Elementary Particle Vacuum Functionals

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Contents

Abstract

Genus-generation correspondence is one of the basic ideas of TGD approach. In order to answer various questions concerning the plausibility of the idea, one should know something about the dependence of the elementary particle vacuum functionals on the vibrational degrees of freedom for the partonic 2-surface.

The construction of the elementary particle vacuum functionals based on Diff invariance, 2-dimensional conformal symmetry, modular invariance plus natural stability requirements indeed leads to an essentially unique form of the vacuum functionals and one can understand why $g > 0$ bosonic families are experimentally absent and why lepton numbers are conserved separately.

An argument suggesting that the number of the light fermion families is three, is developed. The crux of the argument is that the partonic 2-surfaces coding for quantum states are for the maxima of Kähler action hyper-elliptic, that is possess Z_2 conformal symmetry, which for $g > 2$ implies that elementary particle vacuum functional vanishes.

Although the the original model of elementary particle have been modified and replaced with more complex one, the basic idea about the origin of three generations remains intact.

1 Introduction

One of the basic ideas of TGD approach has been genus-generation correspondence: boundary components of the 3-surface should be carriers of elementary particle numbers and the observed fermion families should correspond to various boundary topologies. The details of the assumed correspondence have evolved during years.

1. The first proposal indeed indeed that both fermions and bosons correspond to boundary components so that the genus of the boundary component would classify the particles topologically. At this time I still believed that stringy diagrams would have a direct generalization in TGD framework implying that g would define additive quantum number effectively. Later it became clear that it is Feynman diagrams which must be generalized and the partons at primary vertices must have same genus. Stringy diagrams are still there but have totally different interpretation.

2. Boundary component was later replaced with the light-like surface at which the signature of the induced metric changes and it was natural to identify bosons as wormhole contacts carrying fermion and anti-fermion quantum numbers at opposite light-like worm-hole throats. Hence bosons would be labeled by pairs (g_1, g_2) of genera. For gravitons one had to assume pairs of wormhole contacts in order to obtain spin 2. Already at this stage it became clear that SU(3) should act as a dynamical symmetry with fermions in triplet representation and bosons in octet and singlet representations. The light bosons would correspond to singlets which would guarantee universality of the couplings to fermion families.

3. For long time fermions were identified as single throats but twistorial program and the properties of Chern-Simons Dirac operator suggesting strongly that the fundamental entities must be massless, forced to replace physical fermion with a wormhole contact characterized by (g, g) and transforming like triplet with respect to SU(3) as far as vertices are considered. The hypothesis that SU(3) acts as dynamical symmetry for the reaction vertices has very powerful implications and allows only BFF type vertices required also by bosonic emergence and SUSY symmetry.

4. A further step in the evolution of ideas was the realization that electric-magnetic duality forces to identify all elementary particles as "weak" string like objects consisting of Kähler magnetic flux tubes with opposite magnetic charges at ends. This meant that all elementary particles - not only gravitons- are described by "weak" strings. Note that this stringy character should not be confused with that for wormhole contacts for which throats effectively play the role of string ends. One can say that fundamental objects are massless states at wormhole throats and that all elementary particles as well as string like objects emerge from them.

One might hope that this picture is not too far from the final one as far elementary particles are considered. If one accepts this picture the remaining question is why the number of genera is just three. Could this relate to the fact that $g \leq 2$ Riemann surfaces are always hyper-elliptic (have global Z_2 conformal symmetry) unlike $g > 2$ surfaces? Why the complete bosonic de-localization of the light families should be restricted inside the hyper-elliptic sector? Does the Z_2 conformal symmetry make these states light and make possible de-localization and dynamical $SU(3)$ symmetry? Could it be that for $g > 2$ elementary particle vacuum functionals vanish for hyper-elliptic surfaces? If this the case and if the time evolution for partonic 2-surfaces changing g commutes with Z_2 symmetry then the vacuum functionals localized to $g \leq 2$ surfaces do not disperse to $g > 2$ sectors.

In order to provide answers to either series of questions one must know something about the dependence of the elementary particle state functionals on the geometric properties of the boundary component and in the sequel an attempt to construct what might be called elementary particle vacuum functionals, is made. Irrespective of what identification of interaction vertices is adopted, the arguments involved with the construction involve only the string model type vertices so that the previous discussion seems to apply more or less as such.

The basic assumptions underlying the construction are the following ones:

1. Elementary particle vacuum functionals depend on the geometric properties of the two-surface X^2 representing elementary particle.

2. Vacuum functionals possess extended Diff invariance: all 2-surfaces on the orbit of the 2-surface X^2 correspond to the same value of the vacuum functional. This condition is satisfied if vacuum functionals have as their argument, not X^2 as such, but some 2- surface Y^2 belonging to the unique orbit of X^2 (determined by the principle selecting preferred extremal of the Kähler action as a generalized Bohr orbit [K3]) and determined in $Diff^3$ invariant way.

3. Zero energy ontology allows to select uniquely the partonic two surface as the intersection of the wormhole throat at which the signature of the induced 4-metric changes with either the upper or lower boundary of $CD \times CP_2$. This is essential since otherwise one one could not specify the vacuum functional uniquely.

4. Vacuum functionals possess conformal invariance and therefore for a given genus depend on a finite number of variables specifying the conformal equivalence class of Y^2.

5. Vacuum functionals satisfy the cluster decomposition property: when the surface Y^2 degenerates to a union of two disjoint surfaces (particle decay in string model inspired picture), vacuum functional decomposes into a product of the vacuum functionals associated with disjoint surfaces.

6. Elementary particle vacuum functionals are stable against the two-particle decay $g \rightarrow g_1 + g_2$ and one particle decay $g \rightarrow g - 1$.

In the following the construction will be described in more detail.

1. Some basic concepts related to the description of the space of the conformal equivalence classes of Riemann surfaces are introduced and the concept of hyper-ellipticity is introduced. Since theta functions will play a central role in the construction of the vacuum functionals, also their basic properties are discussed.

2. After these preliminaries the construction of elementary particle vacuum functionals is carried out.

3. Possible explanations for the experimental absence of the higher fermion families are considered.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [?].

2 Identification Of Elementary Particles

The developments in the formulation of quantum TGD which have taken place during the period 2005-2007 [K2, K1] suggest dramatic simplifications of the general picture discussed in the earlier version of this chapter. p-Adic mass calculations [K4, K7, K5] leave a lot of freedom concerning the detailed identification of elementary particles.

2.1 The Evolution Of The Topological Ideas About Elementary Particles

One of the basic ideas of TGD approach has been genus-generation correspondence: boundary components of the 3-surface should be carriers of elementary particle numbers and the observed fermion families should correspond to various boundary topologies.

With the advent of zero energy ontology this picture changed somewhat. It is the wormhole throats identified as light-like 3-surfaces at with the induced metric of the space-time surface changes its signature from Minkowskian to Euclidian, which correspond to the light-like orbits of partonic 2-surfaces. One cannot of course exclude the possibility that also boundary components could allow to satisfy boundary conditions without assuming vacuum extremal property of nearby space-time surface. The intersections of the wormhole throats with the light-like boundaries of causal diamonds (CDs) identified as intersections of future and past directed light cones ($CD \times CP_2$ is actually in question but I will speak about CDs) define special partonic 2-surfaces and it is the moduli of these partonic 2-surfaces which appear in the elementary particle vacuum functionals naturally.

The first modification of the original simple picture comes from the identification of physical particles as bound states of pairs of wormhole contacts (see **Fig.** `http://tgdtheory.fi/appfigures/wormholecontact.jpg` or **Fig. ??** in the appendix of this book) and from the assumption that for generalized Feynman diagrams stringy trouser vertices are replaced with vertices at which the ends of light-like wormhole throats meet. In this picture the interpretation of the analog of trouser vertex is in terms of propagation of same particle along two different paths. This interpretation is mathematically natural since vertices correspond to 2-manifolds rather than singular 2-manifolds which are just splitting to two disjoint components. Second complication comes from the weak form of electric-magnetic duality forcing to identify physical particles as weak strings with magnetic monopoles at their ends and one should understand also the possible complications caused by this generalization.

These modifications force to consider several options concerning the identification of light fermions and bosons and one can end up with a unique identification only by making some assumptions. Masslessness of all wormhole throats- also those appearing in internal lines- and dynamical $SU(3)$ symmetry for particle generations are attractive general enough assumptions of this kind. This means that bosons and their super-partners correspond to wormhole contacts with fermion and anti-fermion at the throats of the contact. Free fermions and their superpartners could correspond to CP_2 type vacuum extremals with single wormhole throat. It turns however that dynamical $SU(3)$ symmetry forces to identify massive (and possibly topologically condensed) fermions as (g, g) type wormhole contacts.

2.1.1 Do free fermions correspond to single wormhole throat or (g, g) wormhole?

The original interpretation of genus-generation correspondence was that free fermions correspond to wormhole throats characterized by genus. The idea of $SU(3)$ as a dynamical symmetry suggested that gauge bosons correspond to octet and singlet representations of $SU(3)$. The further idea that all lines of generalized Feynman diagrams are massless poses a strong additional constraint and it is not clear whether this proposal as such survives.

1. Twistorial program assumes that fundamental objects are massless wormhole throats carrying collinearly moving many-fermion states and also bosonic excitations generated by super-symplectic algebra. In the following consideration only purely bosonic and single fermion throats are considered since they are the basic building blocks of physical particles. The reason is that propagators for high excitations behave like p^{-n}, n the number of fermions associated with the wormhole throat. Therefore single throat allows only spins 0, 1/2, 1 as elementary particles in the usual sense of the word.

2. The identification of massive fermions (as opposed to free massless fermions) as wormhole contacts follows if one requires that fundamental building blocks are massless since at least two massless throats are required to have a massive state. Therefore the conformal excitations with CP_2 mass scale should be assignable to wormhole contacts also in the case of fermions. As already noticed this is not the end of the story: weak strings are required by the weak form of electric-magnetic duality.

3. If free fermions corresponding to single wormhole throat, topological condensation is an essential element of the formation of stringy states. The topological condensation of fermions by topological sum (fermionic CP_2 type vacuum extremal touches another space-time sheet) suggest $(g, 0)$ wormhole contact. Note however that the identification of wormhole throat is as 3-surface at which the signature of the induced metric changes so that this conclusion might be wrong. One can indeed consider also the possibility of (g, g) pairs as an outcome of topological condensation. This is suggested also by the idea that wormhole throats are analogous to string like objects and only this option turns out to be consistent with the BFF vertex based on the requirement of dynamical $SU(3)$ symmetry to be discussed later. The structure of reaction vertices makes it possible to interpret (g, g) pairs as $SU(3)$ triplet. If bosons are obtained as fusion of fermionic and anti-fermionic throats (touching of corresponding CP_2 type vacuum extremals) they correspond naturally to (g_1, g_2) pairs.

4. p-Adic mass calculations distinguish between fermions and bosons and the identification of fermions and bosons should be consistent with this difference. The maximal p-adic temperature $T = 1$ for fermions could relate to the weakness of the interaction of the fermionic wormhole throat with the wormhole throat resulting in topological condensation. This wormhole throat would however carry momentum and 3-momentum would in general be non-parallel to that of the fermion, most naturally in the opposite direction.

   p-Adic mass calculations suggest strongly that for bosons p-adic temperature $T = 1/n$, $n > 1$, so that thermodynamical contribution to the mass squared is negligible. The low p-adic temperature could be due to the strong interaction between fermionic and anti-fermionic wormhole throat leading to the "freezing" of the conformal degrees of freedom related to the relative motion of wormhole throats.

5. The weak form of electric-magnetic duality forces second wormhole throat with opposite magnetic charge and the light-like momenta could sum up to massive momentum. In this case string tension corresponds to electroweak length scale. Therefore p-adic thermodynamics must be assigned to wormhole contacts and these appear as basic units connected by Kähler magnetic flux tube pairs at the two space-time sheets involved. Weak stringy degrees of freedom are however expected to give additional contribution to the mass, perhaps by modifying the ground state conformal weight.

2.1.2 Dynamical $SU(3)$ fixes the identification of fermions and bosons and fundamental interaction vertices

For 3 light fermion families $SU(3)$ suggests itself as a dynamical symmetry with fermions in fundamental $N = 3$-dimensional representation and $N \times N = 9$ bosons in the adjoint representation and singlet representation. The known gauge bosons have same couplings to fermionic families so that they must correspond to the singlet representation. The first challenge is to understand whether it is possible to have dynamical $SU(3)$ at the level of fundamental reaction vertices.

This is a highly non-trivial constraint. For instance, the vertices in which n wormhole throats with same (g_1, g_2) glued along the ends of lines are not consistent with this symmetry. The splitting of the fermionic worm-hole contacts before the proper vertices for throats might however allow the realization of dynamical $SU(3)$. The condition of $SU(3)$ symmetry combined with the requirement that virtual lines resulting also in the splitting of wormhole contacts are always massless, leads to the conclusion that massive fermions correspond to (g, g) type wormhole contacts transforming naturally like $SU(3)$ triplet. This picture conformal with the identification of free fermions as throats but not with the naïve expectation that their topological condensation gives rise to $(g, 0)$ wormhole contact.

The argument leading to these conclusions runs as follows.

1. The question is what basic reaction vertices are allowed by dynamical $SU(3)$ symmetry. FFB vertices are in principle all that is needed and they should obey the dynamical symmetry. The meeting of entire wormhole contacts along their ends is certainly not possible. The splitting of fermionic wormhole contacts before the vertices might be however consistent with $SU(3)$ symmetry. This would give two a pair of 3-vertices at which three wormhole lines meet along partonic 2-surfaces (rather than along 3-D wormhole contacts).

2. Note first that crossing gives all possible reaction vertices of this kind from $F(g_1)\overline{F}(g_2) \rightarrow B(g_1, g_2)$ annihilation vertex, which is relatively easy to visualize. In this reaction $F(g_1)$ and $\overline{F}(g_2)$ wormhole contacts split first. If one requires that all wormhole throats involved are massless, the two wormhole throats resulting in splitting and carrying no fermion number must carry light-like momentum so that they cannot just disappear. The ends of the wormhole throats of the boson must glued together with the end of the fermionic wormhole throat and its companion generated in the splitting of the wormhole. This means that fermionic wormhole first splits and the resulting throats meet at the partonic 2-surface.

   This requires that topologically condensed fermions correspond to (g, g) pairs rather than $(g, 0)$ pairs. The reaction mechanism allows the interpretation of (g, g) pairs as a triplet of dynamical $SU(3)$. The fundamental vertices would be just the splitting of wormhole contact and 3-vertices for throats since $SU(3)$ symmetry would exclude more complex reaction vertices such as n-boson vertices corresponding the gluing of n wormhole contact lines along their 3-dimensional ends. The couplings of singlet representation for bosons would have same coupling to all fermion families so that the basic experimental constraint would be satisfied.

3. Both fermions and bosons cannot correspond to octet and singlet of $SU(3)$. In this case reaction vertices should correspond algebraically to the multiplication of matrix elements e_{ij}: $e_{ij}e_{kl} = \delta_{jk}e_{il}$ allowing for instance $F(g_1, g_2)+\overline{F}(g_2, g_3) \rightarrow B(g_1, g_3)$. Neither the fusion of entire wormhole contacts along their ends nor the splitting of wormhole throats before the fusion of partonic 2-surfaces allows this kind of vertices so that BFF vertex is the only possible one. Also the construction of QFT limit starting from bosonic emergence led to the formulation of perturbation theory in terms of Dirac action allowing only BFF vertex as fundamental vertex [?].

4. Weak electric-magnetic duality brings in an additional complication. $SU(3)$ symmetry poses also now strong constraints and it would seem that the reactions must involve copies of basic BFF vertices for the pairs of ends of weak strings. The string ends with the same Kähler magnetic charge should meet at the vertex and give rise to BFF vertices. For instance, $F\overline{F}B$ annihilation vertex would in this manner give rise to the analog of stringy diagram in which strings join along ends since two string ends disappear in the process.

5. This picture means that all elementary particles - not only gravitons- are described by "weak" strings involving four wormhole throats. Fundamental objects would be partonic 2-surfaces, which in principle can carry arbitrary high fermion numbers N but only $N = 1, 2$ correspond to particles with fermionic and bosonic propagators and the remaining ones correspond to propagators behaving like p^{-n}, $n > 2$, and having interpretation in terms of broken SUSY with a large value of $\mathcal{N}$ identified as the number of fermionic modes. This compositeness of elementary particles should become manifest below weak length scale. Note that this stringy character should not be confused with that for the wormhole contacts for which conformal invariance implies that throats effectively play the role of string ends. One can say that fundamental objects are massless wormhole throats and that all elementary particles as well as string like objects emerge from them.

2.2 Graviton And Other Stringy States

Fermion and anti-fermion can give rise to only single unit of spin since it is impossible to assign angular momentum with the relative motion of wormhole throats. Hence the identification of graviton as single wormhole contact is not possible. The only conclusion is that graviton must be a superposition of fermion-anti-fermion pairs and boson-anti-boson pairs with coefficients determined by the coupling of the parton to graviton. Graviton-graviton pairs might emerge in higher orders. Fermion and anti-fermion would reside at the same space-time sheet and would have a non-vanishing relative angular momentum. Also bosons could have non-vanishing relative angular momentum and Higgs bosons must indeed possess it.

Gravitons are stable if the throats of wormhole contacts carry non-vanishing gauge fluxes so that the throats of wormhole contacts are connected by flux tubes carrying the gauge flux. The mechanism producing gravitons would the splitting of partonic 2-surfaces via the basic vertex. A connection with string picture emerges with the counterpart of string identified as the flux tube connecting the wormhole throats. Gravitational constant would relate directly to the value of the string tension.

The development of the understanding of gravitational coupling has had many twists and it is perhaps to summarize the basic misunderstandings.

1. CP_2 length scale R, which is roughly $10^{3.5}$ times larger than Planck length $l_P = \sqrt{\hbar G}$, defines a fundamental length scale in TGD. The challenge is to predict the value of Planck length $\sqrt{\hbar G}$. The outcome was an identification of a formula for $R^2/\hbar G$ predicting that the magnitude of Kähler coupling strength α_K is near to fine structure constant in electron length scale (for ordinary value of Planck constant should be added here).

2. The emergence of the parton level formulation of TGD finally demonstrated that G actually appears in the fundamental parton level formulation of TGD as a fundamental constant characterizing the M^4 part of CP_2 Kähler gauge potential [K11, K8]. This part is pure gauge in the sense of standard gauge theory but necessary to guarantee that the theory does not reduce to topological QFT. Quantum criticality requires that G remains invariant under p-adic coupling constant evolution and is therefore predictable in principle at least.

3. The TGD view about coupling constant evolution [K9] predicts the proportionality $G \propto L_p^2$, where L_p is p-adic length scale. Together with input from p-adic mass calculations one ends up to two conclusions. The correct conclusion was that Kähler coupling strength is equal to the fine structure constant in the p-adic length scale associated with Mersenne prime $p = M_{127} = 2^{127} - 1$ assignable to electron [K9]. I have considered also the possibility that α_K would be equal to electro-weak $U(1)$ coupling in this scale.

4. The additional - wrong- conclusion was that gravitons must always correspond to the p-adic prime M_{127} since G would otherwise vary as function of p-adic length scale. As a matter fact, the question was for years whether it is G or g_K^2 which remains invariant under p-adic coupling constant evolution. I found both options unsatisfactory until I realized that RG invariance is possible for both g_K^2 and G! The point is that the exponent of the Kähler action associated with the piece of CP_2 type vacuum extremal assignable with the elementary particle is exponentially sensitive to the volume of this piece and logarithmic dependence on the volume fraction is enough to compensate the $L_p^2 \propto p$ proportionality of G and thus guarantee the constancy of G.

The explanation for the small value of the gravitational coupling strength serves as a test for the proposed picture. The exchange of ordinary gauge boson involves the exchange of single CP_2 type extremal giving the exponent of Kähler action compensated by state normalization. In the case of graviton exchange two wormhole contacts are exchanged and this gives second power for the exponent of Kähler action which is not compensated. It would be this additional exponent that would give rise to the huge reduction of gravitational coupling strength from the naïve estimate $G \sim L_p^2$.

2.3 Spectrum Of Non-Stringy States

The 1-throat character of fermions is consistent with the generation-genus correspondence. The 2-throat character of bosons predicts that bosons are characterized by the genera (g_1, g_2) of the wormhole throats. Note that the interpretation of fundamental fermions as wormhole contacts with second throat identified as a Fock vacuum is excluded.

The general bosonic wave-function would be expressible as a matrix M_{g_1,g_2} and ordinary gauge bosons would correspond to a diagonal matrix $M_{g_1,g_2} = \delta_{g_1,g_2}$ as required by the absence of neutral flavor changing currents (say gluons transforming quark genera to each other). 8 new gauge bosons are predicted if one allows all 3×3 matrices with complex entries orthonormalized with respect to trace meaning additional dynamical $SU(3)$ symmetry. Ordinary gauge bosons would be $SU(3)$ singlets in this sense. The existing bounds on flavor changing neutral currents give bounds on the masses of the boson octet. The 2-throat character of bosons should relate to the low value $T = 1/n \ll 1$ for the p-adic temperature of gauge bosons as contrasted to $T = 1$ for fermions.

If one forgets the complications due to the stringy states (including graviton), the spectrum of elementary fermions and bosons is amazingly simple and almost reduces to the spectrum of standard model. In the fermionic sector one would have fermions of standard model. By simple counting leptonic wormhole throat could carry $2^3 = 8$ states corresponding to 2 polarization states, 2 charge states, and sign of lepton number giving 8+8=16 states altogether. Taking into account phase conjugates gives 16+16=32 states.

In the non-stringy boson sector one would have bound states of fermions and phase conjugate fermions. Since only two polarization states are allowed for massless states, one obtains $(2+1) \times (3+1) = 12$ states plus phase conjugates giving 12+12=24 states. The addition of color singlet states for quarks gives 48 gauge bosons with vanishing fermion number and color quantum numbers. Besides 12 electro-weak bosons and their 12 phase conjugates there are 12 exotic bosons and their 12 phase conjugates. For the exotic bosons the couplings to quarks and leptons are determined by the orthogonality of the coupling matrices of ordinary and boson states. For exotic counterparts of W bosons and Higgs the sign of the coupling to quarks is opposite. For photon and Z^0 also the relative magnitudes of the couplings to quarks must change. Altogether this makes 48+16+16=80 states. Gluons would result as color octet states. Family replication would extend each elementary boson state into $SU(3)$ octet and singlet and elementary fermion states into $SU(3)$ triplets.

3 Basic Facts About Riemann Surfaces

In the following some basic aspects about Riemann surfaces will be summarized. The basic topological concepts, in particular the concept of the mapping class group, are introduced, and the Teichmueller parameters are defined as conformal invariants of the Riemann surface, which in fact specify the conformal equivalence class of the Riemann surface completely.

3.1 Mapping Class Group

The first homology group $H_1(X^2)$ of a Riemann surface of genus g contains $2g$ generators [A4, A6, A5]: this is easy to understand geometrically since each handle contributes two homology generators. The so called canonical homology basis can be defined (see **Fig. 1**).

One can define the so called intersection $J(a,b)$ for two elements a and b of the homology group as the number of intersection points for the curves a and b counting the orientation. Since $J(a,b)$ depends on the homology classes of a and b only, it defines an antisymmetric quadratic form in $H_1(X^2)$. In the canonical homology basis the non-vanishing elements of the intersection matrix are:

$$J(a_i, b_j) \quad = \quad -J(b_j, a_i) = \delta_{i,j} \quad . \tag{3.1}$$

J clearly defines symplectic structure in the homology group.

The dual to the canonical homology basis consists of the harmonic one-forms $\alpha_i, \beta_i, i = 1,..,g$ on X^2. These 1-forms satisfy the defining conditions

$$\begin{array}{ll} \int_{a_i} \alpha_j = \delta_{i,j} & \int_{b_i} \alpha_j = 0 \ , \\ \int_{a_i} \beta_j = 0 & \int_{b_i} \beta_j = \delta_{i,j} \ . \end{array} \tag{3.2}$$

The following identity helps to understand the basic properties of the Teichmueller parameters

$$\int_{X^2} \theta \wedge \eta \quad = \quad \sum_{i=1,..,g} [\int_{a_i} \theta \int_{b_i} \eta - \int_{b_i} \theta \int_{a_i} \eta] \quad . \tag{3.3}$$

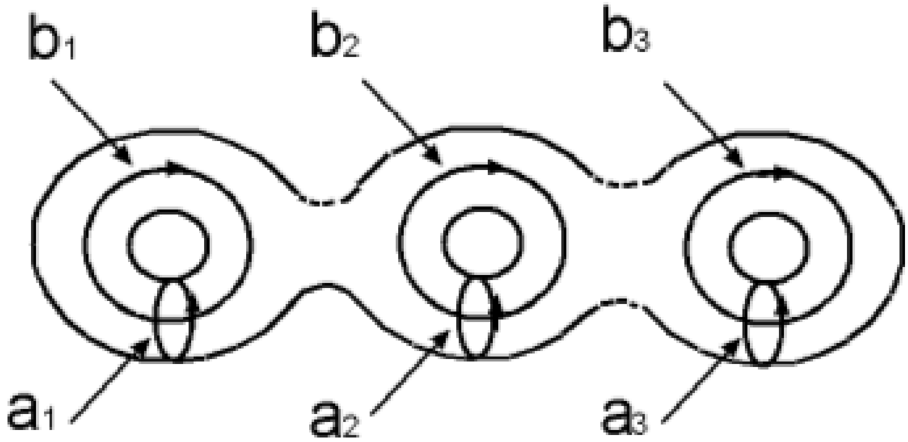

Figure 1: Definition of the canonical homology basis

The existence of topologically nontrivial diffeomorphisms, when X^2 has genus $g > 0$, plays an important role in the sequel. Denoting by $Diff$ the group of the diffeomorphisms of X^2 and by $Diff_0$ the normal subgroup of the diffeomorphisms homotopic to identity, one can define the mapping class group M as the coset group

$$M \quad = \quad Diff/Diff_0 \ . \tag{3.4}$$

The generators of M are so called Dehn twists along closed curves a of X^2. Dehn twist is defined by excising a small tubular neighborhood of a, twisting one boundary of the resulting tube by 2π and gluing the tube back into the surface: see **Fig. 2**.

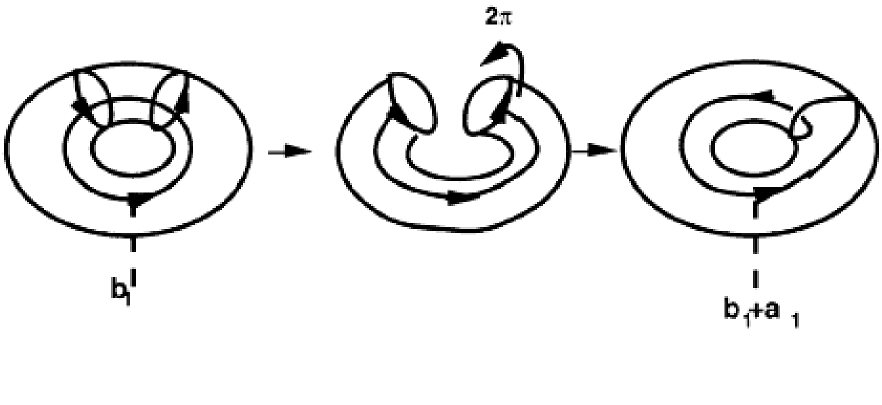

Figure 2: Definition of the Dehn twist

It can be shown that a minimal set of generators is defined by the following curves

$$a_1, b_1, a_1^{-1}a_2^{-1}, a_2, b_2, a_2^{-1}a_3^{-11}, ..., a_g, b_g \quad . \tag{3.5}$$

The action of these transformations in the homology group can be regarded as a symplectic linear transformation preserving the symplectic form defined by the intersection matrix. Therefore the matrix representing the action of $Diff$ on $H_1(X^2)$ is $2g \times 2g$ matrix M with integer entries leaving J invariant: $MJM^T = J$. Mapping class group is often referred also and denoted by $Sp(2g, Z)$. The matrix representing the action of M in the canonical homology basis decomposes into four $g \times g$ blocks A, B, C and D

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad , \tag{3.6}$$

where A and D operate in the subspaces spanned by the homology generators a_i and b_i respectively and C and D map these spaces to each other. The notation $D = [A, B; C, D]$ will be used in the sequel: in this notation the representation of the symplectic form J is $J = [0, 1; -1, 0]$.

3.2 Teichmueller Parameters

The induced metric on the two-surface X^2 defines a unique complex structure. Locally the metric can always be written in the form

$$ds^2 \quad = \quad e^{2\phi} dz d\bar{z} \quad . \tag{3.7}$$

where z is local complex coordinate. When one covers X^2 by coordinate patches, where the line element has the above described form, the transition functions between coordinate patches are holomorphic and therefore define a complex structure.

The conformal transformations ξ of X^2 are defined as the transformations leaving invariant the angles between the vectors of X^2 tangent space invariant: the angle between the vectors X and Y at point x is same as the angle between the images of the vectors under Jacobian map at the image point $\xi(x)$. These transformations need not be globally defined and in each coordinate patch they correspond to holomorphic (anti-holomorphic) mappings as is clear from the diagonal form of the metric in the local complex coordinates. A distinction should be made between local conformal transformations and globally defined conformal transformations, which will be referred to as conformal symmetries: for instance, for hyper-elliptic surfaces the group of the conformal symmetries contains two-element group Z_2.

Using the complex structure one can decompose one-forms to linear combinations of one-forms of type $(1,0)$ $(f(z, \bar{z})dz)$ and $(0,1)$ $(f(z, \bar{z})d\bar{z})$. $(1,0)$ form ω is holomorphic if the function f is holomorphic: $\omega = f(z)dz$ on each coordinate patch.

There are g independent holomorphic one forms ω_i known also as Abelian differentials Alvarez, Farkas, Mumford and one can fix their normalization by the condition

$$\int_{a_i} \omega_j \quad = \quad \delta_{ij} \quad . \tag{3.8}$$

This condition completely specifies ω_i.

Teichmueller parameters Ω_{ij} are defined as the values of the forms ω_i for the homology generators b_j

$$\Omega_{ij} \quad = \quad \int_{b_j} \omega_i \quad . \tag{3.9}$$

The basic properties of Teichmueller parameters are the following:

1. The $g \times g$ matrix Ω is symmetric: this is seen by applying the formula (3.3) for $\theta = \omega_i$ and $\eta = \omega_j$.

2. The imaginary part of Ω is positive: $Im(\Omega) > 0$. This is seen by the application of the same formula for $\theta = \eta$. The space of the matrices satisfying these conditions is known as Siegel upper half plane.

3. The space of Teichmueller parameters can be regarded as a coset space $Sp(2g, R)/U(g)$ [A5]: the action of $Sp(2g, R)$ is of the same form as the action of $Sp(2g, Z)$ and $U(g) \subset Sp(2g, R)$ is the isotropy group of a given point of Teichmueller space.

4. Teichmueller parameters are conformal invariants as is clear from the holomorphy of the defining one-forms.

5. Teichmueller parameters specify completely the conformal structure of Riemann surface [A6]

Although Teichmueller parameters fix the conformal structure of the 2-surface completely, they are not in one-to-one correspondence with the conformal equivalence classes of the two-surfaces:
i) The dimension for the space of the conformal equivalence classes is $D = 3g - 3$, when $g > 1$ and smaller than the dimension of Teichmueller space given by $d = (g \times g + g)/2$ for $g > 3$: all Teichmueller matrices do not correspond to a Riemann surface. In TGD approach this does not produce any problems as will be found later.
ii) The action of the topologically nontrivial diffeomorphisms on Teichmueller parameters is non-trivial and can be deduced from the action of the diffeomorphisms on the homology ($Sp(2g, Z)$ transformation) and from the defining condition $\int_{a_i} \omega_j = \delta_{i,j}$: diffeomorphisms correspond to elements $[A, B; C, D]$ of $Sp(2g, Z)$ and act as generalized Möbius transformations

$$\Omega \to (A\Omega + B)(C\Omega + D)^{-1} \quad . \tag{3.10}$$

All Teichmueller parameters related by $Sp(2g, Z)$ transformations correspond to the same Riemann surface.
iii) The definition of the Teichmueller parameters is not unique since the definition of the canonical homology basis involves an arbitrary numbering of the homology basis. The permutation S of the handles is represented by same $g \times g$ orthogonal matrix both in the basis $\{a_i\}$ and $\{b_i\}$ and induces a similarity transformation in the space of the Teichmueller parameters

$$\Omega \to S\Omega S^{-1} \quad . \tag{3.11}$$

Clearly, the Teichmueller matrices related by a similarity transformations correspond to the same conformal equivalence class. It is easy to show that handle permutations in fact correspond to $Sp(2g, Z)$ transformations.

3.3 Hyper-Ellipticity

The motivation for considering hyper-elliptic surfaces comes from the fact, that $g > 2$ elementary particle vacuum functionals turn out to be vanishing for hyper-elliptic surfaces and this in turn will be later used to provide a possible explanation the non-observability of $g > 2$ particles.

Hyper-elliptic surface X can be defined abstractly as two-fold branched cover of the sphere having the group Z_2 as the group of conformal symmetries (see [A7, A6, A5]. Thus there exists a map $\pi : X \to S^2$ so that the inverse image $\pi^{-1}(z)$ for a given point z of S^2 contains two points except at a finite number (say p) of points z_i (branch points) for which the inverse image contains only one point. Z_2 acts as conformal symmetries permuting the two points in $\pi^{-1}(z)$ and branch points are fixed points of the involution.

The concept can be generalized [A7]: g-hyper-elliptic surface can be defined as a 2-fold covering of genus g surface with a finite number of branch points. One can consider also p-fold coverings instead of 2-fold coverings: a common feature of these Riemann surfaces is the existence of a discrete group of conformal symmetries.

A concrete representation for the hyper-elliptic surfaces [A5] is obtained by studying the surface of C^2 determined by the algebraic equation

$$w^2 - P_n(z) \quad = \quad 0 \ , \tag{3.12}$$

where w and z are complex variables and $P_n(z)$ is a complex polynomial. One can solve w from the above equation

$$w_{\pm} \quad = \quad \pm\sqrt{P_n(z)} \ , \tag{3.13}$$

where the square root is determined so that it has a cut along the positive real axis. What happens that w has in general two roots (two-fold covering property), which coincide at the roots z_i of $P_n(z)$ and if n is odd, also at $z = \infty$: these points correspond to branch points of the hyper-elliptic surface and their number r is always even: $r = 2k$. w is discontinuous at the cuts associated with the square root in general joining two roots of $P_n(z)$ or if n is odd, also some root of P_n and the point $z = \infty$. The representation of the hyper-elliptic surface is obtained by identifying the two branches of w along the cuts. From the construction it is clear that the surface obtained in this manner has genus $k-1$. Also it is clear that Z_2 permutes the different roots $w_{\pm}$ with each other and that $r = 2k$ branch points correspond to fixed points of the involution.

The following facts about the hyper-elliptic surfaces [A6, A5] turn out to be important in the sequel:
i) All $g < 3$ surfaces are hyper-elliptic.
ii) $g \geq 3$ hyper-elliptic surfaces are not in general hyper-elliptic and form a set of codimension 2 in the space of the conformal equivalence classes [A5].

3.4 Theta Functions

An extensive and detailed account of the theta functions and their applications can be found in the book of Mumford [A5] . Theta functions appear also in the loop calculations of string [J1] [A4]. In the following the so called Riemann theta function and theta functions with half integer characteristics will be defined as sections (not strictly speaking functions) of the so called Jacobian variety.

For a given Teichmueller matrix Ω, Jacobian variety is defined as the $2g$-dimensional torus obtained by identifying the points z of C^g (vectors with g complex components) under the equivalence

$$z \sim z + \Omega m + n \ , \tag{3.14}$$

where m and n are points of Z^g (vectors with g integer valued components) and Ω acts in Z^g by matrix multiplication.

The definition of Riemann theta function reads as

$$\Theta(z|\Omega) \quad = \quad \sum_n exp(i\pi n \cdot \Omega \cdot n + i2\pi n \cdot z) \ . \tag{3.15}$$

Here $\cdot$ denotes standard inner product in C^g. Theta functions with half integer characteristics are defined in the following manner. Let a and b denote vectors of C^g with half integer components (component either vanishes or equals to 1/2). Theta function with characteristics $[a,b]$ is defined through the following formula

$$\Theta[a,b](z|\Omega) \quad = \quad \sum_n exp\left[i\pi(n+a)\cdot\Omega\cdot(n+a) + i2\pi(n+a)\cdot(z+b)\right] \ . \tag{3.16}$$

A brief calculation shows that the following identity is satisfied

$$\Theta[a,b](z|\Omega) \quad = \quad exp(i\pi a\cdot\Omega\cdot a + i2\pi a\cdot b)\times\Theta(z+\Omega a+b|\Omega) \tag{3.17}$$

Theta functions are not strictly speaking functions in the Jacobian variety but rather sections in an appropriate bundle as can be seen from the identities

$$\begin{aligned}
\Theta[a,b](z+m|\Omega) &= exp(i2\pi a\cdot m)\Theta[a,b](z\Omega)\ ,\\
\Theta[a,b](z+\Omega m|\Omega) &= exp(\alpha)\Theta[a,b](z|\Omega)\ ,\\
exp(\alpha) &= exp(-i2\pi b\cdot m)exp(-i\pi m\cdot\Omega\cdot m-2\pi m\cdot z)\ .
\end{aligned} \tag{3.18}$$

The number of theta functions is 2^{2g} and same as the number of nonequivalent spinor structures defined on two-surfaces. This is not an accident [A4]: theta functions with given characteristics turn out to be in a close relation to the functional determinants associated with the Dirac operators defined on the two-surface. It is useful to divide the theta functions to even and odd theta functions according to whether the inner product $4a\cdot b$ is even or odd integer. The numbers of even and odd theta functions are $2^{g-1}(2^g+1)$ and $2^{g-1}(2^g-1)$ respectively.

The values of the theta functions at the origin of the Jacobian variety understood as functions of Teichmueller parameters turn out to be of special interest in the following and the following notation will be used:

$$\Theta[a,b](\Omega) \quad \equiv \quad \Theta[a,b](0|\Omega)\ , \tag{3.19}$$

$\Theta[a,b](\Omega)$ will be referred to as theta functions in the sequel. From the defining properties of odd theta functions it can be found that they are odd functions of z and therefore vanish at the origin of the Jacobian variety so that only even theta functions will be of interest in the sequel.

An important result is that also some *even* theta functions vanish for $g>2$ hyper-elliptic surfaces: in fact one can characterize $g>2$ hyper-elliptic surfaces by the vanishing properties of the theta functions [A6, A5]. The vanishing property derives from conformal symmetry (Z_2 in the case of hyper-elliptic surfaces) and the vanishing phenomenon is rather general [A7]: theta functions tend to vanish for Riemann surfaces possessing discrete conformal symmetries. It is not clear (to the author) whether the presence of a conformal symmetry is in fact equivalent with the vanishing of some theta functions. As already noticed, spinor structures and the theta functions with half integer characteristics are in one-to-one correspondence and the vanishing of theta function with given half integer characteristics is equivalent with the vanishing of the Dirac determinant associated with the corresponding spinor structure or equivalently: with the existence of a zero mode for the Dirac operator Alvarez. For odd characteristics zero mode exists always: for even characteristics zero modes exist, when the surface is hyper-elliptic or possesses more general conformal symmetries.

4 Elementary Particle Vacuum Functionals

The basic assumption is that elementary particle families correspond to various elementary particle vacuum functionals associated with the 2-dimensional boundary components of the 3-surface. These functionals need not be localized to a single boundary topology. Neither need their dependence on the boundary component be local. An important role in the following considerations is played by the fact that the preferred extremal property associates a unique 3-surface to each boundary component, the “Bohr orbit” of the boundary and this surface provides a considerable (and necessarily needed) flexibility in the definition of the elementary particle vacuum functionals. There are several natural constraints to be satisfied by elementary particle vacuum functionals.

4.1 Extended Diff Invariance And Lorentz Invariance

Extended Diff invariance is completely analogous to the extension of 3-dimensional Diff invariance to four-dimensional Diff invariance in the interior of the 3-surface. Vacuum functional must be invariant not only under diffeomorphisms of the boundary component but also under the diffeomorphisms of the 3- dimensional "orbit" Y^3 of the boundary component. In other words: the value of the vacuum functional must be same for any time slice on the orbit the boundary component. This is guaranteed if vacuum functional is functional of some two-surface Y^2 belonging to the orbit and defined in $Diff^3$ invariant manner.

An additional natural requirement is Poincare invariance. In the original formulation of the theory only Lorentz transformations of the light cone were exact symmetries of the theory. In this framework the definition of Y^2 as the intersection of the orbit with the hyperboloid $\sqrt{m_{kl}m^km^l} = a$ is $Diff^3$ and Lorentz invariant.

1. *Interaction vertices as generalization of stringy vertices*

For stringy diagrams Poincare invariance of conformal equivalence class and general coordinate invariance are far from being a trivial issues. Vertices are now not completely unique since there is an infinite number of singular 3-manifolds which can be identified as vertices even if one assumes space-likeness. One should be able to select a unique singular 3-manifold to fix the conformal equivalence class.

One might hope that Lorentz invariant invariant and general coordinate invariant definition of Y^2 results by introducing light cone proper time a as a height function specifying uniquely the point at which 3-surface is singular (stringy diagrams help to visualize what is involved), and by restricting the singular 3-surface to be the intersection of $a = constant$ hyperboloid of M^4 containing the singular point with the space-time surface. There would be non-uniqueness of the conformal equivalence class due to the choice of the origin of the light cone but the decomposition of the configuration space of 3-surfaces to a union of WCW s characterized by unions of future and past light cones could resolve this difficulty.

2. *Interaction vertices as generalization of ordinary ones*

If the interaction vertices are identified as intersections for the ends of space-time sheets representing particles, the conformal equivalence class is naturally identified as the one associated with the intersection of the boundary component or light like causal determinant with the vertex. Poincare invariance of the conformal equivalence class and generalized general coordinate invariance follow trivially in this case.

4.2 Conformal Invariance

Conformal invariance implies that vacuum functionals depend on the conformal equivalence class of the surface Y^2 only. What makes this idea so attractive is that for a given genus g WCW becomes effectively finite-dimensional. A second nice feature is that instead of trying to find coordinates for the space of the conformal equivalence classes one can construct vacuum functionals as functions of the Teichmueller parameters.

That one can construct this kind of functions as suitable functions of the Teichmueller parameters is not trivial. The essential point is that the boundary components can be regarded as sub-manifolds of $M^4_+ \times CP_2$: as a consequence vacuum functional can be regarded as a composite function:

2-surface $\rightarrow$ Teichmueller matrix Ω determined by the induced metric $\rightarrow \Omega_{vac}(\Omega)$

Therefore the fact that there are Teichmueller parameters, which do not correspond to any Riemann surface, doesn't produce any trouble. It should be noticed that the situation differs from that in the Polyakov formulation of string models, where one doesn't assume that the metric of the two-surface is induced metric (although classical equations of motion imply this).

4.3 Diff Invariance

Since several values of the Teichmueller parameters correspond to the same conformal equivalence class, one must pose additional conditions on the functions of the Teichmueller parameters in order to obtain single valued functions of the conformal equivalence class.

The first requirement of this kind is the invariance under topologically nontrivial Diff transformations inducing $Sp(2g, Z)$ transformation $(A, B; C, D)$ in the homology basis. The action of these transformations on Teichmueller parameters is deduced by requiring that holomorphic one-forms satisfy the defining conditions in the transformed homology basis. It turns out that the action of the topologically nontrivial diffeomorphism on Teichmueller parameters can be regarded as a generalized Möbius transformation:

$$\Omega \rightarrow (A\Omega + B)(C\Omega + D)^{-1} \quad . \tag{4.1}$$

Vacuum functional must be invariant under these transformations. It should be noticed that the situation differs from that encountered in the string models. In TGD the integration measure over WCW is Diff invariant: in string models the integration measure is the integration measure of the Teichmueller space and this is not invariant under $Sp(2g, Z)$ but transforms like a density: as a consequence the counterpart of the vacuum functional must be also modular covariant since it is the product of vacuum functional and integration measure, which must be modular invariant.

It is possible to show that the quantities

$$(\Theta[a, b]/\Theta[c, d])^4 \quad . \tag{4.2}$$

and their complex conjugates are $Sp(2g, Z)$ invariants [A5] and therefore can be regarded as basic building blocks of the vacuum functionals.

Teichmueller parameters are not uniquely determined since one can always perform a permutation of the g handles of the Riemann surface inducing a redefinition of the canonical homology basis (permutation of g generators). These transformations act as similarities of the Teichmueller matrix:

$$\Omega \rightarrow S\Omega S^{-1} \quad , \tag{4.3}$$

where S is the $g \times g$ matrix representing the permutation of the homology generators understood as orthonormal vectors in the g- dimensional vector space. Therefore the Teichmueller parameters related by these similarity transformations correspond to the same conformal equivalence class of the Riemann surfaces and vacuum functionals must be invariant under these similarities.

It is easy to find out that these similarities permute the components of the theta characteristics: $[a, b] \rightarrow [S(a), S(b)]$. Therefore the invariance requirement states that the handles of the Riemann surface behave like bosons: the vacuum functional constructed from the theta functions is invariant under the permutations of the theta characteristics. In fact, this requirement brings in nothing new. Handle permutations can be regarded as $Sp(2g, Z)$ transformations so that the modular invariance alone guarantees invariance under handle permutations.

4.4 Cluster Decomposition Property

Consider next the behavior of the vacuum functional in the limit, when boundary component with genus g splits to two separate boundary components of genera g_1 and g_2 respectively. The splitting into two separate boundary components corresponds to the reduction of the Teichmueller matrix Ω^g to a direct sum of $g_1 \times g_1$ and $g_2 \times g_2$ matrices $(g_1 + g_2 = g)$:

$$\Omega^g = \Omega^{g_1} \oplus \Omega^{g_2} \quad , \tag{4.4}$$

when a suitable definition of the Teichmueller parameters is adopted. The splitting can also take place without a reduction to a direct sum: the Teichmueller parameters obtained via $Sp(2g, Z)$ transformation from $\Omega^g = \Omega^{g_1} \oplus \Omega^{g_2}$ do not possess direct sum property in general.

The physical interpretation is obvious: the non-diagonal elements of the Teichmueller matrix describe the geometric interaction between handles and at this limit the interaction between the handles belonging to the separate surfaces vanishes. On the physical grounds it is natural to require that vacuum functionals satisfy cluster decomposition property at this limit: that is they reduce to the product of appropriate vacuum functionals associated with the composite surfaces.

Theta functions satisfy cluster decomposition property [A4, A5]. Theta characteristics reduce to the direct sums of the theta characteristics associated with g_1 and g_2 ($a = a_1 \oplus a_2$, $b = b_1 \oplus b_2$) and the dependence on the Teichmueller parameters is essentially exponential so that the cluster decomposition property indeed results:

$$\Theta[a,b](\Omega^g) = \Theta[a_1,b_1](\Omega^{g_1})\Theta[a_2,b_2](\Omega^{g_2}) \ . \tag{4.5}$$

Cluster decomposition property holds also true for the products of theta functions. This property is also satisfied by suitable homogenous polynomials of thetas. In particular, the following quantity playing central role in the construction of the vacuum functional obeys this property

$$Q_0 = \sum_{[a,b]} \Theta[a,b]^4 \bar{\Theta}[a,b]^4 \ , \tag{4.6}$$

where the summation is over all even theta characteristics (recall that odd theta functions vanish at the origin of C^g).

Together with the $Sp(2g,Z)$ invariance the requirement of cluster decomposition property implies that the vacuum functional must be representable in the form

$$\Omega_{vac} = P_{M,N}(\Theta^4, \bar{\Theta}^4)/Q_{MN}(\Theta^4, \bar{\Theta}^4) \tag{4.7}$$

where the homogenous polynomials $P_{M,N}$ and $Q_{M,N}$ have same degrees (M and N as polynomials of $\Theta[a,b]^4$ and $\bar{\Theta}[a,b]^4$.

4.5 Finiteness Requirement

Vacuum functional should be finite. Finiteness requirement is satisfied provided the numerator $Q_{M,N}$ of the vacuum functional is real and positive definite. The simplest quantity of this type is the quantity Q_0 defined previously and its various powers. $Sp(2g,Z)$ invariance and finiteness requirement are satisfied provided vacuum functionals are of the following general form

$$\Omega_{vac} = \frac{P_{N,N}(\Theta^4, \bar{\Theta}^4)}{Q_0^N} \ , \tag{4.8}$$

where $P_{N,N}$ is homogenous polynomial of degree N with respect to $\Theta[a,b]^4$ and $\bar{\Theta}[a,b]^4$. In addition $P_{N,N}$ is invariant under the permutations of the theta characteristics and satisfies cluster decomposition property.

4.6 Stability Against The Decay $G \rightarrow G_1 + G_2$

Elementary particle vacuum functionals must be stable against the genus conserving decays $g \rightarrow g_1 + g_2$. This decay corresponds to the limit at which Teichmueller matrix reduces to a direct sum of the matrices associated with g_1 and g_2 (note however the presence of $Sp(2g,Z)$ degeneracy). In accordance with the topological description of the particle reactions one expects that this decay doesn't occur if the vacuum functional in question vanishes at this limit.

In general the theta functions are non-vanishing at this limit and vanish provided the theta characteristics reduce to a direct sum of the odd theta characteristics. For $g < 2$ surfaces this condition is trivial and gives no constraints on the form of the vacuum functional. For $g = 2$ surfaces the theta function $\Theta(a,b)$, with $a = b = (1/2, 1/2)$ satisfies the stability criterion identically (odd theta functions vanish identically), when Teichmueller parameters separate into a direct sum. One can however perform $Sp(2g, Z)$ transformations giving new points of Teichmueller space describing the decay. Since these transformations transform theta characteristics in a nontrivial manner to each other and since all even theta characteristics belong to same $Sp(2g, Z)$ orbit [A4, A5], the conclusion is that stability condition is satisfied provided $g = 2$ vacuum functional is proportional to the product of fourth powers of all even theta functions multiplied by its complex conjugate.

If $g > 2$ there always exists some theta functions, which vanish at this limit and the minimal vacuum functional satisfying this stability condition is of the same form as in $g = 2$ case, that is proportional to the product of the fourth powers of all even Theta functions multiplied by its complex conjugate:

$$\Omega_{vac} = \prod_{[a,b]} \Theta[a,b]^4 \bar{\Theta}[a,b]^4 / Q_0^N \ , \tag{4.9}$$

where N is the number of even theta functions. The results obtained imply that genus-generation correspondence is one to one for $g > 1$ for the minimal vacuum functionals. Of course, the multiplication of the minimal vacuum functionals with functionals satisfying all criteria except stability criterion gives new elementary particle vacuum functionals: a possible physical identification of these vacuum functionals is most naturally as some kind of excited states.

One of the questions posed in the beginning was related to the experimental absence of $g > 0$, possibly massless, elementary bosons. The proposed stability criterion suggests a nice explanation. The point is that elementary particles are stable against decays $g \to g_1 + g_2$ but not with respect to the decay $g \to g + \ sphere$. As a consequence the direct emission of $g > 0$ gauge bosons is impossible unlike the emission of $g = 0$ bosons: for instance the decay muon $\to$ electron $+(g = 1)$ photon is forbidden.

4.7 Stability Against The Decay $G \to G - 1$

This stability criterion states that the vacuum functional is stable against single particle decay $g \to g - 1$ and, if satisfied, implies that vacuum functional vanishes, when the genus of the surface is smaller than g. In stringy framework this criterion is equivalent to a separate conservation of various lepton numbers: for instance, the spontaneous transformation of muon to electron is forbidden. Notice that this condition doesn't imply that that the vacuum functional is localized to a single genus: rather the vacuum functional of genus g vanishes for all surfaces with genus smaller than g. This hierarchical structure should have a close relationship to Cabibbo-Kobayashi-Maskawa mixing of the quarks.

The stability criterion implies that the vacuum functional must vanish at the limit, when one of the handles of the Riemann surface suffers a pinch. To deduce the behavior of the theta functions at this limit, one must find the behavior of Teichmueller parameters, when i: th handle suffers a pinch. Pinch implies that a suitable representative of the homology generator a_i or b_i contracts to a point.

Consider first the case, when a_i contracts to a point. The normalization of the holomorphic one-form ω_i must be preserved so that that ω_i must behaves as $1/z$, where z is the complex coordinate vanishing at pinch. Since the homology generator b_i goes through the pinch it seems obvious that the imaginary part of the Teichmueller parameter $\Omega_{ii} = \int_{b_i} \omega_i$ diverges at this limit (this conclusion is made also in [A5]): $Im(\Omega_{ii}) \to \infty$.

Of course, this criterion doesn't cover all possible ways the pinch can occur: pinch might take place also, when the components of the Teichmueller matrix remain finite. In the case of torus topology one finds that $Sp(2g, Z)$ element $(A, B; C, D)$ takes $Im(\Omega) = \infty$ to the point C/D of real axis. This suggests that pinch occurs always at the boundary of the Teichmueller space: the imaginary part of Ω_{ij} either vanishes or some matrix element of $Im(\Omega)$ diverges.

Consider next the situation, when b_i contracts to a point. From the definition of the Teichmueller parameters it is clear that the matrix elements Ω_{kl}, with $k, l \neq i$ suffer no change. The matrix element Ω_{ki} obviously vanishes at this limit. The conclusion is that i:th row of Teichmueller matrix vanishes at this limit. This result is obtained also by deriving the $Sp(2g, Z)$ transformation permuting a_i and b_i with each other: in case of torus this transformation reads $\Omega \to -1/\Omega$.

Consider now the behavior of the theta functions, when pinch occurs. Consider first the limit, when $Im(\Omega_{ii})$ diverges. Using the general definition of $\Theta[a, b]$ it is easy to find out that all theta functions for which the i: the component a_i of the theta characteristic is non-vanishing (that is $a_i = 1/2$) are proportional to the exponent $exp(-\pi\Omega_{ii}/4)$ and therefore vanish at the limit. The theta functions with $a_i = 0$ reduce to $g-1$ dimensional theta functions with theta characteristic obtained by dropping i: th components of a_i and b_i and replacing Teichmueller matrix with Teichmueller matrix obtained by dropping i: th row and column. The conclusion is that all theta functions of type $\Theta(a,b)$ with $a = (1/2, 1/2, ..., 1/2)$ satisfy the stability criterion in this case.

What happens for the $Sp(2g,Z)$ transformed points on the real axis? The transformation formula for theta function is given by [A4, A5]

$$\Theta[a,b]((A\Omega+B)(C\Omega+D)^{-1}) = exp(i\phi)det(C\Omega+D)^{1/2}\Theta[c,d](\Omega) \ , \tag{4.10}$$

where

$$\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \left(\begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} (CD^T)_d/2 \\ (AB^T)_d/2 \end{pmatrix} \right) \ . \tag{4.11}$$

Here ϕ is a phase factor irrelevant for the recent purposes and the index d refers to the diagonal part of the matrix in question.

The first thing to notice is the appearance of the diverging square root factor, which however disappears from the vacuum functionals (P and Q have same degree with respect to thetas). The essential point is that theta characteristics transform to each other: as already noticed all even theta characteristics belong to the same $Sp(2g,Z)$ orbit. Therefore the theta functions vanishing at $Im(\Omega_{ii}) = \infty$ do not vanish at the transformed points. It is however clear that for a given Teichmueller parameterization of pinch some theta functions vanish always.

Similar considerations in the case $\Omega_{ik} = 0$, i fixed, show that all theta functions with $b = (1/2, ..., 1/2)$ vanish identically at the pinch. Also it is clear that for $Sp(2g,Z)$ transformed points one can always find some vanishing theta functions. The overall conclusion is that the elementary particle vacuum functionals obtained by using $g \to g_1 + g_2$ stability criterion satisfy also $g \to g-1$ stability criterion since they are proportional to the product of all even theta functions. Therefore the only nontrivial consequence of $g \to g-1$ criterion is that also $g=1$ vacuum functionals are of the same general form as $g > 1$ vacuum functionals.

A second manner to deduce the same result is by restricting the consideration to the hyper-elliptic surfaces and using the representation of the theta functions in terms of the roots of the polynomial appearing in the definition of the hyper-elliptic surface [A5]. When the genus of the surface is smaller than three (the interesting case), this representation is all what is needed since all surfaces of genus $g < 3$ are hyper-elliptic.

Since hyper-elliptic surfaces can be regarded as surfaces obtained by gluing two compactified complex planes along the cuts connecting various roots of the defining polynomial it is obvious that the process $g \to g-1$ corresponds to the limit, when two roots of the defining polynomial coincide. This limit corresponds either to disappearance of a cut or the fusion of two cuts to a single cut. Theta functions are expressible as the products of differences of various roots (Thomae's formula [A5])

$$\Theta[a,b]^4 \propto \prod_{i<j\in T}(z_i - z_j) \prod_{k<l\in CT}(z_k - z_l) \ , \tag{4.12}$$

where T denotes some subset of $\{1, 2, ..., 2g\}$ containing $g+1$ elements and CT its complement. Hence the product of all even theta functions vanishes, when two roots coincide. Furthermore, stability criterion is satisfied only by the product of the theta functions.

Lowest dimensional vacuum functionals are worth of more detailed consideration.

i) $g = 0$ particle family corresponds to a constant vacuum functional: by continuity this vacuum functional is constant for all topologies.

ii) For $g = 1$ the degree of P and Q as polynomials of the theta functions is 24: the critical number of transversal degrees of freedom in bosonic string model! Probably this result is not an accident.
ii) For $g = 2$ the corresponding degree is 80 since there are 10 even genus 2 theta functions.

There are large numbers of vacuum functionals satisfying the relevant criteria, which do not satisfy the proposed stability criteria. These vacuum functionals correspond either to many particle states or to unstable single particle states.

4.8 Continuation Of The Vacuum Functionals To Higher Genus Topologies

From continuity it follows that vacuum functionals cannot be localized to single boundary topology. Besides continuity and the requirements listed above, a natural requirement is that the continuation of the vacuum functional from the sector g to the sector $g+k$ reduces to the product of the original vacuum functional associated with genus g and $g = 0$ vacuum functional at the limit when the surface with genus $g+k$ decays to surfaces with genus g and k: this requirement should guarantee the conservation of separate lepton numbers although different boundary topologies suffer mixing in the vacuum functional. These requirements are satisfied provided the continuation is constructed using the following rule:

Perform the replacement

$$\Theta[a,b]^4 \quad \rightarrow \quad \sum_{c,d} \Theta[a \oplus c, b \oplus d]^4 \tag{4.13}$$

for each fourth power of the theta function. Here c and d are Theta characteristics associated with a surface with genus k. The same replacement is performed for the complex conjugates of the theta function. It is straightforward to check that the continuations of elementary particle vacuum functionals indeed satisfy the cluster decomposition property and are continuous.

To summarize, the construction has provided hoped for answers to some questions stated in the beginning: stability requirements explain the separate conservation of lepton numbers and the experimental absence of $g > 0$ elementary bosons. What has not not been explained is the experimental absence of $g > 2$ fermion families. The vanishing of the $g > 2$ elementary particle vacuum functionals for the hyper-elliptic surfaces however suggest a possible explanation: under some conditions on the surface X^2 the surfaces Y^2 are hyper-elliptic or possess some conformal symmetry so that elementary particle vacuum functionals vanish for them. This conjecture indeed might make sense since the surfaces Y^2 are determined by the asymptotic dynamics and one might hope that the surfaces Y^2 are analogous to the final states of a dissipative system.

5 Explanations For The Absence Of The $g > 2$ Elementary Particles From Spectrum

The decay properties of the intermediate gauge bosons [C1] are consistent with the assumption that the number of the light neutrinos is $N = 3$. Also cosmological considerations pose upper bounds on the number of the light neutrino families and $N = 3$ seems to be favored [C1]. It must be however emphasized that p-adic considerations [K5] encourage the consideration the existence of higher genera with neutrino masses such that they are not produced in the laboratory at present energies. In any case, for TGD approach the finite number of light fermion families is a potential difficulty since genus-generation correspondence suggests that the number of the fermion (and possibly also boson) families is infinite. Therefore one had better to find a good argument showing that the number of the observed neutrino families, or more generally, of the observed elementary particle families, is small also in the world described by TGD.

It will be later found that also TGD inspired cosmology requires that the number of the effectively massless fermion families must be small after Planck time. This suggests that boundary topologies with handle number $g > 2$ are unstable and/or very massive so that they, if present in the spectrum, disappear from it after Planck time, which correspond to the value of the light cone proper time $a \simeq 10^{-11}$ seconds.

In accordance with the spirit of TGD approach it is natural to wonder whether some geometric property differentiating between $g > 2$ and $g < 3$ boundary topologies might explain why only $g < 3$ boundary components are observable. One can indeed find a good candidate for this kind of property: namely hyper-ellipticity, which states that Riemann surface is a two-fold branched covering of sphere possessing two-element group Z_2 as conformal automorphisms. All $g < 3$ Riemann surfaces are hyper-elliptic unlike $g > 2$ Riemann surfaces, which in general do not posses this property. Thus it is natural to consider the possibility that hyper-ellipticity or more general conformal symmetries might explain why only $g < 2$ topologies correspond to the observed elementary particles.

As regards to the present problem the crucial observation is that some even theta functions vanish for the hyper-elliptic surfaces with genus $g > 2$ [A5]. What is essential is that these surfaces have the group Z_2 as conformal symmetries. Indeed, the vanishing phenomenon is more general. Theta functions tend to vanish for $g > 2$ two-surfaces possessing discrete group of conformal symmetries [A7]: for instance, instead of sphere one can consider branched coverings of higher genus surfaces.

From the general expression of the elementary particle vacuum functional it is clear that elementary particle vacuum functionals vanish, when Y^2 is hyper-elliptic surface with genus $g > 2$ and one might hope that this is enough to explain why the number of elementary particle families is three.

5.1 Hyper-Ellipticity Implies The Separation Of $h \leq 2$ And$g > 2$ Sectors To Separate Worlds

If the vertices are defined as intersections of space-time sheets of elementary particles and if elementary particle vacuum functionals are required to have Z_2 symmetry, the localization of elementary particle vacuum functionals to $g \leq 2$ topologies occurs automatically. Even if one allows as limiting case vertices for which 2-manifolds are pinched to topologies intermediate between $g > 2$ and $g \leq 2$ topologies, Z_2 symmetry present for both topological interpretations implies the vanishing of this kind of vertices. This applies also in the case of stringy vertices so that also particle propagation would respect the effective number of particle families. $g > 2$ and $g \leq 2$ topologies would behave much like their own worlds in this approach. This is enough to explain the experimental findings if one can understand why the $g > 2$ particle families are absent as incoming and outgoing states or are very heavy.

5.2 What About $G > 2$ Vacuum Functionals Which Do Not Vanish For Hyper-Elliptic Surfaces?

The vanishing of all $g \geq 2$ vacuum functionals for hyper-elliptic surfaces cannot hold true generally. There must exist vacuum functionals which do satisfy this condition. This suggest that elementary particle vacuum functionals for $g > 2$ states have interpretation as bound states of g handles and that the more general states which do not vanish for hyper-elliptic surfaces correspond to many-particle states composed of bound states $g \leq 2$ handles and cannot thus appear as incoming and outgoing states. Thus $g > 2$ elementary particles would decouple from $g \leq 2$ states.

5.3 Should Higher Elementary Particle Families Be Heavy?

TGD predicts an entire hierarchy of scaled up variants of standard model physics for which particles do not appear in the vertices containing the known elementary particles and thus behave like dark matter [K10]. Also $g > 2$ elementary particles would behave like dark matter and in principle there is no absolute need for them to be heavy.

The safest option would be that $g > 2$ elementary particles are heavy and the breaking of Z_2 symmetry for $g \geq 2$ states could guarantee this. p-Adic considerations lead to a general mass formula for elementary particles such that the mass of the particle is proportional to $\frac{1}{\sqrt{p}}$ [K6]. Also the dependence of the mass on particle genus is completely fixed by this formula. What remains however open is what determines the p-adic prime associated with a particle with given quantum numbers. Of course, it could quite well occur that p is much smaller for $g > 2$ genera than for $g \leq 2$ genera.

5.4 Could Higher Genera Have Interpretation As Many-Particle States?

The topological explanation of family replication phenomenon of fermions in terms of the genus g defined as the number of handles added to sphere to obtain the quantum number carrying partonic 2-surface distinguishes TGD from GUTs and string models. The orbit of the partonic 2-surface defines 3-D light-like orbit identified as wormhole throat at which the induced metric changes its signature. The original model of elementary particle involved only single boundary component replaced later by a wormhole throat. The generalization to the recent situation in which elementary particles correspond to wormhole flux tubes of length of order weak length scales with pairs of wormhole throats at its ends is straight-forward.

The basic objection against the proposal is that it predicts infinite number of particle families unless the $g \leq 3$ topologies are preferred for some reason. Conformal and modular symmetries are basic symmetries of the theory and global conformal symmetries provide an excellent candidate for the sought for reason why.

1. For $g \leq 3$ the 2-surfaces are always hyper-elliptic which means that they have have always Z_2 as global conformal symmetries. For $g \geq 2$ these symmetries are absent in the generic case. Moreover, the ¡modular invariant elementary particle vacuum functionals¡/a¿ vanish for hyper-elliptic surfaces for $g \geq 2$. This leaves several options to consider. The basic idea is however that ground states are usually highly symmetric and that elementary particles correspond to ground states.

2. The simplest guess is that $g \geq 2$ surfaces correspond to very massive states decaying rapidly to states with smaller genus. Due to the conformal symmetry $g \leq 3$ surfaces would be analogous to ground states and would have small masses.

3. The possibility to have partonic 2-surfaces of macroscopic and even astrophysical size identifiable as seats of anyonic macroscopic quantum phases [K8] suggests an alternative interpretation consistent with global conformal symmetries. For partonic 2-surfaces of macroscopic size it seems natural to consider handles as particles glued to a much larger partonic 2-surface by topological sum operation (topological condensation).

   All orientable manifolds can be obtained by topological sum operation from what can be called prime manifolds. In 2-D orientable case prime manifolds are sphere and torus representing in well-defined sense 0 and 1 so that topological sum corresponds to addition of positive integers arithmetically. This would suggest that only sphere and torus appear as single particle states. Particle interpretation however requires that also $g = 0$ and $g = 2$ surfaces topologically condensed to a larger anyonic 2-surface have similar interpretation, at least if they have small enough size. What kind of argument could justify this kind of interpretation?

4. An argument based on symmetries suggests itself. The reduction of degrees of freedom is the generic signature of bound state. Bound state property implies also the reduction of approximate single particle symmetries to an exact overall symmetry. Rotational symmetries of hydrogen atom represent a good example of this. For free many particle states each particle transforms according to a representation of rotation group having total angular momentum defined as sum of its spin and angular momentum. For bound states rotational degrees of freedom are strongly correlated and only overall rotations of the state define rotational symmetries.

   In this spirit one could interpret sphere as vacuum, torus as single handle state, and torus with handle as a bound state of 2 handles in conformal degrees of freedom meaning that the Z_2 symmetries of vacuum and handles are frozen in topological condensation (topological sum) to single overall Z_2. If this interpretation is correct, $g \geq 2$ 2-surfaces would always have a decomposition to many-particle states consisting of spheres, tori and tori with single handle glued to a larger sphere by topological sum. Each of these topologically condensed composites would possess Z_2 as approximate single particle symmetry.

5.5 A new piece to the TGD inspired model of family replication

The TGD vision about family replication phenomenon of fermions is as follows.

1. Fermion families correspond to the genera for partonic 2-surfaces. This predicts generation-genus correspondence. Electron and its neutrino correspond to a sphere with genus $g = 0$; muon and its neutrino to a torus with $g = 1$; τ and its neutrino to to with $g = 2$. Similar picture applies to quarks. CKM mixing corresponds to topological mixings of genera, which are different for different charged states and CKM mixing is the difference of these mixings.

   The problem is that TGD suggests an infinite number of genera. Only 3 fermion families are observed. Why?

2. The first piece of the answer is Z_2 conformal symmetry. It is present for the genera $g = 0, 1, 2$ but only for hyperelliptic Riemann surfaces for $g > 2$.

3. The second piece of the answer is that one regards the genera $g \geq 2$ as many-handle states. For $g \geq 2$ many-handle states would have a continuous mass spectrum and would not be elementary particles. For $g = 2$ a bound state of two handles would be possible by Z_2 symmetry.

Consider now the new building brick for the explanation.

1. Quantum classical correspondence is the basic principle of TGD and requires that quantum states have classical counterparts.

2. Assume that in a suitable region of moduli space it makes sense to talk of a handle as a particle moving in the geometry defined by $g - 1$ handles. One can imagine that the handle is glued by a small wormhole contact to the background defined by $g - 1$ handles and behaves like a free point-like particle moving along a geodesic line of the background.

   This relationship must be symmetric so that the background must move along the geodesic line of the handle. This means that particles and background are glued together along the geodesic lines of both.

3. Consider now various cases.

   (a) The case $g = 0$ is trivial since one has a handle vacuum.

   (b) For $g = 1$, one has the motion of a handle in spherical geometry along a great circle, which corresponds to a geodesic line of the sphere. The torus can rotate like a rigid body and this corresponds to a geodesic line of torus characterized by two winding numbers (m, n). Alternatively, one can say that the sphere rotates along a geodesic of the torus. There is an infinite but discrete number of orbits. The simplest solution is the stationary solution $(m, n) = (0, 0)$.

   (c) For $g = 2$, one has a geodesic motion of a handle in the toric geometry defined by the second handle. Now one can speak of bound states of two handles.

   One would have a gluing of two tori along geodesic lines (m, n) and (r, s). The ratios of these integers are rational so that one obtains a closed orbit. The simplest solution is $(m, n) = (r, s) = 0$.

   Stationary solutions are stable for constant curvature case since curvature of torus vanishes. Locally the stationary solution is like a particle at rest in Euclidian plane.

   (d) For $g = 3$ one has a geodesic motion of the handle in $g = 2$ geometry or vice versa. $g = 2$ geometry has negative total scalar curvature and as a special case a constant negative curvature. This implies that all points are saddle points and therefore unstable geodesics so that two geodesics going through a given point in general diverge. This strongly suggests that only unstable geodesics are possible for $g = 2$ whether it is regarded as background or as a particle. This suggests a butterfly effect and a chaotic behavior. Even if $g = 2$ particle represents a classical bound state the third handle must move along a chaotic geodesics of $g = 2$ geometry.This could explain the absence of bound states at quantum level.

6 Elementary Particle Vacuum Functionals For Dark Matter

One of the open questions is how dark matter hierarchy reflects itself in the properties of the elementary particles. The basic questions are how the quantum phase $q = ep(2i\pi/n)$ makes itself visible in basic theory and how elementary particle vacuum functionals depend on q.

6.1 Hurwitz Zetas Cannot Correspond To Dark Matter In TGD Sense

Intuitively dark matter corresponds to n-sheeted singular coverings of space-time surfaces analogous to corresponding coverings of complex plane. Assume that the consideration can be restricted to string world sheets or partonic 2-surfaces. The complex coordinate is replaced with $w = z^{1/n}$ and the conformal algebra in question has conformal weight spectrum scaled down by $1/n$. Conformal symmetry is broken and only the integer valued conformal weights assignable to $z = w^n$ correspond to gauge symmetries. One can also use w as variable and say that the subalgebra of conformal algebra for covering with conformal weights coming as multiples of n acts as gauge symmetry. The conformal transformations acting as gauge symmetries would not permute the sheets of the covering and space-time sheets would define n conformal equivalence classes.

An important point to notice is that the breaking of conformal symmetry as gauge symmetry would give a justification for p-adic thermodynamics. This breaking could occur for all conformal algebras involved.

Riemann zeta is associated naturally as spectral zeta function $\zeta_{K-D} = \sum 1/\lambda^s$ with the solutions of Kähler-Dirac operator coming as powers of z^m as in string models. What happens for the spectrum and ζ_{K-D} in the replacement of space-time surfaces with its n-fold covering?

(a) To obtain the spectral zeta function characterizing the Kähler-Dirac operator, one just makes the replacement $m \rightarrow m/n$ in the defining formula $\zeta(s) = \sum s^{-m}$ of the spectral zeta function. For the covering Riemann zeta $\zeta(s)$ would be replaced with $\zeta(s)/n^s$ so that zeros would not be affected. The result is not so surprising since sub-algebra of conformal algebra are isomorphic to the algebra itself.

(b) Note that I have also considered the possibility that the conformal weights of the generators of super-symplectic algebra - certainly not of Kähler-Dirac operator - come as zeros of Riemann zeta: this would mean a huge extension of the algebra since the number of generators increase from a finite number to infinite number. Given complex conformal weight for a generator of algebra would correspond to the power r_M^h of the radial light-like coordinate r_M of δM_+^4, h zero of zeta. n-fold covering would correspond to $r_M \rightarrow r_M^{1/n}$ as variable. Orthogonality conditions would allow this spectrum of radial conformal weights. Note that the physical conformal weights (which could have interpretation as mass squared eigen values) would be still integers by what I call conformal confinement.

    Nothing would happen for the spectrum of super-symplectic conformal weights if identified as zeros of the spectral zeta of K-D operator (for which there is however no compelling reason!).

Hurwitz zeta obtained by a shift $m \rightarrow m + a$ in $\zeta = \sum m^{-s}$ to give $\zeta_H(s, a) = \sum 1/(m + a)^s$. Here a could be restricted to be a rational number in the range $(0, 1)$ or inverse integer $1/n$. For integer values of a some of the lowest integers drop from the integer spectrum. For other values something more complex takes place. For $a = 1/2$ Hurwitz zeta is proportional to Riemann zeta: $\zeta(s, 1/2) = (2^s - 1)\zeta(s)$ and its spectrum of zeros includes those or Riemann zeta plus points $s_n = n \times i2\pi/ln(2)$ at imaginary axis. For other value of a the spectrum of zeros is not concentrated on vertical line and does not include zeros of Riemann zeta. The failure of Riemann Hypothesis is an ugly property.

(a) Hurwitz zeta can be identified as spectral zeta if the spectrum of K-D operator consists of functions z^{n+a}. This kind of situation could result from boundary conditions at origin $z = 0$ giving shift rather than scaling of the conformal weights required by the covering space picture. This allows to understand why for integer values of a Hurwitz equals to Riemann zeta apart from few terms.

(b) The spectrum of conformal weights for Ramond type representations comes as integers but for N-S type representations the ground state conformal weight is $\pm 1/2$. Therefore Hurwitz zeta with $a = 1/2$ would be relevant if half odd integer spectrum for conformal weights of K-D operator is allowed. If the zeros of spectral zeta determine the spectrum of generating super-symplectic conformal weights would include also the set $h_n = n \times i2\pi/ln(2)$ in this case.

6.2 Hurwitz Zeta Inspires An Explanation For Why The Number Of Fermion Generations Is Three

It is clear that Hurwitz zeta does not relate to dark matter in TGD sense. The exceptional character of $a = 1/2$ for Hurwitz zeta however inspires an argument for why the number of fermion generations is three. Ironically, Hurwitz zeta is not required by the argument itself!

(a) Dark matter would correspond to n-fold coverings of space-time sheets and also of partonic 2-surfaces. The inclusions of HFFs allow only quantum phases corresponding to $n > 2$ suggesting that dark matter corresponds to $n > 2$ coverings. For this reason there is a temptation to see $n = 2$-sheetedness as a space-time correlate for spin 1/2 property rather than dark matter property. More generally, if n is even, one obtains the same result since Z_2 appears as a factor in Z_n.

(b) Ramond representation ($a = 0$) rather than N-S representation assignable to $a = 1/2$ looks a reasonable candidate for super-conformal representation for fermions. The reason is that two-valuedness is not associated with wave function but with its transformation property under 2π rotation. p-Adic mass calculations indeed assume Ramond representation for fermions.

(c) One expects that for $n > 2$ Z_n acts as a conformal symmetry, which is not gauge symmetry. $n = 2$ can be an exception to this rule. Z_2 symmetry permuting the sheets would act as a global conformal gauge symmetry for hyper-elliptic partonic 2-surfaces. There would be no breaking of conformal symmetry since the degeneracy would be absent.

    The three lowest fermion genera $g \leq 2$ are always hyper-elliptic but for $g > 2$ this would be case only for special values of moduli and for these values of moduli elementary particle vacuum functionals vanish. Thus for $g > 2$ Z_2 could not act as gauge symmetry for physical particles consisting of fundamental fermions. One would obtain something different - perhaps genuine dark matter explaining why higher generations have not been observed! Second interpretation is that the interpretation as single fermion state fails for $g > 2$: handles would behave as particles and one would have the analog of many-particle state.

(d) One can of course criticize this explanation by saying that "dark matter" has replaced the old "very heavy" or more modern "in the second sector of the multiverse", and that one can apply this argument always when theory predicts something which is not observed such as space-time symmetry and colored excitations of quarks and leptons. The basic element of the explanation would be breaking of a conformal symmetry as gauge symmetry and this can indeed take place.

The conclusion is that ζ_H cannot relate to the dark matter in TGD sense. $\zeta_H(s, 1/2)$ cannot appear as spectral zeta function for fundamental fermions and could result only in their bosonization in which wave function is genuinely two valued.

The earlier proposal that Hurwitz zeta could relate to dark matter is wrong. The motivating observation was that one could generalize modular invariance to fractional modular invariance for Riemann surfaces possessing Z_n symmetry and perform a similar generalization for theta functions and elementary particle vacuum functionals. The Hurwitz zetas would form Z_n multiplets assignable to dark matter describable in terms of n-fold coverings. I did not have heart to throw out the mathematical facts related to Hurwitz zetas.

6.3 About Hurwitz Zetas

The action of modular group SL(2, Z) on Riemann zeta [A2] is induced by its action on theta function [A3]. The action of the generator $\tau \rightarrow -1/\tau$ on theta function is essential in providing the functional equation for Riemann Zeta. Usually the action of the generator $\tau \rightarrow \tau + 1$ on Zeta is not considered explicitly. The surprise was that the action of the generator $\tau \rightarrow \tau + 1$ on Riemann Zeta does not give back Riemann zeta but a more general function known as Hurwitz zeta $\zeta(s, z)$ for $z = 1/2$. One finds that Hurwitz zetas for certain rational values of argument define in a well defined sense representations of fractional modular group to which quantum group can be assigned naturally. Could they allow to code the value of the quantum phase $q = exp(i2\pi/n)$ to the solution spectrum of the Kähler-Dirac operator D? As already shown the answer to this question is negative. Despite this Hurwitz zetas deserve a closer examination.

6.3.1 Definition

Hurwitz zeta is obtained by replacing integers m with $m + z$ in the defining sum formula for Riemann Zeta:

$$\zeta(s, z) = \sum_m (m + z)^{-s} \quad . \tag{6.1}$$

Riemann zeta results for $z = n$ apart from finite number of terms.

Hurwitz zeta obeys the following functional equation for rational $z = m/n$ of the second argument [A1]:

$$\zeta(1 - s, \frac{m}{n}) = \frac{2\Gamma(s)}{2\pi n}^s \sum_{k=1}^{n} cos(\frac{\pi s}{2} - \frac{2\pi km}{n})\zeta(s, \frac{k}{n}) \quad . \tag{6.2}$$

The representation of Hurwitz zeta in terms of θ [A1] is given by the equation

$$\int_0^\infty [\theta(z, it) - 1]\, t^{s/2} \frac{dt}{t} = \pi^{(1-s)/2}\Gamma(\frac{1-s}{2})\, [\zeta(1 - s, z) + \zeta(1 - s, 1 - z)] \quad . \tag{6.3}$$

By the periodicity of theta function this gives for $z = n$ Riemann zeta apart from finite number of terms.

6.3.2 The action of $\tau \rightarrow \tau + 1$ transforms $\zeta(s, 0)$ to $\zeta(s, 1/2)$

The action of the transformations $\tau \rightarrow \tau + 1$ on the integral representation of Riemann Zeta [A2] in terms of θ function [A3]

$$\theta(z; \tau) - 1 = 2 \sum_{n=1}^{\infty} [exp(i\pi\tau)]^{n^2} cos(2\pi nz) \tag{6.4}$$

is given by

$$\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) = \int_0^{\infty} \left[\theta(0;it) - 1\right] t^{s/2}\frac{dt}{t} \quad . \qquad (6.5)$$

Using the first formula one finds that the shift $\tau = it \to \tau + 1$ in the argument θ induces the shift $\theta(0;\tau) \to \theta(1/2;\tau)$. Hence the result is Hurwitz zeta $\zeta(s,1/2)$. For $\tau \to \tau + 2$ one obtains Riemann Zeta.

Thus $\zeta(s,0)$ and $\zeta(s,1/2)$ behave like a doublet under modular transformations. Under the subgroup of modular group obtained by replacing $\tau \to \tau + 1$ with $\tau \to \tau + 2$ Riemann Zeta forms a singlet. The functional equation for Hurwitz zeta relates $\zeta(1-s,1/2)$ to $\zeta(s,1/2)$ and $\zeta(s,1) = \zeta(s,0)$ so that also now one obtains a doublet, which is not surprising since the functional equations directly reflects the modular transformation properties of theta functions. This doublet might be the proper object to study instead of singlet if one considers full modular invariance.

6.3.3 Hurwitz zetas form n-plets closed under the action of fractional modular group

The inspection of the functional equation for Hurwitz zeta given above demonstrates that $\zeta(s,m/n)$, $m = 0,1,...,n$, form in a well-defined sense an n-plet under fractional modular transformations obtained by using generators $\tau \to -1/\tau$ and $\tau \to \tau + 2/n$. The latter corresponds to the unimodular matrix $(a,b;c,d) = (1,2/n;0,1)$. These matrices obviously form a group. Note that Riemann zeta is always one member of the multiplet containing n Hurwitz zetas.

These observations bring in mind fractionization of quantum numbers, quantum groups corresponding to the quantum phase $q = exp(i2\pi/n)$, and the inclusions for hyper-finite factors of type II_1 partially characterized by these quantum phases. Fractional modular group obtained using generator $\tau \to \tau + 2/n$ and Hurwitz zetas $\zeta(s,k/n)$ could very naturally relate to these and related structures.

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