. Pitkänen (2006), *Mathematical Aspects of Consciousness Theory.* http://www.helsinki.fi/∼matpitka/mathconsc/mathconsc.html#dnatqc.

[2] A. Khrennikov (2004), *Bell's inequality for conditional probabilities as a test for quantum like behaviour of mind*, arXiv:quant-ph/0402169.

[3] E. Conte, O. Todarello, A. Federici, J. P. Zbilut (2008), *Minds States Follow Quantum Mechanics During Perception and Cognition of Ambigious Figures: A Final Experimental Confirmation*, arXiv:0802.1835v1 [physics.gen-ph].

[4] B. Shipman (1998) *The geometry of momentum mappings on generalized flag manifolds, connections with a dynamical system, quantum mechanics and the dance of honeybee.* http://math.cornell.edu/ oliver/Shipman.gif.
B. Shipman (1998), *On the geometry of certain isospectral sets in the full Kostant-Toda lattice.* http://nyjm.albany.edu:8000/PacJ/1997/Shipman.html.
B. Shipman (1998), *A symmetry of order two in the full Kostant-Toda lattice.* http://www.math.rochester.edu:8080/u/shipman/symmetrypaper/.