Evolution of Ideas about Hyper-finite Factors in TGD

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Contents

Abstract

The work with TGD inspired model for quantum computation led to the realization that von Neumann algebras, in particular hyper-finite factors, could provide the mathematics needed to develop a more explicit view about the construction of M-matrix generalizing the notion of S-matrix in zero energy ontology (ZEO). In this chapter I will discuss various aspects of hyper-finite factors and their possible physical interpretation in TGD framework.

1. Hyper-finite factors in quantum TGD

The following argument suggests that von Neumann algebras known as hyper-finite factors (HFFs) of type III_1 appearing in relativistic quantum field theories provide also the proper mathematical framework for quantum TGD.

1. The Clifford algebra of the infinite-dimensional Hilbert space is a von Neumann algebra known as HFF of type II_1. Therefore also the Clifford algebra at a given point (light-like 3-surface) of world of classical worlds (WCW) is HFF of type II_1. If the fermionic Fock algebra defined by the fermionic oscillator operators assignable to the induced spinor fields (this is actually not obvious!) is infinite-dimensional it defines a representation for HFF of type II_1. Super-conformal symmetry suggests that the extension of the Clifford algebra defining the fermionic part of a super-conformal algebra by adding bosonic super-generators representing symmetries of WCW respects the HFF property. It could however occur that HFF of type II_∞ results.
2. WCW is a union of sub-WCWs associated with causal diamonds (CD) defined as intersections of future and past directed light-cones. One can allow also unions of CDs and the proposal is that CDs within CDs are possible. Whether CDs can intersect is not clear.
3. The assumption that the M^4 proper distance a between the tips of CD is quantized in powers of 2 reproduces p-adic length scale hypothesis but one must also consider the possibility that a can have all possible values. Since $SO(3)$ is the isotropy group of CD, the CDs associated with a given value of a and with fixed lower tip are parameterized by the Lobatchevski space $L(a) = SO(3,1)/SO(3)$. Therefore the CDs with a free position of lower tip are parameterized by $M^4 \times L(a)$. A possible interpretation is in terms of quantum cosmology with a identified as cosmic time. Since Lorentz boosts define a non-compact group, the generalization of so called crossed product construction strongly suggests that the local Clifford algebra of WCW is HFF of type III_1. If one allows all values of a, one ends up with $M^4 \times M^4_+$ as the space of moduli for WCW.
4. An interesting special aspect of 8-dimensional Clifford algebra with Minkowski signature is that it allows an octonionic representation of gamma matrices obtained as tensor products of unit matrix 1 and 7-D gamma matrices γ_k and Pauli sigma matrices by replacing 1 and γ_k by octonions. This inspires the idea that it might be possible to end up with quantum TGD from purely number theoretical arguments. One can start from a local octonionic Clifford algebra in M^8. Associativity (co-associativity) condition is satisfied if one restricts the octonionic algebra to a subalgebra associated with any hyper-quaternionic and thus 4-D sub-manifold of M^8. This means that the induced gamma matrices associated with the Kähler action span a complex quaternionic (complex co-quaternionic) sub-space at each point of the sub-manifold. This associative (co-associative) sub-algebra can be mapped a matrix algebra. Together with $M^8 - H$ duality this leads automatically to quantum TGD and therefore also to the notion of WCW and its Clifford algebra which is however only mappable to an associative (co-associative(algebra and thus to HFF of type II_1.

2. Hyper-finite factors and M-matrix

HFFs of type III_1 provide a general vision about M-matrix.

1. The factors of type III allow unique modular automorphism Δ^{it} (fixed apart from unitary inner automorphism). This raises the question whether the modular automorphism could be used to define the M-matrix of quantum TGD. This is not the case as is obvious already from the fact that unitary time evolution is not a sensible concept in zero energy ontology.
2. Concerning the identification of M-matrix the notion of state as it is used in theory of factors is a more appropriate starting point than the notion modular automorphism but as a generalization of thermodynamical state is certainly not enough for the purposes of

quantum TGD and quantum field theories (algebraic quantum field theorists might disagree!). Zero energy ontology requires that the notion of thermodynamical state should be replaced with its "complex square root" abstracting the idea about M-matrix as a product of positive square root of a diagonal density matrix and a unitary S-matrix. This generalization of thermodynamical state -if it exists- would provide a firm mathematical basis for the notion of M-matrix and for the fuzzy notion of path integral.

3. The existence of the modular automorphisms relies on Tomita-Takesaki theorem, which assumes that the Hilbert space in which HFF acts allows cyclic and separable vector serving as ground state for both HFF and its commutant. The translation to the language of physicists states that the vacuum is a tensor product of two vacua annihilated by annihilation oscillator type algebra elements of HFF and creation operator type algebra elements of its commutant isomorphic to it. Note however that these algebras commute so that the two algebras are not hermitian conjugates of each other. This kind of situation is exactly what emerges in zero energy ontology (ZEO): the two vacua can be assigned with the positive and negative energy parts of the zero energy states entangled by M-matrix.

4. There exists infinite number of thermodynamical states related by modular automorphisms. This must be true also for their possibly existing "complex square roots". Physically they would correspond to different measurement interactions meaning the analog of state function collapse in zero modes fixing the classical conserved charges equal to the quantal counterparts. Classical charges would be parameters characterizing zero modes.

A concrete construction of M-matrix motivated the recent rather precise view about basic variational principles is proposed. Fundamental fermions localized to string world sheets can be said to propagate as massless particles along their boundaries. The fundamental interaction vertices correspond to two fermion scattering for fermions at opposite throats of wormhole contact and the inverse of the conformal scaling generator L_0 would define the stringy propagator characterizing this interaction. Fundamental bosons correspond to pairs of fermion and antifermion at opposite throats of wormhole contact. Physical particles correspond to pairs of wormhole contacts with monopole Kähler magnetic flux flowing around a loop going through wormhole contacts.

3. Connes tensor product as a realization of finite measurement resolution

The inclusions $\mathcal{N} \subset \mathcal{M}$ of factors allow an attractive mathematical description of finite measurement resolution in terms of Connes tensor product but do not fix M-matrix as was the original optimistic belief.

1. In ZEO $\mathcal{N}$ would create states experimentally indistinguishable from the original one. Therefore $\mathcal{N}$ takes the role of complex numbers in non-commutative quantum theory. The space $\mathcal{M}/\mathcal{N}$ would correspond to the operators creating physical states modulo measurement resolution and has typically fractal dimension given as the index of the inclusion. The corresponding spinor spaces have an identification as quantum spaces with non-commutative $\mathcal{N}$-valued coordinates.

2. This leads to an elegant description of finite measurement resolution. Suppose that a universal M-matrix describing the situation for an ideal measurement resolution exists as the idea about square root of state encourages to think. Finite measurement resolution forces to replace the probabilities defined by the M-matrix with their $\mathcal{N}$ "averaged" counterparts. The "averaging" would be in terms of the complex square root of $\mathcal{N}$-state and a direct analog of functionally or path integral over the degrees of freedom below measurement resolution defined by (say) length scale cutoff.

3. One can construct also directly M-matrices satisfying the measurement resolution constraint. The condition that $\mathcal{N}$ acts like complex numbers on M-matrix elements as far as $\mathcal{N}$-"averaged" probabilities are considered is satisfied if M-matrix is a tensor product of M-matrix in $\mathcal{M}(\mathcal{N}$ interpreted as finite-dimensional space with a projection operator to $\mathcal{N}$. The condition that $\mathcal{N}$ averaging in terms of a complex square root of $\mathcal{N}$ state produces this kind of M-matrix poses a very strong constraint on M-matrix if it is assumed to be universal (apart from variants corresponding to different measurement interactions).

4. Analogs of quantum matrix groups from finite measurement resolution?

The notion of quantum group replaces ordinary matrices with matrices with non-commutative elements. In TGD framework I have proposed that the notion should relate to the inclusions of von Neumann algebras allowing to describe mathematically the notion of finite measurement resolution.

In this article I will consider the notion of quantum matrix inspired by recent view about quantum TGD and it provides a concrete representation and physical interpretation of quantum groups in terms of finite measurement resolution. The basic idea is to replace complex matrix elements with operators expressible as products of non-negative hermitian operators and unitary operators analogous to the products of modulus and phase as a representation for complex numbers.

The condition that determinant and sub-determinants exist is crucial for the well-definedness of eigenvalue problem in the generalized sense. The weak definition of determinant meaning its development with respect to a fixed row or column does not pose additional conditions. Strong definition of determinant requires its invariance under permutations of rows and columns. The permutation of rows/columns turns out to have interpretation as braiding for the hermitian operators defined by the moduli of operator valued matrix elements. The commutativity of all sub-determinants is essential for the replacement of eigenvalues with eigenvalue spectra of hermitian operators and sub-determinants define mutually commuting set of operators.

The resulting quantum matrices define a more general structure than quantum group but provide a concrete representation and interpretation for quantum group in terms of finite measurement resolution if q is a root of unity. For $q = \pm 1$ (Bose-Einstein or Fermi-Dirac statistics) one obtains quantum matrices for which the determinant is apart from possible change by sign factor invariant under the permutations of both rows and columns. One could also understand the fractal structure of inclusion sequences of hyper-finite factors resulting by recursively replacing operators appearing as matrix elements with quantum matrices.

5. *Quantum spinors and fuzzy quantum mechanics*

The notion of quantum spinor leads to a quantum mechanical description of fuzzy probabilities. For quantum spinors state function reduction cannot be performed unless quantum deformation parameter equals to $q = 1$. The reason is that the components of quantum spinor do not commute: it is however possible to measure the commuting operators representing moduli squared of the components giving the probabilities associated with "true" and "false". The universal eigenvalue spectrum for probabilities does not in general contain (1,0) so that quantum qbits are inherently fuzzy. State function reduction would occur only after a transition to q=1 phase and decoherence is not a problem as long as it does not induce this transition.

1 Introduction

This chapter has emerged from a splitting of a chapter devote to the possible role of von Neumann algebras known as hyper-finite factors in quantum TGD. Second chapter emerging from the splitting is a representation of basic notions to chapter "Was von Neumann right after all?" [K21] representing only very briefly ideas about application to quantum TGD only briefly.

In the sequel the ideas about TGD applications are reviewed more or less chronologically. A summary about evolution of ideas is in question, not a coherent final structure, and as always the first speculations - in this case roughly for a decade ago - might look rather weird. The vision has however gradually become more realistic looking as deeper physical understanding of factors has evolved slowly.

The mathematics involved is extremely difficult for a physicist like me, and to really learn it at the level of proofs one should reincarnate as a mathematician. Therefore the only practical approach relies on the use of physical intuition to see whether HFFs might the correct structure and what HFFs do mean. What is needed is a concretization of the extremely abstract mathematics involved: mathematics represents only the bones to which physics should add flesh.

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2. Concerning the identification of M-matrix the notion of state as it is used in theory of factors is a more appropriate starting point than the notion modular automorphism but as a generalization of thermodynamical state is certainly not enough for the purposes of quantum TGD and quantum field theories (algebraic quantum field theorists might disagree!). Zero energy ontology requires that the notion of thermodynamical state should be replaced with its "complex square root" abstracting the idea about M-matrix as a product of positive square root of a diagonal density matrix and a unitary S-matrix. This generalization of thermodynamical state -if it exists- would provide a firm mathematical basis for the notion of M-matrix and for the fuzzy notion of path integral.

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1.3 Connes Tensor Product As A Realization Of Finite Measurement Resolution

The inclusions $\mathcal{N} \subset \mathcal{M}$ of factors allow an attractive mathematical description of finite measurement resolution in terms of Connes tensor product but do not fix M-matrix as was the original optimistic belief.

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1.4 Concrete Realization Of The Inclusion Hierarchies

A concrete construction of M-matrix motivated by the recent rather precise view about basic variational principles of TGD allows to identify rather concretely the inclusions of HFFs in TGD framework and relate them to the hierarchies of broken conformal symmetries accompanying quantum criticalities.

1. Fundamental fermions localized to string world sheets can be said to propagate as massless particles along their boundaries. The fundamental interaction vertices correspond to two fermion scattering for fermions at opposite throats of wormhole contact and the inverse of the conformal scaling generator L_0 would define the stringy propagator characterizing this interaction. Fundamental bosons correspond to pairs of fermion and antifermion at opposite throats of wormhole contact. Physical particles correspond to pairs of wormhole contacts with monopole Kähler magnetic flux flowing around a loop going through wormhole contacts.

2. The formulation of scattering amplitudes in terms of Yangian of the super-symplectic algebra leads to a rather detailed view about scattering amplitudes [K19]. In this formulation scattering amplitudes are representations for sequences of algebraic operations connecting collections of elements of Yangian and sequences produce the same result. A huge generalization of the duality symmetry of the hadronic string models is in question.

3. The reduction of the hierarchy of Planck constants $h_{eff}/h = n$ to a hierarchy of quantum criticalities accompanied by a hierarchy of sub-algebras of super-symplectic algebra acting as conformal gauge symmetries leads to the identification of inclusions of HFFs as inclusions of WCW Clifford algebras characterizing by $n(i)$ and $n(i+1) = m(i) \times n(i)$ so that hierarchies of von Neuman algebras, of Planck constants, and of quantum criticalities would be very closely related. In the transition $n(i) \to n(i+1) = m(i) \times n(i)$ the measurement accuracy indeed increases since some conformal gauge degrees of freedom are transformed to physical ones. An open question is whether one could interpret $m(i)$ as the integer characterizing inclusion: the problem is that also $m(i) = 2$ with $\mathcal{M} : \mathcal{N} = 4$ seems to be allowed whereas Jones inclusions allow only $m \geq 3$.

   Even more, number theoretic universality and strong form of holography leads to a detailed vision about the construction of scattering amplitudes suggesting that the hierarchy of algebraic extensions of rationals relates to the above mentioned hierarchies.

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This chapter represents a summary about the development of the ideas with last sections representing the recent latest about the realization and role of HFFs in TGD. I have saved the reader from those speculations that have turned out to reflect my own ignorance or are inconsistent with what I regarded established parts of quantum TGD.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [?].

2 A Vision About The Role Of HFFs In TGD

It is clear that at least the hyper-finite factors of type II_1 assignable to WCW spinors must have a profound role in TGD. Whether also HFFs of type III_1 appearing also in relativistic quantum field theories emerge when WCW spinors are replaced with spinor fields is not completely clear. I have proposed several ideas about the role of hyper-finite factors in TGD framework. In particular, Connes tensor product is an excellent candidate for defining the notion of measurement resolution.

In the following this topic is discussed from the perspective made possible by ZEO and the recent advances in the understanding of M-matrix using the notion of bosonic emergence. The conclusion is that the notion of state as it appears in the theory of factors is not enough for the purposes of quantum TGD. The reason is that state in this sense is essentially the counterpart of thermodynamical state. The construction of M-matrix might be understood in the framework of factors if one replaces state with its "complex square root" natural if quantum theory is regarded as a "complex square root" of thermodynamics. It is also found that the idea that Connes tensor product could fix M-matrix is too optimistic but an elegant formulation in terms of partial trace for the notion of M-matrix modulo measurement resolution exists and Connes tensor product allows interpretation as entanglement between sub-spaces consisting of states not distinguishable in the measurement resolution used. The partial trace also gives rise to non-pure states naturally.

The newest element in the vision is the proposal that quantum criticality of TGD Universe is realized as hierarchies of inclusions of super-conformal algebras with conformal weights coming as multiples of integer n, where n varies. If n_1 divides n_2 then various super-conformal algebras C_{n_2} are contained in C_{n_1}. This would define naturally the inclusion.

2.1 Basic Facts About Factors

In this section basic facts about factors are discussed. My hope that the discussion is more mature than or at least complementary to the summary that I could afford when I started the work with factors for more than half decade ago. I of course admit that this just a humble attempt of a physicist to express physical vision in terms of only superficially understood mathematical notions.

2.1.1 *Basic notions*

First some standard notations. Let $\mathcal{B}(\mathcal{H})$ denote the algebra of linear operators of Hilbert space $\mathcal{H}$ bounded in the norm topology with norm defined by the supremum for the length of the image of a point of unit sphere $\mathcal{H}$. This algebra has a lot of common with complex numbers in that the counterparts of complex conjugation, order structure and metric structure determined by the algebraic structure exist. This means the existence involution -that is *- algebra property. The order structure determined by algebraic structure means following: $A \geq 0$ defined as the condition $(A\xi, \xi) \geq 0$ is equivalent with $A = B^*B$. The algebra has also metric structure $||AB|| \leq ||A||||B|$ (Banach algebra property) determined by the algebraic structure. The algebra is also C^* algebra: $||A^*A|| = ||A||^2$ meaning that the norm is algebraically like that for complex numbers.

A von Neumann algebra $\mathcal{M}$ [A10] is defined as a weakly closed non-degenerate *-subalgebra of $\mathcal{B}(\mathcal{H})$ and has therefore all the above mentioned properties. From the point of view of physicist it is important that a sub-algebra is in question.

In order to define factors one must introduce additional structure.

1. Let $\mathcal{M}$ be subalgebra of $\mathcal{B}(\mathcal{H})$ and denote by $\mathcal{M}'$ its commutant $(\mathcal{H})$ commuting with it and allowing to express $\mathcal{B}(\mathcal{H})$ as $\mathcal{B}(\mathcal{H}) = \mathcal{M} \vee \mathcal{M}'$.

2. A factor is defined as a von Neumann algebra satisfying $\mathcal{M}'' = \mathcal{M}$ $\mathcal{M}$ is called factor. The equality of double commutant with the original algebra is thus the defining condition so that also the commutant is a factor. An equivalent definition for factor is as the condition that the intersection of the algebra and its commutant reduces to a complex line spanned by a unit operator. The condition that the only operator commuting with all operators of the factor is unit operator corresponds to irreducibility in representation theory.

3. Some further basic definitions are needed. $\Omega \in \mathcal{H}$ is cyclic if the closure of $\mathcal{M}\Omega$ is $\mathcal{H}$ and separating if the only element of $\mathcal{M}$ annihilating Ω is zero. Ω is cyclic for $\mathcal{M}$ if and only if it is separating for its commutant. In so called standard representation Ω is both cyclic and separating.

4. For hyperfinite factors an inclusion hierarchy of finite-dimensional algebras whose union is dense in the factor exists. This roughly means that one can approximate the algebra in arbitrary accuracy with a finite-dimensional sub-algebra.

The definition of the factor might look somewhat artificial unless one is aware of the underlying physical motivations. The motivating question is what the decomposition of a physical system to non-interacting sub-systems could mean. The decomposition of $\mathcal{B}(\mathcal{H})$ to $\vee$ product realizes this decomposition.

1. Tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is the decomposition according to the standard quantum measurement theory and means the decomposition of operators in $\mathcal{B}(\mathcal{H})$ to tensor products of mutually commuting operators in $\mathcal{M} = \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{M}' = \mathcal{B}(\mathcal{H}_2)$. The information about $\mathcal{M}$ can be coded in terms of projection operators. In this case projection operators projecting to a complex ray of Hilbert space exist and arbitrary compact operator can be expressed as a sum of these projectors. For factors of type I minimal projectors exist. Factors of type I_n correspond to sub-algebras of $\mathcal{B}(\mathcal{H})$ associated with infinite-dimensional Hilbert space and I_∞ to $\mathcal{B}(\mathcal{H})$ itself. These factors appear in the standard quantum measurement theory where state function reduction can lead to a ray of Hilbert space.

2. For factors of type II no minimal projectors exists whereas finite projectors exist. For factors of type II_1 all projectors have trace not larger than one and the trace varies in the range $(0, 1]$. In this case cyclic vectors Ω exist. State function reduction can lead only to an infinite-dimensional subspace characterized by a projector with trace smaller than 1 but larger than zero. The natural interpretation would be in terms of finite measurement resolution. The tensor product of II_1 factor and I_∞ is II_∞ factor for which the trace for a projector can have arbitrarily large values. II_1 factor has a unique finite tracial state and the set of traces of projections spans unit interval. There is uncountable number of factors of type II but hyper-finite factors of type II_1 are the exceptional ones and physically most interesting.

3. Factors of type III correspond to an extreme situation. In this case the projection operators E spanning the factor have either infinite or vanishing trace and there exists an isometry mapping $E\mathcal{H}$ to $\mathcal{H}$ meaning that the projection operator spans almost all of $\mathcal{H}$. All projectors are also related to each other by isometry. Factors of type III are smallest if the factors are regarded as sub-algebras of a fixed $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ corresponds to isomorphism class of Hilbert spaces. Situation changes when one speaks about concrete representations. Also now hyper-finite factors are exceptional.

4. Von Neumann algebras define a non-commutative measure theory. Commutative von Neumann algebras indeed reduce to $L^\infty(X)$ for some measure space (X, μ) and vice versa.

2.1.2 *Weights, states and traces*

The notions of weight, state, and trace are standard notions in the theory of von Neumann algebras.

1. A weight of von Neumann algebra is a linear map from the set of positive elements (those of form a^*a) to non-negative reals.

2. A positive linear functional is weight with $\omega(1)$ finite.
3. A state is a weight with $\omega(1) = 1$.
4. A trace is a weight with $\omega(aa^*) = \omega(a^*a)$ for all a.
5. A tracial state is a weight with $\omega(1) = 1$.

A factor has a trace such that the trace of a non-zero projector is non-zero and the trace of projection is infinite only if the projection is infinite. The trace is unique up to a rescaling. For factors that are separable or finite, two projections are equivalent if and only if they have the same trace. Factors of type I_n the values of trace are equal to multiples of $1/n$. For a factor of type I_∞ the value of trace are $0, 1, 2, ...$ For factors of type II_1 the values span the range $[0, 1]$ and for factors of type II_∞ n the range $[0, \infty)$. For factors of type III the values of the trace are 0, and ∞.

2.1.3 *Tomita-Takesaki theory*

Tomita-Takesaki theory is a vital part of the theory of factors. First some definitions.

1. Let $\omega(x)$ be a faithful state of von Neumann algebra so that one has $\omega(xx^*) > 0$ for $x > 0$. Assume by Riesz lemma the representation of ω as a vacuum expectation value: $\omega = (\cdot\Omega, \Omega)$, where Ω is cyclic and separating state.
2. Let

$$L^\infty(\mathcal{M}) \equiv \mathcal{M} \ , \quad L^2(\mathcal{M}) = \mathcal{H} \ , \quad L^1(\mathcal{M}) = \mathcal{M}_* \ , \tag{2.1}$$

where $\mathcal{M}_*$ is the pre-dual of $\mathcal{M}$ defined by linear functionals in $\mathcal{M}$. One has ${\mathcal{M}_*}^* = \mathcal{M}$.

3. The conjugation $x \to x^*$ is isometric in $\mathcal{M}$ and defines a map $\mathcal{M} \to L^2(\mathcal{M})$ via $x \to x\Omega$. The map $S_0; x\Omega \to x^*\Omega$ is however non-isometric.
4. Denote by S the closure of the anti-linear operator S_0 and by $S = J\Delta^{1/2}$ its polar decomposition analogous that for complex number and generalizing polar decomposition of linear operators by replacing (almost) unitary operator with anti-unitary J. Therefore $\Delta = S^*S > 0$ is positive self-adjoint and J an anti-unitary involution. The non-triviality of Δ reflects the fact that the state is not trace so that hermitian conjugation represented by S in the state space brings in additional factor $\Delta^{1/2}$.
5. What x can be is puzzling to physicists. The restriction fermionic Fock space and thus to creation operators would imply that Δ would act non-trivially only vacuum state so that $\Delta > 0$ condition would not hold true. The resolution of puzzle is the allowance of tensor product of Fock spaces for which vacua are conjugates: only this gives cyclic and separating state. This is natural in ZEO.

The basic results of Tomita-Takesaki theory are following.

1. The basic result can be summarized through the following formulas

$$\Delta^{it} M \Delta^{-it} = \mathcal{M} \ , J\mathcal{M}J = \mathcal{M}' \ .$$

2. The latter formula implies that $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic algebras. The first formula implies that a one parameter group of modular automorphisms characterizes partially the factor. The physical meaning of modular automorphisms is discussed in [A14, A19] Δ is Hermitian and positive definite so that the eigenvalues of $log(\Delta)$ are real but can be negative. Δ^{it} is however not unitary for factors of type II and III. Physically the non-unitarity must relate to the fact that the flow is contracting so that hermiticity as a local condition is not enough to guarantee unitarity.

3. $\omega \to \sigma_t^\omega = Ad\Delta^{it}$ defines a canonical evolution -modular automorphism- associated with ω and depending on it. The Δ:s associated with different ω:s are related by a unitary inner automorphism so that their equivalence classes define an invariant of the factor.

Tomita-Takesaki theory gives rise to a non-commutative measure theory which is highly non-trivial. In particular the spectrum of Δ can be used to classify the factors of type II and III.

2.1.4 *Modular automorphisms*

Modular automorphisms of factors are central for their classification.

1. One can divide the automorphisms to inner and outer ones. Inner automorphisms correspond to unitary operators obtained by exponentiating Hermitian Hamiltonian belonging to the factor and connected to identity by a flow. Outer automorphisms do not allow a representation as a unitary transformations although $log(\Delta)$ is formally a Hermitian operator.

2. The fundamental group of the type II_1 factor defined as fundamental group group of corresponding II_∞ factor characterizes partially a factor of type II_1. This group consists real numbers λ such that there is an automorphism scaling the trace by λ. Fundamental group typically contains all reals but it can be also discrete and even trivial.

3. Factors of type III allow a one-parameter group of modular automorphisms, which can be used to achieve a partial classification of these factors. These automorphisms define a flow in the center of the factor known as flow of weights. The set of parameter values λ for which ω is mapped to itself and the center of the factor defined by the identity operator (projector to the factor as a sub-algebra of $\mathcal{B}(\mathcal{H})$) is mapped to itself in the modular automorphism defines the Connes spectrum of the factor. For factors of type III_λ this set consists of powers of $\lambda < 1$. For factors of type III_0 this set contains only identity automorphism so that there is no periodicity. For factors of type III_1 Connes spectrum contains all real numbers so that the automorphisms do not affect the identity operator of the factor at all.

The modules over a factor correspond to separable Hilbert spaces that the factor acts on. These modules can be characterized by M-dimension. The idea is roughly that complex rays are replaced by the sub-spaces defined by the action of $\mathcal{M}$ as basic units. M-dimension is not integer valued in general. The so called standard module has a cyclic separating vector and each factor has a standard representation possessing antilinear involution J such that $\mathcal{M}' = J\mathcal{M}J$ holds true (note that J changes the order of the operators in conjugation). The inclusions of factors define modules having interpretation in terms of a finite measurement resolution defined by $\mathcal{M}$.

2.1.5 *Crossed product as a manner to construct factors of type III*

By using so called crossed product crossedproduct for a group G acting in algebra A one can obtain new von Neumann algebras. One ends up with crossed product by a two-step generalization by starting from the semidirect product $G \triangleleft H$ for groups defined as $(g_1, h_1)(g_2, h_2) = (g_1h_1(g_2), h_1h_2)$ (note that Poincare group has interpretation as a semidirect product $M^4 \triangleleft SO(3,1)$ of Lorentz and translation groups). At the first step one replaces the group H with its group algebra. At the second step the the group algebra is replaced with a more general algebra. What is formed is the semidirect product $A \triangleleft G$ which is sum of algebras Ag. The product is given by $(a_1, g_1)(a_2, g_2) = (a_1g_1(a_2), g_1g_2)$. This construction works for both locally compact groups and quantum groups. A not too highly educated guess is that the construction in the case of quantum groups gives the factor $\mathcal{M}$ as a crossed product of the included factor $\mathcal{N}$ and quantum group defined by the factor space $\mathcal{M}/\mathcal{N}$.

The construction allows to express factors of type III as crossed products of factors of type II_∞ and the 1-parameter group G of modular automorphisms assignable to any vector which is cyclic for both factor and its commutant. The ergodic flow θ_λ scales the trace of projector in II_∞ factor by $\lambda > 0$. The dual flow defined by G restricted to the center of II_∞ factor does not depend on the choice of cyclic vector.

The Connes spectrum - a closed subgroup of positive reals - is obtained as the exponent of the kernel of the dual flow defined as set of values of flow parameter λ for which the flow in the center is trivial. Kernel equals to $\{0\}$ for III_0, contains numbers of form $log(\lambda)Z$ for factors of type III$_\lambda$ and contains all real numbers for factors of type III$_1$ meaning that the flow does not affect the center.

2.1.6 *Inclusions and Connes tensor product*

Inclusions $\mathcal{N} \subset \mathcal{M}$ of von Neumann algebras have physical interpretation as a mathematical description for sub-system-system relation. In [K21] there is more extensive TGD colored description of inclusions and their role in TGD. Here only basic facts are listed and the Connes tensor product is explained.

For type I algebras the inclusions are trivial and tensor product description applies as such. For factors of II_1 and III the inclusions are highly non-trivial. The inclusion of type II_1 factors were understood by Vaughan Jones [A1] and those of factors of type III by Alain Connes [A11] .

Formally sub-factor $\mathcal{N}$ of $\mathcal{M}$ is defined as a closed *-stable C-subalgebra of $\mathcal{M}$. Let $\mathcal{N}$ be a sub-factor of type II_1 factor $\mathcal{M}$. Jones index $\mathcal{M} : \mathcal{N}$ for the inclusion $\mathcal{N} \subset \mathcal{M}$ can be defined as $\mathcal{M} : \mathcal{N} = dim_N(L^2(\mathcal{M})) = Tr_{N'}(id_{L^2(\mathcal{M})})$. One can say that the dimension of completion of $\mathcal{M}$ as $\mathcal{N}$ module is in question.

2.1.7 *Basic findings about inclusions*

What makes the inclusions non-trivial is that the position of $\mathcal{N}$ in $\mathcal{M}$ matters. This position is characterized in case of hyper-finite II_1 factors by index $\mathcal{M} : \mathcal{N}$ which can be said to the dimension of $\mathcal{M}$ as $\mathcal{N}$ module and also as the inverse of the dimension defined by the trace of the projector from $\mathcal{M}$ to $\mathcal{N}$. It is important to notice that $\mathcal{M} : \mathcal{N}$ does not characterize either $\mathcal{M}$ or $\mathcal{M}$, only the imbedding.

The basic facts proved by Jones are following [A1] .

1. For pairs $\mathcal{N} \subset \mathcal{M}$ with a finite principal graph the values of $\mathcal{M} : \mathcal{N}$ are given by

$$\begin{aligned} &a)\ \ \mathcal{M} : \mathcal{N} = 4cos^2(\pi/h) \ \ , \quad h \geq 3 \ \ , \\ &b)\ \ \mathcal{M} : \mathcal{N} \geq 4 \ \ . \end{aligned} \tag{2.2}$$

   the numbers at right hand side are known as Beraha numbers [A17] . The comments below give a rough idea about what finiteness of principal graph means.

2. As explained in [B1] , for $\mathcal{M} : \mathcal{N} < 4$ one can assign to the inclusion Dynkin graph of ADE type Lie-algebra g with h equal to the Coxeter number h of the Lie algebra given in terms of its dimension and dimension r of Cartan algebra r as $h = (dimg(g) - r)/r$. For $\mathcal{M} : \mathcal{N} < 4$ ordinary Dynkin graphs of D_{2n} and E_6, E_8 are allowed. The Dynkin graphs of Lie algebras of $SU(n)$, E_7 and D_{2n+1} are however not allowed. $E_6, E_7, andE_8$ correspond to symmetry groups of tetrahedron, octahedron/cube, and icosahedron/dodecahedron. The group for octahedron/cube is missing: what could this mean?

   For $\mathcal{M} : \mathcal{N} = 4$ one can assign to the inclusion an extended Dynkin graph of type ADE characterizing Kac Moody algebra. Extended ADE diagrams characterize also the subgroups of $SU(2)$ and the interpretation proposed in [A22] is following-

   The ADE diagrams are associated with the $n = \infty$ case having $\mathcal{M} : \mathcal{N} \geq 4$. There are diagrams corresponding to infinite subgroups: A_∞ corresponding to $SU(2)$ itself, $A_{-\infty,\infty}$ corresponding to circle group $U(1)$, and infinite dihedral groups (generated by a rotation by a non-rational angle and reflection.

   One can construct also inclusions for which the diagrams corresponding to finite subgroups $G \subset SU(2)$ are extension of A_n for cyclic groups, of D_n dihedral groups, and of E_n with $n = 6, 7, 8$ for tetrahedron, cube, dodecahedron. These extensions correspond to ADE type Kac-Moody algebras.

   The extension is constructed by constructing first factor R as infinite tensor power of $M_2(C)$ (complexified quaternions). Sub-factor R_0 consists elements of of R of form $Id \otimes x$. $SU(2)$

preserves R_0 and for any subgroup G of $SU(2)$ one can identify the inclusion $N \subset M$ in terms of $N = R_0^G$ and $M = R^G$, where $N = R_0^G$ and $M = R^G$ consists of fixed points of R_0 and R under the action of G. The principal graph for $N \subset M$ is the extended Coxeter-Dynk graph for the subgroup G.

Physicist might try to interpret this by saying that one considers only sub-algebras R_0^G and R^G of observables invariant under G and obtains extended Dynkin diagram of G defining an ADE type Kac-Moody algebra. Could the condition that Kac-Moody algebra elements with non-vanishing conformal weight annihilate the physical states state that the state is invariant under R_0 defining measurement resolution. Besides this the states are also invariant under finite group G? Could R_0^G and R^G correspond just to states which are also invariant under finite group G.

2.1.8 *Connes tensor product*

The basic idea of Connes tensor product is that a sub-space generated sub-factor $\mathcal{N}$ takes the role of the complex ray of Hilbert space. The physical interpretation is in terms of finite measurement resolution: it is not possible to distinguish between states obtained by applying elements of $\mathcal{N}$.

Intuitively it is clear that it should be possible to decompose $\mathcal{M}$ to a tensor product of factor space $\mathcal{M}/\mathcal{N}$ and $\mathcal{N}$:

$$\mathcal{M} = \mathcal{M}/\mathcal{N} \otimes \mathcal{N} \ . \tag{2.3}$$

One could regard the factor space $\mathcal{M}/\mathcal{N}$ as a non-commutative space in which each point corresponds to a particular representative in the equivalence class of points defined by $\mathcal{N}$. The connections between quantum groups and Jones inclusions suggest that this space closely relates to quantum groups. An alternative interpretation is as an ordinary linear space obtained by mapping $\mathcal{N}$ rays to ordinary complex rays. These spaces appear in the representations of quantum groups. Similar procedure makes sense also for the Hilbert spaces in which $\mathcal{M}$ acts.

Connes tensor product can be defined in the space $\mathcal{M} \otimes \mathcal{M}$ as entanglement which effectively reduces to entanglement between $\mathcal{N}$ sub-spaces. This is achieved if $\mathcal{N}$ multiplication from right is equivalent with $\mathcal{N}$ multiplication from left so that $\mathcal{N}$ acts like complex numbers on states. One can imagine variants of the Connes tensor product and in TGD framework one particular variant appears naturally as will be found.

In the finite-dimensional case Connes tensor product of Hilbert spaces has a rather simple representation. If the matrix algebra N of $n \times n$ matrices acts on V from right, V can be regarded as a space formed by $m \times n$ matrices for some value of m. If N acts from left on W, W can be regarded as space of $n \times r$ matrices.

1. In the first representation the Connes tensor product of spaces V and W consists of $m \times r$ matrices and Connes tensor product is represented as the product VW of matrices as $(VW)_{mr}e^{mr}$. In this representation the information about N disappears completely as the interpretation in terms of measurement resolution suggests. The sum over intermediate states defined by N brings in mind path integral.

2. An alternative and more physical representation is as a state

$$\sum_n V_{mn}W_{nr}e^{mn} \otimes e^{nr}$$

in the tensor product $V \otimes W$.

3. One can also consider two spaces V and W in which N acts from right and define Connes tensor product for $A^\dagger \otimes_N B$ or its tensor product counterpart. This case corresponds to the modification of the Connes tensor product of positive and negative energy states. Since Hermitian conjugation is involved, matrix product does not define the Connes tensor product now. For $m = r$ case entanglement coefficients should define a unitary matrix commuting with the action of the Hermitian matrices of N and interpretation would be in terms of symmetry. HFF property would encourage to think that this representation has an analog in the case of HFFs of type II_1.

4. Also type I_n factors are possible and for them Connes tensor product makes sense if one can assign the inclusion of finite-D matrix algebras to a measurement resolution.

2.1.9 *Factors in quantum field theory and thermodynamics*

Factors arise in thermodynamics and in quantum field theories [A21, A14, A19] . There are good arguments showing that in HFFs of III_1 appear are relativistic quantum field theories. In non-relativistic QFTs the factors of type I appear so that the non-compactness of Lorentz group is essential. Factors of type III_1 and III_λ appear also in relativistic thermodynamics.

The geometric picture about factors is based on open subsets of Minkowski space. The basic intuitive view is that for two subsets of M^4, which cannot be connected by a classical signal moving with at most light velocity, the von Neumann algebras commute with each other so that $\vee$ product should make sense.

Some basic mathematical results of algebraic quantum field theory [A19] deserve to be listed since they are suggestive also from the point of view of TGD.

1. Let $\mathcal{O}$ be a bounded region of R^4 and define the region of M^4 as a union $\cup_{|x|<\epsilon}(\mathcal{O}+x)$ where $(\mathcal{O}+x)$ is the translate of O and $|x|$ denotes Minkowski norm. Then every projection $E \in \mathcal{M}(\mathcal{O})$ can be written as WW^* with $W \in \mathcal{M}(\mathcal{O}_\epsilon)$ and $W^*W = 1$. Note that the union is not a bounded set of M^4. This almost establishes the type III property.

2. Both the complement of light-cone and double light-cone define HFF of type III_1. Lorentz boosts induce modular automorphisms.

3. The so called split property suggested by the description of two systems of this kind as a tensor product in relativistic QFTs is believed to hold true. This means that the HFFs of type III_1 associated with causally disjoint regions are sub-factors of factor of type I_∞. This means

$$\mathcal{M}_1 \subset \mathcal{B}(\mathcal{H}_1) \times 1 \ \ , \ \ \mathcal{M}_2 \subset 1 \otimes \mathcal{B}(\mathcal{H}_2) \ \ .$$

An infinite hierarchy of inclusions of HFFs of type III_1s is induced by set theoretic inclusions.

2.2 TGD And Factors

The following vision about TGD and factors relies heavily on zero energy ontology, TGD inspired quantum measurement theory, basic vision about quantum TGD, and bosonic emergence.

2.2.1 *The problems*

Concerning the role of factors in TGD framework there are several problems of both conceptual and technical character.

1. Conceptual problems

It is safest to start from the conceptual problems and take a role of skeptic.

1. Under what conditions the assumptions of Tomita-Takesaki formula stating the existence of modular automorphism and isomorphy of the factor and its commutant hold true? What is the physical interpretation of the formula $\mathcal{M}' = J\mathcal{M}J$ relating factor and its commutant in TGD framework?

2. Is the identification $M = \Delta^{it}$ sensible is quantum TGD and ZEO, where M-matrix is "complex square root" of exponent of Hamiltonian defining thermodynamical state and the notion of unitary time evolution is given up? The notion of state ω leading to Δ is essentially thermodynamical and one can wonder whether one should take also a "complex square root" of ω to get M-matrix giving rise to a genuine quantum theory.

3. TGD based quantum measurement theory involves both quantum fluctuating degrees of freedom assignable to light-like 3-surfaces and zero modes identifiable as classical degrees of freedom assignable to interior of the space-time sheet. Zero modes have also fermionic counterparts. State preparation should generate entanglement between the quantal and classical states. What this means at the level of von Neumann algebras?

4. What is the TGD counterpart for causal disjointness. At space-time level different space-time sheets could correspond to such regions whereas at imbedding space level causally disjoint CDs would represent such regions.

2. Technical problems

There are also more technical questions.

1. What is the von Neumann algebra needed in TGD framework? Does one have a a direct integral over factors? Which factors appear in it? Can one construct the factor as a crossed product of some group G with direct physical interpretation and of naturally appearing factor A? Is A a HFF of type II_∞? assignable to a fixed CD? What is the natural Hilbert space $\mathcal{H}$ in which A acts?

2. What are the geometric transformations inducing modular automorphisms of II_∞ inducing the scaling down of the trace? Is the action of G induced by the boosts in Lorentz group. Could also translations and scalings induce the action? What is the factor associated with the union of Poincare transforms of CD? $log(\Delta)$ is Hermitian algebraically: what does the non-unitarity of $exp(log(\Delta)it)$ mean physically?

3. Could Ω correspond to a vacuum which in conformal degrees of freedom depends on the choice of the sphere S^2 defining the radial coordinate playing the role of complex variable in the case of the radial conformal algebra. Does *-operation in $\mathcal{M}$ correspond to Hermitian conjugation for fermionic oscillator operators and change of sign of super conformal weights?

The exponent of the Kähler-Dirac action gives rise to the exponent of Kähler function as Dirac determinant and fermionic inner product defined by fermionic Feynman rules. It is implausible that this exponent could as such correspond to ω or Δ^{it} having conceptual roots in thermodynamics rather than QFT. If one assumes that the exponent of the Kähler-Dirac action defines a "complex square root" of ω the situation changes. This raises technical questions relating to the notion of square root of ω.

1. Does the complex square root of ω have a polar decomposition to a product of positive definite matrix (square root of the density matrix) and unitary matrix and does $\omega^{1/2}$ correspond to the modulus in the decomposition? Does the square root of Δ have similar decomposition with modulus equal equal to $\Delta^{1/2}$ in standard picture so that modular automorphism, which is inherent property of von Neumann algebra, would not be affected?

2. Δ^{it} or rather its generalization is defined modulo a unitary operator defined by some Hamiltonian and is therefore highly non-unique as such. This non-uniqueness applies also to $|\Delta|$. Could this non-uniqueness correspond to the thermodynamical degrees of freedom?

2.2.2 *ZEO and factors*

The first question concerns the identification of the Hilbert space associated with the factors in ZEO. As the positive or negative energy part of the zero energy state space or as the entire space of zero energy states? The latter option would look more natural physically and is forced by the condition that the vacuum state is cyclic and separating.

1. The commutant of HFF given as $\mathcal{M}' = J\mathcal{M}J$, where J is involution transforming fermionic oscillator operators and bosonic vector fields to their Hermitian conjugates. Also conformal weights would change sign in the map which conforms with the view that the light-like boundaries of CD are analogous to upper and lower hemispheres of S^2 in conformal field theory. The presence of J representing essentially Hermitian conjugation would suggest that positive and zero energy parts of zero energy states are related by this formula so that state space decomposes to a tensor product of positive and negative energy states and M-matrix can be regarded as a map between these two sub-spaces.

2. The fact that HFF of type II_1 has the algebra of fermionic oscillator operators as a canonical representation makes the situation puzzling for a novice. The assumption that the vacuum is cyclic and separating means that neither creation nor annihilation operators can annihilate it. Therefore Fermionic Fock space cannot appear as the Hilbert space in the Tomita-Takesaki theorem. The paradox is circumvented if the action of * transforms creation operators acting on the positive energy part of the state to annihilation operators acting on negative energy part of the state. If J permutes the two Fock vacuums in their tensor product, the action of S indeed maps permutes the tensor factors associated with $\mathcal{M}$ and $\mathcal{M}'$.

It is far from obvious whether the identification $M = \Delta^{it}$ makes sense in ZEO.

1. In ZEO M-matrix defines time-like entanglement coefficients between positive and negative energy parts of the state. M-matrix is essentially "complex square root" of the density matrix and quantum theory similar square root of thermodynamics. The notion of state as it appears in the theory of HFFs is however essentially thermodynamical. Therefore it is good to ask whether the "complex square root of state" could make sense in the theory of factors.

2. Quantum field theory suggests an obvious proposal concerning the meaning of the square root: one replaces exponent of Hamiltonian with imaginary exponential of action at $T \to 0$ limit. In quantum TGD the exponent of Kähler-Dirac action giving exponent of Kähler function as real exponent could be the manner to take this complex square root. Kähler-Dirac action can therefore be regarded as a "square root" of Kähler action.

3. The identification $M = \Delta^{it}$ relies on the idea of unitary time evolution which is given up in ZEO based on CDs? Is the reduction of the quantum dynamics to a flow a realistic idea? As will be found this automorphism could correspond to a time translation or scaling for either upper or lower light-cone defining CD and can ask whether Δ^{it} corresponds to the exponent of scaling operator L_0 defining single particle propagator as one integrates over t. Its complex square root would correspond to fermionic propagator.

4. In this framework $J\Delta^{it}$ would map the positive energy and negative energy sectors to each other. If the positive and negative energy state spaces can identified by isometry then $M = J\Delta^{it}$ identification can be considered but seems unrealistic. $S = J\Delta^{1/2}$ maps positive and negative energy states to each other: could S or its generalization appear in M-matrix as a part which gives thermodynamics? The exponent of the Kähler-Dirac action does not seem to provide thermodynamical aspect and p-adic thermodynamics suggests strongly the presence exponent of $exp(-L_0/T_p)$ with T_p chose in such manner that consistency with p-adic thermodynamics is obtained. Could the generalization of $J\Delta^{n/2}$ with Δ replaced with its "square root" give rise to padic thermodynamics and also ordinary thermodynamics at the level of density matrix? The minimal option would be that power of Δ^{it} which imaginary value of t is responsible for thermodynamical degrees of freedom whereas everything else is dictated by the unitary S-matrix appearing as phase of the "square root" of ω.

2.2.3 *Zero modes and factors*

The presence of zero modes justifies quantum measurement theory in TGD framework and the relationship between zero modes and HFFs involves further conceptual problems.

1. The presence of zero modes means that one has a direct integral over HFFs labeled by zero modes which by definition do not contribute to WCW line element. The realization of quantum criticality in terms of Kähler-Dirac action [K6] suggests that also fermionic zero mode degrees of freedom are present and correspond to conserved charges assignable to the critical deformations of the pace-time sheets. Induced Kähler form characterizes the values of zero modes for a given space-time sheet and the symplectic group of light-cone boundary characterizes the quantum fluctuating degrees of freedom. The entanglement between zero modes and quantum fluctuating degrees of freedom is essential for quantum measurement theory. One should understand this entanglement.

2. Physical intuition suggests that classical observables should correspond to longer length scale than quantal ones. Hence it would seem that the interior degrees of freedom outside CD should correspond to classical degrees of freedom correlating with quantum fluctuating degrees of freedom of CD.

3. Quantum criticality means that Kähler-Dirac action allows an infinite number of conserved charges which correspond to deformations leaving metric invariant and therefore act on zero modes. Does this super-conformal algebra commute with the super-conformal algebra associated with quantum fluctuating degrees of freedom? Could the restriction of elements of quantum fluctuating currents to 3-D light-like 3-surfaces actually imply this commutativity. Quantum holography would suggest a duality between these algebras. Quantum measurement theory suggests even 1-1 correspondence between the elements of the two super-conformal algebras. The entanglement between classical and quantum degrees of freedom would mean that prepared quantum states are created by operators for which the operators in the two algebras are entangled in diagonal manner.

4. The notion of finite measurement resolution has become key element of quantum TGD and one should understand how finite measurement resolution is realized in terms of inclusions of hyper-finite factors for which sub-factor defines the resolution in the sense that its action creates states not distinguishable from each other in the resolution used. The notion of finite measurement resolution suggests that one should speak about entanglement between sub-factors and corresponding sub-spaces rather than between states. Connes tensor product would code for the idea that the action of sub-factors is analogous to that of complex numbers and tracing over sub-factor realizes this idea.

5. Just for fun one can ask whether the duality between zero modes and quantum fluctuating degrees of freedom representing quantum holography could correspond to $\mathcal{M}' = J\mathcal{M}J$? This interpretation must be consistent with the interpretation forced by zero energy ontology. If this crazy guess is correct (very probably not!), both positive and negative energy states would be observed in quantum measurement but in totally different manner. Since this identity would simplify enormously the structure of the theory, it deserves therefore to be shown wrong.

2.2.4 *Crossed product construction in TGD framework*

The identification of the von Neumann algebra by crossed product construction is the basic challenge. Consider first the question how HFFs of type II_∞ emerge, how modular automorphisms act on them, and how one can understand the non-unitary character of the Δ^{it} in an apparent conflict with the hermiticity and positivity of Δ.

1. The Clifford algebra at a given point of WCW(CD) (light-like 3-surfaces with ends at the boundaries of CD) defines HFF of type II_1 or possibly a direct integral of them. For a given CD having compact isotropy group $SO(3)$ leaving the rest frame defined by the tips of CD invariant the factor defined by Clifford algebra valued fields in WCW(CD) is most naturally HFF of type II_∞. The Hilbert space in which this Clifford algebra acts, consists of spinor fields in WCW(CD). Also the symplectic transformations of light-cone boundary leaving light-like 3-surfaces inside CD can be included to G. In fact all conformal algebras leaving CD invariant could be included in CD.

2. The downwards scalings of the radial coordinate r_M of the light-cone boundary applied to the basis of WCW (CD) spinor fields could induce modular automorphism. These scalings reduce the size of the portion of light-cone in which the WCW spinor fields are non-vanishing and effectively scale down the size of CD. $exp(iL_0)$ as algebraic operator acts as a phase multiplication on eigen states of conformal weight and therefore as apparently unitary operator. The geometric flow however contracts the CD so that the interpretation of $exp(itL_0)$ as a unitary modular automorphism is not possible. The scaling down of CD reduces the value of the trace if it involves integral over the boundary of CD. A similar reduction is implied by the downward shift of the upper boundary of CD so that also time translations would induce

modular automorphism. These shifts seem to be necessary to define rest energies of positive and negative energy parts of the zero energy state.

3. The non-triviality of the modular automorphisms of II_∞ factor reflects different choices of ω. The degeneracy of ω could be due to the non-uniqueness of conformal vacuum which is part of the definition of ω. The radial Virasoro algebra of light-cone boundary is generated by $L_n = L^*_{-n}$, $n \neq 0$ and $L_0 = L^*_0$ and negative and positive frequencies are in asymmetric position. The conformal gauge is fixed by the choice of $SO(3)$ subgroup of Lorentz group defining the slicing of light-cone boundary by spheres and the tips of CD fix $SO(3)$ uniquely. One can however consider also alternative choices of $SO(3)$ and each corresponds to a slicing of the light-cone boundary by spheres but in general the sphere defining the intersection of the two light-cone does not belong to the slicing. Hence the action of Lorentz transformation inducing different choice of $SO(3)$ can lead out from the preferred state space so that its representation must be non-unitary unless Virasoro generators annihilate the physical states. The non-vanishing of the conformal central charge c and vacuum weight h seems to be necessary and indeed can take place for super-symplectic algebra and Super Kac-Moody algebra since only the differences of the algebra elements are assumed to annihilate physical states.

Modular automorphism of HFFs type III_1 can be induced by several geometric transformations for HFFs of type III_1 obtained using the crossed product construction from II_∞ factor by extending CD to a union of its Lorentz transforms.

1. The crossed product would correspond to an extension of II_∞ by allowing a union of some geometric transforms of CD. If one assumes that only CDs for which the distance between tips is quantized in powers of 2, then scalings of either upper or lower boundary of CD cannot correspond to these transformations. Same applies to time translations acting on either boundary but not to ordinary translations. As found, the modular automorphisms reducing the size of CD could act in HFF of type II_∞.

2. The geometric counterparts of the modular transformations would most naturally correspond to any non-compact one parameter sub-group of Lorentz group as also QFT suggests. The Lorentz boosts would replace the radial coordinate r_M of the light-cone boundary associated with the radial Virasoro algebra with a new one so that the slicing of light-cone boundary with spheres would be affected and one could speak of a new conformal gauge. The temporal distance between tips of CD in the rest frame would not be affected. The effect would seem to be however unitary because the transformation does not only modify the states but also transforms CD.

3. Since Lorentz boosts affect the isotropy group $SO(3)$ of CD and thus also the conformal gauge defining the radial coordinate of the light-cone boundary, they affect also the definition of the conformal vacuum so that also ω is affected so that the interpretation as a modular automorphism makes sense. The simplistic intuition of the novice suggests that if one allows wave functions in the space of Lorentz transforms of CD, unitarity of Δ^{it} is possible. Note that the hierarchy of Planck constants assigns to CD preferred M^2 and thus direction of quantization axes of angular momentum and boosts in this direction would be in preferred role.

4. One can also consider the HFF of type III_λ if the radial scalings by negative powers of 2 correspond to the automorphism group of II_∞ factor as the vision about allowed CDs suggests. $\lambda = 1/2$ would naturally hold true for the factor obtained by allowing only the radial scalings. Lorentz boosts would expand the factor to HFF of type III_1. Why scalings by powers of 2 would give rise to periodicity should be understood.

The identification of M-matrix as modular automorphism Δ^{it}, where t is complex number having as its real part the temporal distance between tips of CD quantized as 2^n and temperature as imaginary part, looks at first highly attractive, since it would mean that M-matrix indeed exists mathematically. The proposed interpretations of modular automorphisms do not support the idea that they could define the S-matrix of the theory. In any case, the identification as modular automorphism would not lead to a magic universal formula since arbitrary unitary transformation is involved.

2.2.5 *Quantum criticality and inclusions of factors*

Quantum criticality fixes the value of Kähler coupling strength but is expected to have also an interpretation in terms of a hierarchies of broken conformal gauge symmetries suggesting hierarchies of inclusions.

1. In ZEO 3-surfaces are unions of space-like 3-surfaces at the ends of causal diamond (CD). Space-time surfaces connect 3-surfaces at the boundaries of CD. The non-determinism of Kähler action allows the possibility of having several space-time sheets connecting the ends of space-time surface but the conditions that classical charges are same for them reduces this number so that it could be finite. Quantum criticality in this sense implies non-determinism analogous to that of critical systems since preferred extremals can co-incide and suffer this kind of bifurcation in the interior of CD. This quantum criticality can be assigned to the hierarchy of Planck constants and the integer n in $h_{eff} = n \times h$ [K13] corresponds to the number of degenerate space-time sheets with same Kähler action and conserved classical charges.

2. Also now one expects a hierarchy of criticalities and since criticality and conformal invariance are closely related, a natural conjecture is that the fractal hierarchy of sub-algebras of conformal algebra isomorphic to conformal algebra itself and having conformal weights coming as multiples of n corresponds to the hierarchy of Planck constants. This hierarchy would define a hierarchy of symmetry breakings in the sense that only the sub-algebra would act as gauge symmetries.

3. The assignment of this hierarchy with super-symplectic algebra having conformal structure with respect to the light-like radial coordinate of light-cone boundary looks very attractive. An interesting question is what is the role of the super-conformal algebra associated with the isometries of light-cone boundary $R_+ \times S^2$ which are conformal transformations of sphere S^2 with a scaling of radial coordinate compensating the scaling induced by the conformal transformation. Does it act as dynamical or gauge symmetries?

4. The natural proposal is that the inclusions of various superconformal algebras in the hierarchy define inclusions of hyper-finite factors which would be thus labelled by integers. Any sequences of integers for which n_i divides n_{i+1} would define a hierarchy of inclusions proceeding in reverse direction. Physically inclusion hierarchy would correspond to an infinite hierarchy of criticalities within criticalities.

2.3 Can One Identify M-Matrix From Physical Arguments?

Consider next the identification of M-matrix from physical arguments from the point of view of factors.

2.3.1 *A proposal for M-matrix*

The proposed general picture reduces the core of U-matrix to the construction of S-matrix possibly having the real square roots of density matrices as symmetry algebra. This structure can be taken as a template as one tries to to imagine how the construction of M-matrix could proceed in quantum TGD proper.

1. At the bosonic sector one would have converging functional integral over WCW . This is analogous to the path integral over bosonic fields in QFTs. The presence of Kähler function would make this integral well-defined and would not encounter the difficulties met in the case of path integrals.

2. In fermionic sector 1-D Dirac action and its bosonic counterpart imply that spinors modes localized at string world sheets are eigenstates of induced Dirac operator with generalized eigenvalue $p^k\gamma_k$ defining light-like 8-D momentum so that one would obtain fermionic propagators

massless in 8-D sense at light-light geodesics of imbedding space. The 8-D generalization of twistor Grassmann approach is suggestive and would mean that the residue integral over fermionic virtual momenta gives only integral over massless momenta and virtual fermions differ from real fermions only in that they have non-physical polarizations so that massless Dirac operator replacing the propagator does not annihilate the spinors at the other end of the line.

3. Fundamental bosons (not elementary particles) correspond to wormhole contacts having fermion and antifermion at opposite throats and bosonic propagators are composite of massless fermion propagators. The directions of virtual momenta are obviously strongly correlated so that the approximation as a gauge theory with gauge symmetry breaking in almost massless sector is natural. Massivation follows necessary from the fact that also elementary particles are bound states of two wormhole contacts.

4. Physical fermions and bosons correspond to pairs of wormhole contacts with throats carrying Kähler magnetic charge equal to Kähler electric charge (dyon). The absence of Dirac monopoles (as opposed to homological magnetic monopoles due to CP_2 topology) implies that wormhole contacts must appear as pairs (also large numbers of them are possible and 3 valence quarks inside baryons could form Kähler magnetic tripole). Hence elementary particles would correspond to pairs of monopoles and are accompanied by Kähler magnetic flux loop running along the two space-time sheets involved as well as fermionic strings connecting the monopole throats.

   There seems to be no specific need to assign string to the wormhole contact and if is a piece of deformed CP_2 type vacuum extremal this might not be even possible: the Kähler-Dirac gamma matrices would not span 2-D space in this case since the CP_2 projection is 4-D. Hence massless fermion propagators would be assigned only with the boundaries of string world sheets at Minkowskian regions of space-time surface. One could say that physical particles are bound states of massless fundamental fermions and the non-collinearity of their four-momenta can make them massive. Therefore the breaking of conformal invariance would be due to the bound state formation and this would also resolve the infrared divergence problems plaguing Grassmann twistor approach by introducing natural length scale assignable to the size of particles defined by the string like flux tube connecting the wormhole contacts. This point is discussed in more detail in [K19].

   The bound states would form representations of super-conformal algebras so that stringy mass formula would emerge naturally. p-Adic mass calculations indeed assume conformal invariance in $CP2$ length scale assignable to wormhole contacts. Also the long flux tube strings contribute to the particle masses and would explain gauge boson masses.

5. The interaction vertices would correspond topologically to decays of 3-surface by splitting in complete analogy with ordinary Feynman diagrams. At the level of orbits of partonic 2-surface the vertices would be represented by partonic 2-surfaces. In [K19] the interpretation of scattering ampiltudes as sequences of algebraic operations for the Yangian of super-symplectic algebra is proposed: product and co-product would define time 3-vertex and its time reversal. At the level of fermions the diagrams reduce to braid diagrams since fermions are "free". At vertices fermions can however reflect in time direction so that fermion-antifermion annihilations in classical fields can be said to appear in the vertices.

   The Yangian is generated by super-symplectic fermionic Noether charges assignable to the strings connecting partonic 2-surfaces. The interpretation of vertices as algebraic operations implies that all sequences of operations connecting given collections of elements of Yangian at the opposite boundaries of CD give rise to the same amplitude. This means a huge generalization of the duality symmetry of hadronic string models that I have proposed already earlier: the chapter [K8] is a remnant of an "idea that came too early". The propagators are associated with the fermionic lines identifiable as boundaries of string world sheets. These lines are light-like geodesics of H and fermion lines correspond topartial wave in the space S^3 of light like 8-momenta with fixed M^4 momentum. For external lines M^8 momentum corresponds to the $M^4 \times CP_2$ quantum numbers of a spinor harmonic.

The amplitudes can be formulated using only partonic 2-surfaces and string world sheets and the algebraic continuation to achieve number theoretic Universality should be rather straightforward: the parameters characterizing 2-surfaces - by conformal invariance various conformal moduli - in the algebraic extension of rationals are replaced with real and various p-adic numbers.

6. Wormhole contacts represent fundamental interaction vertex pairs and propagators between them and one has stringy super-conformal invariance. Therefore there are excellent reasons to expect that the perturbation theory is free of divergences. Without stringy contributions for massive conformal excitations of wormhole contacts one would obtain the usual logarithmic UV divergences of massless gauge theories. The fact that physical particles are bound states of massless particles, gives good hopes of avoiding IR divergences of massless theories.

The figures ??, ?? (`http://tgdtheory.fi/appfigures/elparticletgd.jpg` `http://tgdtheory.fi/appfigures/tgdgrpahs.jpg`) in the appendix of this book illustrate the relationship between TGD diagrammatics, QFT diagrammatics and stringy diagrammatics. In [K19] a more detailed construction based on the generalization of twistor approach and the idea that scattering amplitudes represent sequences of algebraic operation in the Yangian of super-symplectic algebra, is considered.

2.3.2 *Quantum TGD as square root of thermodynamics*

ZEO (ZEO) suggests strongly that quantum TGD corresponds to what might be called square root of thermodynamics. Since fermionic sector of TGD corresponds naturally to a hyper-finite factor of type II_1, and super-conformal sector relates fermionic and bosonic sectors (WCW degrees of freedom), there is a temptation to suggest that the mathematics of von Neumann algebras generalizes: in other worlds it is possible to speak about the complex square root of ω defining a state of von Neumann algebra [A21] [K21]. This square root would bring in also the fermionic sector and realized super-conformal symmetry. The reduction of determinant with WCW vacuum functional would be one manifestation of this supersymmetry.

The exponent of Kähler function identified as real part of Kähler action for preferred extremals coming from Euclidian space-time regions defines the modulus of the bosonic vacuum functional appearing in the functional integral over WCW. The imaginary part of Kähler action coming from the Minkowskian regions is analogous to action of quantum field theories and would give rise to interference effects distinguishing thermodynamics from quantum theory. This would be something new from the point of view of the canonical theory of von Neumann algebra. The saddle points of the imaginary part appear in stationary phase approximation and the imaginary part serves the role of Morse function for WCW.

The exponent of Kähler function depends on the real part of t identified as Minkowski distance between the tips of CD. This dependence is not consistent with the dependence of the canonical unitary automorphism Δ^{it} of von Neumann algebra on t [A21], [K21] and the natural interpretation is that the vacuum functional can be included in the definition of the inner product for spinors fields of WCW. More formally, the exponent of Kähler function would define ω in bosonic degrees of freedom.

Note that the imaginary exponent is more natural for the imaginary part of Kähler action coming from Minkowskian region. In any case, one has combination of thermodynamics and QFT and the presence of thermodynamics makes the functional integral mathematically well-defined.

Number theoretic vision requiring number theoretical universality suggests that the value of CD size scales as defined by the distance between the tips is expected to come as integer multiples of CP_2 length scale - at least in the intersection of real and p-adic worlds. If this is the case the continuous faimily of modular automorphisms would be replaced with a discretize family.

2.3.3 *Quantum criticality and hierarchy of inclusions*

Quantum criticality and related fractal hierarchies of breakings of conformal symmetry could allow to understand the inclusion hierarchies for hyper-finite factors. Quantum criticality - implied by the condition that the Kähler-Dirac action gives rise to conserved currents assignable to the deformations of the space-time surface - means the vanishing of the second variation of Kähler action for these deformations. Preferred extremals correspond to these 4-surfaces and $M^8 - M^4 \times CP_2$ duality would allow to identify them also as associative (co-associative) space-time surfaces.

Quantum criticality is basically due to the failure of strict determinism for Kähler action and leads to the hierarchy of dark matter phases labelled by the effective value of Planck constant $h_{eff} = n \times h$. These phases correspond to space-time surfaces connecting 3-surfaces at the ends of CD which are multi-sheeted having n conformal equivalence classes.

Conformal invariance indeed relates naturally to quantum criticality. This brings in n discrete degrees of freedom and one can technically describe the situation by using n-fold singular covering of the imbedding space [K13]. One can say that there is hierarchy of broken conformal symmetries in the sense that for $h_{eff} = n \times h$ the sub-algebra of conformal algebras with conformal weights coming as multiples of n act as gauge symmetries. This implies that classical symplectic Noether charges vanish for this sub-algebra. The quantal conformal charges associated with induced spinor fields annihilate the physical states. Therefore it seems that the measured quantities are the symplectic charges and there is not need to introduce any measurement interaction term and the formalism simplifies dramatically.

The resolution increases with $h_{eff}/h = n$. Also the number of of strings connecting partonic 2-surfaces (in practice elementary particles and their dark counterparts plus bound states generated by connecting dark strings) characterizes physically the finite measurement resolution. Their presence is also visible in the geometry of the space-time surfaces through the conditions that induced W fields vanish at them (well-definedness of em charge), and by the condition that the canonical momentum currents for Kähler action define an integrable distribution of planes parallel to the string world sheet. In spirit with holography, preferred extremal is constructed by fixing string world sheets and partonic 2-surfaces and possibly also their light-like orbits (should one fix wormhole contacts is not quite clear). If the analog of AdS/CFT correspondence holds true, the value of Kähler function is expressible as the energy of string defined by area in the effective metric defined by the anti-commutators of K-D gamma matrices.

Super-symplectic algebra, whose charges are represented by Noether charges associated with strings connecting partonic 2-surfaces extends to a Yangian algebra with multi-stringy generators [K19]. The better the measurement resolution, the larger the maximal number of strings associated with the multilocal generator.

Kac-Moody type transformations preserving light-likeness of partonic orbits and possibly also the light-like character of the boundaries of string world sheets carrying modes of induced spinor field underlie the conformal gauge symmetry. The minimal option is that only the light-likeness of the string end world line is preserved by the conformal symmetries. In fact, conformal symmetries was originally deduced from the light-likeness condition for the M^4 projection of CP_2 type vacuum extremals.

The inclusions of super-symplectic Yangians form a hierarchy and would naturally correspond to inclusions of hyperfinite factors of type II_1. Conformal symmetries acting as gauge transformations would naturally correspond to degrees of freedom below measurement resolution and would correspond to included subalgebra. As h_{eff} increases, infinite number of these gauge degrees of freedom become dynamical and measurement resolution is increased. This picture is definitely in conflict with the original view but the reduction of criticality in the increase of h_{eff} forces it.

2.3.4 *Summarizing*

On basis of above considerations it seems that the idea about "complex square root" of the state ω of von Neumann algebras might make sense in quantum TGD. Also the discretized versions of modular automorphism assignable to the hierarchy of CDs would make sense and because of its non-uniqueness the generator Δ of the canonical automorphism could bring in the flexibility needed one wants thermodynamics. Stringy picture forces to ask whether Δ could in some situation be proportional $exp(L_0)$, where L_0 represents as the infinitesimal scaling generator of either super-symplectic algebra or super Kac-Moody algebra (the choice does not matter since the differences of the generators annihilate physical states in coset construction). This would allow to reproduce real thermodynamics consistent with p-adic thermodynamics. Note that also p-adic thermodynamics would be replaced by its square root in ZEO.

2.4 Finite Measurement Resolution And HFFs

The finite resolution of quantum measurement leads in TGD framework naturally to the notion of quantum M-matrix for which elements have values in sub-factor $\mathcal{N}$ of HFF rather than being complex numbers. M-matrix in the factor space $\mathcal{M}/\mathcal{N}$ is obtained by tracing over $\mathcal{N}$. The condition that $\mathcal{N}$ acts like complex numbers in the tracing implies that M-matrix elements are proportional to maximal projectors to $\mathcal{N}$ so that M-matrix is effectively a matrix in $\mathcal{M}/\mathcal{N}$ and situation becomes finite-dimensional. It is still possible to satisfy generalized unitarity conditions but in general case tracing gives a weighted sum of unitary M-matrices defining what can be regarded as a square root of density matrix.

2.4.1 *About the notion of observable in ZEO*

Some clarifications concerning the notion of observable in zero energy ontology are in order.

1. As in standard quantum theory observables correspond to hermitian operators acting on either positive or negative energy part of the state. One can indeed define hermitian conjugation for positive and negative energy parts of the states in standard manner.

2. Also the conjugation $A \rightarrow JAJ$ is analogous to hermitian conjugation. It exchanges the positive and negative energy parts of the states also maps the light-like 3-surfaces at the upper boundary of CD to the lower boundary and vice versa. The map is induced by time reflection in the rest frame of CD with respect to the origin at the center of CD and has a well defined action on light-like 3-surfaces and space-time surfaces. This operation cannot correspond to the sought for hermitian conjugation since JAJ and A commute.

3. In order to obtain non-trivial fermion propagator one must add to Dirac action 1-D Dirac action in induced metric with the boundaries of string world sheets at the light-like parton orbits. Its bosonic counterpart is line-length in induced metric. Field equations imply that the boundaries are light-like geodesics and fermion has light-like 8-momentum. This suggests strongly a connection with quantum field theory and an 8-D generalization of twistor Grassmannian approach. By field equations the bosonic part of this action does not contribute to the Kähler action. Chern-Simons Dirac terms to which Kähler action reduces could be responsible for the breaking of CP and T symmetries as they appear in CKM matrix.

4. ZEO gives Cartan sub-algebra of the Lie algebra of symmetries a special status. Only Cartan algebra acting on either positive or negative states respects the zero energy property but this is enough to define quantum numbers of the state. In absence of symmetry breaking positive and negative energy parts of the state combine to form a state in a singlet representation of group. Since only the net quantum numbers must vanish ZEO allows a symmetry breaking respecting a chosen Cartan algebra.

5. In order to speak about four-momenta for positive and negative energy parts of the states one must be able to define how the translations act on CDs. The most natural action is a shift of the upper (lower) tip of CD. In the scale of entire CD this transformation induced Lorentz boost fixing the other tip. The value of mass squared is identified as proportional to the average of conformal weight in p-adic thermodynamics for the scaling generator L_0 for either super-symplectic or Super Kac-Moody algebra.

2.4.2 *Inclusion of HFFs as characterizer of finite measurement resolution at the level of S-matrix*

The inclusion $\mathcal{N} \subset \mathcal{M}$ of factors characterizes naturally finite measurement resolution. This means following things.

1. Complex rays of state space resulting usually in an ideal state function reduction are replaced by $\mathcal{N}$-rays since $\mathcal{N}$ defines the measurement resolution and takes the role of complex numbers in ordinary quantum theory so that non-commutative quantum theory results. Non-commutativity corresponds to a finite measurement resolution rather than something exotic occurring in Planck length scales. The quantum Clifford algebra $\mathcal{M}/\mathcal{N}$ creates physical states

modulo resolution. The fact that $\mathcal{N}$ takes the role of gauge algebra suggests that it might be necessary to fix a gauge by assigning to each element of $\mathcal{M}/\mathcal{N}$ a unique element of $\mathcal{M}$. Quantum Clifford algebra with fractal dimension $\beta = \mathcal{M} : \mathcal{N}$ creates physical states having interpretation as quantum spinors of fractal dimension $d = \sqrt{\beta}$. Hence direct connection with quantum groups emerges.

2. The notions of unitarity, hermiticity, and eigenvalue generalize. The elements of unitary and hermitian matrices and $\mathcal{N}$-valued. Eigenvalues are Hermitian elements of $\mathcal{N}$ and thus correspond entire spectra of Hermitian operators. The mutual non-commutativity of eigenvalues guarantees that it is possible to speak about state function reduction for quantum spinors. In the simplest case of a 2-component quantum spinor this means that second component of quantum spinor vanishes in the sense that second component of spinor annihilates physical state and second acts as element of $\mathcal{N}$ on it. The non-commutativity of spinor components implies correlations between then and thus fractal dimension is smaller than 2.

3. The intuition about ordinary tensor products suggests that one can decompose Tr in $\mathcal{M}$ as

$$Tr_{\mathcal{M}}(X) = Tr_{\mathcal{M}/\mathcal{N}} \times Tr_{\mathcal{N}}(X) \ . \tag{2.4}$$

Suppose one has fixed gauge by selecting basis $|r_k\rangle$ for $\mathcal{M}/\mathcal{N}$. In this case one expects that operator in $\mathcal{M}$ defines an operator in $\mathcal{M}/\mathcal{N}$ by a projection to the preferred elements of $\mathcal{M}$.

$$\langle r_1|X|r_2\rangle = \langle r_1|Tr_{\mathcal{N}}(X)|r_2\rangle \ . \tag{2.5}$$

4. Scattering probabilities in the resolution defined by $\mathcal{N}$ are obtained in the following manner. The scattering probability between states $|r_1\rangle$ and $|r_2\rangle$ is obtained by summing over the final states obtained by the action of $\mathcal{N}$ from $|r_2\rangle$ and taking the analog of spin average over the states created in the similar from $|r_1\rangle$. $\mathcal{N}$ average requires a division by $Tr(P_{\mathcal{N}}) = 1/\mathcal{M} : \mathcal{N}$ defining fractal dimension of $\mathcal{N}$. This gives

$$p(r_1 \to r_2) = \mathcal{M} : \mathcal{N} \times \langle r_1|Tr_{\mathcal{N}}(SP_{\mathcal{N}}S^\dagger)|r_2\rangle \ . \tag{2.6}$$

This formula is consistent with probability conservation since one has

$$\sum_{r_2} p(r_1 \to r_2) = \mathcal{M} : \mathcal{N} \times Tr_N(SS^\dagger) = \mathcal{M} : \mathcal{N} \times Tr(P_N) = 1 \ . \tag{2.7}$$

5. Unitarity at the level of $\mathcal{M}/\mathcal{N}$ can be achieved if the unit operator Id for $\mathcal{M}$ can be decomposed into an analog of tensor product for the unit operators of $\mathcal{M}/\mathcal{N}$ and $\mathcal{N}$ and M decomposes to a tensor product of unitary M-matrices in $\mathcal{M}/\mathcal{N}$ and $\mathcal{N}$. For HFFs of type II projection operators of $\mathcal{N}$ with varying traces are present and one expects a weighted sum of unitary M-matrices to result from the tracing having interpretation in terms of square root of thermodynamics.

6. This argument assumes that $\mathcal{N}$ is HFF of type II_1 with finite trace. For HFFs of type III_1 this assumption must be given up. This might be possible if one compensates the trace over $\mathcal{N}$ by dividing with the trace of the infinite trace of the projection operator to $\mathcal{N}$. This probably requires a limiting procedure which indeed makes sense for HFFs.

2.4.3 *Quantum M-matrix*

The description of finite measurement resolution in terms of inclusion $\mathcal{N} \subset \mathcal{M}$ seems to boil down to a simple rule. Replace ordinary quantum mechanics in complex number field C with that in $\mathcal{N}$. This means that the notions of unitarity, hermiticity, Hilbert space ray, etc.. are replaced with their $\mathcal{N}$ counterparts.

The full M-matrix in $\mathcal{M}$ should be reducible to a finite-dimensional quantum M-matrix in the state space generated by quantum Clifford algebra $\mathcal{M}/\mathcal{N}$ which can be regarded as a finite-dimensional matrix algebra with non-commuting $\mathcal{N}$-valued matrix elements. This suggests that full M-matrix can be expressed as M-matrix with $\mathcal{N}$-valued elements satisfying $\mathcal{N}$-unitarity conditions.

Physical intuition also suggests that the transition probabilities defined by quantum S-matrix must be commuting hermitian $\mathcal{N}$-valued operators inside every row and column. The traces of these operators give $\mathcal{N}$-averaged transition probabilities. The eigenvalue spectrum of these Hermitian matrices gives more detailed information about details below experimental resolution. $\mathcal{N}$-hermicity and commutativity pose powerful additional restrictions on the M-matrix.

Quantum M-matrix defines $\mathcal{N}$-valued entanglement coefficients between quantum states with $\mathcal{N}$-valued coefficients. How this affects the situation? The non-commutativity of quantum spinors has a natural interpretation in terms of fuzzy state function reduction meaning that quantum spinor corresponds effectively to a statistical ensemble which cannot correspond to pure state. Does this mean that predictions for transition probabilities must be averaged over the ensemble defined by "quantum quantum states"?

2.4.4 *Quantum fluctuations and inclusions*

Inclusions $\mathcal{N} \subset \mathcal{M}$ of factors provide also a first principle description of quantum fluctuations since quantum fluctuations are by definition quantum dynamics below the measurement resolution. This gives hopes for articulating precisely what the important phrase "long range quantum fluctuations around quantum criticality" really means mathematically.

1. Phase transitions involve a change of symmetry. One might hope that the change of the symmetry group $G_a \times G_b$ could universally code this aspect of phase transitions. This need not always mean a change of Planck constant but it means always a leakage between sectors of imbedding space. At quantum criticality 3-surfaces would have regions belonging to at least two sectors of H.

2. The long range of quantum fluctuations would naturally relate to a partial or total leakage of the 3-surface to a sector of imbedding space with larger Planck constant meaning zooming up of various quantal lengths.

3. For M-matrix in $\mathcal{M}/\mathcal{N}$ regarded as $calN$ module quantum criticality would mean a special kind of eigen state for the transition probability operator defined by the M-matrix. The properties of the number theoretic braids contributing to the M-matrix should characterize this state. The strands of the critical braids would correspond to fixed points for $G_a \times G_b$ or its subgroup.

2.4.5 *M-matrix in finite measurement resolution*

The following arguments relying on the proposed identification of the space of zero energy states give a precise formulation for M-matrix in finite measurement resolution and the Connes tensor product involved. The original expectation that Connes tensor product could lead to a unique M-matrix is wrong. The replacement of ω with its complex square root could lead to a unique hierarchy of M-matrices with finite measurement resolution and allow completely finite theory despite the fact that projectors have infinite trace for HFFs of type III_1.

1. In ZEO the counterpart of Hermitian conjugation for operator is replaced with $\mathcal{M} \to J\mathcal{M}J$ permuting the factors. Therefore $N \in \mathcal{N}$ acting to positive (negative) energy part of state corresponds to $N \to N' = JNJ$ acting on negative (positive) energy part of the state.

2. The allowed elements of N much be such that zero energy state remains zero energy state. The superposition of zero energy states involved can however change. Hence one must have that the counterparts of complex numbers are of form $N = JN_1J \vee N_2$, where N_1 and N_2 have same quantum numbers. A superposition of terms of this kind with varying quantum numbers for positive energy part of the state is possible.

3. The condition that N_{1i} and N_{2i} act like complex numbers in $\mathcal{N}$-trace means that the effect of $JN_{1i}J \vee N_{2i}$ and $JN_{2i}Ji \vee N_{1i}$ to the trace are identical and correspond to a multiplication by a constant. If $\mathcal{N}$ is HFF of type II$_1$ this follows from the decomposition $\mathcal{M} = \mathcal{M}/\mathcal{N} \otimes \mathcal{N}$ and from $Tr(AB) = Tr(BA)$ assuming that M is of form $M = M_{\mathcal{M}/\mathcal{N}} \times P_{\mathcal{N}}$. Contrary to the original hopes that Connes tensor product could fix the M-matrix there are no conditions on $M_{\mathcal{M}/\mathcal{N}}$ which would give rise to a finite-dimensional M-matrix for Jones inclusions. One can replaced the projector $P_{\mathcal{N}}$ with a more general state if one takes this into account in * operation.

4. In the case of HFFs of type III_1 the trace is infinite so that the replacement of Tr_N with a state ω_N in the sense of factors looks more natural. This means that the counterpart of * operation exchanging N_1 and N_2 represented as $SA\Omega = A^*\Omega$ involves Δ via $S = J\Delta^{1/2}$. The exchange of N_1 and N_2 gives altogether Δ. In this case the KMS condition $\omega_{\mathcal{N}}(AB) = \omega_{\mathcal{N}}\Delta A)$ guarantees the effective complex number property [A4] .

5. Quantum TGD more or less requires the replacement of ω with its "complex square root" so that also a unitary matrix U multiplying Δ is expected to appear in the formula for S and guarantee the symmetry. One could speak of a square root of KMS condition [A4] in this case. The QFT counterpart would be a cutoff involving path integral over the degrees of freedom below the measurement resolution. In TGD framework it would mean a cutoff in the functional integral over WCW and for the modes of the second quantized induced spinor fields and also cutoff in sizes of causal diamonds. Discretization in terms of braids replacing light-like 3-surfaces should be the counterpart for the cutoff.

6. If one has M-matrix in $\mathcal{M}$ expressible as a sum of M-matrices of form $M_{\mathcal{M}/\mathcal{N}} \times M_{\mathcal{N}}$ with coefficients which correspond to the square roots of probabilities defining density matrix the tracing operation gives rise to square root of density matrix in M.

2.4.6 *Is universal M-matrix possible?*

The realization of the finite measurement resolution could apply only to transition probabilities in which $\mathcal{N}$-trace or its generalization in terms of state ω_N is needed. One might however dream of something more.

1. Maybe there exists a universal M-matrix in the sense that the same M-matrix gives the M-matrices in finite measurement resolution for all inclusions $\mathcal{N} \subset \mathcal{M}$. This would mean that one can write

$$M = M_{\mathcal{M}/\mathcal{N}} \otimes M_{\mathcal{N}} \tag{2.8}$$

   for any physically reasonable choice of $\mathcal{N}$. This would formally express the idea that M is as near as possible to M-matrix of free theory. Also fractality suggests itself in the sense that $M_{\mathcal{N}}$ is essentially the same as $M_{\mathcal{M}}$ in the same sense as $\mathcal{N}$ is same as $\mathcal{M}$. It might be that the trivial solution $M = 1$ is the only possible solution to the condition.

2. $M_{\mathcal{M}/\mathcal{N}}$ would be obtained by the analog of $Tr_{\mathcal{N}}$ or ω_N operation involving the "complex square root" of the state ω in case of HFFs of type III$_1$. The QFT counterpart would be path integration over the degrees of freedom below cutoff to get effective action.

3. Universality probably requires assumptions about the thermodynamical part of the universal M-matrix. A possible alternative form of the condition is that it holds true only for canonical choice of "complex square root" of ω or for the S-matrix part of M:

$$S = S_{\mathcal{M}/\mathcal{N}} \otimes S_{\mathcal{N}} \quad (2.9)$$

for any physically reasonable choice $\mathcal{N}$.

4. In TGD framework the condition would say that the M-matrix defined by the Kähler-Dirac action gives M-matrices in finite measurement resolution via the counterpart of integration over the degrees of freedom below the measurement resolution.

An obvious counter argument against the universality is that if the M-matrix is "complex square root of state" cannot be unique and there are infinitely many choices related by a unitary transformation induced by the flows representing modular automorphism giving rise to new choices. This would actually be a well-come result and make possible quantum measurement theory.

In the section "Handful of problems with a common resolution" it was found that one can add to both Kähler action and Kähler-Dirac action a measurement interaction term characterizing the values of measured observables. The measurement interaction term in Kähler action is Lagrange multiplier term at the space-like ends of space-time surface fixing the value of classical charges for the space-time sheets in the quantum superposition to be equal with corresponding quantum charges. The term in Kähler-Dirac action is obtained from this by assigning to this term canonical momentum densities and contracting them with gamma matrices to obtain Kähler-Dirac gamma matrices appearing in 3-D analog of Dirac action. The constraint terms would leave Kähler function and Kähler metric invariant but would restrict the vacuum functional to the subset of 3-surfaces with fixed classical conserved charges (in Cartan algebra) equal to their quantum counterparts.

2.4.7 *Connes tensor product and space-like entanglement*

Ordinary linear Connes tensor product makes sense also in positive/negative energy sector and also now it makes sense to speak about measurement resolution. Hence one can ask whether Connes tensor product should be posed as a constraint on space-like entanglement. The interpretation could be in terms of the formation of bound states. The reducibility of HFFs and inclusions means that the tensor product is not uniquely fixed and ordinary entanglement could correspond to this kind of entanglement.

Also the counterpart of p-adic coupling constant evolution would makes sense. The interpretation of Connes tensor product would be as the variance of the states with respect to some subgroup of $U(n)$ associated with the measurement resolution: the analog of color confinement would be in question.

2.4.8 *2-vector spaces and entanglement modulo measurement resolution*

John Baez and collaborators [A18] are playing with very formal looking formal structures obtained by replacing vectors with vector spaces. Direct sum and tensor product serve as the basic arithmetic operations for the vector spaces and one can define category of n-tuples of vectors spaces with morphisms defined by linear maps between vectors spaces of the tuple. n-tuples allow also element-wise product and sum. They obtain results which make them happy. For instance, the category of linear representations of a given group forms 2-vector spaces since direct sums and tensor products of representations as well as n-tuples make sense. The 2-vector space however looks more or less trivial from the point of physics.

The situation could become more interesting in quantum measurement theory with finite measurement resolution described in terms of inclusions of hyper-finite factors of type II_1. The reason is that Connes tensor product replaces ordinary tensor product and brings in interactions via irreducible entanglement as a representation of finite measurement resolution. The category in question could give Connes tensor products of quantum state spaces and describing interactions. For instance, one could multiply M-matrices via Connes tensor product to obtain category of M-matrices having also the structure of 2-operator algebra.

1. The included algebra represents measurement resolution and this means that the infinite-D sub-Hilbert spaces obtained by the action of this algebra replace the rays. Sub-factor

takes the role of complex numbers in generalized QM so that one obtains non-commutative quantum mechanics. For instance, quantum entanglement for two systems of this kind would not be between rays but between infinite-D subspaces corresponding to sub-factors. One could build a generalization of QM by replacing rays with sub-spaces and it would seem that quantum group concept does more or less this: the states in representations of quantum groups could be seen as infinite-dimensional Hilbert spaces.

2. One could speak about both operator algebras and corresponding state spaces modulo finite measurement resolution as quantum operator algebras and quantum state spaces with fractal dimension defined as factor space like entities obtained from HFF by dividing with the action of included HFF. Possible values of the fractal dimension are fixed completely for Jones inclusions. Maybe these quantum state spaces could define the notions of quantum 2-Hilbert space and 2-operator algebra via direct sum and tensor production operations. Fractal dimensions would make the situation interesting both mathematically and physically.

Suppose one takes the fractal factor spaces as the basic structures and keeps the information about inclusion.

1. Direct sums for quantum vectors spaces would be just ordinary direct sums with HFF containing included algebras replaced with direct sum of included HFFs.

2. The tensor products for quantum state spaces and quantum operator algebras are not anymore trivial. The condition that measurement algebras act effectively like complex numbers would require Connes tensor product involving irreducible entanglement between elements belonging to the two HFFs. This would have direct physical relevance since this entanglement cannot be reduced in state function reduction. The category would defined interactions in terms of Connes tensor product and finite measurement resolution.

3. The sequences of super-conformal symmetry breakings identifiable in terms of inclusions of super-conformal algebras and corresponding HFFs could have a natural description using the 2-Hilbert spaces and quantum 2-operator algebras.

2.5 Questions About Quantum Measurement Theory In Zero Energy Ontology

The following summary about quantum measurement theory in ZEO is somewhat out-of-date and somewhat sketchy. For more detailed view see [K4, K5, K1].

2.5.1 *Fractal hierarchy of state function reductions*

In accordance with fractality, the conditions for the Connes tensor product at a given time scale imply the conditions at shorter time scales. On the other hand, in shorter time scales the inclusion would be deeper and would give rise to a larger reducibility of the representation of $\mathcal{N}$ in $\mathcal{M}$. Formally, as $\mathcal{N}$ approaches to a trivial algebra, one would have a square root of density matrix and trivial S-matrix in accordance with the idea about asymptotic freedom.

M-matrix would give rise to a matrix of probabilities via the expression $P(P_+ \rightarrow P_-) = Tr[P_+ M^\dagger P_- M]$, where P_+ and P_- are projectors to positive and negative energy energy $\mathcal{N}$-rays. The projectors give rise to the averaging over the initial and final states inside $\mathcal{N}$ ray. The reduction could continue step by step to shorter length scales so that one would obtain a sequence of inclusions. If the U-process of the next quantum jump can return the M-matrix associated with $\mathcal{M}$ or some larger HFF, U process would be kind of reversal for state function reduction.

Analytic thinking proceeding from vision to details; human life cycle proceeding from dreams and wild actions to the age when most decisions relate to the routine daily activities; the progress of science from macroscopic to microscopic scales; even biological decay processes: all these have an intriguing resemblance to the fractal state function reduction process proceeding to shorter and shorter time scales. Since this means increasing thermality of M-matrix, U process as a reversal of state function reduction might break the second law of thermodynamics.

The conservative option would be that only the transformation of intentions to action by U process giving rise to new zero energy states can bring in something new and is responsible for evolution. The non-conservative option is that the biological death is the U-process of the next quantum jump leading to a new life cycle. Breathing would become a universal metaphor for what happens in quantum Universe. The 4-D body would be lived again and again.

2.5.2 *How quantum classical correspondence is realized at parton level?*

Quantum classical correspondence must assign to a given quantum state the most probable space-time sheet depending on its quantum numbers. The space-time sheet $X^4(X^3)$ defined by the Kähler function depends however only on the partonic 3-surface X^3, and one must be able to assign to a given quantum state the most probable X^3 - call it X^3_{max} - depending on its quantum numbers.

$X^4(X^3_{max})$ should carry the gauge fields created by classical gauge charges associated with the Cartan algebra of the gauge group (color isospin and hypercharge and electromagnetic and Z^0 charge) as well as classical gravitational fields created by the partons. This picture is very similar to that of quantum field theories relying on path integral except that the path integral is restricted to 3-surfaces X^3 with exponent of Kähler function bringing in genuine convergence and that 4-D dynamics is deterministic apart from the delicacies due to the 4-D spin glass type vacuum degeneracy of Kähler action.

Stationary phase approximation selects X^3_{max} if the quantum state contains a phase factor depending not only on X^3 but also on the quantum numbers of the state. A good guess is that the needed phase factor corresponds to either Chern-Simons type action or an action describing the interaction of the induced gauge field with the charges associated with the braid strand. This action would be defined for the induced gauge fields. YM action seems to be excluded since it is singular for light-like 3-surfaces associated with the light-like wormhole throats (not only $\sqrt{det(g_3)}$ but also $\sqrt{det(g_4)}$ vanishes).

The challenge is to show that this is enough to guarantee that $X^4(X^3_{max})$ carries correct gauge charges. Kind of electric-magnetic duality should relate the normal components F_{ni} of the gauge fields in $X^4(X^3_{max})$ to the gauge fields F_{ij} induced at X^3. An alternative interpretation is in terms of quantum gravitational holography.

One is forced to introduce gauge couplings and also electro-weak symmetry breaking via the phase factor. This is in apparent conflict with the idea that all couplings are predictable. The essential uniqueness of M-matrix in the case of HFFs of type II_1 (at least) however means that their values as a function of measurement resolution time scale are fixed by internal consistency. Also quantum criticality leads to the same conclusion. Obviously a kind of bootstrap approach suggests itself.

2.5.3 *Quantum measurements in ZEO*

ZEO based quantum measurement theory leads directly to a theory of conscious entities. The basic idea is that state function reduction localizes the second boundary of CD so that it becomes a piece of light-cone boundary (more precisely $\delta M^4_\pm \times CP_2$).

Repeated reductions are possible as in standard quantum measurement theory and leave the passive boundary of CD. Repeated reduction begins with U process generating a superposition of CDs with the active boundary of CD being de-localized in the moduli space of CDs, and is followed by a localization in this moduli space so that single CD is the outcome. This process tends to increase the distance between the ends of the CD and has interpretation as a space-time correlate for the flow of subjective time.

Self as a conscious entity corresponds to this sequence of repeated reductions on passive boundary of CD. The first reduction at opposite boundary means death of self and its re-incarnation at the opposite boundary of CD. Also the increase of Planck constant and generation of negentropic entanglement is expected to be associated with this state function reduction.

Weak form of NMP is the most plausible variational principle to characterize the state function reduction. It does not require maximal negentropy gain for state function reductions but allows it. In other words, the outcome of reduction is n-dimensional eigen space of density matrix space but this space need not have maximum possible dimension and even 1-D ray is possible in which case the entanglement negentropy vanishes for the final state and system becomes isolated from the rest of the world. Weak form of NMP brings in free will and can allow also larger negentropy gain than the strong form if n is a product of primes. The beauty of this option is that one can understand how the generalization of p-adic length scale hypothesis emerges.

2.5.4 *Hyper-finite factors of type II_1 and quantum measurement theory with a finite measurement resolution*

The realization that the von Neumann algebra known as hyper-finite factor of type II_1 is tailor made for quantum TGD has led to a considerable progress in the understanding of the mathematical structure of the theory and these algebras provide a justification for several ideas introduced earlier on basis of physical intuition.

Hyper-finite factor of type II_1 has a canonical realization as an infinite-dimensional Clifford algebra and the obvious guess is that it corresponds to the algebra spanned by the gamma matrices of WCW. Also the local Clifford algebra of the imbedding space $H = M^4 \times CP_2$ in octonionic representation of gamma matrices of H is important and the entire quantum TGD emerges from the associativity or co-associativity conditions for the sub-algebras of this algebra which are local algebras localized to maximal associative or co-associate sub-manifolds of the imbedding space identifiable as space-time surfaces.

The notion of inclusion for hyper-finite factors provides an elegant description for the notion of measurement resolution absent from the standard quantum measurement theory.

1. The included sub-factor creates in ZEO states not distinguishable from the original one and the formally the coset space of factors defining quantum spinor space defines the space of physical states modulo finite measurement resolution.

2. The quantum measurement theory for hyperfinite factors differs from that for factors of type I since it is not possible to localize the state into single ray of state space. Rather, the ray is replaced with the sub-space obtained by the action of the included algebra defining the measurement resolution. The role of complex numbers in standard quantum measurement theory is taken by the non-commutative included algebra so that a non-commutative quantum theory is the outcome.

3. This leads also to the notion of quantum group. For instance, the finite measurement resolution means that the components of spinor do not commute anymore and it is not possible to reduce the state to a precise eigenstate of spin. It is however perform a reduction to an eigenstate of an observable which corresponds to the probability for either spin state.

4. For HFFs the dimension of infinite-dimensional state space is finite and 1 by convention. For included HFF $\mathcal{N} \subset \mathcal{M}$ the dimension of the tensor factor space containing only the degrees of freedom which are above measurement resolution is given by the index of inclusion $d = \mathcal{M} : \mathcal{N}$. One can say that the dimension associated with degrees of freedom below measurement resolution is $D = 1/d$. This number is never large than 1 for the inclusions and contains a set of discrete values $d = 4cos^2(2\pi/n)$, $n \geq 3$, plus the continuum above it. The fractal generalization of the formula for entanglement entropy gives $S = -log(1/D) = -log(d) \leq 0$ so that one can say that the entanglement negentropy assignable to the projection operators to the sub-factor is positive except for $n = 3$ for which it vanishes. The non-measured degrees of freedom carry information rather than entropy.

5. Clearly both HFFs of type I and II allow entanglement negentropy and allow to assign it with finite measurement resolution. In the case of factors its is not clear whether the weak form of NMP allows makes sense.

As already explained, the topology of the many-sheeted space-time encourages the generalization of the notion of quantum entanglement in such a manner that unentangled systems can possess entangled sub-systems. One can say that the entanglement between sub-selves is not visible in the resolution characterizing selves. This makes possible sharing and fusion of mental images central for TGD inspired theory of consciousness. These concepts find a deeper justification from the quantum measurement theory for hyper-finite factors of type II_1 for which the finite measurement resolution is basic notion.

2.5.5 *Hierarchies of conformal symmetry breakings, Planck constants, and inclusions of HFFs*

The basic almost prediction of TGD is a fractal hierarchy of breakings of symplectic symmetry as a gauge symmetry.

It is good to briefly summarize the basic facts about the symplectic algebra assigned with $\delta M^4_\pm \times CP_2$ first.

1. Symplectic algebra has the structure of Virasoro algebra with respect to the light-like radial coordinate r_M of the light-cone boundary taking the role of complex coordinate for ordinary conformal symmetry. The Hamiltonians generating symplectic symmetries can be chosen to be proportional to functions $f_n(r_M)$. What is the natural choice for $f_n(r_M)$ is not quite clear. Ordinary conformal invariance would suggests $f_n(r_M) = r_M^n$. A more adventurous possibility is that the algebra is generated by Hamiltonians with $f_n(r_M) = r^{-s}$, where s is a root of Riemann Zeta so that one has either $s = 1/2 + iy$ (roots at critical line) or $s = -2n$, $n > 0$ (roots at negative real axis).

2. The set of conformal weights would be linear space spanned by combinations of all roots with integer coefficients $s = n - iy$, $s = \sum n_i y_i$, $n > -n_0$, where $-n_0 \geq 0$ is negative conformal weight. Mass squared is proportional to the total conformal weight and must be real demanding $y = \sum y_i = 0$ for physical states: I call this conformal confinement analogous to color confinement. One could even consider introducing the analog of binding energy as "binding conformal weight".

   Mass squared must be also non-negative (no tachyons) giving $n_0 \geq 0$. The generating conformal weights however have negative real part -1/2 and are thus tachyonic. Rather remarkably, p-adic mass calculations force to assume negative half-integer valued ground state conformal weight. This plus the fact that the zeros of Riemann Zeta has been indeed assigned with critical systems forces to take the Riemannian variant of conformal weight spectrum with seriousness. The algebra allows also now infinite hierarchy of conformal sub-algebras with weights coming as n-ples of the conformal weights of the entire algebra.

3. The outcome would be an infinite number of hierarchies of symplectic conformal symmetry breakings. Only the generators of the sub-algebra of the symplectic algebra with radial conformal weight proportional to n would act as gauge symmetries at given level of the hierarchy. In the hierarchy n_i divides n_{i+1}. In the symmetry breaking $n_i \to n_{i+1}$ the conformal charges, which vanished earlier, would become non-vanishing. Gauge degrees of freedom would transform to physical degrees of freedom.

4. What about the conformal Kac-Moody algebras associated with spinor modes. It seems that in this case one can assume that the conformal gauge symmetry is exact just as in string models.

The natural interpretation of the conformal hierarchies $n_i \to n_{i+1}$ would be in terms of increasing measurement resolution.

1. Conformal degrees of freedom below measurement resolution would be gauge degrees of freedom and correspond to generators with conformal weight proportional to n_i. Conformal hierarchies and associated hierarchies of Planck constants and n-fold coverings of space-time surface connecting the 3-surfaces at the ends of causal diamond would give a concrete realization of the inclusion hierarchies for hyper-finite factors of type II_1 [K21].

   n_i could correspond to the integer labelling Jones inclusions and associating with them the quantum group phase factor $U_n = exp(i2\pi/n)$, $n \geq 3$ and the index of inclusion given by $|M : N| = 4cos^2(2\pi/n)$ defining the fractal dimension assignable to the degrees of freedom above the measurement resolution. The sub-algebra with weights coming as n-multiples of the basic conformal weights would act as gauge symmetries realizing the idea that these degrees of freedom are below measurement resolution.

2. If $h_{eff} = n \times h$ defines the conformal gauge sub-algebra, the improvement of the resolution would scale up the Compton scales and would quite concretely correspond to a zoom analogous to that done for Mandelbrot fractal to get new details visible. From the point of view

of cognition the improving resolution would fit nicely with the recent view about h_{eff}/h as a kind of intelligence quotient.

This interpretation might make sense for the symplectic algebra of $\delta M^4_\pm \times CP_2$ for which the light-like radial coordinate r_M of light-cone boundary takes the role of complex coordinate. The reason is that symplectic algebra acts as isometries.

3. If Kähler action has vanishing total variation under deformations defined by the broken conformal symmetries, the corresponding conformal charges are conserved. The components of WCW Kähler metric expressible in terms of second derivatives of Kähler function can be however non-vanishing and have also components, which correspond to WCW coordinates associated with different partonic 2-surfaces. This conforms with the idea that conformal algebras extend to Yangian algebras generalizing the Yangian symmetry of $\mathcal{N} = 4$ symmetric gauge theories. The deformations defined by symplectic transformations acting gauge symmetries the second variation vanishes and there is not contribution to WCW Kähler metric.

4. One can interpret the situation also in terms of consciousness theory. The larger the value of h_{eff}, the lower the criticality, the more sensitive the measurement instrument since new degrees of freedom become physical, the better the resolution. In p-adic context large n means better resolution in angle degrees of freedom by introducing the phase $exp(i2\pi/n)$ to the algebraic extension and better cognitive resolution. Also the emergence of negentropic entanglement characterized by $n \times n$ unitary matrix with density matrix proportional to unit matrix means higher level conceptualization with more abstract concepts.

The extension of the super-conformal algebra to a larger Yangian algebra is highly suggestive and gives and additional aspect to the notion of measurement resolution.

1. Yangian would be generated from the algebra of super-conformal charges assigned with the points pairs belonging to two partonic 2-surfaces as stringy Noether charges assignable to strings connecting them. For super-conformal algebra associated with pair of partonic surface only single string associated with the partonic 2-surface. This measurement resolution is the almost the poorest possible (no strings at all would be no measurement resolution at all!).

2. Situation improves if one has a collection of strings connecting set of points of partonic 2-surface to other partonic 2-surface(s). This requires generalization of the super-conformal algebra in order to get the appropriate mathematics. Tensor powers of single string super-conformal charges spaces are obviously involved and the extended super-conformal generators must be multi-local and carry multi-stringy information about physics.

3. The generalization at the first step is simple and based on the idea that co-product is the "time inverse" of product assigning to single generator sum of tensor products of generators giving via commutator rise to the generator. The outcome would be expressible using the structure constants of the super-conformal algebra schematically a $Q^1_A = f_A^{BC} Q_B \otimes Q_C$. Here Q_B and Q_C are super-conformal charges associated with separate strings so that 2-local generators are obtained. One can iterate this construction and get a hierarchy of n-local generators involving products of n stringy super-conformal charges. The larger the value of n, the better the resolution, the more information is coded to the fermionic state about the partonic 2-surface and 3-surface. This affects the space-time surface and hence WCW metric but not the 3-surface so that the interpretation in terms of improved measurement resolution makes sense. This super-symplectic Yangian would be behind the quantum groups and Jones inclusions in TGD Universe.

4. n gives also the number of space-time sheets in the singular covering. One possible interpretation is in terms measurement resolution for counting the number of space-time sheets. Our recent quantum physics would only see single space-time sheet representing visible manner and dark matter would become visible only for $n > 1$.

It is not an accident that quantum phases are assignable to Yangian algebras, to quantum groups, and to inclusions of HFFs. The new deep notion added to this existing complex of high level mathematical concepts are hierarchy of Planck constants, dark matter hierarchy, hierarchy of criticalities, and negentropic entanglement representing physical notions. All these aspects represent new physics.

2.6 Planar Algebras And Generalized Feynman Diagrams

Planar algebras [A6] are a very general notion due to Vaughan Jones and a special class of them is known to characterize inclusion sequences of hyper-finite factors of type II_1 [A15] . In the following an argument is developed that planar algebras might have interpretation in terms of planar projections of generalized Feynman diagrams (these structures are metrically 2-D by presence of one light-like direction so that 2-D representation is especially natural). In [K9] the role of planar algebras and their generalizations is also discussed.

2.6.1 *Planar algebra very briefly*

First a brief definition of planar algebra.

1. One starts from planar k-tangles obtained by putting disks inside a big disk. Inner disks are empty. Big disk contains $2k$ braid strands starting from its boundary and returning back or ending to the boundaries of small empty disks in the interior containing also even number of incoming lines. It is possible to have also loops. Disk boundaries and braid strands connecting them are different objects. A black-white coloring of the disjoint regions of k-tangle is assumed and there are two possible options (photo and its negative). Equivalence of planar tangles under diffeomorphisms is assumed.

2. One can define a product of k-tangles by identifying k-tangle along its outer boundary with some inner disk of another k-tangle. Obviously the product is not unique when the number of inner disks is larger than one. In the product one deletes the inner disk boundary but if one interprets this disk as a vertex-parton, it would be better to keep the boundary.

3. One assigns to the planar k-tangle a vector space V_k and a linear map from the tensor product of spaces V_{k_i} associated with the inner disks such that this map is consistent with the decomposition k-tangles. Under certain additional conditions the resulting algebra gives rise to an algebra characterizing multi-step inclusion of HFFs of type II_1.

4. It is possible to bring in additional structure and in TGD framework it seems necessary to assign to each line of tangle an arrow telling whether it corresponds to a strand of a braid associated with positive or negative energy parton. One can also wonder whether disks could be replaced with closed 2-D surfaces characterized by genus if braids are defined on partonic surfaces of genus g. In this case there is no topological distinction between big disk and small disks. One can also ask why not allow the strands to get linked (as suggested by the interpretation as planar projections of generalized Feynman diagrams) in which case one would not have a planar tangle anymore.

2.6.2 *General arguments favoring the assignment of a planar algebra to a generalized Feynman diagram*

There are some general arguments in favor of the assignment of planar algebra to generalized Feynman diagrams.

1. Planar diagrams describe sequences of inclusions of HFF:s and assign to them a multi-parameter algebra corresponding indices of inclusions. They describe also Connes tensor powers in the simplest situation corresponding to Jones inclusion sequence. Suppose that also general Connes tensor product has a description in terms of planar diagrams. This might be trivial.

2. Generalized vertices identified geometrically as partonic 2-surfaces indeed contain Connes tensor products. The smallest sub-factor N would play the role of complex numbers meaning that due to a finite measurement resolution one can speak only about N-rays of state space and the situation becomes effectively finite-dimensional but non-commutative.

3. The product of planar diagrams could be seen as a projection of 3-D Feynman diagram to plane or to one of the partonic vertices. It would contain a set of 2-D partonic 2-surfaces. Some of them would correspond vertices and the rest to partonic 2-surfaces at future and past directed light-cones corresponding to the incoming and outgoing particles.

4. The question is how to distinguish between vertex-partons and incoming and outgoing partons. If one does not delete the disk boundary of inner disk in the product, the fact that lines arrive at it from both sides could distinguish it as a vertex-parton whereas outgoing partons would correspond to empty disks. The direction of the arrows associated with the lines of planar diagram would allow to distinguish between positive and negative energy partons (note however line returning back).

5. One could worry about preferred role of the big disk identifiable as incoming or outgoing parton but this role is only apparent since by compactifying to say S^2 the big disk exterior becomes an interior of a small disk.

2.6.3 *A more detailed view*

The basic fact about planar algebras is that in the product of planar diagrams one glues two disks with identical boundary data together. One should understand the counterpart of this in more detail.

1. The boundaries of disks would correspond to 1-D closed space-like stringy curves at partonic 2-surfaces along which fermionic anti-commutators vanish.

2. The lines connecting the boundaries of disks to each other would correspond to the strands of number theoretic braids and thus to braidy time evolutions. The intersection points of lines with disk boundaries would correspond to the intersection points of strands of number theoretic braids meeting at the generalized vertex.

   [Number theoretic braid belongs to an algebraic intersection of a real parton 3-surface and its p-adic counterpart obeying same algebraic equations: of course, in time direction algebraicity allows only a sequence of snapshots about braid evolution].

3. Planar diagrams contain lines, which begin and return to the same disk boundary. Also "vacuum bubbles" are possible. Braid strands would disappear or appear in pairwise manner since they correspond to zeros of a polynomial and can transform from complex to real and vice versa under rather stringent algebraic conditions.

4. Planar diagrams contain also lines connecting any pair of disk boundaries. Stringy decay of partonic 2-surfaces with some strands of braid taken by the first and some strands by the second parton might bring in the lines connecting boundaries of any given pair of disks (if really possible!).

5. There is also something to worry about. The number of lines associated with disks is even in the case of k-tangles. In TGD framework incoming and outgoing tangles could have odd number of strands whereas partonic vertices would contain even number of k-tangles from fermion number conservation. One can wonder whether the replacement of boson lines with fermion lines could imply naturally the notion of half-k-tangle or whether one could assign half-k-tangles to the spinors of WCW ("world of classical worlds") whereas corresponding Clifford algebra defining HFF of type II_1 would correspond to k-tangles.

2.7 Miscellaneous

The following considerations are somewhat out-of-date: hence the title "Miscellaneous".

2.7.1 *Connes tensor product and fusion rules*

One should demonstrate that Connes tensor product indeed produces an M-matrix with physically acceptable properties.

The reduction of the construction of vertices to that for n-point functions of a conformal field theory suggest that Connes tensor product is essentially equivalent with the fusion rules for conformal fields defined by the Clifford algebra elements of $CH(CD)$ (4-surfaces associated with 3-surfaces at the boundary of causal diamond CD in M^4), extended to local fields in M^4 with gamma matrices acting on WCW spinor s assignable to the partonic boundary components.

Jones speculates that the fusion rules of conformal field theories can be understood in terms of Connes tensor product [A22] and refers to the work of Wassermann about the fusion of loop group representations as a demonstration of the possibility to formula the fusion rules in terms of Connes tensor product [A12] .

Fusion rules are indeed something more intricate that the naive product of free fields expanded using oscillator operators. By its very definition Connes tensor product means a dramatic reduction of degrees of freedom and this indeed happens also in conformal field theories.

1. For non-vanishing n-point functions the tensor product of representations of Kac Moody group associated with the conformal fields must give singlet representation.

2. The ordinary tensor product of Kac Moody representations characterized by given value of central extension parameter k is not possible since k would be additive.

3. A much stronger restriction comes from the fact that the allowed representations must define integrable representations of Kac-Moody group [A13] . For instance, in case of $SU(2)_k$ Kac Moody algebra only spins $j \leq k/2$ are allowed. In this case the quantum phase corresponds to $n = k + 2$. $SU(2)$ is indeed very natural in TGD framework since it corresponds to both electro-weak $SU(2)_L$ and isotropy group of particle at rest.

Fusion rules for localized Clifford algebra elements representing operators creating physical states would replace naive tensor product with something more intricate. The naivest approach would start from M^4 local variants of gamma matrices since gamma matrices generate the Clifford algebra Cl associated with $CH(CD)$. This is certainly too naive an approach. The next step would be the localization of more general products of Clifford algebra elements elements of Kac Moody algebras creating physical states and defining free on mass shell quantum fields. In standard quantum field theory the next step would be the introduction of purely local interaction vertices leading to divergence difficulties. In the recent case one transfers the partonic states assignable to the light-cone boundaries $\delta M^4_{\pm}(m_i) \times CP_2$ to the common partonic 2-surfaces X^2_V along $X^3_{L,i}$ so that the products of field operators at the same space-time point do not appear and one avoids infinities.

The remaining problem would be the construction an explicit realization of Connes tensor product. The formal definition states that left and right $\mathcal{N}$ actions in the Connes tensor product $\mathcal{M} \otimes_{\mathcal{N}} \mathcal{M}$ are identical so that the elements $nm_1 \otimes m_2$ and $m_1 \otimes m_2 n$ are identified. This implies a reduction of degrees of freedom so that free tensor product is not in question. One might hope that at least in the simplest choices for $\mathcal{N}$ characterizing the limitations of quantum measurement this reduction is equivalent with the reduction of degrees of freedom caused by the integrability constraints for Kac-Moody representations and dropping away of higher spins from the ordinary tensor product for the representations of quantum groups. If fusion rules are equivalent with Connes tensor product, each type of quantum measurement would be characterized by its own conformal field theory.

In practice it seems safest to utilize as much as possible the physical intuition provided by quantum field theories. In [K11] a rather precise vision about generalized Feynman diagrams is developed and the challenge is to relate this vision to Connes tensor product.

2.7.2 *Connection with topological quantum field theories defined by Chern-Simons action*

There is also connection with topological quantum field theories (TQFTs) defined by Chern- Simons action [A16] .

1. The light-like 3-surfaces X_l^3 defining propagators can contain unitary matrix characterizing the braiding of the lines connecting fermions at the ends of the propagator line. Therefore the modular S-matrix representing the braiding would become part of propagator line. Also incoming particle lines can contain similar S-matrices but they should not be visible in the M-matrix. Also entanglement between different partonic boundary components of a given incoming 3-surface by a modular S-matrix is possible.

2. Besides CP_2 type extremals MEs with light-like momenta can appear as brehmstrahlung like exchanges always accompanied by exchanges of CP_2 type extremals making possible momentum conservation. Also light-like boundaries of magnetic flux tubes having macroscopic size could carry light-like momenta and represent similar brehmstrahlung like exchanges. In this case the modular S-matrix could make possible topological quantum computations in $q \neq 1$ phase [**?**] . Notice the somewhat counter intuitive implication that magnetic flux tubes of macroscopic size would represent change in quantum jump rather than quantum state. These quantum jumps can have an arbitrary long geometric duration in macroscopic quantum phases with large Planck constant [K2] .

There is also a connection with topological QFT defined by Chern-Simons action allowing to assign topological invariants to the 3-manifolds [A16] . If the light-like CDs $X_{L,i}^3$ are boundary components, the 3-surfaces associated with particles are glued together somewhat like they are glued in the process allowing to construct 3-manifold by gluing them together along boundaries. All 3-manifold topologies can be constructed by using only torus like boundary components.

This would suggest a connection with 2+1-dimensional topological quantum field theory defined by Chern-Simons action allowing to define invariants for knots, links, and braids and 3-manifolds using surgery along links in terms of Wilson lines. In these theories one consider gluing of two 3-manifolds, say three-spheres S^3 along a link to obtain a topologically non-trivial 3-manifold. The replacement of link with Wilson lines in $S^3\#S^3 = S^3$ reduces the calculation of link invariants defined in this manner to Chern-Simons theory in S^3.

In the recent situation more general structures are possible since arbitrary number of 3-manifolds are glued together along link so that a singular 3-manifolds with a book like structure are possible. The allowance of CDs which are not boundaries, typically 3-D light-like throats of wormhole contacts at which induced metric transforms from Minkowskian to Euclidian, brings in additional richness of structure. If the scaling factor of CP_2 metric can be arbitrary large as the quantization of Planck constant predicts, this kind of structure could be macroscopic and could be also linked and knotted. In fact, topological condensation could be seen as a process in which two 4-manifolds are glued together by drilling light-like CDs and connected by a piece of CP_2 type extremal.

3 Fresh View About Hyper-Finite Factors In TGD Framework

In the following I will discuss the basic ideas about the role of hyper-finite factors in TGD with the background given by a work of more than half decade. First I summarize the input ideas which I combine with the TGD inspired intuitive wisdom about HFFs of type II_1 and their inclusions allowing to represent finite measurement resolution and leading to notion of quantum spaces with algebraic number valued dimension defined by the index of the inclusion.

Also an argument suggesting that the inclusions define "skewed" inclusions of lattices to larger lattices giving rise to quasicrystals is proposed. The core of the argument is that the included HFF of type II_1 algebra is a projection of the including algebra to a subspace with dimension $D \leq 1$. The projection operator defines the analog of a projection of a bigger lattice to the included lattice. Also the fact that the dimension of the tensor product is product of dimensions of factors just like the number of elements in finite group is product of numbers of elements of coset space and subgroup, supports this interpretation.

One also ends up with a detailed identification of the hyper-finite factors in orbital degrees of freedom in terms of symplectic group associated with $\delta M^4_\pm \times CP_2$ and the group algebras of their discrete subgroups define what could be called "orbital degrees of freedom" for WCW spinor fields. By very general argument this group algebra is HFF of type II, maybe even II_1.

3.1 Crystals, Quasicrystals, Non-Commutativity And Inclusions Of Hyperfinite Factors Of Type II_1

I list first the basic ideas about non-commutative geometries and give simple argument suggesting that inclusions of HFFs correspond to "skewed" inclusions of lattices as quasicrystals.

1. Quasicrystals (see `http://tinyurl.com/67kz3qo`) (say Penrose tilings) [A8] can be regarded as subsets of real crystals and one can speak about "skewed" inclusion of real lattice to larger lattice as quasicrystal. What this means that included lattice is obtained by projecting the larger lattice to some lower-dimensional subspace of lattice.

2. The argument of Connes concerning definition of non-commutative geometry can be found in the book of Michel Lapidus at page 200. Quantum space is identified as a space of equivalence classes. One assigns to pairs of elements inside equivalence class matrix elements having the element pair as indices (one assumes numerable equivalence class). One considers irreducible representations of the algebra defined by the matrices and identifies the equivalent irreducible representations. If I have understood correctly, the equivalence classes of irreps define a discrete point set representing the equivalence class and it can often happen that there is just single point as one might expect. This I do not quite understand since it requires that all irreps are equivalent.

3. It seems that in the case of linear spaces - von Neumann algebras and accompanying Hilbert spaces - one obtains a connection with the inclusions of HFFs and corresponding quantum factor spaces that should exist as analogs of quantum plane. One replaces matrices with elements labelled by element pairs with linear operators in HFF of type II_1. Index pairs correspond to pairs in linear basis for the HFF or corresponding Hilbert space.

4. Discrete infinite enumerable basis for these operators as a linear space generates a lattice in summation. Inclusion $N \subset M$ defines inclusion of the lattice/crystal for N to the corresponding lattice of M. Physical intuition suggests that if this inclusion is "skewed" one obtains quasicrystal. The fact the index of the inclusion is algebraic number suggests that the coset space M/N is indeed analogous to quasicrystal.

   More precisely, the index of inclusion is defined for hyper-finite factors of type II_1 using the fact that quantum trace of unit matrix equals to unity $Tr(Id(M)) = 1$, and from the tensor product composition $M = (M/N) \times N$ given $Tr(Id(M)) = 1 = Ind(M/N)Tr(P(M \to N))$, where $P(M \to N$ is projection operator from M to N. Clearly, $Ind(M/N) = 1/Tr(P(M \to N))$ defines index as a dimension of quantum space M/N.

   For Jones inclusions characterized by quantum phases $q = exp(i2\pi/n)$, $n = 3, 4, ...$ the values of index are given by $Ind(M/N) = 4cos^2(\pi/n)$, $n = 3, 4, ...$ There is also another range inclusions $Ind(M/N) \geq 4$: note that $Tr(P(M \to N))$ defining the dimension of N as included sub-space is never larger than one for HFFs of type II_1. The projection operator $P(M \to N)$ is obviously the counterpart of the projector projecting lattice to some lower-dimensional sub-space of the lattice.

5. Jones inclusions are between linear spaces but there is a strong analogy with non-linear coset spaces since for the tensor product the dimension is product of dimensions and for discrete coset spaces G/H one has also the product formula $n(G) = n(H) \times n(G/H)$ for the numbers of elements. Noticing that space of quantum amplitudes in discrete space has dimension equal to the number of elements of the space, one could say that Jones inclusion represents quantized variant for classical inclusion raised from the level of discrete space to the level of space of quantum states with the number of elements of set replaced by dimension. In fact, group algebras of infinite and enumerable groups defined HFFs of type II under rather general conditions (see below).

   Could one generalize Jones inclusions so that they would apply to non-linear coset spaces analogs of the linear spaces involved ? For instance, could one think of infinite-dimensional groups G and H for which Lie-algebras defining their tangent spaces can be regarded as HFFs of type II_1? The dimension of the tangent space is dimension of the non-linear manifold: could this mean that the non-linear infinite-dimensional inclusions reduce to tangent space level and thus to the inclusions for Lie-algebras regarded hyper-finite factors of type II_1 or more generally, type II? This would would rise to quantum spaces which have finite but algebraic valued quantum dimension and in TGD framework take into account the finite measurement resolution.

6. To concretize this analogy one can check what is the number of points map from 5-D space containing aperiodic lattice as a projection to a 2-D irrational plane containing only origin as common point with the 5-D lattice. It is easy to get convinced that the projection is 1-to-1 so that the number of points projected to a given point is 1. By the analogy with Jones inclusions this would mean that the included space has same von Neumann dimension 1 - just like the including one. In this case quantum phase equals $q = exp(i2\pi/n)$, $n = 3$ - the lowest possible value of n. Could one imagine the analogs of $n > 3$ inclusions for which the number of points projected to a given point would be larger than 1? In 1-D case the rational lines $y = (k/l)x$ define 1-D rational analogs of quasi crystals. The points $(x, y) = (m, n)$, $m \ mod \ l = 0$ are projected to the same point. The number of points is now infinite and the ratio of points of 2-D lattice and 1-D crystal like structure equals to l and serves as the analog for the quantum dimension $d_q = 4cos^2(\pi/n)$.

To sum up, this this is just physicist's intuition: it could be wrong or something totally trivial from the point of view of mathematician. The main message is that the inclusions of HFFs might define also inclusions of lattices as quasicrystals.

3.2 HFFs And Their Inclusions In TGD Framework

In TGD framework the inclusions of HFFs have interpretation in terms of finite measurement resolution. If the inclusions define quasicrystals then finite measurement resolution would lead to quasicrystals.

1. The automorphic action of N in $M \supset N$ and in associated Hilbert space H_M where N acts generates physical operators and accompanying stas (operator rays and rays) not distinguishable from the original one. States in finite measurement resolution correspond to N-rays rather than complex rays. It might be natural to restrict to unitary elements of N.

   This leads to the need to construct the counterpart of coset space M/N and corresponding linear space H_M/H_N. Physical intuition tells that the indices of inclusions defining the "dimension" of M/N are algebraic numbers given by Jones index formula.

2. Here the above argument would assign to the inclusions also inclusions of lattices as quasicrystals.

3.2.1 *Degrees of freedom for WCW spinor field*

Consider first the identification of various kinds of degrees of freedom in TGD Universe.

1. Very roughly, WCW ("world of classical worlds") spinor is a state generated by fermionic creation operators from vacuum at given 3-surface. WCW spinor field assigns this kind of spinor to each 3-surface. WCW spinor fields decompose to tensor product of spin part (Fock state) and orbital part ("wave" in WCW) just as ordinary spinor fields.

2. The conjecture motivated by super-symmetry has been that both WCW spinors and their orbital parts (analogs of scalar field) define HFFs of type II_1 in quantum fluctuating degrees of freedom.

3. Besides these there are zero modes, which by definition do not contribute to WCW Kähler metric.

   (a) If the zero zero modes are symplectic invariants, they appear only in conformal factor of WCW metric. Symplectically invariant zero modes represent purely classical degrees of freedom - direction of a pointer of measurement apparatus in quantum measurement - and in given experimental arrangement they entangle with quantum fluctuating degrees of freedom in one-one manner so that state function reduction assigns to the outcome of state function reduction position of pointer. I forget symplectically invariant zero modes and other analogous variables in the following and concentrate to the degrees of freedom contributing WCW line-element.

(b) There are also zero modes which are not symplectic invariants and are analogous to degrees of freedom generated by the generators of Kac-Moody algebra having vanishing conformal weight. They represent "center of mass degrees of freedom" and this part of symmetric algebra creates the representations representing the ground states of Kac-Moody representations. Restriction to these degrees of freedom gives QFT limit in string theory. In the following I will speak about "cm degrees of freedom".

The general vision about symplectic degrees of freedom (the analog of "orbital degrees of freedom" for ordinary spinor field) is following.

1. WCW (assignable to given CD) is a union over the sub-WCWs labeled by zero modes and each sub-WCW representing quantum fluctuating degrees of freedom and "cm degrees of freedom" is infinite-D symmetric space. If symplectic group assignable to $\delta M^4_+ \times CP_2$ acts as as isometries of WCW then "orbital degrees of freedom" are parametrized by the symplectic group or its coset space (note that light-cone boundary is 3-D but radial dimension is light-like so that symplectic - or rather contact structure - exists).

   Let S^2 be $r_M = constant$ sphere at light-cone boundary (r_M is the radial light-like coordinate fixed apart from Lorentz transformation). The full symplectic group would act as isometries of WCW but does not - nor cannot do so - act as symmetries of Kähler action except in the huge vacuum sector of the theory correspond to vacuum extremals.

2. WCW Hamiltonians can be deduced as "fluxes" of the Hamiltonians of $\delta M^4_+ \times CP_2$ taken over partonic 2-surfaces. These Hamiltonians expressed as products of Hamiltonians of S^2 and CP_2 multiplied by powers r_M^n. Note that r_M plays the role of the complex coordinate z for Kac-Moody algebras and the group G defining KM is replaced with symplectic group of $S^2 \times CP_2$. Hamiltonians can be assumed to have well-defined spin (SO(3)) and color (SU(3)) quantum numbers.

3. The generators with vanishing radial conformal weight ($n = 0$) correspond to the symplectic group of $S^2 \times CP_2$. They are not symplectic invariants but are zero modes. They would correspond to "cm degrees of freedom" characterizing the ground states of representations of the full symplectic group.

3.2.2 *Discretization at the level of WCW*

The general vision about finite measurement resolution implies discretization at the level of WCW.

1. Finite measurement resolution at the level of WCW means discretization. Therefore the symplectic groups of $\delta M^4_+ \times CP_2$ *resp.* $S^2 \times CP_2$ are replaced by an enumerable discrete subgroup. WCW is discretized in both quantum fluctuating degrees of freedom and "center of mass" degrees of freedom.

2. The elements of the group algebras of these discrete groups define the "orbitals parts" of WCW spinor fields in discretization. I will later develop an argument stating that they are HFFs of type II - maybe even II_1. Note that also function spaces associated with the coset spaces of these discrete subgroups could be considered.

3. Discretization applies also in the spin degrees of freedom. Since fermionic Fock basis generates quantum counterpart of Boolean algebra the interpretation in terms of the physical correlates of Boolean cognition is motivated (fermion number 1/0 and various spins in decomposition to a tensor product of lower-dimensional spinors represent bits). Note that in ZEO fermion number conservation does not pose problems and zero states actually define what might be regarded as quantum counterparts of Boolean rules $A \to B$.

4. Note that 3-surfaces correspond by the strong form of GCI/holography to collections of partonic 2-surfaces and string world sheets of space-time surface intersecting at discrete set of points carrying fermionic quantum numbers. WCW spinors are constructed from second quantized induced spinor fields and fermionic Fock algebra generates HFF of type II_1.

3.2.3 *Does WCW spinor field decompose to a tensor product of two HFFs of type II_1?*

The group algebras associated with infinite discrete subgroups of the symplectic group define the discretized analogs of waves in WCW having quantum fluctuating part and cm part. The proposal is that these group algebras are HFFs of type II_1. The spinorial degrees of freedom correspond to fermionic Fock space and this is known to be HFF. Therefore WCW spinor fields would defined tensor product of HFFs of type II_1. The interpretation would be in terms of supersymmetry at the level of WCW. Super-conformal symmetry is indeed the basic symmetry of TGD so that this result is a physical "must". The argument goes as follows.

1. In non-zero modes WCW is symplectic group of $\delta M^4_+ \times CP_2$ (call this group just *Sympl*) reduces to the analog of Kac-Moody group associated with $S^2 \times CP_2$, where S^2 is $r_M =$ constant sphere of light-cone boundary and z is replaced with radial coordinate. The Hamiltonians, which do not depend on r_M would correspond to zero modes and one could not assign metric to them although symplectic structure is possible. In "cm degrees of freedom" one has symplectic group associated with $S^2 \times CP_2$.

2. Finite measurement resolution, which seems to be coded already in the structure of the preferred extremals and of the solutions of the Kähler-Dirac equation, suggests strongly that this symplectic group is replaced by its discrete subgroup or symmetric coset space. What this group is, depends on measurement resolution defined by the cutoffs inherent to the solutions. These subgroups and coset spaces would define the analogs of Platonic solids in WCW!

3. Why the discrete infinite subgroups of *Sympl* would lead naturally to HFFs of type II? There is a very general result stating that group algebra of an enumerable discrete group, which has infinite conjugacy classes, and is amenable so that its regular representation in group algebra decomposes to all unitary irreducibles is HFF of type II. See for examples about HFFs of type II listed in Wikipedia article (see `http://tinyurl.com/y8445w8q`) [A3].

4. Suppose that the group algebras associated the discrete subgroups *Sympl* are indeed HFFs of type II or even type II_1. Their inclusions would define finite measurement resolution the orbital degrees of freedom for WCW spinor fields. Included algebra would create rays of state space not distinguishable experimentally. The inclusion would be characterized by the inclusion of the lattice defined by the generators of included algebra by linearity. One would have inclusion of this lattice to a lattice associated with a larger discrete group. Inclusions of lattices are however known to give rise to quasicrystals (Penrose tilings are basic example), which define basic non-commutative structures. This is indeed what one expects since the dimension of the coset space defined by inclusion is algebraic number rather than integer.

5. Also in fermionic degrees of freedom finite measurement resolution would be realized in terms of inclusions of HFFs- now certainly of type II_1. Therefore one could obtain hierarchies of lattices included as quasicrystals.

What about zero modes which are symplectic invariants and define classical variables? They are certainly discretized too. One might hope that one-one correlation between zero modes (classical variables) and quantum fluctuating degrees of freedom suggested by quantum measurement theory allows to effectively eliminate them. Besides zero modes there are also modular degrees of freedom associated with partonic 2-surfaces defining together with their 4-D tangent space data basis objects by strong form of holography. Also these degrees of freedom are automatically discretized. But could one consider finite measurement resolution also in these degrees of freedom. If the symplectic group of $S^2 \times CP_2$ defines zero modes then one could apply similar argument also in these degrees of freedom to discrete subgroups of $S^2 \times CP_2$.

3.3 Little Appendix: Comparison Of WCW Spinor Fields With Ordinary Second Quantized Spinor Fields

In TGD one identifies states of Hilbert space as WCW spinor fields. The analogy with ordinary spinor field helps to understand what they are. I try to explain by comparison with QFT.

3.3.1 *Ordinary second quantized spinor fields*

Consider first ordinary fermionic QFT in fixed space-time. Ordinary spinor is attached to an space-time point and there is $2^{D/2}$ dimensional space of spin degrees of freedom. Spinor field attaches spinor to every point of space-time in a continuous/smooth manner. Spinor fields satisfying Dirac equation define in Euclidian metric a Hilbert space with a unitary inner product. In Minkowskian case this does not work and one must introduce second quantization and Fock space to get a unitary inner product. This brings in what is essentially a basic realization of HFF of type II_1 as allowed operators acting in this Fock space. It is operator algebra rather than state space which is HFF of type II_1 but they are of course closely related.

3.3.2 *Classical WCW spinor fields as quantum states*

What happens TGD where one has quantum superpositions of 4-surface/3-surfaces by GCI/partonic 2-surfaces with 4-D tangent space data by strong form of GCI.

1. First guess: space-time point is replaced with 3-surface. Point like particle becomes 3-surface representing particle. WCW spinors are fermionic Fock states at this surface. WCW spinor fields are Fock state as a functional of 3-surface. Inner product decomposes to Fock space inner product plus functional integral over 3-surfaces (no path integral!). One could speak of quantum multiverse. Not single space-time but quantum superposition of them. This quantum multiverse character is something new as compared to QFT.

2. Second guess: forced by ZEO, by geometrization of Feynman diagrams, etc.

   (a) 3-surfaces are actually not connected 3-surfaces. They are collections of components at both ends of CD and connected to single connected structure by 4-surface. Components of 3-surface are like incoming and outgoing particles in connected Feynman diagrams. Lines are identified as regions of Euclidian signature or equivalently as the 3-D light-like boundaries between Minkowskian and Euclidian signature of the induced metric.

   (b) Spinors(!!) are defined now by the fermionic Fock space of second quantized induced spinor fields at these 3-surfaced and by holography at 4-surface. This fermionic Fock space is assigned to all multicomponent 3-surfaces defined in this manner and WCW spinor fields are defined as in the first guess. This brings integration over WCW to the inner product.

3. Third, even more improved guess: motivated by the solution ansatz for preferred extremals and for Kähler-Dirac equation [K6] giving a connection with string models.

   The general solution ansatz restricts all spinor components but right-handed neutrino to string world sheets and partonic 2-surfaces: this means effective 2-dimensionality. String world sheets and partonic 2-surfaces intersect at the common ends of light-like and space-like braids at ends of CD and at along wormhole throat orbits so that effectively discretization occurs. This fermionic Fock space replaces the Fock space of ordinary second quantization.

4 MIP*= RE: What could this mean physically?

I received a very interesting link to a popular article (`https://cutt.ly/sfd5UQF`) explaining a recently discovered deep result in mathematics having implications also in physics. The article [A20] (`https://cutt.ly/rffiYdc`) by Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen has a rather concise title "**MIP*=RE**". In the following I try to express the impressions of a (non-mainstream) physicist about the result.

The following is the result expressed using the concepts of computer science about which I know very little at the hard technical level. The results are however told to state something highly non-trivial about physics.

1. RE (recursively enumerable languages) denotes all problems solvable by computer. P denotes the problems solvable in a polynomial time. NP does not refer to a non-polynomial time but to "non-deterministic polynomial acceptable problems" - I hope this helps the reader- I am a little bit confused! It is not known whether P = NP is true.

2. IP problems (P is now for "prover" that can be solved by a collaboration of an interrogator and prover who tries to convince the interrogator that her proof is convincing with high enough probability. MIP involves multiple l provers treated as criminals trying to prove that they are innocent and being not allowed to communicate. MIP* is the class of solvable problems in which the provers are allowed to entangle.

The finding, which is characterized as shocking, is that *all* problems solvable by a Turing computer belong to this class: MIP*=RE. All problems solvable by computer would reduce to problems in the class MIP*! Quantum computation would indeed add something genuinely new to the classical computation.

Quantum entanglement would play an essential role in quantum computation. Also the implications for physics are highly non-trivial.

1. Connes imbedding problem asking whether all infinite-D matrices can always be approximated by finite-Dmatrices has a negative solution.Therefore MIP*= RE does not hold true for hyperfinite factors of type II_1 (HFFs) central in quantum TGD. Also the Tirelson problem finds a solution. The measurements of commuting observers performed by two observers are equivalent to the measurements of tensor products of observables only in finite-D case and for HFFs. That quantum entanglement would not have any role in HFFs is in conflict with intuition.

2. In the TGD framework finite measurement resolution is realized in terms of HFFs at Hilbert space level and in terms of cognitive representations at space-time level defined purely number-theoretically. This leads to a hierarchy of adeles defined by extensions of rationals and the Hilbert spaces must have algebraic extensions of rationals as a coefficient field. Therefore one cannot in general case find a unitary transformation mapping the entangled situation to an unentangled one, and quantum entanglement plays a key role. It seems that computationalism formulated in terms of recursive functions of natural numbers must be formulated for the hierarchy of extensions of rationals in terms of algebraic integers.

3. In TGD inspired theory of consciousness entanglement between observers could be seen as a kind of telepathy helping to perform conscious quantum computations. Zero energy ontology also suggests a modification of the views about quantum computation. TGD can be formulated also for real and p-adic continua identified as correlates of sensory experience and cognition, and it seems that computational paradigm need not apply in the continuum cases.

4.1 Two physically interesting applications

There are two physically interesting applications of the theorem interesting also from the TGD point of view and force to make explicit the assumptions involved.

4.1.1 About the quantum physical interpretation of MP*

To proceed one must clarify the quantum physical interpretation of MIP*.

Quantum measurement requires entanglement of the observer O with the measured system M. What is basically measured is the density matrix of M (or equivalently that of O).

State function reduction gives as an outcome a state, which corresponds to an eigenvalue of the density matrix. Note that this state can be an entangled state if the density matrix has degenerate eigenvalues. Quantum measurement can be regarded as a question to the measured system: "What are the values of given commuting observables?". The final state of quantum measurement provides an eigenstate of the observables as the answer to this question. M would be in the role of the prover and O_i would serve as interrogators.

In the first case multiple interrogators measurements would entangle M with unentangled states of the tensor product $H_1 \otimes H_2$ for O followed by a state function reduction splitting the state of M to un-entangled state in the tensor product $M_1 \otimes M_2$.

In the second case the entire$M$ would be interrogated using entanglement of M with entangled states of $H_1 \otimes H_2$ using measurements of several commuting observables. The theorem would state that interrogation in this manner is more efficient in infinite-D case unless HFFs are involved.

3. This interpretation differs from the interpretation in terms of computational problem solving in which one would have several provers and one interrogator. Could these interpretations be dual as the complete symmetry of the quantum measurement with respect to O and M suggests? In the case of multiple provers (analogous to accused criminals) it is advantageous to isolate them. In the case of multiple interrogators the best result is obtained if the interrogator does not effectively split itself into several ones.

4.1.2 Connes embedding problem and the notion of finite measurement/cognitive resolution

Alain Connes formulated what has become known as Connes embedding problem. The question is whether infinite matrices forming factor of type II_1 can be *always* approximated by finite-D matrices that is imbedded in a *hyperfinite* factor of type II_1 (HFF). Factors of type II and their HFFs are special classes of von Neumann algebras possibly relevant for quantum theory.

This result means that if one has measured of a complete set of for a product of commuting observables acting in the full space, one can find in the finite-dimensional case a unitary transformation transforming the observables to tensor products of observables associated with the factors of a tensor product. In the infinite-D case this is not true.

What seems to put alarms ringing is that in TGD there are excellent arguments suggesting that the state space has HFFs as building bricks. Does the result mean that entanglement cannot help in quantum computation in TGD Universe? I do not want to live in this kind of Universe!

4.1.3 Tsirelson problem

Tsirelson problem (see this) is another problem mentioned in the popular article as a physically interesting application. The problem relates to the mathematical description of quantum measurement.

Three systems are considered. There are two systems O_1 and O_2 representing observers and the third representing the measured system M. The measurement reducing the entanglement between M and O_1 or O_2 can regarded as producing correspondence between state of M and O_1 or O_2, and one can think that O_1 or O_2 measures only obserservables in its own state space as a kind of image of M. There are two manners to see the situation. The provers correspond now to the observers and the two situations correspond to provers without and with entanglement.

Consider first a situation in which one has single Hilbert space H and single observer O. This situation is analogous to IP.

1. The state of the system is described statistically by a density matrix - not necessarily pure state -, whose diagonal elements have interpretation as reduction probabilities of states in this bases. The measurement situation fixes the state basis used. Assume an ensemble of identical copies of the system in this state. Assume that one has a complete set of commuting observables.

2. By measuring all observables for the members of the ensemble one obtains the probabilities as diagonal elements of the density matrix. If the observable is the density matrix having no- degenerate eigenvalues, the situation is simplified dramatically. It is enough to use the density matrix as an observable. TGD based quantum measurement theory assumes that the density matrix describing the entanglement between two subsystems is in a universal observable measure in state function reductions reducing their entanglement.

3. Can one deduce also the state of M as a superposition of states in the basic chosen by the observer? This basis need not be the same as the basis defined by - say density matrix if the system has interacted with some system and this ineracton has led to an eigenstate of the density matrix. Assume that one can prepare the latter basis by a physical process such as this kind of interaction.

   The coefficients of the state form a set of N^2 complex numbers defining a unitary $N \times N$ matrix. Unitarity conditions give N conditions telling that the complex rows and unit vectors: these numbers are given by the measurement of all observables. There are also $N(N-1)$ conditions telling that the rows are orthogonal. Together these $N + N(N-1) = N^2$ numbers fix the elements of the unitary matrix and therefore the complex coefficients of the state basis of the system can be deduced from a complete set of measurements for all elements of the basis.

Consider now the analog of the MIS* involving more than one observer. For simplicity consider two observers.

1. Assume that the state space H of M decomposes to a tensor product $H = H_1 \otimes H_2$ of state spaces H_1 and H_2 such that O_1 measures observables X_1 in H_1 and O_2 measuresobservables X_2 in H_2. The observables X_1 and X_2 commute since they act in different tensor factors. The absence of interaction between the factors corresponds to the inability of the provers to communicate. As in the previous case, one can deduce the probabilities for the various outcomes of the joint measurements interpreted as measurements of a complete set of observables $X_1 \otimes X_2$.

2. One can also think that the two systems form a single system O so that O_1 and O_2 can entangle. This corresponds to a situation in which entanglement between the provers is allowed. Now X_1 and X_2 are not in general independent but also now they must commute. One can deduce the probabilities for various outcomes as eigenstates of observables X_1X_2 and deduce the diagonal elements of the density matrix as probabilities.

Are these manners to see the situation equivalent? Tsirelson demonstrated that this is the case for finite-dimensional Hilbert spaces, which can indeed be decomposed to a tensor product of factors associated with O_1 and O_2. This means that one finds a unitary transformation transforming the entangled situation to an unentangled one and to tensor product observables.

For the infinite-dimensional case the situation remained open. According to the article, the new result implies that this is not the case. For hyperfinite factors the situation can be approximated with a finite-D Hilbert space so that the situations are equivalent in arbitrary precise approximation.

4.2 The connection with TGD

The result looks at first a bad news from the TGD point of view, where HFFs are highly suggestive. One must be however very careful with the basic definitions.

4.2.1 Measurement resolution

Measurement resolution is the basic notion.

1. There are intuitive physicist's arguments demonstrating that in TGD the operator algebras involved with TGD are HFFs provides a description of finite measurement resolution. The inclusion of HFFs defines the notion of resolution: included factor represents the degrees

of freedom not seen in the resolution used [K21, K14] (http://tgdtheoryd.fi/pfpool/vNeumann.pdf) and http://tgdtheoryd.fi/pfpool/vNeumannnew.pdf).

Hyperfinite factors involve new structures like quantum groups and quantum algebras reflecting the presence of additional symmetries: actually the "world of classical worlds" (WCW) as the space of space-time surfaces as maximal group of isometries and this group has a fractal hierarchy of isomorphic groups imply inclusion hierarchies of HFFs. By the analogs of gauge conditions this infinite-D group reduces to a hierarchy of effectively finite-D groups. For quantum groups the infinite number of irreps of the corresponding compact group effectively reduces to a finite number of them, which conforms with the notion of hyper-finiteness.

It looks that the reduction of the most general quantum theory to TGD-like theory relying on HFFs is not possible. This would not be surprising taking into account gigantic symmetries responsible for the cancellation of infinities in TGD framework and the very existence of WCW geometry.

2. Second TGD based approach to finite resolution is purely number theoretic [?] and involves adelic physics as a fusion of the real physics with various p-adic physics as correlates of cognition. Cognitive representations are purely number theoretic and unique discretizations of space-time surfaces defined by a given extension of rationals forming an evolutionary hierarchy: the coordinates for the points of space-time as a 4-surface of the imbedding space $H = M^4 \times CP_2$ or of its dual M^8 are in the extension of rationals defining the adele. In the case of M^8 the preferred coordinates are unique apart from time translation. These two views would define descriptions of the finite resolution at the level of space-time and Hilbert space. In particular, the hierarchies of extensions of rationals should define hierarchies of inclusions of HFFs.

For hyperfinite factors the analog of MIP*=RE cannot hold true. Doesn't the TGD Universe allow a solution of all the problems solvable by Turing Computer? There is a loophole in this argument.

1. The point is that for the hierarchy of extensions of rationals also Hilbert spaces have as a coefficient field the extension of rationals! Unitary transformations are restricted to matrices with elements in the extension. In general it is not possible to realize the unitary transformation mapping the entangled situation to an un-entangled one! The weakening of the theorem would hold true for the hierarchy of adeles and entanglement would give something genuinely new for quantum computation!

2. A second deep implication is that the density matrix characterizing the entanglement between two systems cannot in general be diagonalized such that all diagonal elements identifiable as probabilities would be in the extension considered. One would have stable or partially stable entanglement (could the projection make sense for the states or subspaces with entanglement probability in the extension). For these bound states the binding mechanism is purely number theoretical. For a given extension of p-adic numbers one can assign to algebraic entanglement also information measure as a generalization of Shannon entropy as a p-adic entanglement entropy (real valued). This entropy can be negative and the possible interpretation is that the entanglement carries conscious information.

4.2.2 What about transcendental extensions?

Duringthewriting of this article an interesting question popped up.

1. Also transcendental extensions of rationals are possible, and one can consider the generalization of the computationalism by also allowing functions in transcendental extensions. Could the hierarchy of algebraic extensions could continue with transcendental extensions? Could one even play with the idea that the discovery of transcendentals meant a quantum leap leading to an extension involving for instance e and π as basic transcendentals? Could one generalize the notion of polynomial root to a root of a function allowing Taylor expansion $f(x) = \sum q_n x^n$ with rational coefficients so that the roots of $f(x) = 0$ could be used define transcendental extensions of rationals?

2. Powers of e or its root define and infinite-D extensions having the special property that they are finite-D for p-adic number fields because e^p is ordinary p-adic number. In the p-adic context e can be defined as a root of the equation $x^p - \sum p^n/n! = 0$ making sense also for rationals. The numbers $log(p_i)$ such that p_i appears a factor for integers smaller than p define infinite-D extension of both rationals and p-adic numbers. They are obtained as roots of $e^x - p_i = 0$.

3. The numbers $(2n+1)\pi$ $(2n\pi)$ can be defined as roots of $sin(x) = 0$ $(cos(x) = 0$. The extension by π is infinite-dimensional and the conditions defining it would serve as consistency conditions when the extension contains roots of unity and effectively replaces them.

4. What about other transcendentals appearing in mathematical physics? The values $\zeta(n)$ of Riemann Zeta appearing in scattering amplitudes are for even values of n given by $\zeta(2n) = (-1)^{n+1}B_{2n}(2\pi)^{2n}/2(2n+1)!$. This follows from the functional identity for Riemann zeta and from the expression $\zeta(-n) = (-1)^n B_{n+1}/(n+1)$ ($(B(-1/2) = -1/2)$ (`https://cutt.ly/dfgtgmw`). The Bernoulli numbers B_n are rational and vanish for odd values of n. An open question is whether also the odd values are proportional toπ^n with a rational coefficient or whether they represent "new" transcendentals.

4.2.3 What about the situation for the continuum version of TGD?

At least the cognitively finitely representable physics would have the HFF property with coefficient field of Hilbert spaces replaced by an extension of rationals. Number theoretical universality would suggest that HFF property characterizes also the physics of continuum TGD. This leads to a series of questions.

1. Does the new theorem imply that in the continuum version of TGD all quantum computations allowed by the Turing paradigm for real coefficients field for quantum states are not possible: $MIP* \subset RE$? The hierarchy of extensions of rationals allows utilization of entanglement, and one can even wonder whether one could have $MIP* = RE$ at the limit of algebraic numbers.

2. Could the number theoretic vision force change also the view about quantum computation? What does RE actually mean in this framework? Can one really assume complex entanglement coefficients in computation. Does the computational paradigm makes sense at all in the continuum picture?

   Are both real and p-adic continuum theories unreachable by computation giving rise to cognitive representations in the algebraic intersubsection of the sensory and cognitive worlds? I have indeed identified real continuum physics as a correlate for sensory experience and various p-adic physics as correlates of cognition in TGD: They would represent the computionally unreachable parts of existence.

   Continuum physics involves transcendentals and in mathematics this brings in analytic formulas and partial differential equations. At least at the level of mathematical consciousness the emergence of the notion of continuum means a gigantic step. Also this suggests that transcendentality is something very real and that computation cannot catch all of it.

3. Adelic theorem allows to express the norm of a rational number as a product of inverses of its p-adic norms. Very probably this representation holds true also for the analogs of rationals formed from algebraic integeres. Reals can be approximated by rationals. Could extensions of all p-adic numbers fields restricted to the extension of rationals say about real physics only what can be expressed using language?

Also fermions are highly interesting in the recent context. In TGD spinor structure can be seen as a square root of Kähler geometry, in particular for the "world of classical worlds" (WCW). Fermions are identified as correlates of Boolean cognition. The continuum case for fermions does not follow as a naive limit of algebraic picture.

1. The quantization of the induced spinors in TGD looks different in discrete and continuum cases. Discrete case is very simple since equal-time anticommutators give discrete Kronecker

deltas. In the continuum case one has delta functions possibly causing infinite vacuum energy like divergences in conserved Noether charges (Dirac sea).

2. In [?] (`https://cutt.ly/zfftoK6`) I have proposed how these problems could be avoided by avoiding anticommutators giving delta-function. The proposed solution is based on zero energy ontology and TGD based view about space-time. One also obtains a long-sought-for concrete realization for the idea that second quantized induce spinor fields are obtained as restrictions of second quantized free spinor fields in $H = M^4 \times CP_2$ to space-time surface. The fermionic variant of $M^8 - H$-duality [?] provides further insights and gives a very concrete picture about the dynamics of fermions in TGD.

These considerations relate in an interesting manner to consciousness. Quantum entanglement makes in the TGD framework possible telepathic sharing of mental images represented by subselves of self. For the series of discretizations of physics by HFFs and cognitive representations associated with extensions of rationals, the result indeed means something new.

4.2.4 What does one mean with quantum computation in TGD Universe?

The TGD approach raises some questions about computation.

1. The ordinary computational paradigm is formulated for Turing machines manipulating natural numbers by recursive algorithms. Programs would essentially represent a recursive function $n \rightarrow f(n)$. What happens to this paradigm when extensions of rationals define cognitive representations as unique space-time discretizations with algebraic numbers as the limit giving rise to a dense in the set of reals.

   The usual picture would be that since reals can be approximated by rationals, the situation is not changed. TGD however suggests that one should replace at least the quantum version of the Turing paradigm by considering functions mapping algebraic integers (algebraic rational) to algebraic integers.

   Quite concretely, one can manipulate algebraic numbers without approximation as a rational and only at the end perform this approximation and computations would construct recursive functions in this manner. This would raise entanglement to an active role even if one has HFFs and even if classical computations could still look very much like ordinary computation using integers.

   This suggests that computationalism usually formulated in terms of recursive functions of natural or rational numbers could be replaced with a hierarchy of computationalisms for the hierarchy of extensions of rationals. One would have recursively definable functions defined and having values in the extensions of rationals. These functions would be analogs of analytic functions (or polynomials) with the complex variable replaced with an integer or a rational of the extension. This poses very powerful constraints and there are good reasons to expect an increase of computational effectiveness. One can hope that at the limit of algebraic numbers of these functions allow arbitrarily precise approximations to real functions. If the real world phenomena can be indeed approximated by cognitive representations in the TGD sense, one can imaginea highly interesting approach to AI.

2. ZEO brings in also time reversal occurring in "big" (ordinary) quantum jumps and this modifies the views about quantum computation. In ZEO based conscious quantum computation halting means "death" and "reincarnation" of conscious entity, self? How the processes involving series of haltings in this sense differs from ordinary quantum computation: could one shorten the computation time by going forth and back in time.

There are many interesting questions to be considered. $M^8 - H$ duality gives justifications for the vision about algebraic physics. TGD leads also to the notion of infinite prime and I have considered the possibility that infinite primes could give a precise meaning for the dimension of infinite-D Hilbert space. Could the number-theoretic view about infinite be considerably richer than the idea about infinity as limit would suggest [K17].

The construction of infinite primes is analogous to a repeated second quantization of arithmetic supersymmetric quantum field theory allowing also bound states at each level and a concrete correspondence with the hierarchy of space-time sheets is suggestive. For the infinite primes at the lowest level of the hierarchy single particle states correspond to rationals and bound states to polynomials and therefore to the sets of their roots. This strongly suggests a connection with M^8 picture.

5 Analogs Of Quantum Matrix Groups From Finite Measurement Resolution?

The notion of quantum group [?]eplaces ordinary matrices with matrices with non-commutative elements. This notion is physically very interesting, and in TGD framework I have proposed that it should relate to the inclusions of von Neumann algebras allowing to describe mathematically the notion of finite measurement resolution [?] These ideas have developed slowly through various side tracks.

In the sequel I will consider the notion of quantum matrix inspired by the recent view about quantum TGD relying on the notion of finite measurement resolution and show that under some additional conditions it provides a concrete representation and physical interpretation of quantum groups in terms of finite measurement resolution.

1. The basic idea is to replace complex matrix elements with operators, which are products of non-negative hermitian operators and unitary operators analogous to the products of modulus and phase as a representation for complex numbers. Modulus and phase would be non-commuting and have commutation relation analogous to that between momentum and plane-wave in accordance with the idea about quantization of complex numbers.

2. The condition that determinant and sub-determinants exist is crucial for the well-definedness of eigenvalue problem in the generalized sense. Strong/weak permutation symmetry of determinant requires its invariance apart from sign change under permutations of rows and/or columns. Weak permutation symmetry means development of determinant with respect to a fixed row or column and does not pose additional conditions. For weak permutation symmetry the permutation of rows/columns would however have a natural interpretation as braiding for the hermitian operators defined by the moduli of operator valued matrix elements and here quantum group structure emerges.

3. The commutativity of all sub-determinants is essential for the replacement of eigenvalues with eigenvalue spectra of hermitian operators and sub-determinants define mutually commuting set of operators.

Quantum matrices define a more general structure than quantum group but provide a concrete representation for them in terms of finite measurement resolution, in particular when q is a root of unity. For $q = \pm 1$ (Bose-Einstein or Fermi-Dirac statistics) one obtains quantum matrices for which the determinant is apart from possible change by a sign factor invariant under the permutations of both rows and columns. One can also understand the recursive fractal structure of inclusion sequences of hyper-finite factors resulting by replacing operators appearing as matrix elements with quantum matrices and a concrete connection with quantum groups emerges.

In Zero Energy Ontology (ZEO) M-matrix serving as the basic building brick of unitary U-matrix and identified as a hermitian square root of density matrix provides a possible application for this vision. Especially fascinating is the possibility of hierarchies of measurement resolutions represented as inclusion sequences realized as recursive construction of M-matrices. Quantization would emerge already at the level of complex numbers appearing as M-matrix elements.

This approach might allow to unify various ideas behind TGD. For instance, Yangian algebras emerging naturally in twistor approach are examples of quantum algebras. The hierarchy of Planck constants should have close relationship with inclusions and fractal hierarchy of sub-algebras of super-symplectic and other conformal algebras.

5.1 Well-definedness Of The Eigenvalue Problem As A Constraint To Quantum Matrices

Intuition suggests that the presence of degrees of freedom below measurement resolution implies that one must use density matrix description obtained by taking trace over the unobserved degrees of freedom. One could argue that in state function reduction with finite measurement resolution the outcome is not a pure state, or not even negentropically entangled state (possible in TGD framework) but a state described by a density matrix. The challenge is to describe the situation mathematically in an elegant manner.

1. There is present an infinite number of degrees of freedom below measurement resolution with which measured degrees of freedom entangle so that their presence affects the situation. One has a system with finite number degrees of freedom such as two-state system described by a quantum spinor. In this case observables as hermitian operators described by 2×2 matrices would be replaced by quantum matrices with elements, which in general do not commute.

   An attractive generalization of complex numbers appearing as elements of matrices is obtained by replacing them with products $H_{ij} = h_{ij}u_{ij}$ of hermitian operators h_{ij} with non-negative spectrum (modulus of complex number) and unitary operators u_{ij} (phase of complex number) suggests itself. The commutativity of h_{ij} and u_{ij} would look nice but is not necessary and is in conflict with the idea that modulus and phase of an amplitudes do not commute in quantum mechanics.

   Very probably this generalization is trivial for mathematician. One could indeed interpret the generalization in terms of a tensor product of finite-dimensional matrices with possibly infinite-dimensional space of operators of Hilbert space. For the physicist the situation might be different as the following proposal for what hermitian quantum matrices could be suggests.

2. The modulus of complex number is replaced with a hermitian operator having non-negative eigenvalues. The representation as $h = AA^\dagger + A^\dagger A$ is would guarantee this. The phase of complex number would be replaced by a unitary operator U possibly allowing the representation $U = exp(iT)$, T hermitian. The commutativity condition

   $$[h_{ij}, u_{ij}] = 0 \tag{5.1}$$

   for a given matrix element is also suggestive but as already noticed, Uncertainty Principle suggests that modulus and phase do not commute as operators. The commutator of modulus and phase would naturally be equal to that between momentum operator and plane wave:

   $$[h_{ij}, u_{ij}] = i\hbar \times u_{ij} \quad , \tag{5.2}$$

   Here $\hbar = h/2\pi$ can be chosen to be unity in standard quantum theory. In TGD it can be generalized to a hermitian operator H_{eff}/h with an integer valued spectrum of eigenvalues given by $h_{eff}/h = n$ so that ordinary and dark matter sectors would be unified to single structure mathematically.

3. The notions of eigenvalues and eigenvectors for a hermitian operator should generalize. Now hermitian operator H would be a matrix with formally the same structure as $N \times N$ hermitian matrix in commutative number field - say complex numbers - possibly satisfying additional conditions.

   Hermitian matrix can be written as

   $$H_{ij} = h_{ij}u_{ij} \quad \text{for i>j} \quad H_{ij} = u_{ij}h_{ij} \quad \text{for i<j} \quad , \quad H_{ii} = h_i \quad . \tag{5.3}$$

   Hermiticity conditions $H_{ij} = H_{ji}^\dagger$ give

$$h_{ij} = h_{ji} \quad , \quad u_{ij} = u_{ji}^{\dagger} \quad . \tag{5.4}$$

Here it has been assumed that one has quantum SU(2). For quantum U(2) one would have $U_{11} = U_{22}^{\dagger} = h_a u_a$ with u_a commuting with other operators. The form of the conditions is same as for ordinary hermitian matrices and it is not necessary to assume commutativity $[h_{ij}, u_{ij}] = 0$. Generalization of Pauli spin matrices provides a simple illustration.

4. The well-definedness of eigenvalue problem gives a strong constraint on the notion of hermitian quantum matrix. Eigenvalues of hermitian operator are determined by the vanishing of determinant $det(H - \lambda I)$. Its expression involves sub-determinants and one must decide whether to demand that the definition of determinant is independent of which column or row one chooses to develop the determinant.

   For ordinary matrix the determinant is expressible as sum of symmetric functions:

$$det(H - \lambda I) = \sum \lambda^n S_n(H) \quad . \tag{5.5}$$

   Elementary symmetric functions S_n - n-functions in following - have the property that they are sums of contributions from to n-element paths along the matrix with the property that path contains no vertical or horizontal steps. One has a discrete analog of path integral in which time increases in each step by unit. The analogy with fermionic path integral is also obvious. In the non-commutative case non-commutativity poses problems since different orderings of rows (or columns) along the same n-path give different results.

   (a) For the first option one gives up the condition that determinant can be developed with respect to any row or column and defines determinant by developing it with respect to say first row or first column. If one developing with respect to the column (row) the permutations of rows (columns) do not affect the value of determinant or sub-determinants but permutations of columns (rows) do so unless one poses additional conditions stating that the permutations do not affect given contribution to the determinant or sub-determinant. It turns out that this option must be applied in the case of ordinary quantum group. For quantum phase $q = \pm 1$ the determinant is invariant under permutations of both rows and columns.

   (b) Second manner to get rid of difficulty would be that n-path does not depend on the ordering of the rows (columns) differ only by the usual sign factor. For 2×2 case this would give

$$ad - bc = da - cb \quad , \quad \text{(Option 2)} \tag{5.6}$$

   These conditions state the invariance of the n-path under permutation group S_n permuting rows or columns.

   (c) For the third option the elements along n-paths commute: paths could be said to be "classical". The invariance of N-path in this sense guarantees the invariance of all n-paths. In 2-D case this gives

$$[a, d] = 0 \quad , \quad [b, c] = 0 \quad . \quad \text{(Option 3)} \tag{5.7}$$

5. One should have a well-defined eigenvalue problem. If the n-functions commute, one can diagonalize the corresponding operators simultaneously and the eigenvalues problem reduces to possibly infinite number of ordinary eigenvalue problems corresponding to restrictions to given set of eigenvalues associated with $N - 1$ symmetric functions. This gives an additional constraint on quantum matrices.

In 2-dimensional case one would have the condition

$$[ad - bc, a + d] = 0 \ . \tag{5.8}$$

Depending on how strong S_2 invariance one requires, one obtains 0, 1, 2 nontrivial conditions for 2×2 quantum matrices and 1 condition from the commutativity of n-functions besides hermiticity conditions.

For $N \times N$-matrices one would have $N! - 1$ non-trivial conditions from the strong form of permutation invariance guaranteeing the permutation symmetry of n-functions and $N(N-1)/2$ conditions from the commutativity of n-functions.

6. The eigenvectors of the density matrix are obtained in the usual manner for each eigenvalue contributing to quantum eigenvalue. Also the diagonalization can be carried out by a unitary transformation for each eigenvalue separately. Hence the standard approach seems to generalize almost trivially.

   What makes the proposal non-trivial and possibly physically interesting is that the hermitian operators are not assumed to be just tensor products of $N \times N$ hermitian matrices with hermitian operators in Hilbert space.

The notion of unitary quantum matrix should also make sense. The naive guess is that the exponentiation of a linear combination of ordinary hermitian matrices with coefficients, which are hermitian matrices gives quantum unitary matrices. In the case of U(1) the replacement of exponentiation parameter t in $exp(itX)$ with a hermitian operator gives standard expression for the exponent and it is trivial to see that unitary conditions are satisfied also in this case. Also in the case of SU(2) it is easy to verify that the guess is correct. One must also check that one indeed obtains a group: it could also happen that only semi-group is obtained.

In any case, one could speak of quantum matrix groups with coordinates replaced by hermitian matrices. These quantum matrix group need not be identical with quantum groups in the standard sense of the word. Maybe this could provide one possible meaning for quantization in the case of groups and perhaps also in the case of coset spaces G/H.

5.2 The Relationship To Quantum Groups And Quantum Lie Algebras

It is interesting to find out whether quantum matrices give rise to quantum groups under suitable additional conditions. The child's guess for these conditions is that the permutation of rows and columns correspond to braiding for the hermitian moduli h_{ij} defined by unitary operators U_{ij}.

5.2.1 *Quantum groups and quantum matrices*

The conditions for hermiticity and unitary do not involve quantum parameter q, which suggests that the naive generalization of the notion of unitary matrix gives unitary group obtained by replacing complex number field with operator algebra gives group with coordinates defined by hermitian operators rather than standard quantum group. This turns out to be the case and it seems that quantum matrices provide a concrete representation for quantum group. The notion of braiding as that for operators h_{ij} can be said to emerge from the notion of quantum matrix.

1. Exponential of quantum hermitian matrix is excellent candidate for quantum unitary matrix. One should check the exponentiation indeed gives rise to a quantum unitary matrix. For $q = \pm 1$ this seems obvious but one should check this separately for other roots of unity. Instead of considering the general case, we consider explicit ansatz for unitary U(2) quantum matrix as $U = [a, b; -b^\dagger, a^\dagger]$. The conditions for unitary quantum group in the proposed sense would state the orthonormality and unit norm property of rows/columns.

   The explicit form of the conditions reads as

$$\begin{array}{ll} ab - ba = 0 \ , & ab^\dagger = b^\dagger a \ , \\ aa^\dagger + bb^\dagger = 1 \ , & a^\dagger a + b^\dagger b = 1 \ . \end{array} \tag{5.9}$$

The orthogonality conditions are unique and reduce to the vanishing of commutators.

Normalization conditions involve a choice of ordering. One possible manner to avoid the problem is to assume that both orderings give same unit length for row or column (as done above). If only the other option is assumed then only third or fourth equations is needed. The invariance of determinant under permutation of rows would imply $[a, a^\dagger] = [b, b^\dagger] = 0$ and the ordering problem would disappear.

2. One can look what conditions the explicit representation $U_{ij} = h_{ij}u_{ij}$ or equivalently $[h_a u_a, h_b u_b; -u_b^\dagger h_b, u_a^\dagger h_a]$ gives. The intuitive expectation is that U(2) matrix decomposes to a product of commutating SU(2) matrix and U(1) matrices. This implies that u_a commutes with the other matrices involved. One obtains the conditions

$$h_a h_b = h_b(u_b h_a u_b^\dagger) \ , \quad h_b h_a = (u_b h_a u_b^\dagger)h_b \ . \tag{5.10}$$

These conditions state that the permutation of h_a and h_b analogous to braiding operation is a unitary operation.

For the purposes of comparison consider now the corresponding conditions for $SU(2)_q$ matrix.

1. The $\mathrm{SU}(2)_q$ matrix $[a, b; b^\dagger, a^\dagger]$ with *real* value of q (see `http://tinyurl.com/yb8tycag`) satisfies the conditions

$$\begin{array}{lll} ba = qab \ , & b^\dagger a = qab^\dagger , & bb^\dagger = b^\dagger b \ , \\ a^\dagger a + q^2 b^\dagger b = 1 \ , & aa^\dagger + bb^\dagger = 1 \ . & \end{array} \tag{5.11}$$

This gives $[a^\dagger, a] = (1 - q^2)b^\dagger b$. The above conditions would correspond to $q = \pm 1$ but with complex numbers replaced with operator algebra. q-commutativity obviously replaces ordinary commutativity in the conditions and one can speak of q-orthonormality.

For complex values of q - in particular roots of unity - the condition $a^\dagger a + q^2 b^\dagger b = 1$ is in general not self-consistent since hermitian conjugation transforms q^2 to its complex conjugate. Hence this condition must be dropped for complex roots of unity.

2. Only for $q = \pm 1$ corresponding to Bose-Einstein and Fermi-Dirac statistics the conditions are consistent with the invariance of n-functions (determinant) under permutations of both rows and columns. Indeed, if 2×2 q-determinant is developed with respect to column, the permutation of rows does not affect its value. This is trivially true also in $N \times N$ dimensional case since the permutation of rows does not affect the n-paths at all.

If the symmetry under permutations is weakened, nothing prevents from posing quantum orthogonality conditions also now and the decomposition to a product of positive and hermitian matrices give a concrete meaning to the notion of quantum group.

Do various $n-$functions commute with each other for $\mathrm{SU}(2)_q$? The only commutator of this kind is that for the trace and determinant and should vanish:

$$\left[b + b^\dagger, aa^\dagger + bb^\dagger\right] = 0 \ . \tag{5.12}$$

Since $a^\dagger a$ and $aa^\dagger$ are linear combinations of $b^\dagger b = b^\dagger b$, they vanish. Hence it seems that TGD based view about quantum groups is consistent with the standard view.

3. One can look these conditions in TGD framework by restricting the consideration to the case of SU(2) ($u_a = 1$) and using the ansatz $U = [h_a, h_b u_b; -u_b^\dagger h_b, h_a]$. Orthogonality conditions read as

$$h_a h_b = q h_b(u_b h_a u_b^\dagger) \ , \quad h_b h_a = q(u_b h_a u_b^\dagger)h_b \ .$$

If q is root of unity, these conditions state that the permutation of h_a and h_b analogous to a unitary braiding operation apart from a multiplication with quantum phase q. For $q = \pm 1$ the sign-factor is that in standard statistics. Braiding picture could help guess the commutators of h_{ij} in the case of $N \times N$ quantum matrices. The permutations of rows and columns would have interpretation as braidings and one could say that braided commutators of matrix elements vanish.

The conditions from the normalization give

$$h_a^2 + h_b^2 = 1 \ , \quad h_a^2 + q^2(u_b^\dagger h_b^2 u_b) = 1 \ . \tag{5.13}$$

For complex q the latter condition does not make sense since $h_a^2 - 1$ and $u_b^\dagger h_b^2 u_b$ are hermitian matrices with real eigenvalues. Also for real values of $q \neq \pm 1$ one obtains contradicion since the spectra of unitarily related hermitian operators would differ by scaling factor q^2. Hence one must give up the condition involving q^2 unless one has $q = \pm 1$. Note that the term proportional to q^2 does not allow interpretation in terms of braiding.

4. Roots of unity are natural number theoretically as values of q but number theoretical universality allows the generic value of q would be a complex number existing simultaneously in all p-adic number properly extended. This would suggest the spectrum of q to come as

$$q(m,n) = e^{1/m} exp(\frac{i2\pi}{n}) \ . \tag{5.14}$$

The motivation comes from the fact that e^p is ordinary p-adic number for all p-adic number fields so e and also any root of e defines a finite-dimensional extension of p-adic numbers [K20] [?]. The roots of unity would be associated to the discretization of the ordinary angles in case of compact matrix groups. Roots of e would be associated with the discretization of hyperbolic angles needed in the case of non-compact matrix groups such as SL(2,C).

Also now unification of various values of q to single single operator Q, which is product of *commuting* hermitian and unitary operators and commuting with the hermitian operator H representing the spectrum of Planck constant would code the spectrum. Skeptic can of course wonder, whether the modulus and phase of Q can be assumed to commute. The relationship between integers associated with H and Q is interesting.

5.2.2 *Quantum Lie algebras and quantum matrices*

What about quantum Lie algebras? There are many notions of quantum Lie algebra and quantum group. General formulas for the commutation relations are well-known for Drinfeld-Jimbo type quantum groups (see `http://tinyurl.com/yb8tycag`). The simplest guess is that one just poses the defining conditions for quantum group, replaces complex numbers as coefficient module with operator algebra, and poses the above described conditions making possible to speak about eigenvalues and eigen vectors. One might however hope that this representation allows to realize the non-commutativity of matrix elements of quantum Lie algebra in a concrete manner.

1. For SU(2) the commutation relations for the elements X_+, X_-, h read as

$$[h, X_\pm] = \pm X_\pm \ , \quad [X_+, X_-] = h \ . \tag{5.15}$$

Here one can use the 2×2 matrix representations for the ladder operators $X^\pm$ and diagonal angular momentum generator h.

2. For SU(2)$_q$ one has

$$[h, X_\pm] = \pm X_\pm \ , \quad [X_+, X_-] = \frac{q^h - q^{-h}}{q - q^{-1}} \ . \tag{5.16}$$

3. Using the ansatz for the generators but allowing hermitian operator coefficients in non-diagonal generators $X_{\pm}$, one obtains the condition

   For $SU(2)_q$ one would have

$$[X_+, X_-] = h_+^2 = h_-^2 = \frac{q^h - q^{-h}}{q - q^{-1}} \quad . \tag{5.17}$$

   Clearly, the proposal might make possible to have concrete representations for the quantum Lie algebras making the decomposition to measurable and directly non-measurable degrees of freedom explicit.

The conclusion is that finite measurement resolution does not lead automatically to standard quantum groups although the proposed realization is consistent with them. Also the quantum phases $q = \pm 1$ $n = 1, 2$ are realized and correspond to strong permutation symmetry and Bose-Einstein and Fermi statistics.

5.3 About Possible Applications

The realization for the notion of finite measurement resolution is certainly the basic application but one can imagine also other applications where hermitian and unitary matrices appear.

5.3.1 *Density matrix description of degrees of freedom below measurement resolution*

Density matrix ρ obtained by tracing over non-observable degrees of freedom is a fundamental example about a hermitian matrix satisfying the additional condition $Tr(\rho) = 1$.

1. A state function reduction with a finite measurement resolution would lead to a non-pure state. This state would be describable using $N \times N$-dimensional quantum hermitian quantum density matrix satisfying the condition $Tr(\rho) = 1$ (or more generally $Tr_q(\rho) = 1$), and satisfying the additional conditions allowing to reduce its diagonalization to that for a collection of ordinary density matrices so that the eigenvalues of ordinary density matrix would be replaced by N quantum eigenvalues defined by infinite-dimensional diagonalized density matrices.

2. One would have N quantum eigenvalues - quantum probabilities - each decomposing to possibly infinite set of ordinary probabilities assignable to the degrees of freedom below measurement resolution and defining density matrix for non-pure states resulting in state function reduction.

5.3.2 *Some questions*

Some further questions pop up naturally.

1. One might hope that the quantum counterparts of hermitian operators are in some sense universal, at least in TGD framework (by quantum criticality). Could the condition that the commutator of hermitian generators is proportional to $i\hbar$ times hermitian generator pose additional constraints? In 2-D case this condition is satisfied for quantum SU(2) generators and very probably the same is true also in the general case. The possible problems result from the non-commutativity but $(XY)^\dagger = Y^\dagger X^\dagger$ identity takes care that there are no problems.

2. One can also raise physics related questions. What one can say about most general quantum Hamiltonians and their energy spectra, say quantum hydrogen atom? What about quantum angular momentum? If the proposed construction is only a concretization of abstract quantum group construction, then nothing new is expected at the level of representations of quantum groups.

3. Could the spectrum of h_{eff} define a quantum h as a hermitian positive definite operator? Could this allow a description for the presence of dark matter, which is not directly observable? Same question applies to the quantum parameter q.

4. M-matrices are basic building bricks of scattering amplitudes in ZEO. M-matrix is produce of hermitian "complex" square root H of density matrix satisfying $H^2 = \rho$ and unitary S-matrix S. It has been proposed that these matrices commute. The previous consideration relying on basic quantum thinking suggests that they relate like translation generator in radial direction and phase defined by angle and thus satisfy $[H, S] = i(H_{eff}/h) \times S$. This would give enormously powerful additional condition to S-matrix. One can also ask whether M-matrices in presence of degrees of freedom below measurement resolution is quantum version of M-matrix in the proposed sense.

5. Fractality is of of the key notions of TGD and characterizes also hyperfinite factors. I have proposed some realizations of fractality such as infinite primes and finite-dimensional Hilbert spaces taking the role of natural numbers and ordinary sum and product replaced with direct sum and tensor product. One could also imagine a fractal hierarchy of quantum matrices obtained by replacing the operators appearing as matrix elements of quantum matrix element by quantum matrices. This hierarchy could relate to the sequence of inclusions of HFFs.

6 Jones Inclusions And Cognitive Consciousness

WCW spinors have a natural interpretation in terms of a quantum version of Boolean algebra. Beliefs of various kinds are the basic element of cognition and obviously involve a representation of the external world or part of it as states of the system defining the believer. Jones inclusions mediating unitary mappings between the spaces of WCWs spinors of two systems are excellent candidates for these maps, and it is interesting to find what one kind of model for beliefs this picture leads to.

The resulting quantum model for beliefs provides a cognitive interpretation for quantum groups and predicts a universal spectrum for the probabilities that a given belief is true. This spectrum depends only on the integer n characterizing the quantum phase $q = exp(i2\pi/n)$ characterizing the Jones inclusion. For $n \neq \infty$ the logic is inherently fuzzy so that absolute knowledge is impossible. $q = 1$ gives ordinary quantum logic with qbits having precise truth values after state function reduction.

6.1 Does One Have A Hierarchy Of U- And M-Matrices?

U-matrix describes scattering of zero energy states and since zero energy states can be illustrated in terms of Feynman diagrams one can say that scattering of Feynman diagrams is in question. The initial and final states of the scattering are superpositions of Feynman diagrams characterizing the corresponding M-matrices which contain also the positive square root of density matrix as a factor.

The hypothesis that U-matrix is the tensor product of S-matrix part of M-matrix and its Hermitian conjugate would make U-matrix an object deducible by physical measurements. One cannot of course exclude that something totally new emerges. For instance, the description of quantum jumps creating zero energy state from vacuum might require that U-matrix does not reduce in this manner. One can assign to the U-matrix a square like structure with S-matrix and its Hermitian conjugate assigned with the opposite sides of a square.

One can imagine of constructing higher level physical states as composites of zero energy states by replacing the S-matrix with M-matrix in the square like structure. These states would provide a physical representation of U-matrix. One could define U-matrix for these states in a similar manner. This kind of hierarchy could be continued indefinitely and the hierarchy of higher level U and M-matrices would be labeled by a hierarchy of n-cubes, $n = 1, 2,...$ TGD inspired theory of consciousness suggests that this hierarchy can be interpreted as a hierarchy of abstractions represented in terms of physical states. This hierarchy brings strongly in mind also the hierarchies of n-algebras and n-groups and this forces to consider the possibility that something genuinely new emerges at each step of the hierarchy. A connection with the hierarchies of infinite primes [K17] and Jones inclusions are suggestive.

6.2 Feynman Diagrams As Higher Level Particles And Their Scattering As Dynamics Of Self Consciousness

The hierarchy of inclusions of hyper-finite factors of II_1 as counterpart for many-sheeted space-time lead inevitably to the idea that this hierarchy corresponds to a hierarchy of generalized Feynman diagrams for which Feynman diagrams at a given level become particles at the next level. Accepting this idea, one is led to ask what kind of quantum states these Feynman diagrams correspond, how one could describe interactions of these higher level particles, what is the interpretation for these higher level states, and whether they can be detected.

6.2.1 *Jones inclusions as analogs of space-time surfaces*

The idea about space-time as a 4-surface replicates itself at the level of operator algebra and state space in the sense that Jones inclusion can be seen as a representation of the operator algebra $\mathcal{N}$ as infinite-dimensional linear sub-space (surface) of the operator algebra $\mathcal{M}$. This encourages to think that generalized Feynman diagrams could correspond to image surfaces in II_1 factor having identification as kind of quantum space-time surfaces.

Suppose that the modular S-matrices are representable as the inner automorphisms $\Delta(\mathcal{M}_k^{it}$ assigned to the external lines of Feynman diagrams. This would mean that $\mathcal{N} \subset \mathcal{M}_k$ moves inside $calM_k$ along a geodesic line determined by the inner automorphism. At the vertex the factors $calM_k$ to fuse along $\mathcal{N}$ to form a Connes tensor product. Hence the copies of $\mathcal{N}$ move inside $\mathcal{M}_k$ like incoming 3-surfaces in H and fuse together at the vertex. Since all $\mathcal{M}_k$ are isomorphic to a universal factor $\mathcal{M}$, many-sheeted space-time would have a kind of quantum image inside II_1 factor consisting of pieces which are $d = \mathcal{M} : \mathcal{N}/2$-dimensional quantum spaces according to the identification of the quantum space as subspace of quantum group to be discussed later. In the case of partonic Clifford algebras the dimension would be indeed $d \leq 2$.

6.2.2 *The hierarchy of Jones inclusions defines a hierarchy of S-matrices*

It is possible to assign to a given Jones inclusion $\mathcal{N} \subset \mathcal{M}$ an entire hierarchy of Jones inclusions $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 ...$, $\mathcal{M}_0 = N$, $\mathcal{M}_1 = M$. A possible interpretation for these inclusions would be as a sequence of topological condensations.

This sequence also defines a hierarchy of Feynman diagrams inside Feynman diagrams. The factor $\mathcal{M}$ containing the Feynman diagram having as its lines the unitary orbits of $\mathcal{N}$ under $\Delta_{\mathcal{M}}$ becomes a parton in $\mathcal{M}_1$ and its unitary orbits under $\Delta_{\mathcal{M}_1}$ define lines of Feynman diagrams in M_1. The concrete representation for M-matrix or projection of it to some subspace as entanglement coefficients of partons at the ends of a braid assignable to the space-like 3-surface representing a vertex of a higher level Feynman diagram. In this manner quantum dynamics would be coded and simulated by quantum states.

The outcome can be said to be a hierarchy of Feynman diagrams within Feynman diagrams, a fractal structure for which many particle scattering events at a given level become particles at the next level. The particles at the next level represent dynamics at the lower level: they have the property of "being about" representing perhaps the most crucial element of conscious experience. Since net conserved quantum numbers can vanish for a system in TGD Universe, this kind of hierarchy indeed allows a realization as zero energy states. Crossing symmetry can be understood in terms of this picture and has been applied to construct a model for M-matrix at high energy limit [K11] .

One might perhaps say that quantum space-time corresponds to a double inclusion and that further inclusions bring in N-parameter families of space-time surfaces.

6.2.3 *Higher level Feynman diagrams*

The lines of Feynman diagram in $\mathcal{M}_{n+1}$ are geodesic lines representing orbits of $\mathcal{M}_n$ and this kind of lines meet at vertex and scatter. The evolution along lines is determined by $\Delta_{\mathcal{M}_{n+1}}$. These lines contain within themselves $\mathcal{M}_n$ Feynman diagrams with similar structure and the hierarchy continues down to the lowest level at which ordinary elementary particles are encountered.

For instance, the generalized Feynman diagrams at the second level are ribbon diagrams obtained by thickening the ordinary diagrams in the new time direction. The interpretation as ribbon diagrams crucial for topological quantum computation and suggested to be realizable in terms of zero energy states in [**?**] is natural. At each level a new time parameter is introduced so that the dimension of the diagram can be arbitrarily high. The dynamics is not that of ordinary surfaces but the dynamics induced by the $\Delta_{\mathcal{M}_n}$.

6.2.4 *Quantum states defined by higher level Feynman diagrams*

The intuitive picture is that higher level quantum states corresponds to the self reflective aspect of existence and must provide representations for the quantum dynamics of lower levels in their own structure. This dynamics is characterized by M-matrix whose elements have representation in terms of Feynman diagrams.

1. These states correspond to zero energy states in which initial states have "positive energies" and final states have "negative energies". The net conserved quantum numbers of initial and final state partons compensate each other. Gravitational energies, and more generally gravitational quantum numbers defined as absolute values of the net quantum numbers of initial and final states do not vanish. One can say that thoughts have gravitational mass but no inertial mass.

2. States in sub-spaces of positive and negative energy states are entangled with entanglement coefficients given by M-matrix at the level below.

To make this more concrete, consider first the simplest non-trivial case. In this case the particles can be characterized as ordinary Feynman diagrams, or more precisely as scattering events so that the state is characterized by $\hat{S} = P_{in}SP_{out}$, where S is S-matrix and P_{in} *resp.* P_{out} is the projection to a subspace of initial *resp.* final states. An entangled state with the projection of S-matrix giving the entanglement coefficients is in question.

The larger the domains of projectors P_{in} and P_{out}, the higher the representative capacity of the state. The norm of the non-normalized state $\hat{S}$ is $Tr(\hat{S}\hat{S}^\dagger) \leq 1$ for II_1 factors, and at the limit $\hat{S} = S$ the norm equals to 1. Hence, by II_1 property, the state always entangles infinite number of states, and can in principle code the entire S-matrix to entanglement coefficients.

The states in which positive and negative energy states are entangled by a projection of S-matrix might define only a particular instance of states for which conserved quantum numbers vanish. The model for the interaction of Feynman diagrams discussed below applies also to these more general states.

6.2.5 *The interaction of $\mathcal{M}_n$ Feynman diagrams at the second level of hierarchy*

What constraints can one pose to the higher level reactions? How Feynman diagrams interact? Consider first the scattering at the second level of hierarchy ($\mathcal{M}_1$), the first level $\mathcal{M}_0$ being assigned to the interactions of the ordinary matter.

1. Conservation laws pose constraints on the scattering at level $\mathcal{M}_1$. The Feynman diagrams can transform to new Feynman diagrams only in such a manner that the net quantum numbers are conserved separately for the initial positive energy states and final negative energy states of the diagram. The simplest assumption is that positive energy matter and negative energy matter know nothing about each other and effectively live in separate worlds. The scattering matrix form Feynman diagram like states would thus be simply the tensor product $S \otimes S^\dagger$, where S is the S-matrix characterizing the lowest level interactions and identifiable as unitary factor of M-matrix for zero energy states. Reductionism would be realized in the sense that, apart from the new elements brought in by $\Delta_{\mathcal{M}_n}$ defining single particle free dynamics, the lowest level would determine in principle everything occurring at the higher level providing representations about representations about... for what occurs at the basic level. The lowest level would represent the physical world and higher levels the theory about it.

2. The description of hadronic reactions in terms of partons serves as a guide line when one tries to understand higher level Feynman diagrams. The fusion of hadronic space-time sheets corresponds to the vertices $\mathcal{M}_1$. In the vertex the analog of parton plasma is formed by a process known as parton fragmentation. This means that the partonic Feynman diagrams belonging to disjoint copies of $\mathcal{M}_0$ find themselves inside the same copy of $\mathcal{M}_0$. The standard description would apply to the scattering of the initial *resp.* final state partons.

3. After the scattering of partons hadronization takes place. The analog of hadronization in the recent case is the organization of the initial and final state partons to groups I_i and F_i such that the net conserved quantum numbers are same for I_i and F_i. These conditions can be satisfied if the interactions in the plasma phase occur only between particles belonging to the clusters labeled by the index i. Otherwise only single particle states in $\mathcal{M}_1$ would be produced in the reactions in the generic case. The cluster decomposition of S-matrix to a direct sum of terms corresponding to partitions of the initial state particles to clusters which do not interact with each other obviously corresponds to the "hadronization". Therefore no new dynamics need to be introduced.

4. One cannot avoid the question whether the parton picture about hadrons indeed corresponds to a higher level physics of this kind. This would require that hadronic space-time sheets carry the net quantum numbers of hadrons. The net quantum numbers associated with the initial state partons would be naturally identical with the net quantum numbers of hadron. Partons and they negative energy conjugates would provide in this picture a representation of hadron about hadron. This kind of interpretation of partons would make understandable why they cannot be observed directly. A possible objection is that the net gravitational mass of hadron would be three times the gravitational mass deduced from the inertial mass of hadron if partons feed their gravitational fluxes to the space-time sheet carrying Earth's gravitational field.

5. This picture could also relate to the suggested duality between string and parton pictures [K18] . In parton picture hadron is formed from partons represented by space-like 2-surfaces X_i^2 connected by join along boundaries bonds. In string picture partonic 2-surfaces are replaced with string orbits. If one puts positive and negative energy particles at the ends of string diagram one indeed obtains a higher level representation of hadron. If these pictures are dual then also in parton picture positive and negative energies should compensate each other. Interestingly, light-like 3-D causal determinants identified as orbits of partons could be interpreted as orbits of light like string word sheets with "time" coordinate varying in space-like direction.

6.2.6 *Scattering of Feynman diagrams at the higher levels of hierarchy*

This picture generalizes to the description of higher level Feynman diagrams.

1. Assume that higher level vertices have recursive structure allowing to reduce the Feynman diagrams to ordinary Feynman diagrams by a procedure consisting of finite steps.

2. The lines of diagrams are classified as incoming or outgoing lines according to whether the time orientation of the line is positive or negative. The time orientation is associated with the time parameter t_n characterizing the automorphism $\Delta_{\mathcal{M}_\backslash}^{it_n}$. The incoming and outgoing net quantum numbers compensate each other. These quantum numbers are basically the quantum numbers of the state at the lowest level of the hierarchy.

3. In the vertices the $\mathcal{M}_{n+1}$ particles fuse and $\mathcal{M}_n$ particles form the analog of quark gluon plasma. The initial and final state particles of $\mathcal{M}_n$ Feynman diagram scatter independently and the S-matrix S_{n+1} describing the process is tensor product $S_n \otimes S_n^\dagger$. By the clustering property of S-matrix, this scattering occurs only for groups formed by partons formed by the incoming and outgoing particles $\mathcal{M}_n$ particles and each outgoing $\mathcal{M}_{n+1}$ line contains and irreducible $\mathcal{M}_n$ diagram. By continuing the recursion one finally ends down with ordinary Feynman diagrams.

6.3 Logic, Beliefs, And Spinor Fields In The World Of Classical Worlds

Beliefs can be characterized as Boolean value maps $\beta_i(p)$ telling whether i believes in proposition p or not. Additional structure is brought in by introducing the map $\lambda_i(p)$ telling whether p is true or not in the environment of i. The task is to find quantum counterpart for this model.

6.3.1 *The spectrum of probabilities for outcomes in state function reduction with finite measurement resolution is universal*

Consider quantum two-spinor as a model of a system with finite measurement resolution implying that state function reduction do not anymore lead to a spin state with a precise value but that one can only predict the probability distribution for the outcome of measurement. These probabilities can be also interpreted as truth values of a belief in finite cognitive resolution.

It is actually possible to calculate the spectrum of the probabilities of truth values with rather mild additional assumptions.

1. Since the Hermitian operators $X_1 = (z^1\overline{z^1} + \overline{z^1}z^1)/2$ and $X_2 = (z^2\overline{z^2} + \overline{z^2}z^2)/2$ commute, physical states can be chosen to be eigen states of these operators and it is possible to assign to the truth values probabilities given by $p_1 = X_1/R^2$ and $p_2 = X_2/R^2$, $R^2 = X_1 + X_2$.

2. By introducing the analog of the harmonic oscillator vacuum as a state $|0\rangle$ satisfying $z^1|0\rangle = z^2|0\rangle = 0$, one obtains eigen states of X_1 and X_2 as states $|n_1, n_2\rangle = \overline{z^1}^{n_1}\overline{z^2}^{n_2}|0\rangle$, $n_1 \geq 0, n_2 \geq 0$. The eigenvalues of X_1 and X_2 are given by a modified harmonic oscillator spectrum as

$$X_1 = (1/2 + n_1 q^{n_2})r \ \ , \ \ X_2 = (1/2 + n_2 q^{n_1})r \ \ .$$

   The reality of eigenvalues (hermiticity) is guaranteed if one has $n_1 = N_1 n$ and $n_1 = N_2 n$ and implies that the spectrum of eigen states gets increasingly thinner for $n \to \infty$. This must somehow reflect the fractal dimension. The fact that large values of oscillator quantum numbers n_1 and n_2 correspond to the classical limit suggests that modulo condition guarantees approximate classicality of the logic for $n \to \infty$.

3. The probabilities p_1 and p_2 for the truth values given by $(p_1, p_2) = (1/2 + N_1 n, 1/2 + N_2 n)/[1 + (N_1 + N_2)n]$ are rational and allow an interpretation as both real and p-adic numbers. This also conforms with the frequency interpretation for probabilities. All states are are inherently fuzzy and only at the limits $N_1 \gg N_2$ and $N_2 \gg N_1$ non-fuzzy states result. As noticed, $n = \infty$ must be treated separately and corresponds to an ordinary non-fuzzy qbit logic. At $n \to \infty$ limit one has $(p_1, p_2) = (N_1, N_2)/(N_1, N_2)$: at this limit $N_1 = 0$ or $N_2 = 0$ states are non-fuzzy.

4. A possible interpretation for the fuzziness is in terms of finite measurement resolution. The quantized probabilities could be assigned with diagonalized density matrix regarded as matrix with elements which are commuting hermitian operators. The generalized eigenvalues would be eigenvalues spectra. States would not be pure expect at the limits$N_1 \gg N_2$ and $N_2 \gg N_1$. The non-purity of the state could be understood in terms of entanglement with the degrees of freedom below measurement resolution describable in terms of inclusion of von Neumann algebras. One could perhaps say that in finite measurement resolution the outcome of state function reduction is always non-pure state characterized by a universal density matrix obtained by tracing over non-visible degrees of freedom.

6.3.2 *WCW spinors as logic statements*

In TGD framework the infinite-dimensional WCW (CH) spinor fields defined in CH, the "world of classical worlds", describe quantum states of the Universe [K6] . WCW spinor field can be regarded as a state in infinite-dimensional Fock space and are labeled by a collection of various two valued indices like spin and weak isospin. The interpretation is as a collection of truth values of logic statements one for each fermionic oscillator operator in the state. For instance, spin up and down would correspond to two possible truth values of a proposition characterized by other quantum numbers of the mode.

The hierarchy of space-time sheet could define a physical correlate for the hierarchy of higher order logics (statements about statements about...). The space-time sheet containing N fermions topologically condensed at a larger space-time sheet behaves as a fermion or boson depending on whether N is odd or even. This hierarchy has also a number theoretic counterpart: the construction of infinite primes [K17] corresponds to a repeated second quantization of a super-symmetric quantum field theory.

6.3.3 *Quantal description of beliefs*

The question is whether TGD inspired theory of consciousness allows a fundamental description of beliefs.

1. Beliefs define a model about some subsystem of universe constructed by the believer. This model can be understood as some kind of representation of real word in the state space representing the beliefs.

2. One can wonder what is the difference between real and p-adic variants of WCW spinor fields and whether they could represent reality and beliefs about reality. WCW spinors (as opposed to spinor fields) are constructible in terms of fermionic oscillator operators and seem to be universal in the sense that one cannot speak about p-adic and real WCW spinors as different objects. Real/p-adic spinor fields however have real/p-adic space-time sheets as arguments. This would suggest that there is no fundamental difference between the logic statements represented by p-adic and real WCW spinors.

3. This vision is realized if the intersection of reality and various p-adicities corresponds to an algebraically universal set of consisting of partonic 2-surfaces and string world sheets for which defining parameters are WCW coordinates in an algebraic extension of rationals defining that for p-adic number fields. Induced spinor fields would be localized at string world sheets and their intersections with partonic 2-surfaces and would be number theoretically universal. If second quantized induced spinor fields are correlates of Boolean cognition, which is behind the entire mathematics, their number theoretical universality is indeed a highly natural condition. Also fermionic anticommutation relations are number theoretically universal. By conformal invariance the conformal moduli of string world sheets and partonic 2-surface would be the natural WCW coordinates for the 2-surfaces in question and I proposed their p-adicization already in p-adic mass calculations for two decades ago.

   This picture would provide an elegant realization for the p-adicization. There would be ne need to map real space-time surfaces directly to p-adic ones and vice versa and one would avoid problems related to general coordinate invariance (GCI) completely. Strong form of holography would assign to partonic surfaces the real and p-adic variants. Already p-adic mass calculations support the presence of cognition in all length scales.

These observations suggest a more concrete view about how beliefs emerge physically.

The idea that p-adic WCW spinor fields could serve as representations of beliefs and real WCW spinor fields as representations of reality looks very nice and conforms with the adelic vision that space-time is adele - a book-like structure contains space-time sheets in various number fields as pages glued together along back for which the parameters characterizing space-time surface are numbers in an algebraic expansion of rationals. Real space-time surfaces would be correlates for sensory experience and p-adic space-time sheets for cognition.

6.4 Jones Inclusions For Hyperfinite Factors Of Type II_1 As A Model For Symbolic And Cognitive Representations

Consider next a more detailed model for how cognitive representations and beliefs are realized at quantum level. This model generalizes trivially to symbolic representations.

The Clifford algebra of gamma matrices associated with WCW spinor fields corresponds to a von Neumann algebra known as hyper-finite factor of type II_1. The mathematics of these algebras is extremely beautiful and reproduces basic mathematical structures of modern physics (conformal field theories, quantum groups, knot and braid groups,....) from the mere assumption that the world of classical worlds possesses infinite-dimensional Kähler geometry and allows spinor structure.

The almost defining feature is that the infinite-dimensional unit matrix of the Clifford algebra in question has by definition unit trace. Type II_1 factors allow also what are known as Jones inclusions of Clifford algebras $\mathcal{N} \subset \mathcal{M}$. What is special to II_1 factors is that the induced unitary mappings between spinor spaces are genuine inclusions rather than 1-1 maps.

The S-matrix associated with the real-to-p-adic quantum transition inducing belief from reality would naturally define Jones inclusion of CH Clifford algebra $\mathcal{N}$ associated with the real space-time sheet to the Clifford algebra $\mathcal{M}$ associated with the p-adic space-time sheet. The moduli squared of S-matrix elements would define probabilities for pairs or real and belief states.

In Jones inclusion $\mathcal{N} \subset \mathcal{M}$ the factor $\mathcal{N}$ is included in factor $\mathcal{M}$ such that $\mathcal{M}$ can be expressed as $\mathcal{N}$-module over quantum space $\mathcal{M}/\mathcal{N}$ which has fractal dimension given by Jones index $\mathcal{M} : \mathcal{N} = 4cos^2(\pi/n) \leq 4$, $n = 3, 4, ...$ varying in the range $[1, 4]$. The interpretation is as the fractal dimension corresponding to a dimension of Clifford algebra acting in $d = \sqrt{\mathcal{M} : \mathcal{N}}$-dimensional spinor space: d varies in the range $[1, 2]$. The interpretation in terms of a quantal variant of logic is natural.

6.4.1 *Probabilistic beliefs*

For $\mathcal{M} : \mathcal{N} = 4$ ($n = \infty$) the dimension of spinor space is $d = 2$ and one can speak about ordinary 2-component spinors with $\mathcal{N}$-valued coefficients representing generalizations of qubits. Hence the inclusion of a given $\mathcal{N}$-spinor as M-spinor can be regarded as a belief on the proposition and for the decomposition to a spinor in N-module $\mathcal{M}/\mathcal{N}$ involves for each index a choice $\mathcal{M}/\mathcal{N}$ spinor component selecting super-position of up and down spins. Hence one has a superposition of truth values in general and one can speak only about probabilistic beliefs. It is not clear whether one can choose the basis in such a manner that $\mathcal{M}/\mathcal{N}$ spinor corresponds always to truth value 1. Since WCW spinor field is in question and even if this choice might be possible for a single 3-surface, it need not be possible for deformations of it so that at quantum level one can only speak about probabilistic beliefs.

6.4.2 *Fractal probabilistic beliefs*

For $d < 2$ the spinor space associated with $\mathcal{M}/\mathcal{N}$ can be regarded as quantum plane having complex quantum dimension d with two non-commuting complex coordinates z^1 and z^2 satisfying $z^1z^2 = qz^2z^1$ and $\overline{z^1z^2} = \overline{q}\overline{z^2z^1}$. These relations are consistent with hermiticity of the real and imaginary parts of z^1 and z^2 which define ordinary quantum planes. Hermiticity also implies that one can identify the complex conjugates of z^i as Hermitian conjugates.

The further commutation relations $[z^1, \overline{z^2}] = [z^2, \overline{z^1}] = 0$ and $[z^1, \overline{z^1}] = [z^2, \overline{z^2}] = r$ give a closed algebra satisfying Jacobi identities. One could argue that $r \geq 0$ should be a function $r(n)$ of the quantum phase $q = exp(i2\pi/n)$ vanishing at the limit $n \to \infty$ to guarantee that the algebra becomes commutative at this limit and truth values can be chosen to be non-fuzzy. $r = sin(\pi/n)$ would be the simplest choice. As will be found, the choice of $r(n)$ does not however affect at all the spectrum for the probabilities of the truth values. $n = \infty$ case corresponding to non-fuzzy quantum logic is also possible and must be treated separately: it corresponds to Kac Moody algebra instead of quantum groups.

The non-commutativity of complex spinor components means that z^1 and z^2 are not independent coordinates: this explains the reduction of the number of the effective number of truth values to $d < 2$. The maximal reduction occurs to $d = 1$ for $n = 3$ so that there is effectively only single truth value and one could perhaps speak about taboo or dogma or complete disappearance of the notions of truth and false (this brings in mind reports about meditative states: in fact $n = 3$ corresponds to a phase in which Planck constant becomes infinite so that the system is maximally quantal).

As non-commuting operators the components of d-spinor are not simultaneously measurable for $d < 2$. It is however possible to measure simultaneously the operators describing the probabilities $z^1\overline{z^1}$ and $z^2\overline{z^2}$ for truth values since these operators commute. An inherently fuzzy Boolean logic would be in question with the additional feature that the spinorial counterparts of statement and its negation cannot be regarded as independent observables although the corresponding probabilities satisfy the defining conditions for commuting observables.

If one can speak of a measurement of probabilities for $d < 2$, it differs from the ordinary quantum measurement in the sense that it cannot involve a state function reduction to a pure qubit meaning irreducible quantal fuzziness. One could speak of fuzzy qbits or fqbits (or quantum qbits) instead of qbits. This picture would provide the long sought interpretation for quantum groups.

The previous picture applies to all representations $M_1 \subset M_2$, where M_1 and M_2 denote either real or p-adic Clifford algebras for some prime p. For instance, real-real Jones inclusion could be interpreted as symbolic representations assignable to a unitary mapping of the states of a subsystem M_1 of the external world to the state space M_2 of another real subsystem. $p_1 \rightarrow p_2$ unitary inclusions would in turn map cognitive representations to cognitive representations. There is a strong temptation to assume that these Jones inclusions define unitary maps realizing universe as a universal quantum computer mimicking itself at all levels utilizing cognitive and symbolic representations. Subsystem-system inclusion would naturally define one example of Jones inclusion.

6.4.3 *The spectrum of probabilities of truth values is universal*

It is actually possible to calculate the spectrum of the probabilities of truth values with rather mild additional assumptions.

1. Since the Hermitian operators $X_1 = (z^1\overline{z^1} + \overline{z^1}z^1)/2$ and $X_2 = (z^2\overline{z^2} + \overline{z^2}z^2)/2$ commute, physical states can be chosen to be eigen states of these operators and it is possible to assign to the truth values probabilities given by $p_1 = X_1/R^2$ and $p_2 = X_2/R^2$, $R^2 = X_1 + X_2$.

2. By introducing the analog of the harmonic oscillator vacuum as a state $|0\rangle$ satisfying $z^1|0\rangle = z^2|0\rangle = 0$, one obtains eigen states of X_1 and X_2 as states $|n_1, n_2\rangle = \overline{z^1}^{n_1}\overline{z^2}^{n_2}|0\rangle$, $n_1 \geq 0, n_2 \geq 0$. The eigenvalues of X_1 and X_2 are given by a modified harmonic oscillator spectrum as

$$X_1 = (1/2 + n_1q^{n_2})r \ \ , \ \ X_2 = (1/2 + n_2q^{n_1})r \ \ .$$

   The reality of eigenvalues (hermiticity) is guaranteed if one has $n_1 = N_1n$ and $n_1 = N_2n$ and implies that the spectrum of eigen states gets increasingly thinner for $n \rightarrow \infty$. This must somehow reflect the fractal dimension. The fact that large values of oscillator quantum numbers n_1 and n_2 correspond to the classical limit suggests that modulo condition guarantees approximate classicality of the logic for $n \rightarrow \infty$.

3. The probabilities p_1 and p_2 for the truth values given by $(p_1, p_2) = (1/2+N_1n, 1/2+N_2n)/[1+(N_1 + N_2)n]$ are rational and allow an interpretation as both real and p-adic numbers. This also conforms with the frequency interpretation for probabilities. All states are are inherently fuzzy and only at the limits $N_1 \gg N_2$ and $N_2 \gg N_1$ non-fuzzy states result. As noticed, $n = \infty$ must be treated separately and corresponds to an ordinary non-fuzzy qbit logic. At $n \rightarrow \infty$ limit one has $(p_1, p_2) = (N_1, N_2)/(N_1, N_2)$: at this limit $N_1 = 0$ or $N_2 = 0$ states are non-fuzzy.

4. A possible interpretation for the fuzziness is in terms of finite measurement resolution. The quantized probabilities could be assigned with diagonalized density matrix regarded as matrix with elements which are commuting hermitian operators. The generalized eigenvalues would be eigenvalues spectra. States would not be pure expect at the limits$N_1 \gg N_2$ and $N_2 \gg N_1$. The non-purity of the state could be understood in terms of entanglement with the degrees of freedom below measurement resolution describable in terms of inclusion of von Neumann algebras. One could perhaps say that in finite measurement resolution the outcome of state function reduction is always non-pure state characterized by a universal density matrix obtained by tracing over non-visible degrees of freedom.

6.4.4 *How to define variants of belief quantum mechanically?*

Probabilities of true and false for Jones inclusion characterize the plausibility of the belief and one can ask whether this description is enough to characterize states such as knowledge, misbelief, doubt, delusion, and ignorance. The truth value of $\beta_i(p)$ is determined by the measurement of probability assignable to Jones inclusion on the p-adic side. The truth value of $\lambda_i(p)$ is determined by a similar measurement on the real side. β and λ appear completely symmetrically and one can consider all kinds of triplets $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \mathcal{M}_3$ assuming that there exist unitary S-matrix like maps mediating a sequence $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \mathcal{M}_3$ of Jones inclusions. Interestingly, the hierarchies of Jones inclusions are a key concept in the theory of hyper-finite factors of type II_1 and pair of inclusions plays a fundamental role.

Let us restrict the consideration to the situation when $\mathcal{M}_1$ corresponds to a real subsystem of the external world, $\mathcal{M}_2$ its real representation by a real subsystem, and $\mathcal{M}_3$ to p-adic cognitive representation of M_3. Assume that both real and p-adic sides involve a preferred state basis for qubits representing truth and false.

Assume first that both $\mathcal{M}_1 \subset \mathcal{M}_2$ and $\mathcal{M}_2 \subset \mathcal{M}_3$ correspond to $d = 2$ case for which ordinary quantum measurement or truth value is possible giving outcome true or false. Assume further that the truth values have been measured in both M_2 and M_3.

1. Knowledge corresponds to the proposition $\beta_i(p) \wedge \lambda_i(p)$.

2. Misbelief to the proposition $\beta_i(p) \wedge \neq \lambda_i(p)$.
   Knowledge and misbelief would involve both the measurement of real and p-adic probabilities
   .

3. Assume next that one has $d < 2$ form $\mathcal{M}_2 \subset \mathcal{M}_3$. Doubt can be regarded neither belief or disbelief: $\beta_i(p) \wedge \neq \beta_i(\neq p)$: belief is inherently fuzzy although proposition can be non-fuzzy.

   Assume next that truth values in $\mathcal{M}_1 \subset \mathcal{M}_2$ inclusion corresponds to $d < 2$ so that the basic propositions are inherently fuzzy.

4. Delusion is a belief which cannot be justified: $\beta_i(p) \wedge \lambda_i(p) \wedge \neq \lambda(\neq p))$. This case is possible if $d = 2$ holds true for $\mathcal{M}_2 \subset \mathcal{M}_3$. Note that also misbelief that cannot be shown wrong is possible.
   In this case truth values cannot be quantum measured for $\mathcal{M}_1 \subset \mathcal{M}_2$ but can be measured for $\mathcal{M}_2 \subset \mathcal{M}_3$. Hence the states are products of pure $\mathcal{M}_3$ states with fuzzy $\mathcal{M}_2$ states.

5. Ignorance corresponds to the proposition $\beta_i(p) \wedge \neq \beta_i(\neq p) \wedge \lambda_i(p) \wedge \neq \lambda(\neq p))$. Both real representational states and belief states are inherently fuzzy.

Quite generally, only for $d_1 = d_2 = 2$ ideal knowledge and ideal misbelief are possible. Fuzzy beliefs and logics approach to ordinary one at the limit $n \rightarrow \infty$, which according to the proposal of [K15] corresponds to the ordinary value of Planck constant. For other cases these notions are only approximate and quantal approach allows to characterize the goodness of the approximation. A new kind of inherent quantum uncertainty of knowledge is in question and one could speak about a Uncertainty Principle for cognition and symbolic representations. Also the unification of symbolic and various kinds of cognitive representations deserves to be mentioned.

6.5 Intentional Comparison Of Beliefs By Topological Quantum Computation?

Intentional comparison would mean that for a given initial state also the final state of the quantum jump is fixed. This requires the ability to engineer S-matrix so that it leads from a given state to single state only. Any S-matrix representing permutation of the initial states fulfills these conditions. This condition is perhaps unnecessarily strong.

Quantum computation is basically the engineering of S-matrix so that it represents a superposition of parallel computations. In TGD framework topological quantum computation based on the braiding of magnetic flux tubes would be represented as an evolution characterized by braid [**?**] . The dynamical evolution would be associated with light-like boundaries of braids. This evolution has dual interpretations either as a limit of time evolution of quantum state (program running) or a quantum state satisfying conformal invariance constraints (program code).

The dual interpretation would mean that conformally invariant states are equivalent with engineered time evolutions and topological computation realized as braiding connecting the quantum states to be compared (beliefs represented as many-fermion states at the boundaries of magnetic flux tubes) could give rise to conscious computational comparison of beliefs. The complexity of braiding would give a measure for how much the states to be compared differ.

Note that quantum computation is defined by a unitary map which could also be interpreted as symbolic representation of states of system M_1 as states of system M_2 mediated by the braid of join along boundaries bonds connecting the two space-time sheets in question and having light-like boundaries. These considerations suggest that the idea about S-matrix of the Universe should be generalized so that the dynamics of the Universe is dynamics of mimicry described by an infinite collection of fermionic S-matrices representable in terms of Jones inclusions.

6.6 The Stability Of Fuzzy Qbits And Quantum Computation

The stability of fqbits against state function reduction might have deep implications for quantum computation since quantum spinors would be stable against state function reduction induced by the perturbations inducing de-coherence in the normal situation. If this is really true, and if the only dangerous perturbations are those inducing the phase transition to qbits, the implications for quantum computation could be dramatic. Of course, the rigidity of qbits could be just another way to say that topological quantum computations are stable against thermal perturbations not destroying anyons [**?**] .

The stability of fqbits could also be another manner to state the stability of rational, or more generally algebraic, bound state entanglement against state function reduction, which is one of the basic hypothesis of TGD inspired theory of consciousness [K3] . For sequences of Jones inclusions or equivalently, for multiple Connes tensor products, one would obtain tensor products of quantum spinors making possible arbitrary complex configurations of fqbits. Anyonic braids in topological quantum computation would have interpretation as representations for this kind of tensor products.

6.7 Fuzzy Quantum Logic And Possible Anomalies In The Experimental Data For The EPR-Bohm Experiment

The experimental data for EPR-Bohm experiment [J3] excluding hidden variable interpretations of quantum theory. What is less known that the experimental data indicates about possibility of an anomaly challenging quantum mechanics [J1] . The obvious question is whether this anomaly might provide a test for the notion of fuzzy quantum logic inspired by the TGD based quantum measurement theory with finite measurement resolution.

6.7.1 *The anomaly*

The experimental situation involves emission of two photons from spin zero system so that photons have opposite spins. What is measured are polarizations of the two photons with respect to polarization axes which differ from standard choice of this axis by rotations around the axis of photon momentum characterized by angles α and β. The probabilities for observing polarizations (i,j), where i,j is taken Z_2 valued variable for a convenience of notation are $P_{ij}(\alpha,\beta)$, are predicted to be $P_{00} = P_{11} = cos^2(\alpha-\beta)/2$ and $P_{01} = P_{10} = sin^2(\alpha-\beta)/2$.

Consider now the discrepancies.

1. One has four identities $P_{i,i} + P_{i,i+1} = P_{ii} + P_{i+1,i} = 1/2$ having interpretation in terms of probability conservation. Experimental data of [J3] are not consistent with this prediction [J2] and this is identified as the anomaly.

2. The QM prediction $E(\alpha,\beta) = \sum_i (P_{i,i} - P_{i,i+1}) = cos(2(\alpha-\beta)$ is not satisfied neither: the maxima for the magnitude of E are scaled down by a factor $\simeq .9$. This deviation is not discussed in [J2] .

Both these findings raise the possibility that QM might not be consistent with the data. It turns out that fuzzy quantum logic predicted by TGD and implying that the predictions for the probabilities and correlation must be replaced by ensemble averages, can explain anomaly b) but not anomaly a). A "mundane" explanation for anomaly a) is proposed.

6.7.2 *Predictions of fuzzy quantum logic for the probabilities and correlations*

1. The description of fuzzy quantum logic in terms statistical ensemble

The fuzzy quantum logic implies that the predictions $P_{i,j}$ for the probabilities should be replaced with ensemble averages over the ensembles defined by fuzzy quantum logic. In practice this means that following replacements should be carried out:

$$\begin{aligned} P_{i,j} &\rightarrow P^2 P_{i,j} + (1-P)^2 P_{i+1,j+1} \\ &+ P(1-P)\left[P_{i,j+1} + P_{i+1,j}\right] \ . \end{aligned} \tag{6.1}$$

Here P is one of the state dependent universal probabilities/fuzzy truth values for some value of n characterizing the measurement situation. The concrete predictions would be following

$$\begin{aligned} P_{0,0} = P_{1,1} &\rightarrow A\frac{cos^2(\alpha-\beta)}{2} + B\frac{sin^2(\alpha-\beta)}{2} \\ &= (A-B)\frac{cos^2(\alpha-\beta)}{2} + \frac{B}{2} \ , \\ P_{0,1} = P_{1,0} &\rightarrow A\frac{sin^2(\alpha-\beta)}{2} + B\frac{cos^2(\alpha-\beta)}{2} \\ &= (A-B)\frac{sin^2(\alpha-\beta)}{2} + \frac{B}{2} \ , \\ A &= P^2 + (1-P)^2 \ , \ B = 2P(1-P) \ . \end{aligned} \tag{6.2}$$

The prediction is that the graphs of probabilities as a function as function of the angle $\alpha - \beta$ are scaled by a factor $1 - 4P(1 - P)$ and shifted upwards by $P(1 - P)$. The value of P, and one might hope even the value of n labeling Jones inclusion and the integer m labeling the quantum state might be deducible from the experimental data as the upward shift. The basic prediction is that the maxima of curves measuring probabilities $P_(i, j)$ have minimum at $B/2 = P(1 - P)$ and maximum is scaled down to $(A - B)/2 = 1/2 - 2P(1 - P)$.

If the P is same for all pairs i, j, the correlation $E = \sum_i(P_{ii} - P_{i,i+1})$ transforms as

$$E(\alpha, \beta) \rightarrow [1 - 4P(1-P)]\, E(\alpha, \beta) \ . \tag{6.3}$$

Only the normalization of $E(\alpha, \beta)$ as a function of $\alpha - \beta$ reducing the magnitude of E occurs. In particular the maximum/minimum of E are scaled down from $E = \pm 1$ to $E = \pm(1 - 4P(1 - P))$.

From the figure 1b) of [J2] the scaling down indeed occurs for magnitudes of E with same amount for minimum and maximum. Writing $P = 1 - \epsilon$ one has $A - B \simeq 1 - 4\epsilon$ and $B \simeq 2\epsilon$ so that the maximum is in the first approximation predicted to be at $1 - 4\epsilon$. The graph would give $1 - P \simeq \epsilon \simeq .025$. Thus the model explains the reduction of the magnitude for the maximum and minimum of E which was not however considered to be an anomaly in [J1, J2] .

A further prediction is that the identities $P(i, i) + P(i + 1, i) = 1/2$ should still hold true since one has $P_{i,i} + P_{i,i+1} = (A - B)/2 + B = 1$. This is implied also by probability conservation. The four curves corresponding to these identities do not however co-incide as the figure 6 of [J2] demonstrates. This is regarded as the basic anomaly in [J1, J2] . From the same figure it is also clear that below $\alpha - \beta < 10$ degrees $P_{++} = P_{--}$ $\Delta P_{+-} = -\Delta P_{-+}$ holds true in a reasonable approximation. After that one has also non-vanishing ΔP_{ii} satisfying $\Delta P_{++} = -\Delta P_{--}$. This kind of splittings guarantee the identity $\sum_{ij} P_{ij} = 1$. These splittings are not visible in E.

Since probability conservation requires $P_{ii} + P_{ii+1} = 1$, a mundane explanation for the discrepancy could be that the failure of the conditions $P_{i,i} + P_{ii+1} = 1$ means that the measurement efficiency is too low for P_{+-} and yields too low values of $P_{+-}+P_{--}$ and $P_{+-}+P_{++}$. The constraint $\sum_{ij} P_{ij}=1$ would then yield too high value for P_{-+}. Similar reduction of measurement efficiency for P_{++} could explain the splitting for $\alpha-\beta>10$ degrees.

Clearly asymmetry with respect to exchange of photons or of detectors is in question.

1. The asymmetry of two photon state with respect to the exchange of photons could be considered as a source of asymmetry. This would mean that the photons are not maximally entangled. This could be seen as an alternative "mundane" explanation.

2. The assumption that the parameter P is different for the detectors does not change the situation as is easy to check.

3. One manner to achieve splittings which resemble observed splittings is to assume that the value of the probability parameter P depends on the *polarization pair*: $P=P(i,j)$ so that one has $(P(-,+),P(+,-))=(P+\Delta,P-\Delta)$ and $(P(-,-),P(+,+))=(P+\Delta_1,P-\Delta_1)$. $\Delta\simeq .025$ and $\Delta_1\simeq\Delta/2$ could produce the observed splittings qualitatively. One would however always have $P(i,i)+P(i,i+1)\geq 1/2$. Only if the procedure extracting the correlations uses the constraint $\sum_{i,j} P_{ij}=1$ effectively inducing a constant shift of P_{ij} downwards an asymmetry of observed kind can result. A further objection is that there are no special reason for the values of $P(i,j)$ to satisfy the constraints.

2. Is it possible to say anything about the value of n in the case of EPR-Bohm experiment?

To explain the reduction of the maximum magnitudes of the correlation E from 1 to $\sim .9$ in the experiment discussed above one should have $p_1\simeq .9$. It is interesting to look whether this allows to deduce any information about the valued of n. At the limit of large values of N_in one would have $(N_1-N_2)/(N_1+N_2)\simeq .4$ so that one cannot say anything about n in this case. $(N_1,N_2)=(3,1)$ satisfies the condition exactly. For $n=3$, the smallest possible value of n, this would give $p_1\simeq .88$ and for $n=4$ $p_1=.41$. With high enough precision it might be possible to select between $n=3$ and $n=4$ options if small values of N_i are accepted.

6.8 Category Theoretic Formulation For Quantum Measurement Theory With Finite Measurement Resolution?

I have been trying to understand whether category theory might provide some deeper understanding about quantum TGD, not just as a powerful organizer of fuzzy thoughts but also as a tool providing genuine physical insights. Marni Dee Sheppeard (or Kea in her blog Arcadian Functor at `http://tinyurl.com/yb3lsbjq`) is also interested in categories but in much more technical sense. Her dream is to find a category theoretical formulation of M-theory as something, which is not the 11-D something making me rather unhappy as a physicist with second foot still deep in the muds of low energy phenomenology.

6.8.1 *Locales, frames, Sierpinski topologies and Sierpinski space*

The ideas below popped up when Kea mentioned in M-theory lesson 51 the notions of locale and frame [A2] . In Wikipedia I learned that complete Heyting algebras, which are fundamental to category theory, are objects of three categories with differing arrows. CHey, Loc and its opposite category Frm (arrows reversed). Complete Heyting algebras are partially ordered sets which are complete lattices. Besides the basic logical operations there is also algebra multiplication (I have considered the possible role of categories and Heyting algebras in TGD in [K10]). From Wikipedia I also learned that locales and the dual notion of frames form the foundation of pointless topology [A7] . These topologies are important in topos theory which does not assume axiom of choice.

The so called particular point topology [A5] assumes a selection of single point but I have the physicist's feeling that it is otherwise rather near to pointless topology. Sierpinski topology [A9] is this kind of topology. Sierpinski topology is defined in a simple manner: the set is open only if it contains a given preferred point p. The dual of this topology defined in the obvious sense exists also. Sierpinski space consisting of just two points 0 and 1 is the universal building block of these topologies in the sense that a map of an arbitrary space to Sierpinski space provides it with

Sierpinski topology as the induced topology. In category theoretical terms Sierpinski space is the initial object in the category of frames and terminal object in the dual category of locales. This category theoretic reductionism looks highly attractive.

6.8.2 *Particular point topologies, their generalization, and number theoretical braids*

Pointless, or rather particular point topologies might be very interesting from physicist's point of view. After all, every classical physical measurement has a finite space-time resolution. In TGD framework discretization by number theoretic braids replaces partonic 2-surface with a discrete set consisting of algebraic points in some extension of rationals: this brings in mind something which might be called a topology with a set of particular algebraic points. Could this preferred set belongs to any open set in the particular point topology appropriate in this situation?

Perhaps the physical variant for the axiom of choice could be restricted so that only sets of algebraic points in some extension of rationals can be chosen freely and the choices is defined by the intersection of p-adic and real partonic 2-surfaces and in the framework of TGD inspired theory of consciousness would thus involve the interaction of cognition with the material world. The extension would depend on the position of the physical system in the algebraic evolutionary hierarchy defining also a cognitive hierarchy. Certainly this would fit very nicely to the formulation of quantum TGD unifying real and p-adic physics by gluing real and p-adic number fields to single super-structure via common algebraic points.

6.8.3 *Analogs of particular point topologies at the level of state space: finite measurement resolution*

There is also a finite measurement resolution in Hilbert space sense not taken into account in the standard quantum measurement theory based on factors of type I. In TGD framework one indeed introduces quantum measurement theory with a finite measurement resolution so that complex rays become included hyper-finite factors of type II_1 (HFFs).

1. Could topology with particular algebraic points have a generalization allowing a category theoretic formulation of the quantum measurement theory without states identified as complex rays?

2. How to achieve this? In the transition of ordinary Boolean logic to quantum logic in the old fashioned sense (von Neuman again!) the set of subsets is replaced with the set of subspaces of Hilbert space. Perhaps this transition has a counterpart as a transition from Sierpinski topology to a structure in which sub-spaces of Hilbert space are quantum subspaces with complex rays replaced with the orbits of subalgebra defining the measurement resolution. Sierpinski space {0,1} would in this generalization be replaced with the quantum counterpart of the space of 2-spinors. Perhaps one should also introduce q-category theory with Heyting algebra being replaced with q-quantum logic.

6.8.4 *Fuzzy quantum logic as counterpart for Sierpinksi space*

The program formulated above might indeed make sense. The lucky association induced by Kea's blog was to the ideas about fuzzy quantum logic realized in terms of quantum 2-spinor that I had developed a couple of years ago. Fuzzy quantum logic would reflect the finite measurement resolution. I just list the pieces of the argument.

Spinors and qbits: Spinors define a quantal variant of Boolean statements, qbits. One can however go further and define the notion of quantum qbit, qqbit. I indeed did this for couple of years ago (the last section of this chapter).

Q-spinors and qqbits: For q-spinors the two components a and b are not commuting numbers but non-Hermitian operators: $ab = qba$, q a root of unity. This means that one cannot measure both a and b simultaneously, only either of them. $aa^\dagger$ and $bb^\dagger$ however commute so that probabilities for bits 1 and 0 can be measured simultaneously. State function reduction is not possible to a state in which a or b gives zero. The interpretation is that one has q-logic is inherently fuzzy: there are no absolute truths or falsehoods. One can actually predict the spectrum of eigenvalues of probabilities for say 1. Obviously quantum spinors would be state space counterparts of Sierpinski space and for $q \neq 1$ the choice of preferred spinor component is very natural. Perhaps this fuzzy quantum logic replaces the logic defined by the Heyting algebra.

Q-locale: Could one think of generalizing the notion of locale to quantum locale by using the idea that sets are replaced by sub-spaces of Hilbert space in the conventional quantum logic. Q-openness would be defined by identifying quantum spinors as the initial object, q-Sierpinski space. a (resp. b for the dual category) would define q-open set in this space. Q-open sets for other quantum spaces would be defined as inverse images of a (resp. b) for morphisms to this space. Only for q=1 one could have the q-counterpart of rather uninteresting topology in which all sets are open and every map is continuous.

Q-locale and HFFs: The q-Sierpinski character of q-spinors would conform with the very special role of Clifford algebra in the theory of HFFs, in particular, the special role of Jones inclusions to which one can assign spinor representations of $SU(2)$. The Clifford algebra and spinors of the world of classical worlds identifiable as Fock space of quark and lepton spinors is the fundamental example in which 2-spinors and corresponding Clifford algebra serves as basic building brick although tensor powers of any matrix algebra provides a representation of HFF.

Q-measurement theory: Finite measurement resolution (q-quantum measurement theory) means that complex rays are replaced by sub-algebra rays. This would force the Jones inclusions associated with $SU(2)$ spinor representation and would be characterized by quantum phase q and bring in the q-topology and q-spinors. Fuzzyness of qqbits of course correlates with the finite measurement resolution.

Q-n-logos: For other q-representations of SU(2) and for representations of compact groups (Appendix) one would obtain something which might have something to do with quantum n-logos, quantum generalization of n-valued logic. All of these would be however less fundamental and induced by q-morphisms to the fundamental representation in terms of spinors of the world of classical worlds. What would be however very nice that if these q-morphisms are constructible explicitly it would become possible to build up q-representations of various groups using the fundamental physical realization - and as I have conjectured [K7] - McKay correspondence and huge variety of its generalizations would emerge in this manner.

The analogs of Sierpinski spaces: The discrete subgroups of $SU(2)$, and quite generally, the groups Z_n associated with Jones inclusions and leaving the choice of quantization axes invariant, bring in mind the n-point analogs of Sierpinski space with unit element defining the particular point. Note however that $n \geq 3$ holds true always so that one does not obtain Sierpinski space itself. If all these n preferred points belong to any open set it would not be possible to decompose this preferred set to two subsets belonging to disjoint open sets. Recall that the generalized imbedding space related to the quantization of Planck constant is obtained by gluing together coverings $M^4 \times CP_2 \rightarrow M^4 \times CP_2/G_a \times G_b$ along their common points of base spaces. The topology in question would mean that if some point in the covering belongs to an open set, all of them do so. The interpretation would be that the points of fiber form a single inseparable quantal unit.

Number theoretical braids identified as as subsets of the intersection of real and p-adic variants of algebraic partonic 2-surface define a second candidate for the generalized Sierpinski space with a set of preferred points.

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