Category Theory and Quantum TGD

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Contents

Abstract

Possible applications of category theory to quantum TGD are discussed. The so called 2-plectic structure generalizing the ordinary symplectic structure by replacing symplectic 2-form with 3-form and Hamiltonians with Hamiltonian 1-forms has a natural place in TGD since the dynamics of the light-like 3-surfaces is characterized by Chern-Simons type action. The notion of planar operad was developed for the classification of hyper-finite factors of type II_1 and its mild generalization allows to understand the combinatorics of the generalized Feynman diagrams obtained by gluing 3-D light-like surfaces representing the lines of Feynman diagrams along their 2-D ends representing the vertices.

The fusion rules for the symplectic variant of conformal field theory, whose existence is strongly suggested by quantum TGD, allow rather precise description using the basic notions of category theory and one can identify a series of finite-dimensional nilpotent algebras as discretized versions of field algebras defined by the fusion rules. These primitive fusion algebras can be used to construct more complex algebras by replacing any algebra element by a primitive fusion algebra. Trees with arbitrary numbers of branches in any node characterize the resulting collection of fusion algebras forming an operad. One can say that an exact solution of symplectic scalar field theory is obtained.

Conformal fields and symplectic scalar field can be combined to form symplecto-formal fields. The combination of symplectic operad and Feynman graph operad leads to a construction of Feynman diagrams in terms of n-point functions of conformal field theory. M-matrix elements with a finite measurement resolution are expressed in terms of a hierarchy of symplecto-conformal n-point functions such that the improvement of measurement resolution corresponds to an algebra homomorphism mapping conformal fields in given resolution to composite conformal fields in improved resolution. This expresses the idea that composites behave as independent conformal fields. Also other applications are briefly discussed.

Years after writing this chapter a very interesting new TGD related candidate for a category emerged. The preferred extremals of Kähler action would form a category if the proposed duality mapping associative (co-associative) 4-surfaces of embedding space respects associativity (co-associativity). The duality would allow to construct new preferred extremals of Kähler action.

1 Introduction

TGD predicts several hierarchical structures involving a lot of new physics. These structures look frustratingly complex and category theoretical thinking might help to build a bird's eye view about the situation. I have already earlier considered the question how category theory might be applied in TGD [K2, K1]. Besides the far from complete understanding of the basic mathematical structure of TGD also my own limited understanding of category theoretical ideas have been a serious limitation. During last years considerable progress in the understanding of quantum TGD proper has taken place and the recent formulation of TGD is in terms of light-like 3-surfaces, zero energy ontology and number theoretic braids [K6, **?**]. There exist also rather detailed formulations for the fusion of p-adic and real physics and for the dark matter hierarchy. This motivates a fresh look to how category theory might help to understand quantum TGD.

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Years after writing this chapter a very interesting new TGD related candidate for a category emerged. The preferred extremals would form a category if the proposed duality mapping associative (co-associative) 4-surfaces of embedding space respects associativity (co-associativity) [K5]. The duality would allow to construct new preferred extremals of Kähler action.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://tgdtheory.fi/cmaphtml.html` [L2]. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L3].

2 S-Matrix As A Functor

John Baez's [A13] discusses in a physicist friendly manner the possible application of category theory to physics. The lessons obtained from the construction of topological quantum field theories (TQFTs) suggest that category theoretical thinking might be very useful in attempts to construct theories of quantum gravitation.

The point is that the Hilbert spaces associated with the initial and final state n-1-manifold of n-cobordism indeed form in a natural manner category. Morphisms of Hilb in turn are unitary or possibly more general maps between Hilbert spaces. TQFT itself is a functor assigning to a cobordism the counterpart of S-matrix between the Hilbert spaces associated with the initial and final n-1-manifold. The surprising result is that for $n \leq 4$ the S-matrix can be unitary S-matrix only if the cobordism is trivial. This should lead even string theorist to raise some worried questions.

In the hope of feeding some category theoretic thinking into my spine, I briefly summarize some of the category theoretical ideas discussed in the article and relate it to the TGD vision, and after that discuss the worried questions from TGD perspective. That space-time makes sense only relative to embedding space would conform with category theoretic thinking.

2.1 The *-Category Of Hilbert Spaces

Baez considers first the category of Hilbert spaces. Intuitively the definition of this category looks obvious: take linear spaces as objects in category Set, introduce inner product as additional structure and identify morphisms as maps preserving this inner product. In finite-D case the category with inner product is however identical to the linear category so that the inner product does not seem to be absolutely essential. Baez argues that in infinite-D case the morphisms need not be restricted to unitary transformations: one can consider also bounded linear operators as morphisms since they play key role in quantum theory (consider only observables as Hermitian operators). For hyper-finite factors of type II_1 inclusions define very important morphisms which are not unitary transformations but very similar to them. This challenges the belief about the fundamental role of unitarity and raises the question about how to weaken the unitarity condition without losing everything.

The existence of the inner product is essential only for the metric topology of the Hilbert space. Can one do without inner product as an inherent property of state space and reduce it to a morphism? One can indeed express inner product in terms of morphisms from complex numbers to Hilbert space and their conjugates. For any state Ψ of Hilbert space there is a unique morphisms T_Ψ from C to Hilbert space satisfying $T_\Psi(1) = \Psi$. If one assumes that these morphisms have conjugates T_Ψ^* mapping Hilbert space to C, inner products can be defined as morphisms $T_\Phi^* T_\Psi$. The Hermitian conjugates of operators can be defined with respect to this inner product so that one obtains *-category. Reader has probably realized that T_Ψ and its conjugate correspond to ket and bra in Dirac's formalism.

Note that in TGD framework based on hyper-finite factors of type II_1 (HFFs) the inclusions of complex rays might be replaced with inclusions of HFFs with included factor representing the finite measurement resolution. Note also the analogy of inner product with the representation of space-times as 4-surfaces of the embedding space in TGD.

2.2 The Monoidal *-Category Of Hilbert Spaces And Its Counterpart At The Level Of Ncob

One can give the category of Hilbert spaces a structure of monoid by introducing explicitly the tensor products of Hilbert spaces. The interpretation is obvious for physicist. Baez describes the details of this identification, which are far from trivial and in the theory of quantum groups very interesting things happen. A non-commutative quantum version of the tensor product implying braiding is possible and associativity condition leads to the celebrated Yang-Baxter equations: inclusions of HFFs lead to quantum groups too.

At the level of nCob the counterpart of the tensor product is disjoint union of n-1-manifolds. This unavoidably creates the feeling of cosmic loneliness. Am I really a disjoint 3-surface in emptiness which is not vacuum even in the geometric sense? Cannot be true!

This horrifying sensation disappears if n-1-manifolds are n-1-surfaces in some higher-dimensional embedding space so that there would be at least something between them. I can emit a little baby manifold moving somewhere perhaps being received by some-one somewhere and I can receive radiation from some-one at some distance and in some direction as small baby manifolds making gentle tosses on my face!

This consoling feeling could be seen as one of the deep justifications for identifying fundamental objects as light-like partonic 3-surfaces in TGD framework. Their ends correspond to 2-D partonic surfaces at the boundaries of future or past directed light-cones (states of positive and negative energy respectively) and are indeed disjoint but not in the desperately existential sense as 3-geometries of General Relativity.

This disjointness has also positive aspect in TGD framework. One can identify the color degrees of freedom of partons as those associated with CP_2 degrees of freedom. For instance, SU(3) analogs for rotational states of rigid body become possible. 4-D space-time surfaces as preferred extremals of Kähler action connect the partonic 3-surfaces and bring in classical representation of correlations and thus of interactions. The representation as sub-manifolds makes it also possible to speak about positions of these sub-Universes and about distances between them. The habitants of TGD Universe are maximally free but not completely alone.

2.3 TSFT As A Functor

The category theoretic formulation of TQFT relies on a very elegant and general idea. Quantum transition has as a space-time correlate an n-dimensional surface having initial final states as its n-1-dimensional ends. One assigns Hilbert spaces of states to the ends and S-matrix would be a unitary morphism between the ends. This is expressed in terms of the category theoretic language by introducing the category nCob with objects identified as n-1-manifolds and morphisms as cobordisms and *-category Hilb consisting of Hilbert spaces with inner product and morphisms which are bounded linear operators which do not however preserve the unitarity. Note that the morphisms of nCob cannot anymore be identified as maps between n-1-manifolds interpreted as sets with additional structure so that in this case category theory is more powerful than set theory.

TQFT is identified as a functor nCob $\rightarrow$ Hilb assigning to n-1-manifolds Hilbert spaces, and to cobordisms unitary S-matrices in the category Hilb. This looks nice but the surprise is that for $n \leq 4$ unitary S-matrix exists only if the cobordism is trivial so that topology changing transitions are not possible unless one gives up unitarity.

This raises several worried questions.

1. Does this result mean that in TQFT sense unitary S-matrix for topology changing transitions from a state containing n_i closed strings to a state containing $n_f \neq n_i$ strings does not exist? Could the situation be same also for more general non-topological stringy S-matrices? Could the non-converging perturbation series for S-matrix with finite individual terms matrix fail to no non-perturbative counterpart? Could it be that M-theory is doomed to remain a dream with no hope of being fulfilled?

2. Should one give up the unitarity condition and require that the theory predicts only the relative probabilities of transitions rather than absolute rates? What the proper generalization of the S-matrix could be?

3. What is the relevance of this result for quantum TGD?

2.4   The Situation Is In TGD Framework

The result about the non-existence of unitary S-matrix for topology changing cobordisms allows new insights about the meaning of the departures of TGD from string models.

2.4.1   Cobordism cannot give interesting selection rules

When I started to work with TGD for more than 28 years ago, one of the first ideas was that one could identify the selection rules of quantum transitions as topological selection rules for cobordisms. Within week or two came the great disappointment: there were practically no selection rules. Could one revive this naïve idea? Could the existence of unitary S-matrix force the topological selection rules after all? I am skeptic. If I have understood correctly the discussion of what happens in 4-D case [A11] only the exotic diffeo-structures modify the situation in 4-D case.

2.4.2   Light-like 3-surfaces allow cobordism

In the physically interesting GRT like situation one would expect the cobordism to be mediated by a space-time surface possessing Lorentz signature. This brings in metric and temporal distance. This means complications since one must leave the pure TQFT context. Also the classical dynamics of quantum gravitation brings in strong selection rules related to the dynamics in metric degrees of freedom so that TQFT approach is not expected to be useful from the point of view of quantum gravity and certainly not the limit of a realistic theory of quantum gravitation.

In TGD framework situation is different. 4-D space-time sheets can have Euclidian signature of the induced metric so that Lorentz signature does not pose conditions. The counterparts of cobordisms correspond at fundamental level to light-like 3-surfaces, which are arbitrarily except for the light-likeness condition (the effective 2-dimensionality implies generalized conformal invariance and analogy with 3-D black-holes since 3-D vacuum Einstein equations are satisfied). Field equations defined by the Chern-Simons action imply that CP_2 projection is at most 2-D but this condition holds true only for the extremals and one has functional integral over all light-like 3-surfaces. The temporal distance between points along light-like 3-surface vanishes. The constraints from light-likeness bring in metric degrees of freedom but in a very gentle manner and just to make the theory physically interesting.

2.4.3   Feynman cobordism as opposed to ordinary cobordism

In string model context the discouraging results from TQFT hold true in the category of nCob, which corresponds to trouser diagrams for closed strings or for their open string counterparts. In TGD framework these diagrams are replaced with a direct generalization of Feynman diagrams for which 3-D light-like partonic 3-surfaces meet along their 2-D ends at the vertices. In honor of Feynman one could perhaps speak of Feynman cobordisms. These surfaces are singular as 3-manifolds but vertices are nice 2-manifolds. I contrast to this, in string models diagrams are nice 2-manifolds but vertices are singular as 1-manifolds (say eye-glass type configurations for closed strings).

This picture gains a strong support for the interpretation of fermions as light-like throats associated with connected sums of CP_2 type extremals with space-time sheets with Minkowski signature and of bosons as pairs of light-like wormhole throats associated with CP_2 type extremal connecting two space-time sheets with Minkowski signature of induced metric. The space-time sheets have opposite time orientations so that also zero energy ontology emerges unavoidably. There is also consistency TGD based explanation of the family replication phenomenon in terms of genus of light-like partonic 2-surfaces.

One can wonder what the 4-D space-time sheets associated with the generalized Feynman diagrams could look like? One can try to gain some idea about this by trying to assign 2-D surfaces to ordinary Feynman diagrams having a subset of lines as boundaries. In the case of $2\rightarrow2$ reaction open string is pinched to a point at vertex. $1\rightarrow2$ vertex, and quite generally, vertices with odd number of lines, are impossible. The reason is that 1-D manifolds of finite size can have either 0 or 2 ends whereas in higher-D the number of boundary components is arbitrary. What one expects to happen in TGD context is that wormhole throats which are at distance characterized by CP_2 fuse together in the vertex so that some kind of pinches appear also now.

2.4.4   Zero energy ontology

Zero energy ontology gives rise to a second profound distinction between TGD and standard QFT. Physical states are identified as states with vanishing net quantum numbers, in particular energy. Everything is creatable from vacuum - and one could add- by intentional action so that zero energy ontology is profoundly Eastern. Positive *resp.* negative energy parts of states can be identified as states associated with 2-D partonic surfaces at the boundaries of future *resp.* past directed light-cones, whose tips correspond to the arguments of n-point functions. Each incoming/outgoing particle would define a mini-cosmology corresponding to not so big bang/crunch. If the time scale of perception is much shorter than time interval between positive and zero energy states, the ontology looks like the Western positive energy ontology. Bras and kets correspond naturally to the positive and negative energy states and phase conjugation for laser photons making them indeed something which seems to travel in opposite time direction is counterpart for bra-ket duality.

The new element would be quantum measurements performed separately for observables assignable to positive and negative energy states. These measurements would be characterized in terms of Jones inclusions. The state function reduction for the negative energy states could be interpreted as a detection of a particle reaction.

2.4.5   Finite temperature S-matrix defines genuine quantum state in zero energy ontology

In TGD framework one encounters two S-matrix like operators.

1. U-matrix is the analog of the ordinary S-matrix and constructible in terms of it and orthonormal basis of square roots of density matrices expressible as products of hermitian operators multiplied by unitary S-matrix [K4].

2. The S-matrix like operator describing what happens in laboratory corresponds to the time-like entanglement coefficients between positive and negative energy parts of the state. Measurement of reaction rates would be a measurement of observables reducing time like entanglement and very much analogous to an ordinary quantum measurement reducing space-like entanglement. There is a finite measurement resolution described by inclusion of HFFs and this means that situation reduces effectively to a finite-dimensional one.

p-Adic thermodynamics strengthened with p-adic length scale hypothesis predicts particle masses with an amazing success. At first the thermodynamical approach seems to be in contradiction with the idea that elementary particles are quantal objects. Unitarity is however *not* necessary if one accepts that only relative probabilities for reductions to pairs of initial and final states interpreted as particle reactions can be measured.

The beneficial implications of unitarity are not lost if one replaces QFT with thermal QFT. Category theoretically this would mean that the time-like entanglement matrix associated with the product of cobordisms is a product of these matrices for the factors. The time parameter in S-matrix would be replaced with a complex time parameter with the imaginary part identified as inverse temperature. Hence the interpretation in terms of time evolution is not lost.

In the theory of hyper-finite factors of type III_1 the partition function for thermal equilibrium states and S-matrix can be neatly fused to a thermal S-matrix for zero energy states and one could introduce p-adic thermodynamics at the level of quantum states. It seems that this picture applies to HFFs by restriction. Therefore the loss of unitarity S-matrix might after all turn to a victory by more or less forcing both zero energy ontology and p-adic thermodynamics. Note that also the presence of factor of type I coming from embedding space degrees of freedom forces thermal S-matrix.

2.4.6   Time-like entanglement coefficients as a square root of density matrix?

All quantum states do not correspond to thermal states and one can wonder what might be the most general identification of the quantum state in zero energy ontology. Density matrix formalism defines a very general formulation of quantum theory. Since the quantum states in zero energy ontology are analogous to operators, the idea that time-like entanglement coefficients in some sense define a square root of density matrix is rather natural. This would give the defining conditions

$$\rho^+ = SS^\dagger \ , \rho^- = S^\dagger S \ ,$$
$$Tr(\rho^\pm) = 1 \ . \tag{2.1}$$

$\rho^\pm$ would define density matrix for positive/negative energy states. In the case HFFs of type II_1 one obtains unitary S-matrix and also the analogs of pure quantum states are possible for factors of type I. The numbers $p^+_{m,n} = |S^2_{m,n}|/\rho^+_{m,m}$ and $p^-_{m,n} = |S^2_{n,m}|/\rho^-_{m,m}$ give the counterparts of the usual scattering probabilities.

A physically well-motivated hypothesis would be that S has expression $S = \sqrt{\rho}S_0$ such that S_0 is a universal unitary S-matrix, and $\sqrt{\rho}$ is square root of a state dependent density matrix. Note that in general S is not diagonalizable in the algebraic extension involved so that it is not possible to reduce the scattering to a mere phase change by a suitable choice of state basis.

What makes this kind of hypothesis aesthetically attractive is the unification of two fundamental matrices of quantum theory to single one. This unification is completely analogous to the combination of modulus squared and phase of complex number to a single complex number: complex valued Schrödinger amplitude is replaced with operator valued one.

2.4.7   S-matrix as a functor and the groupoid structure formed by S-matrices

In zero energy ontology S-matrix can be seen as a functor from the category of Feynman cobordisms to the category of operators. S-matrix can be identified as a "square root" of the positive energy density matrix $S = \rho^{1/2}_+ S_0$, where S_0 is a unitary matrix and ρ_+ is the density matrix for positive energy part of the zero energy state. Obviously one has $SS^\dagger = \rho_+$. $S^\dagger S = \rho_-$ gives the density matrix for negative energy part of zero energy state. Clearly, S-matrix can be seen as matrix valued generalization of Schrödinger amplitude. Note that the "indices" of the S-matrices correspond to WCW spinor s (fermions and their bound states giving rise to gauge bosons and gravitons) and to WCW degrees of freedom. For hyper-finite factor of II_1 it is not strictly speaking possible to speak about indices since the matrix elements are traces of the S-matrix multiplied by projection operators to infinite-dimensional subspaces from right and left.

The functor property of S-matrices implies that they form a multiplicative structure analogous but not identical to groupoid [A1]. Recall that groupoid has associative product and there exist always right and left inverses and identity in the sense that ff^{-1} and $f^{-1}f$ are always defined but not identical and one has $fgg^{-1} = f$ and $f^{-1}fg = g$.

The reason for the groupoid like property is that S-matrix is a map between state spaces associated with initial and final sets of partonic surfaces and these state spaces are different so that inverse must be replaced with right and left inverse. The defining conditions for groupoid are replaced with more general ones. Also now associativity holds but the role of inverse is taken by hermitian conjugate. Thus one has the conditions $fgg^\dagger = f\rho_{g,+}$ and $f^\dagger fg = \rho_{f,-}g$, and the conditions $ff^\dagger = \rho_+$ and $f^\dagger f = \rho_-$ are satisfied. Here $\rho_\pm$ is density matrix associated with positive/negative energy parts of zero energy state. If the inverses of the density matrices exist, groupoid axioms hold true since $f^{-1}_L = f^\dagger \rho^{-1}_{f,+}$ satisfies $ff^{-1}_L = Id_+$ and $f^{-1}_R = \rho^{-1}_{f,-}f^\dagger$ satisfies $f^{-1}_R f = Id_-$.

There are good reasons to believe that also tensor product of its appropriate generalization to the analog of co-product makes sense with non-triviality characterizing the interaction between the systems of the tensor product. If so, the S-matrices would form very beautiful mathematical structure bringing in mind the corresponding structures for 2-tangles and N-tangles. Knowing how incredibly powerful the group like structures have been in physics one has good reasons to hope that groupoid like structure might help to deduce a lot of information about the quantum dynamics of TGD.

A word about nomenclature is in order. S has strong associations to unitarity and it might be appropriate to replace S with some other letter. The interpretation of S-matrix as a generalized Schrödinger amplitude would suggest Ψ-matrix. Since the interaction with Kea's M-theory blog at (see `http://tinyurl.com/yb3lsbjq` (M denotes Monad or Motif in this context) was led ot the realization of the connection with density matrix, also M-matrix might be considered. S-matrix as a functor from the category of Feynman cobordisms in turn suggests C or F. Or could just Matrix denoted by M in formulas be enough? Certainly it would inspire feeling of awe!

3   Further Ideas

The work of John Baez and students has inspired also the following ideas about the role of category theory in TGD.

3.1   Operads, Number Theoretical Braids, And Inclusions Of HFFs

The description of braids leads naturally to category theory and quantum groups when the braiding operation, which can be regarded as a functor, is not a mere permutation. Discreteness is a natural notion in the category theoretical context. To me the most natural manner to interpret discreteness is - not something emerging in Planck scale- but as a correlate for a finite measurement resolution and quantum measurement theory with finite measurement resolution leads naturally to number theoretical braids as fundamental discrete structures so that category theoretic approach becomes well-motivated. Discreteness is also implied by the number theoretic approach to quantum TGD from number theoretic associativity condition [L1] central also for category theoretical thinking as well as from the realization of number theoretical universality by the fusion of real and p-adic physics to single coherent whole.

Operads are formally single object multi-categories [A4, A14]. This object consist of an infinite sequence of sets of n-ary operations. These operations can be composed and the compositions are associative (operations themselves need not be associative) in the sense that the is natural isomorphism (symmetries) mapping differently bracketed compositions to each other. The coherence laws for operads formulate the effect of permutations and bracketing (association) as functors acting as natural isomorphisms. A simple manner to visualize the composition is as an addition of $n_1, ... n_k$ leaves to the leaves $1, ..., k$ of k-leaved tree.

An interesting example of operad is the braid operad formulating the combinatorics for a hierarchy of braids formed from braids by grouping subsets of braids having $n_1, ... n_k$ strands and defining the strands of a k-braid. In TGD framework this grouping can be identified in terms of the formation bound states of particles topologically condensed at larger space-time sheet and coherence laws allow to deduce information about scattering amplitudes. In conformal theories braided categories indeed allow to understand duality of stringy amplitudes in terms of associativity condition.

Planar operads [A10] define an especially interesting class of operads. The reason is that the inclusions of HFFs give rise to a special kind of planar operad [A5]. The object of this multi-category [A3] consists of planar k-tangles. Planar operads are accompanied by planar algebras. It will be found that planar operads allow a generalization which could provide a description for the combinatorics of the generalized Feynman diagrams and also rigorous formulation for how the arrow of time emerges in TGD framework and related heuristic ideas challenging the standard views.

3.2   Generalized Feynman Diagram As Category?

John Baez has proposed a category theoretical formulation of quantum field theory as a functor from the category of n-cobordisms to the category of Hilbert spaces [A13, A9]. The attempt to generalize this formulation looks well motivated in TGD framework because TGD can be regarded as almost topological quantum field theory in a well defined sense and braids appear as fundamental structures. It however seems that formulation as a functor from nCob to Hilb is not general enough.

In zero energy ontology events of ordinary ontology become quantum states with positive and negative energy parts of quantum states localizable to the upper and lower light-like boundaries of causal diamond (CD).

1. Generalized Feynman diagrams associated with a given CD involve quantum superposition of light-like 3-surfaces corresponding to given generalized Feynman diagram. These superpositions could be seen as categories with 3-D light-like surfaces containing braids as arrows and 2-D vertices as objects. Zero energy states would represent quantum superposition of categories (different topologies of generalized Feynman diagram) and M-matrix defined as Connes tensor product would define a functor from this category to the Hilbert space of zero energy states for given CD (tensor product defines quite generally a functor).

2. What is new from the point of view of physics that the sequences of generalized lines would
   define compositions of arrows and morphisms having identification in terms of braids which
   replicate in vertices. The possible interpretation of the replication is in terms of copying of
   information in classical sense so that even elementary particles would be information carrying
   and processing structures. This structure would be more general than the proposal of John
   Baez that S-matrix corresponds to a function from the category of n-dimensional cobordisms
   to the category Hilb.

3. p-Adic length scale hypothesis follows if the temporal distance between the tips of CD mea-
   sured as light-cone proper time comes as an octave of CP_2 time scale: $T = 2^n T_0$. This
   assumption implies that the p-adic length scale resolution interpreted in terms of a hierarchy
   of increasing measurement resolutions comes as octaves of time scale. A weaker condition
   would be $T_p = pT_0$, p prime, and would assign all p-adic time scales to the size scale hierarchy
   of CDs.

This preliminary picture is of course not far complete since it applies only to single CD. There
are several questions. Can one allow CDs within CDs and is every vertex of generalized Feynman
diagram surrounded by this kind of CD. Can one form unions of CDs freely?

1. Since light-like 3-surfaces in 8-D embedding space have no intersections in the generic posi-
   tion, one could argue that the overlap must be allowed and makes possible the interaction of
   between zero energy states belonging to different CDs. This interaction would be something
   new and present also for sub-CDs of a given CD.

2. The simplest guess is that the unrestricted union of CDs defines the counterpart of tensor
   product at geometric level and that extended M-matrix is a functor from this category to
   the tensor product of zero energy state spaces. For non-overlapping CDs ordinary tensor
   product could be in question and for overlapping CDs tensor product would be non-trivial.
   One could interpret this M-matrix as an arrow between M-matrices of zero energy states
   at different CDs: the analog of natural transformation mapping two functors to each other.
   This hierarchy could be continued ad infinitum and would correspond to the hierarchy of
   n-categories.

This rough heuristics represents of course only one possibility among many since the notion of
category is extremely general and the only limits are posed by the imagination of the mathemati-
cian. Also the view about zero energy states is still rather primitive.

4 Planar Operads, The Notion Of Finite Measurement Res-olution, And Arrow Of Geometric Time

In the sequel the idea that planar operads or their appropriate generalization might allow to
formulate generalized Feynman diagrammatics in zero energy ontology will be considered. Also a
description of measurement resolution and arrow of geometric time in terms of operads is discussed.

4.1   Zeroth Order Heuristics About Zero Energy States

Consider now the existing heuristic picture about the zero energy states and coupling constant
evolution provided by CDs.

1. The tentative description for the increase of the measurement resolution in terms CDs is
   that one inserts to the upper and/or lower light-like boundary of CD smaller CDs by gluing
   them along light-like radial ray from the tip of CD. It is also possible that the vertices of
   generalized Feynman diagrams belong inside smaller CD: s and it turns out that these CD:
   s must be allowed.

2. The considerations related to the arrow of geometric time suggest that there is asymmetry
   between upper and lower boundaries of CD. The minimum requirement is that the measure-
   ment resolution is better at upper light-like boundary.

3. In zero energy ontology communications to the direction of geometric past are possible and phase conjugate laser photons represent one example of this.

4. Second law of thermodynamics must be generalized in such a way that it holds with respect to subjective time identified as sequence of quantum jumps. The arrow of geometric time can however vary so that apparent breaking of second law is possible in shorter time scales at least. One must however understand why second law holds true in so good an approximation.

5. One must understand also why the contents of sensory experience is concentrated around a narrow time interval whereas the time scale of memories and anticipation are much longer. The proposed mechanism is that the resolution of conscious experience is higher at the upper boundary of CD. Since zero energy states correspond to light-like 3-surfaces, this could be a result of self-organization rather than a fundamental physical law.

   (a) CDs define the perceptive field for self. Selves are curious about the space-time sheets outside their perceptive field in the geometric future of the embedding space and perform quantum jumps tending to shift the superposition of the space-time sheets to the direction of geometric past (past defined as the direction of shift!). This creates the illusion that there is a time=snapshot front of consciousness moving to geometric future in fixed background space-time as an analog of train illusion.

   (b) The fact that news come from the upper boundary of CD implies that self concentrates its attention to this region and improves the resolutions of sensory experience and quantum measurement here. The sub-CD: s generated in this manner correspond to mental images with contents about this region. As a consequence, the contents of conscious experience, in particular sensory experience, tend to be about the region near the upper boundary.

   (c) This mechanism in principle allows the arrow of the geometric time to vary and depend on p-adic length scale and the level of dark matter hierarchy. The occurrence of phase transitions forcing the arrow of geometric time to be same everywhere are however plausible for the reason that the lower and upper boundaries of given CD must possess the same arrow of geometric time.

   (d) If this is the mechanism behind the arrow of time, planar operads can provide a description of the arrow of time but not its explanation.

This picture is certainly not general enough, can be wrong at the level of details, and at best relates to the whole like single particle wave mechanics to quantum field theory.

4.2 Planar Operads

The geometric definition of planar operads [A6, A4, A5, A10] without using the category theoretical jargon goes as follows.

1. There is an external disk and some internal disks and a collection of disjoint lines connecting disk boundaries.

2. To each disk one attaches a non-negative integer k, called the color of disk. The disk with color k has k points at each boundary with the labeling $1, 2, ...k$ running clockwise and starting from a distinguished marked point, decorated by "*". A more restrictive definition is that disk colors are correspond to even numbers so that there are $k = 2n$ points lines leaving the disk boundary boundary. The planar tangles with $k = 2n$ correspond to inclusions of HFFs.

3. Each curve is either closed (no common points with disk boundaries) or joins a marked point to another marked point. Each marked point is the end point of exactly one curve.

4. The picture is planar meaning that the curves cannot intersect and diks cannot overlap.

5. Disks differing by isotopies preserving *'s are equivalent.

Given a planar k-tangle-one of whose internal disks has color k_i- and a k_i-tangle S, one can define the tangle $T \circ_i S$ by isotoping S so that its boundary, together with the marked points and the *'s co-incides with that of D_i and after that erase the boundary of D_i. The collection of planar tangle together with the composition defined in this manner- is called the colored operad of planar tangles.

One can consider also generalizations of planar operads.

1. The composition law is not affected if the lines of operads branch outside the disks. Branching could be allowed even at the boundaries of the disks although this does not correspond to a generic situation. One might call these operads branched operads.

2. The composition law could be generalized to allow additional lines connecting the points at the boundary of the added disk so that each composition would bring in something genuinely new. Zero energy insertion could correspond to this kind of insertions.

3. TGD picture suggests also the replacement of lines with braids. In category theoretical terms this means that besides association one allows also permutations of the points at the boundaries of the disks.

The question is whether planar operads or their appropriate generalizations could allow a characterization of the generalized Feynman diagrams representing the combinatorics of zero energy states in zero energy ontology and whether also the emergence of arrow of time could be described (but probably not explained) in this framework.

4.3 Planar Operads And Zero Energy States

Are planar operads sufficiently powerful to code the vision about the geometric correlates for the increase of the measurement resolution and coupling constant evolution formulated in terms of CDs? Or perhaps more realistically, could one improve this formulation by assuming that zero energy states correspond to wave functions in the space of planar tangles or of appropriate modifications of them? It seems that the answer to the first question is almost affirmative.

1. Disks are analogous to the white regions of a map whose details are not visible in the measurement resolution used. Disks correspond to causal diamonds (CDs) in zero energy ontology. Physically the white regions relate to the vertices of the generalized Feynman diagrams and possibly also to the initial and final states (strictly speaking, the initial and final states correspond to the legs of generalized Feynman diagrams rather than their ends).

2. The composition of tangles means addition of previously unknown details to a given white region of the map and thus to an increase of the measurement resolution. This conforms with the interpretation of inclusions of HFFs as a characterization of finite measurement resolution and raises the hope that planar operads or their appropriate generalization could provide the proper language to describe coupling constant evolution and their perhaps even generalized Feynman diagrams.

3. For planar operad there is an asymmetry between the outer disk and inner disks. One might hope that this asymmetry could explain or at least allow to describe the arrow of time. This is not the case. If the disks correspond to causal diamonds (CDs) carrying positive *resp.* negative energy part of zero energy state at upper *resp.* lower light-cone boundary, the TGD counterpart of the planar tangle is CD containing smaller CD: s inside it. The smaller CD: s contain negative energy particles at their upper boundary and positive energy particles at their lower boundary. In the ideal resolution vertices represented 2-dimensional partonic at which light-like 3-surfaces meet become visible. There is no inherent asymmetry between positive and negative energies and no inherent arrow of geometric time at the fundamental level. It is however possible to model the arrow of time by the distribution of sub-CD: s. By previous arguments self-organization of selves can lead to zero energy states for which the measurement resolution is better near the upper boundary of the CD.

4. If the lines carry fermion or anti-fermion number, the number of lines entering to a given CD must be even as in the case of planar operads as the following argument shows.

(a) In TGD framework elementary fermions correspond to single wormhole throat associated with topologically condensed CP_2 type extremal and the signature of the induced metric changes at the throat.

(b) Elementary bosons correspond to pairs of wormhole throats associated with wormhole contacts connecting two space-time sheets of opposite time orientation and modellable as a piece of CP_2 type extremal. Each boson therefore corresponds to 2 lines within CP_2 radius.

(c) As a consequence the total number of lines associated with given CD is even and the generalized Feynman diagrams can correspond to a planar algebra associated with an inclusion of HFFs.

5. This picture does not yet describe zero energy insertions.

   (a) The addition of zero energy insertions corresponds intuitively to the allowance of new lines inside the smaller CD: s not coming from the exterior. The addition of lines connecting points at the boundary of disk is possible without losing the basic geometric composition of operads. In particular one does not lose the possibility to color the added tangle using two colors (colors correspond to two groups G and H which characterize an inclusion of HFFs [A10]).

   (b) There is however a problem. One cannot remove the boundaries of sub-CD after the composition of CDs since this would give lines beginning from and ending to the interior of disk and they are invisible only in the original resolution. Physically this is of course what one wants but the inclusion of planar tangles is expected to fail in its original form, and one must generalize the composition of tangles to that of CD: s so that the boundaries of sub-CD: s are not thrown away in the process.

   (c) It is easy to see that zero energy insertions are inconsistent with the composition of planar tangles. In the inclusion defining the composition of tangles both sub-tangle and tangle induce a color to a given segment of the inner disk. If these colors are identical, one can forget the presence of the boundary of the added tangle. When zero energy insertions are allowed, situation changes as is easy to see by adding a line connecting points in a segment of given color at the boundary of the included tangle. There exists no consistent coloring of the resulting structure by using only two colors. Coloring is however possible using four colors, which by four-color theorem is the minimum number of colors needed for a coloring of planar map: this however requires that the color can change as one moves through the boundary of the included disk - this is in accordance with the physical picture.

   (d) Physical intuition suggests that zero energy insertion as an improvement of measurement resolution maps to an improved color resolution and that the composition of tangles generalizes by requiring that the included disk is colored by using new nuances of the original colors. The role of groups in the definition of inclusions of HFFs is consistent with idea that G and H describe color resolution in the sense that the colors obtained by their action cannot be resolved. If so, the improved resolution means that G and H are replaced by their subgroups $G_1 \subset G$ and $H_1 \subset H$. Since the elements of a subgroup have interpretation as elements of group, there are good hopes that by representing the inclusion of tangles as inclusion of groups, one can generalize the composition of tangles.

6. Also CD: s glued along light-like ray to the upper and lower boundaries of CD are possible in principle and -according the original proposal- correspond to zero energy insertions according. These CD: s might be associated with the phase transitions changing the value of $\hbar$ leading to different pages of the book like structure defined by the generalized embedding space.

7. p-Adic length scale hypothesis is realized if the hierarchy of CDs corresponds to a hierarchy of temporal distances between tips of CDs given as $a = T_n = 2^{-n}T_0$ using light-cone proper time.

8. How this description relates to braiding? Each line corresponds to an orbit of a partonic boundary component and in principle one must allow internal states containing arbitrarily

high fermion and anti-fermion numbers. Thus the lines decompose into braids and one must allow also braids of braids hierarchy so that each line corresponds to a braid operad in improved resolution.

4.4 Relationship To Ordinary Feynman Diagrammatics

The proposed description is not equivalent with the description based on ordinary Feynman diagrams.

1. In standard physics framework the resolution scale at the level of vertices of Feynman diagrams is something which one is forced to pose in practical calculations but cannot pose at will as opposed to the measurement resolution. Light-like 3-surfaces can be however regarded only locally orbits of partonic 2-surfaces since generalized conformal invariance is true only in 3-D patches of the light-like 3-surface. This means that light-like 3-surfaces are in principle the fundamental objects so that zero energy states can be regarded only locally as a time evolutions. Therefore measurement resolution can be applied also to the distances between vertices of generalized Feynman diagrams and calculational resolution corresponds to physical resolution. Also the resolution can be better towards upper boundary of CD so that the arrow of geometric time can be understood. This is a definite prediction which can in principle kill the proposed scenario.

2. A further counter argument is that generalized Feynman diagrams are identified as light-like 3-surfaces for which Kähler function defined by a preferred extremal of Kähler action is maximum. Therefore one cannot pose any ad hoc rules on the positions of the vertices. One can of course insist that maximum of Kähler function with the constraint posed by $T_n = 2^n T_0$ (or $T_p = p^n T_0$) hierarchy is in question.

It would be too optimistic to believe that the details of the proposal are correct. However, if the proposal is on correct track, zero energy states could be seen as wave functions in the operad of generalized tangles (zero energy insertions and braiding) as far as combinatorics is involved and the coherence rules for these operads would give strong constraints on the zero energy state and fix the general structure of coupling constant evolution.

5 Category Theory And Symplectic QFT

Besides the counterpart of the ordinary Kac-Moody invariance quantum TGD possesses so called super-symplectic conformal invariance. This symmetry leads to the proposal that a symplectic variant of conformal field theory should exist. The n-point functions of this theory defined in S^2 should be expressible in terms of symplectic areas of triangles assignable to a set of n-points and satisfy the duality rules of conformal field theories guaranteeing associativity. The crucial prediction is that symplectic n-point functions vanish whenever two arguments co-incide. This provides a mechanism guaranteeing the finiteness of quantum TGD implied by very general arguments relying on non-locality of the theory at the level of 3-D surfaces.

The classical picture suggests that the generators of the fusion algebra formed by fields at different point of S^2 have this point as a continuous index. Finite quantum measurement resolution and category theoretic thinking in turn suggest that only the points of S^2 corresponding the strands of number theoretic braids are involved. It turns out that the category theoretic option works and leads to an explicit hierarchy of fusion algebras forming a good candidate for so called little disk operad whereas the first option has difficulties.

5.1 Fusion Rules

Symplectic fusion rules are non-local and express the product of fields at two points s_k an s_l of S^2 as an integral over fields at point s_r, where integral can be taken over entire S^2 or possibly also over a 1-D curve which is symplectic invariant in some sense. Also discretized version of fusion rules makes sense and is expected serve as a correlate for finite measurement resolution.

By using the fusion rules one can reduce n-point functions to convolutions of 3-point functions involving a sequence of triangles such that two subsequent triangles have one vertex in common.

For instance, 4-point function reduces to an expression in which one integrates over the positions of the common vertex of two triangles whose other vertices have fixed. For n-point functions one has n-3 freely varying intermediate points in the representation in terms of 3-point functions.

The application of fusion rules assigns to a line segment connecting the two points s_k and s_l a triangle spanned by s_k, s_l and s_r. This triangle should be symplectic invariant in some sense and its symplectic area A_{klm} would define the basic variable in terms of which the fusion rule could be expressed as $C_{klm} = f(A_{klm})$, where f is fixed by some constraints. Note that A_{klm} has also interpretations as solid angle and magnetic flux.

5.2 What Conditions Could Fix The Symplectic Triangles?

The basic question is how to identify the symplectic triangles. The basic criterion is certainly the symplectic invariance: if one has found N-D symplectic algebra, symplectic transformations of S^2 must provide a new one. This is guaranteed if the areas of the symplectic triangles remain invariant under symplectic transformations. The questions are how to realize this condition and whether it might be replaced with a weaker one. There are two approaches to the problem.

5.2.1 Physics inspired approach

In the first approach inspired by classical physics symplectic invariance for the edges is interpreted in the sense that they correspond to the orbits of a charged particle in a magnetic field defined by the Kähler form. Symplectic transformation induces only a $U(1)$ gauge transformation and leaves the orbit of the charged particle invariant if the vertices are not affected since symplectic transformations are not allowed to act on the orbit directly in this approach. The general functional form of the structure constants C_{klm} as a function $f(A_{klm})$ of the symplectic area should guarantee fusion rules.

If the action of the symplectic transformations does not affect the areas of the symplectic triangles, the construction is invariant under general symplectic transformations. In the case of uncharged particle this is not the case since the edges are pieces of geodesics: in this case however fusion algebra however trivializes so that one cannot conclude anything. In the case of charged particle one might hope that the area remains invariant under general symplectic transformations whose action is induced from the action on vertices. The equations of motion for a charged particle involve a Kähler metric determined by the symplectic structure and one might hope that this is enough to achieve this miracle. If this is not the case - as it might well be - one might hope that although the areas of the triangles are not preserved, the triangles are mapped to each other in such a way that the fusion algebra rules remain intact with a proper choice of the function $f(A_{klm})$. One could also consider the possibility that the function $f(A_{klm})$ is dictated from the condition that the it remains invariant under symplectic transformations. It however turns that this approach does not work as such.

5.2.2 Category theoretical approach

The second realization is guided by the basic idea of category theoretic thinking: the properties of an object are determined its relationships to other objects. Rather than postulating that the symplectic triangle is something which depends solely on the three points involved via some geometric notion like that of geodesic line of orbit of charged particle in magnetic field, one assumes that the symplectic triangle reflects the properties of the fusion algebra, that is the relations of the symplectic triangle to other symplectic triangles. Thus one must assign to each triplet (s_1, s_2, s_3) of points of S^2 a triangle just from the requirement that braided associativity holds true for the fusion algebra.

All symplectic transformations leaving the N points fixed and thus generated by Hamiltonians vanishing at these points would give new gauge equivalent realizations of the fusion algebra and deform the edges of the symplectic triangles without affecting their area. One could even say that symplectic triangulation defines a new kind geometric structure in S^2. The quantum fluctuating degrees of freedom are parameterized by the symplectic group of $S^2 \times CP_2$ in TGD so that symplectic the geometric representation of the triangulation changes but its inherent properties remain invariant.

The elegant feature of category theoretical approach is that one can in principle construct the fusion algebra without any reference to its geometric realization just from the braided associativity and nilpotency conditions and after that search for the geometric realizations. Fusion algebra has also a hierarchy of discrete variants in which the integral over intermediate points in fusion is replaced by a sum over a fixed discrete set of points and this variant is what finite measurement resolution implies. In this case it is relatively easy to see if the geometric realization of a given abstract fusion algebra is possible.

5.2.3 The notion of number theoretical braid

Braids -not necessary number theoretical- provide a realization discretization as a space-time correlate for the finite measurement resolution. The notion of braid was inspired by the idea about quantum TGD as almost topological quantum field theory. Although the original form of this idea has been buried, the notion of braid has survived: in the decomposition of space-time sheets to string world sheets, the ends of strings define representatives for braid strands at light-like 3-surfaces.

The notion of number theoretic universality inspired the much more restrictive notion of number theoretic braid requiring that the points in the intersection of the braid with the partonic 2-surface correspond to rational or at most algebraic points of H in preferred coordinates fixed by symmetry considerations. The challenge has been to find a unique identification of the number theoretic braid or at least of the end points of the braid. The following consideration suggest that the number theoretic braids are not a useful notion in the generic case but make sense and are needed in the intersection of real and p-adic worlds which is in crucial role in TGD based vision about living matter [K3].

It is only the braiding that matters in topological quantum field theories used to classify braids. Hence braid should require only the fixing of the end points of the braids at the intersection of the braid at the light-like boundaries of CDs and the braiding equivalence class of the braid itself. Therefore it is enough is to specify the topology of the braid and the end points of the braid in accordance with the attribute "number theoretic". Of course, the condition that all points of the strand of the number theoretic braid are algebraic is impossible to satisfy.

The situation in which the equations defining X^2 make sense both in real sense and p-adic sense using appropriate algebraic extension of p-adic number field is central in the TGD based vision about living matter [K3]. The reason is that in this case the notion of number entanglement theoretic entropy having negative values makes sense and entanglement becomes information carrying. This motivates the identification of life as something in the intersection of real and p-adic worlds. In this situation the identification of the ends of the number theoretic braid as points belonging to the intersection of real and p-adic worlds is natural. These points -call them briefly algebraic points- belong to the algebraic extension of rationals needed to define the algebraic extension of p-adic numbers. This definition however makes sense also when the equations defining the partonic 2-surfaces fail to make sense in both real and p-adic sense. In the generic case the set of points satisfying the conditions is discrete. For instance, according to Fermat's theorem the set of rational points satisfying $X^n + Y^n = Z^n$ reduces to the point $(0, 0, 0)$ for $n = 3, 4, ...$ Hence the constraint might be quite enough in the intersection of real and p-adic worlds where the choice of the algebraic extension is unique.

One can however criticize this proposal.

1. One must fix the number of points of the braid and outside the intersection and the non-uniquencess of the algebraic extension makes the situation problematic. Physical intuition suggests that the points of braid define carriers of quantum numbers assignable to second quantized induced spinor fields so that the total number of fermions anti-fermions would define the number of braids. In the intersection the highly non-trivial implication is that this number cannot exceed the number of algebraic points.

2. In the generic case one expects that even the smallest deformation of the partonic 2-surface can change the number of algebraic points and also the character of the algebraic extension of rational numbers needed. The restriction to rational points is not expected to help in the generic case. If the notion of number theoretical braid is meant to be practical, must be able to decompose WCW to open sets inside which the numbers of algebraic points of braid at its

ends are constant. For real topology this is expected to be impossible and it does not make sense to use p-adic topology for WCW whose points do not allow interpretation as p-adic partonic surfaces.

3. In the intersection of real and p-adic worlds which corresponds to a discrete subset of WCW , the situation is different. Since the coefficients of polynomials involved with the definition of the partonic 2-surface must be rational or at most algebraic, continuous deformations are not possible so that one avoids the problem.

4. This forces to ask the reason why for the number theoretic braids. In the generic case they seem to produce only troubles. In the intersection of real and p-adic worlds they could however allow the construction of the elements of M-matrix describing quantum transitions changing p-adic to real surfaces and vice versa as realizations of intentions and generation of cognitions. In this the case it is natural that only the data from the intersection of the two worlds are used. In [K3] I have sketched the idea about number theoretic quantum field theory as a description of intentional action and cognition.

There is also the problem of fixing the interior points of the braid modulo deformations not affecting the topology of the braid.

1. Infinite number of non-equivalent braidings are possible. Should one allow all possible braidings for a fixed light-like 3-surface and say that their existence is what makes the dynamics essentially three-dimensional even in the topological sense? In this case there would be no problems with the condition that the points at both ends of braid are algebraic.

2. Or should one try to characterize the braiding uniquely for a given partonic 2-surfaces and corresponding 4-D tangent space distributions? The slicing of the space-time sheet by partonic 2-surfaces and string word sheets suggests that the ends of string world sheets could define the braid strands in the generic context when there is no algebraicity condition involved. This could be taken as a very natural manner to fix the topology of braid but leave the freedom to choose the representative for the braid. In the intersection of real and p-adic worlds there is no good reason for the end points of strands in this case to be algebraic at both ends of the string world sheet. One can however start from the braid defined by the end points of string world sheets, restrict the end points to be algebraic at the end with a smaller number of algebraic pointsandthen perform a topologically non-trivial deformation of the braid so that also the points at the other end are algebraic? Non-trivial deformations need not be possible for all possible choices of algebraic braid points at the other end of braid and different choices of the set of algebraic points would give rise to different braidings. A further constraint is that only the algebraic points at which one has assign fermion or anti-fermion are used so that the number of braid points is not always maximal.

3. One can also ask whether one should perform the gauge fixing for the strands of the number theoretic braid using algebraic functions making sense both in real and p-adic context. This question does not seem terribly relevant since since it is only the topology of the braid that matters.

5.2.4   Symplectic triangulations and braids

The identification of the edges of the symplectic triangulation as the end points of the braid is favored by conceptual economy. The nodes of the symplectic triangulation would naturally correspond to the points in the intersection of the braid with the light-like boundaries of CD carrying fermion or anti-fermion number. The number of these points could be arbitrarily large in the generic case but in the intersection of real and p-adic worlds these points correspond to subset of algebraic points belonging to the algebraic extension of rationals associated with the definition of partonic 2-surfaces so that the sum of fermion and anti-fermion numbers would be bounded above. The presence of fermions in the nodes would be the physical prerequisite for measuring the phase factors defined by the magnetic fluxes. This could be understood in terms of gauge invariance forcing to assign to a pair of points of triangulation the non-integrable phase factor defined by the Kähler gauge potential.

The remaining problem is how uniquely the edges of the triangulation can be determined.

1. The allowance of all possible choices for edges would bring in an infinite number of degrees of freedom. These curves would be analogous to freely vibrating strings. This option is not attractive. One should be able to pose conditions on edges and whatever the manner to specify the edges might be, it must make sense also in the intersection of real and p-adic worlds. In this case the total phase factor must be a root of unity in the algebraic extension of rationals involved and this poses quantization rules analogous to those for magnetic flux. The strongest condition is that the edges are such that the non-integrable phase factor is a root of unity for each edge. It will be found that similar quantization is implied also by the associativity conditions and this justifies the interpretation of phase factors defining the fusion algebra in terms of the Kähler magnetic fluxes. This would pose strong constraints on the choice of edges but would not fix completely the phase factors, and it seems that one must allow all possible triangulations consistent with this condition and the associativity conditions so that physical state is a quantum superposition over all possible symplectic triangulations characterized by the fusion algebras.

2. In the real context one would have an infinite hierarchy of symplectic triangulations and fusion algebras satisfying the associativity conditions with the number of edges equal to the total number N of fermions and anti-fermions. Encouragingly, this hierarchy corresponds also to a hierarchy of $\mathcal{N} = N$ SUSY algebras [?] (large values of $\mathcal{N}$ are not a catastrophe in TGD framework since the physical content of SUSY symmetry is not the same as that in the standard approach). In the intersection of real and p-adic worlds the value of $\mathcal{N}$ would be bounded by the total number of algebraic points. Hence the notion of finite measurement resolution, cutoff in $\mathcal{N}$ and bound on the total fermion number would make physics very simple in the intersection of real and p-adic worlds.

Two kinds of symplectic triangulations are possible since one can use the symplectic forms associated with CP_2 and $r_M = constant$ sphere S^2 of light-cone boundary. For a given collection of nodes the choices of edges could be different for these two kinds of triangulations. Physical state would be proportional to the product of the phase factors assigned to these triangulations.

5.3 Associativity Conditions And Braiding

The generalized fusion rules follow from the associativity condition for n-point functions modulo phase factor if one requires that the factor assignable to n-point function has interpretation as n-point function. Without this condition associativity would be trivially satisfied by using a product of various bracketing structures for the n fields appearing in the n-point function. In conformal field theories the phase factor defining the associator is expressible in terms of the phase factor associated with permutations represented as braidings and the same is expected to be true also now.

1. Already in the case of 4-point function there are three different choices corresponding to the 4 possibilities to connect the fixed points s_k and the varying point s_r by lines. The options are (1-2, 3-4), (1-3, 2-4), and (1-4, 2-3) and graphically they correspond to s-, t-, and u-channels in string diagrams satisfying also this kind of fusion rules. The basic condition would be that same amplitude results irrespective of the choice made. The duality conditions guarantee associativity in the formation of the n-point amplitudes without any further assumptions. The reason is that the writing explicitly the expression for a particular bracketing of n-point function always leads to some bracketing of one particular 4-point function and if duality conditions hold true, the associativity holds true in general. To be precise, in quantum theory associativity must hold true only in projective sense, that is only modulo a phase factor.

2. This framework encourages category theoretic approach. Besides different bracketing there are different permutations of the vertices of the triangle. These permutations can induce a phase factor to the amplitude so that braid group representations are enough. If one has representation for the basic braiding operation as a quantum phase $q = exp(i2\pi/N)$, the phase factors relating different bracketings reduce to a product of these phase factors since $(AB)C$ is obtained from $A(BC)$ by a cyclic permutation involving to permutations

represented as a braiding. Yang-Baxter equations express the reduction of associator to braidings. In the general category theoretical setting associators and braidings correspond to natural isomorphisms leaving category theoretical structure invariant.

3. By combining the duality rules with the condition that 4-point amplitude vanishes, when any two points co-incide, one obtains from $s_k = s_l$ and $s_m = s_n$ the condition stating that the sum (or integral in possibly existing continuum version) of $U^2(A_{klm})|f|^2(x_{kmr})$ over the third point s_r vanishes. This requires that the phase factor U is non-trivial so that Q must be non-vanishing if one accepts the identification of the phase factor as Bohm-Aharonov phase.

4. Braiding operation gives naturally rise to a quantum phase. A good guess is that braiding operation maps triangle to its complement since only in this manner orientation is preserved so that area is A_{klm} is mapped to $A_{klm} - 4\pi$. If the f is proportional to the exponent $exp(-A_{klm}Q)$, braiding operation induces a complex phase factor $q = exp(-i4\pi Q)$.

5. For half-integer values of Q the algebra is commutative. For $Q = M/N$, where M and N have no common factors, only braided commutativity holds true for $N \geq 3$ just as for quantum groups characterizing also Jones inclusions of HFFs. For $N = 4$ anti-commutativity and associativity hold true. Charge fractionization would correspond to non-trivial braiding and presumably to non-standard values of Planck constant and coverings of M^4 or CP_2 depending on whether S^2 corresponds to a sphere of light-cone boundary or homologically trivial geodesic sphere of CP_2.

5.4   Finite-Dimensional Version Of The Fusion Algebra

Algebraic discretization due to a finite measurement resolution is an essential part of quantum TGD. In this kind of situation the symplectic fields would be defined in a discrete set of N points of S^2: natural candidates are subsets of points of p-adic variants of S^2. Rational variant of S^2 has as its points points for which trigonometric functions of θ and ϕ have rational values and there exists an entire hierarchy of algebraic extensions. The interpretation for the resulting breaking of the rotational symmetry would be a geometric correlate for the choice of quantization axes in quantum measurement and the book like structure of the embedding space would be direct correlate for this symmetry breaking. This approach gives strong support for the category theory inspired philosophy in which the symplectic triangles are dictated by fusion rules.

5.4.1   General observations about the finite-dimensional fusion algebra

1. In this kind of situation one has an algebraic structure with a finite number of field values with integration over intermediate points in fusion rules replaced with a sum. The most natural option is that the sum is over all points involved. Associativity conditions reduce in this case to conditions for a finite set of structure constants vanishing when two indices are identical. The number $M(N)$ of non-vanishing structure constants is obtained from the recursion formula $M(N) = (N-1)M(N-1) + (N-2)M(N-2) + ... + 3M(3) = NM(N-1)$, $M(3) = 1$ given $M(4) = 4$, $M(5) = 20$, $M(6) = 120$, ... With a proper choice of the set of points associativity might be achieved. The structure constants are necessarily complex so that also the complex conjugate of the algebra makes sense.

2. These algebras resemble nilpotent algebras ($x^n = 0$ for some n) and Grassmann algebras ($x^2 = 0$ always) in the sense that also the products of the generating elements satisfy $x^2 = 0$ as one can find by using duality conditions on the square of a product $x = yz$ of two generating elements. Also the products of more than N generating elements necessary vanish by braided commutativity so that nilpotency holds true. The interpretation in terms of measurement resolution is that partonic states and vertices can involve at most N fermions in this measurement resolution. Elements anti-commute for $q = -1$ and commute for $q = 1$ and the possibility to express the product of two generating elements as a sum of generating elements distinguishes these algebras from Grassman algebras. For $q = -1$ these algebras resemble Lie-algebras with the difference that associativity holds true in this particular case.

3. I have not been able to find whether this kind of hierarchy of algebras corresponds to some well-known algebraic structure with commutativity and associativity possibly replaced with their braided counterparts. Certainly these algebras would be category theoretical generalization of ordinary algebras for which commutativity and associativity hold true in strict sense.

4. One could forget the representation of structure constants in terms of triangles and think these algebras as abstract algebras. The defining equations are $x_i^2 = 0$ for generators plus braided commutativity and associativity. Probably there exists solutions to these conditions. One can also hope that one can construct braided algebras from commutative and associative algebras allowing matrix representations. Note that the solution the conditions allow scalings of form $C_{klm} \to \lambda_k \lambda_l \lambda_m C_{klm}$ as symmetries.

5.4.2   Formulation and explicit solution of duality conditions in terms of inner product

Duality conditions can be formulated in terms of an inner product in the function space associated with N points and this allows to find explicit solutions to the conditions.

1. The idea is to interpret the structure constants C_{klm} as wave functions C_{kl} in a discrete space consisting of N points with the standard inner product

$$\langle C_{kl}, C_{mn} \rangle = \sum_r C_{klr} \overline{C}_{mnr} \ . \tag{5.1}$$

2. The associativity conditions for a trivial braiding can be written in terms of the inner product as

$$\langle C_{kl}, \overline{C}_{mn} \rangle = \langle C_{km}, \overline{C}_{ln} \rangle = \langle C_{kn}, \overline{C}_{ml} \rangle \ . \tag{5.2}$$

3. Irrespective of whether the braiding is trivial or not, one obtains for $k = m$ the orthogonality conditions

$$\langle C_{kl}, \overline{C}_{kn} \rangle = 0 \ . \tag{5.3}$$

For each k one has basis of $N - 1$ wave functions labeled by $l \neq k$, and the conditions state that the elements of basis and conjugate basis are orthogonal so that conjugate basis is the dual of the basis. The condition that complex conjugation maps basis to a dual basis is very special and is expected to determine the structure constants highly uniquely.

4. One can also find explicit solutions to the conditions. The most obvious trial is based on orthogonality of function basis of circle providing representation for Z_{N-2} and is following:

$$C_{klm} = E_{klm} \times exp(i\phi_k + \phi_l + \phi_m) \ , \quad \phi_m = \frac{n(m)2\pi}{N-2} \ . \tag{5.4}$$

Here E_{klm} is non-vanishing only if the indices have different values. The ansatz reduces the conditions to the form

$$\sum_r E_{klr} E_{mnr} exp(i2\phi_r) = \sum_r E_{kmr} E_{lnr} exp(i2\phi_r) = \sum_r E_{knr} E_{mlr} exp(i2\phi_r) \ . \tag{5.5}$$

In the case of braiding one can allow overall phase factors. Orthogonality conditions reduce to

$$\sum_r E_{klr} E_{knr} exp(i2\phi_r) = 0 \ \ . \tag{5.6}$$

If the integers $n(m)$, $m \neq k, l$ span the range $(0, N - 3)$ ortogonality conditions are satisfied if one has $E_{klr} = 1$ when the indices are different. This guarantees also duality conditions since the inner products involving k, l, m, n reduce to the same expression

$$\sum_{r \neq k,l,m,n} exp(i2\phi_r) \ \ . \tag{5.7}$$

5. For a more general choice of phases the coefficients E_{klm} must have values differing from unity and it is not clear whether the duality conditions can be satisfied in this case.

5.4.3   Do fusion algebras form little disk operad?

The improvement of measurement resolution means that one adds further points to an existing set of points defining a discrete fusion algebra so that a small disk surrounding a point is replaced with a little disk containing several points. Hence the hierarchy of fusion algebras might be regarded as a realization of a little disk operad [A2] and there would be a hierarchy of homomorphisms of fusion algebras induced by the fusion. The inclusion homomorphism should map the algebra elements of the added points to the algebra element at the center of the little disk.

  A more precise prescription goes as follows.

1. The replacement of a point with a collection of points in the little disk around it replaces the original algebra element ϕ_{k_0} by a number of new algebra elements ϕ_K besides already existing elements ϕ_k and brings in new structure constants C_{KLM}, C_{KLk} for $k \neq k_0$, and C_{Klm}.

2. The notion of improved measurement resolution allows to conclude

$$C_{KLk} = 0 \ \ , \ \ k \neq k_0 \ \ , \ \ C_{Klm} = C_{k_0 lm} \ \ . \tag{5.8}$$

3. In the homomorphism of new algebra to the original one the new algebra elements and their products should be mapped as follows:

$$\begin{aligned} \phi_K &\to \phi_{k_0} \ , \\ \phi_K \phi_L &\to \phi_{k_0}^2 = 0 \ \ , \ \ \phi_K \phi_l \to \phi_{k_0} \phi_l \ \ . \end{aligned} \tag{5.9}$$

Expressing the products in terms of structure constants gives the conditions

$$\sum_M C_{KLM} = 0 \ \ , \ \ \sum_r C_{Klr} = \sum_r C_{k_0 lr} = 0 \ \ . \tag{5.10}$$

The general ansatz for the structure constants based on roots of unity guarantees that the conditions hold true.

4. Note that the resulting algebra is more general than that given by the basic ansatz since the improvement of the measurement resolution at a given point can correspond to different value of N as that for the original algebra given by the basic ansatz. Therefore the original ansatz gives only the basic building bricks of more general fusion algebras. By repeated local improvements of the measurement resolution one obtains an infinite hierarchy of algebras labeled by trees in which each improvement of measurement resolution means the splitting of the branch with arbitrary number N of branches. The number of improvements of the measurement resolution defining the height of the tree is one invariant of these algebras. The fusion algebra operad has a fractal structure since each point can be replaced by any fusion algebra.

5.4.4   How to construct geometric representation of the discrete fusion algebra?

Assuming that solutions to the fusion conditions are found, one could try to find whether they allow geometric representations. Here the category theoretical philosophy shows its power.

1. Geometric representations for C_{klm} would result as functions $f(A_{klm})$ of the symplectic area for the symplectic triangles assignable to a set of N points of S^2.

2. If the symplectic triangles can be chosen freely apart from the area constraint as the category theoretic philosophy implies, it should be relatively easy to check whether the fusion conditions can be satisfied. The phases of C_{klm} dictate the areas A_{klm} rather uniquely if one uses Bohm-Aharonov ansatz for a fixed the value of Q. The selection of the points s_k would be rather free for phases near unity since the area of the symplectic triangle associated with a given triplet of points can be made arbitrarily small. Only for the phases far from unity the points s_k cannot be too close to each other unless Q is very large. The freedom to chose the points rather freely conforms with the general view about the finite measurement resolution as the origin of discretization.

3. The remaining conditions are on the moduli $|f(A_{klm})|$. In the discrete situation it is rather easy to satisfy the conditions just by fixing the values of f for the particular triangles involved: $|f(A_{klm})| = |C_{klm}|$. For the exact solution to the fusion conditions $|f(A_{klm})| = 1$ holds true.

4. Constraints on the functional form of $|f(A_{klm})|$ for a fixed value of Q can be deduced from the correlation between the modulus and phase of C_{klm} without any reference to geometric representations. For the exact solution of fusion conditions there is no correlation.

5. If the phase of C_{klm} has A_{klm} as its argument, the decomposition of the phase factor to a sum of phase factors means that the A_{klm} is sum of contributions labeled by the vertices. Also the symplectic area defined as a magnetic flux over the triangle is expressible as sum of the quantities $\int A_\mu dx^\mu$ associated with the edges of the triangle. These fluxes should correspond to the fluxes assigned to the vertices deduced from the phase factors of $\Psi(s_k)$. The fact that vertices are ordered suggest that the phase of $\Psi(s_j)$ fixes the value of $\int A_\mu dx^\mu$ for an edge of the triangle starting from s_k and ending to the next vertex in the ordering. One must find edges giving a closed triangle and this should be possible. The option for which edges correspond to geodesics or to solutions of equations of motion for a charged particle in magnetic field is not flexible enough to achieve this purpose.

6. The quantization of the phase angles as multiples of $2\pi/(N-2)$ in the case of N-dimensional fusion algebra has a beautiful geometric correlate as a quantization of symplecto-magnetic fluxes identifiable as symplectic areas of triangles defining solid angles as multiples of $2\pi/(N-2)$. The generalization of the fusion algebra to p-adic case exists if one allows algebraic extensions containing the phase factors involved. This requires the allowance of phase factors $exp(i2\pi/p)$, p a prime dividing $N-2$. Only the exponents $exp(i\int A_\mu dx^\mu) = exp(in2\pi/(N-2))$ exist p-adically. The p-adic counterpart of the curve defining the edge of triangle exists if the curve can be defined purely algebraically (say as a solution of polynomial equations with rational coefficients) so that p-adic variant of the curve satisfies same equations.

5.4.5   Does a generalization to the continuous case exist?

The idea that a continuous fusion algebra could result as a limit of its discrete version does not seem plausible. The reason is that the spatial variation of the phase of the structure constants increases as the spatial resolution increases so that the phases $exp(i\phi(s)$ cannot be continuous at continuum limit. Also the condition $E_{klm} = 1$ for $k \neq l \neq m$ satisfied by the explicit solutions to fusion rules fails to have direct generalization to continuum case.

To see whether the continuous variant of fusion algebra can exist, one can consider an approximate generalization of the explicit construction for the discrete version of the fusion algebra by the effective replacement of points s_k with small disks which are not allowed to intersect. This would mean that the counterpart $E(s_k, s_l, s_m)$ vanishes whenever the distance between two arguments is below a cutoff a small radius d. Puncturing corresponds physically to the cutoff implied by the finite measurement resolution.

1. The ansatz for C_{klm} is obtained by a direct generalization of the finite-dimensional ansatz:

$$C_{klm} \quad = \quad \kappa_{s_k,s_l,s_m} \Psi(s_k)\Psi(s_l)\Psi(s_m) \quad . \tag{5.11}$$

where κ_{s_k,s_l,s_m} vanishes whenever the distance of any two arguments is below the cutoff distance and is otherwise equal to 1.

2. Orthogonality conditions read as

$$\Psi(s_k)\Psi(s_l) \int \kappa_{s_k,s_l,s_r} \kappa_{s_k,s_n,s_r} \Psi^2(s_m) d\mu(s_r) = \Psi(s_k)\Psi(s_l) \int_{S^2(s_k,sl,s_n)} \Psi^2(s_r) d\mu(s_r) = 0 \tag{5.12}$$

The resulting condition reads as

$$\int_{S^2(s_k,sl,s_n)} \Psi^2(s_r) d\mu(s_r) \quad = \quad 0 \tag{5.13}$$

This condition holds true for any pair s_k, s_l and this might lead to difficulties.

3. The general duality conditions are formally satisfied since the expression for all fusion products reduces to

$$\Psi(s_k)\Psi(s_l)\Psi(s_m)\Psi(s_n)X \quad ,$$
$$X \quad = \quad \int_{S^2} \kappa_{s_k,s_l,s_m,s_n} \Psi(s_r) d\mu(s_r)$$
$$= \quad \int_{S^2(s_k,s_l,s_m,s_n)} \Psi(s_m) d\mu(s_r)$$
$$= \quad -\int_{D^2(s_i)} \Psi^2(s_r) d\mu(s_r) \quad , \quad i = k,l,s,m \quad . \tag{5.14}$$

These conditions state that the integral of Ψ^2 any disk of fixed radius d is same: this result follows also from the orthogonality condition. This condition might be difficult to satisfy exactly and the notion of finite measurement resolution might be needed. For instance, it might be necessary to restrict the consideration to a discrete lattice of points which would lead back to a discretized version of algebra. Thus it seems that the continuum generalization of the proposed solution to fusion rules does not work.

6   Could Operads Allow The Formulation Of The Generalized Feynman Rules?

The previous discussion of symplectic fusion rules leaves open many questions.

1. How to combine symplectic and conformal fields to what might be called symplecto-conformal fields?

2. The previous discussion applies only in super-symplectic degrees of freedom and the question is how to generalize the discussion to super Kac-Moody degrees of freedom. One must of course also try to identify more precisely what Kac-Moody degrees of freedom are!

3. How four-momentum and its conservation in the limits of measurement resolution enters this picture? Could the phase factors assocaited with the symplectic triangulation carry information about four-momentum?

4. At least two operads related to measurement resolution seem to be present: the operads formed by the symplecto-conformal fields and by generalized Feynman diagrams. For generalized Feynman diagrams causal diamond (CD) is the basic object whereas disks of S^2 are the basic objects in the case of symplecto-conformal QFT with a finite measurement resolution. Could these two different views about finite measurement resolution be more or less equivalent and could one understand this equivalence at the level of details.

5. Is it possible to formulate generalized Feynman diagrammatics and improved measurement resolution algebraically?

6.1 How To Combine Conformal Fields With Symplectic Fields?

The conformal fields of conformal field theory should be somehow combined with symplectic scalar field to form what might be called symplecto-conformal fields.

1. The simplest thing to do is to multiply ordinary conformal fields by a symplectic scalar field so that the fields would be restricted to a discrete set of points for a given realization of N-dimensional fusion algebra. The products of these symplecto-conformal fields at different points would define a finite-dimensional algebra and the products of these fields at same point could be assumed to vanish.

2. There is a continuum of geometric realizations of the symplectic fusion algebra since the edges of symplectic triangles can be selected rather freely. The integrations over the coordinates z_k (most naturally the complex coordinate of S^2 transforming linearly under rotations around quantization axes of angular momentum) restricted to the circle appearing in the definition of simplest stringy amplitudes would thus correspond to the integration over various geometric realizations of a given N-dimensional symplectic algebra.

Fusion algebra realizes the notion of finite measurement resolution. One implication is that all n-point functions vanish for $n > N$. Second implication could be that the points appearing in the geometric realizations of N-dimensional symplectic fusion algebra have some minimal distance. This would imply a cutoff to the multiple integrals over complex coordinates z_k varying along circle giving the analogs of stringy amplitudes. This cutoff is not absolutely necessary since the integrals defining stringy amplitudes are well-defined despite the singular behavior of n-point functions. One can also ask whether it is wise to introduce a cutoff that is not necessary and whether fusion algebra provides only a justification for the $1 + i\epsilon$ prescription to avoid poles used to obtain finite integrals.

The fixed values for the quantities $\int A_\mu dx^\mu$ along the edges of the symplectic triangles could indeed pose a lower limit on the distance between the vertices of symplectic triangles. Whether this occurs depends on what one precisely means with symplectic triangle.

1. The conformally invariant condition that the angles between the edges at vertices are smaller than π for triangle and larger than π for its conjugate is not enough to exclude loopy edges and one would obtain ordinary stringy amplitudes multiplied by the symplectic phase factors. The outcome would be an integral over arguments $z_1, z_2, ..z_n$ for standard stringy n-point amplitude multiplied by a symplectic phase factor which is piecewise constant in the integration domain.

2. The condition that the points at different edges of the symplectic triangle can be connected by a geodesic segment belonging to the interior of the triangle is much stronger and would induce a length scale cutoff since loops cannot be used to create large enough value of $\int A_\mu dx^\mu$ for a given side of triangle. Symplectic invariance would be obtained for small enough symplectic transformations. How to realize this cutoff at the level of calculations is not clear. One could argue that this problem need not have any nice solution and since finite measurement resolution requires only finite calculational resolution, the approximation allowing loopy edges is acceptable.

3. The restriction of the edges of the symplectic triangle within a tubular neighborhood of a geodesic -more more generally an orbit of charged particle - with thickness determined by

the length scale resolution in S^2 would also introduce the length scale cutoff with symplectic invariance within measurement resolution.

Symplecto-conformal should form an operad. This means that the improvement of measurement resolution should correspond also to an algebra homomorphism in which super-symplectic symplecto-conformal fields in the original resolution are mapped by algebra homomorphism into fields which contain sum over products of conformal fields at different points: for the symplectic parts of field the products reduces always to a sum over the values of field. For instance, if the field at point s is mapped to an average of fields at points s_k, nilpotency condition $x^2 = 0$ is satisfied.

6.2 Symplecto-Conformal Fields In Super-Kac-Moody Sector

The picture described above applies only in super-symplectic degrees of freedom. The vertices of generalized Feynman diagrams are absent from the description and CP_2 Kähler form induced to space-time surface which is absolutely essential part of quantum TGD is nowhere visible in the treatment.

How should one bring in Super Kac-Moody (SKM) algebra? The condition that the basic building bricks are same for the treatment of these degrees of freedom is a valuable guideline.

6.2.1 What does SKM algebra mean?

The first thing to consider is what SKM could mean. The recent view is that symplectic algebra corresponds to symplectic transformations for the boundary of causal diamond CD which looks locally like $\delta M_\pm^4 \times CP_2$. For this super-algebra fermionic generators would be contractions of covariantly constant right-handed neutrino with the second quantized induced spinor field to which the contraction $j_A^k \Gamma_k$ of symplectic vector field with gamma matrices acts. For SKM algebra corresponding generators would be similar contractions of other spinor modes but involving only Killing vectors fields that is symplectic isometries.

The recent view about quantum criticality strongly suggests that the conformal symmetries act as almost gauge symmetries producing from a given preferred extremal new ones with same action and conserved charges. "Almost" means that sub-algebra of conformal algebra annihilates the physical states. The subalgebras in question form a fractal hierarchy and are isomorphic with the conformal algebra itself. They contain generators for which the conformal weight is multiple of integer n characterizing also the value of Planck constant given by $h_{eff} = n \times h$. n defines the number of conformal equivalence classes of space-time surfaces connecting fixed 3-surfaces at the boundaries of CD (see **Fig. http://tgdtheory.fi/appfigures/planckhierarchy.jpg** or **Fig. ??** in the appendix of this book).

Since Kähler action reduces for the general ansatz for the preferred extremals to 3-D Chern-Simons terms, the action of the conformal symmetries reduces also to the 3-D space-like surfaces where it is trivial by definition and to non-trivial action to the light-like 3-surfaces at which the signature of the induced metric changes: I have used to call this surface partonic orbits.

It must be however observed that one can consider also the possibility that SKM algebra corresponds to the isometries of $\delta M^4 \pm \times CP_2$ continued to the space-time surface by field equations. These isometries are conformal transformations of S^2 ($\delta M_\pm^4 = S^2 \times R_+$) with conformal scaling compensated by the local scaling of the light-like radial coordinate r_M to guarantee that the metric reducing to that for S^2 apart from conformal scaling factor R_M^2 remains invariant. If this is the case the SKM contains also other than symplectic isometries.

6.2.2 Attempt to formulate symplectic triangulation for SKM algebra

The analog of symplectic triangulation for SKM algebra obviously requires that SKM algebra corresponds to symplectic isometries rather than including all $\delta M_\pm^4 = S^2 \times R_+$ isometries in one-one correspondence with conformal transformations of S^2.

1. In the transition from super-symplectic to SKM degrees of freedom the light-cone boundary is naturally replaced with the light-like 3-surface X^3 representing the light-like random orbit of parton and serving as the basic dynamical object of quantum TGD. The sphere S^2 of light-cone boundary is in turn replaced with a partonic 2-surface X^2. This suggests how to proceed.

2. In the case of SKM algebra the symplectic fusion algebra is represented geometrically as points of partonic 2-surface X^2 by replacing the symplectic form of S^2 with the induced CP_2 symplectic form at the partonic 2-surface and defining $U(1)$ gauge field. This gives similar hierarchy of symplecto-conformal fields as in the super-symplectic case. This also realizes the crucial aspects of the classical dynamics defined by Kähler action. In particular, for vacuum 2-surfaces symplectic fusion algebra trivializes since Kähler magnetic fluxes vanish identically and 2-surfaces near vacua require a large value of N for the dimension of the fusion algebra since the available Kähler magnetic fluxes are small.

3. In super-symplectic case the projection along light-like ray allows to map the points at the light-cone boundaries of CD to points of same sphere S^2. In the case of light-like 3-surfaces light-like geodesics representing braid strands allow to map the points of the partonic two-surfaces at the future and past light-cone boundaries to the partonic 2-surface representing the vertex. The earlier proposal was that the ends of strands meet at the partonic 2-surface so that braids would replicate at vertices. The properties of symplectic fields would however force identical vanishing of the vertices if this were the case. There is actually no reason to assume this condition and with this assumption vertices involving total number N of incoming and outgoing strands correspond to symplecto-conformal N-point function as is indeed natural. Also now Kähler magnetic flux induces cutoff distance.

4. SKM braids reside at light-like 3-surfaces representing lines of generalized Feynman diagrams. If super-symplectic braids are needed at all, they must be assigned to the two light-like boundaries of CD meeting each other at the sphere S^2 at which future and past directed light-cones meet.

6.3   The Treatment Of Four-Momentum

Four-momentum enjoys a special role in super-symplectic and SKM representations in that it does not correspond to a quantum number assignable to the generators of these algebras. It would be nice if the somewhat mysterious phase factors associated with the representation of the symplectic algebra could code for the four-momentum - or rather the analogs of plane waves representing eigenstates of four-momentum at the points associated with the geometric representation of the symplectic fusion algebra.

   Also the vision about TGD as almost topological QFT suggests that the symplectic degrees of freedom added to the conformal degrees of freedom defining alone a mere topological QFt somehow code for the physical degrees of freedom should and also four-momentum. If so, the symplectic triangulation might somehow code for four-momentum.

6.3.1   The representation of longitudinal momentum in terms of phase factors

The following argument suggests that S^2 and X^2 triangulations cannot naturally represent four-momentum and that one needs extension into 3-D light-like triangulation to achieve this.

1. The basic question is whether four-momentum could be coded in terms of non-integrable phase factors appearing in the representations of the symplectic fusion algebras.

2. In the symplectic case S^2 triangulation suggests itself as a representation of angular momentum only: it would be kind of spin network. In the SKM case X^2 would suggest representation of color hyper charge and isospin in terms of phases since CP_2 symmetries act non-trivially in Chern-Simons action. Does this mean that symplectic and SKM triangulations must be extended so that they are 3-D and defined for space-like 3-surface and the light-like orbit of partonic 2-surface. This would give additional phase factors assignable to presumably light-like edges. Ligh-like momentum would be natural and the recent twistorial formulation of quantum TGD indeed assigns massless momenta to fermion lines.

   Suppose that one has 3-D light-like triangulation eith at δCD or at light-like orbits of partonic 2-surface. Consider first coding of four-momentum assuming only Kähler gauge potential of CP_2 possibly having M^4 part which is pure gauge.

1. Four different phase factors are needed if all components of four-momentum are to be coded. Both number theoretical vision about quantum TGD and the realization of the hierarchy of Planck constants assign to each point of space-time surface the same plane $M^2 \subset M^4$ having as the plane of non-physical polarizations. This condition allows to assign to a given light-like partonic 3-surface unique extremal of Kähler action defining the Kähler function as the value of Kähler action.

   Also p-adic mass calculations support the view that the physical states correspond to eigen states for the components of longitudinal momentum only (also the parton model for hadrons assumes this). This encourages to think that only M^2 part of four-momentum is coded by the phase factors. Transversal momentum squared would be a well defined quantum number and determined from mass shell conditions for the representations of super-symplectic (or equivalently SKM) conformal algebra much like in string model.

2. The phase factors associated with the 3-D symplectic fusion algebra in $S^2 \times R_+$ mean a deviation from conformal n-point functions, and the innocent question is whether these phase factors could be identified as plane-wave phase factors in S^2 could be associated with the transversal part of the four-momentum so that the n-point functions would be strictly analogous with stringy amplitudes. Alternative, and perhaps more natural, interpretation is in terms of spin and angular momentum.

3. Suppose one allows a gauge transformation of Kähler gauge potential inducing a pure gauge M^4 component to the Kähler gauge potential expressible as scalar function of M^4 coordinates. This kind of term might allow to achieve the vanishing of $j^\alpha A_\alpha$ term of at least its integral over space-time surface in Kähler action implying reduction of Kähler action to Chern-Simons terms if weak form of electric magnetic duality holds true. The scalar function can be interpreted as integral of a position dependent momentum along curve defined by $S^2 \times R_+$ triangulation and gives hopes of coding four-momentum in terms of Kähler gauge potential.

   In fact, the identification of the phase factors $exp(i \int A_\mu dx^\mu / \hbar)$ along a path as phase factors $exp(ip_{L,k} \Delta m^k)$ defined by the ends of the path and associated with the longitudinal part of four-momentum would correspond to an integral form of covariant constancy condition $\frac{dx^\mu}{ds}(\partial_\mu - iA_\mu)\Psi = 0$ along the edge of the symplectic triangle of more general path.

4. For the SKM triangulation associated with the light-like orbit X_l^3 of partonic 2-surface analogous phase factor would come from the integral along the (most naturally) light-like curve defining braid strand associated with the point in question. A geometric representation for the two projections of the four-momentum would thus result in SKM degrees of freedom and apart from the non-uniqueness related to the multiples of a 2π the components of M^2 momentum could be deduced from the phase factors. If one is satisfied with the projection of momentum in M^2, this is enough.

5. Neither of these phase factors is able to code all components of four-momentum. One might however hope that together they could give enough information to deduce the four-momentum if it is assumed to correspond to the rest system.

6. The phase factors assignable to the symplectic triangles in S^2 and X^2 have nothing to do with momentum. Because the space-like phase factor $exp(iS_z\Delta\phi/\hbar)$ associated with the edge of the symplectic triangle is completely analogous to that for momentum, one can argue that the symplectic triangulation could define a kind of spin network utilized in discretized approaches to quantum gravity. The interpretation raises the question about the interpretation of the quantum numbers assignable to the Lorentz invariant phase factors defined by the CP_2 Kähler gauge potential.

6.3.2   The quantum numbers associated with phase factors for CP_2 parts of Kähler gauge potentials

Suppose that it is possible to assign two independent and different phase factors to the same geometric representation, in other words have two independent symplectic fields with the same

geometric representation. The product of two symplectic fields indeed makes sense and satisfies the defining conditions. One can define prime symplectic algebras and decompose symplectic algebras to prime factors. Since one can allow permutations of elements in the products it becomes possible to detect the presence of product structure experimentally by detecting different combinations for products of phases caused by permutations realized as different combinations of quantum numbers assigned with the factors. The geometric representation for the product of n symplectic fields would correspond to the assignment of n edges to any pair of points. The question concerns the interpretation of the phase factors assignable to the CP_2 parts of Kähler gauge potentials of S^2 and CP, Kähler form.

1. The natural interpretation for the two additional phase factors would be in terms of color quantum numbers. Color hyper charge and isospin are mathematically completely analogous to the components of four-momentum so that a possible identification of the phase factors is as a representation of these quantum numbers. The representation of plane waves as phase factors $exp(ip_k \Delta m^k/\hbar)$ generalizes to the representation $exp(iQ_A \Delta \Phi^A/\hbar)$, where Φ_A are the angle variables conjugate to the Hamiltonians representing color hyper charge and isospin. This representation depends on end points only so that the crucial symplectic invariance with respect to the symplectic transformations respecting the end points of the edge is not lost ($U(1)$ gauge transformation is induced by the scalar $j^k A_k$, where j^k is the symplectic vector field in question).

2. One must be cautious with the interpretation of the phase factors as a representation for color hyper charge and isospin since a breaking of color gauge symmetry would result since the phase factors associated with different values of color isospin and hypercharge would be different and could not correspond to same edge of symplectic triangle. This is questionable since color group itself represents symplectic transformations. The construction of CP_2 as a coset space $SU(3)/U(2)$ identifies $U(2)$ as the holonomy group of spinor connection having interpretation as electro-weak group. Therefore also the interpretation of the phase factors in terms of em charge and weak charge can be considered. In TGD framework electro-weak gauge potential indeed suffer a non-trivial gauge transformation under color rotations so that the correlation between electro-weak quantum numbers and non-integrable phase factors in Cartan algebra of the color group could make sense. Electro-weak symmetry breaking would have a geometric correlate in the sense that different values of weak isospin cannot correspond to paths with same values of phase angles $\Delta \Phi^A$ between end points.

3. If the phase factors associated with the M^4 and CP_2 are assumed to be identical, the existence of geometric representation is guaranteed. This however gives constraints between rest mass, spin, and color (or electro-weak) quantum numbers.

6.3.3   Some general comments

Some further comments about phase factors are in order.

1. By number theoretical universality the plane wave factors associated with four-momentum must have values coming as roots of unity (just as for a particle in box consisting of discrete lattice of points). At light-like boundary the quantization conditions reduce to the condition that the value of light-like coordinate is rational of form m/N, if N: th roots of unity are allowed.

2. In accordance with the finite measurement resolution of four-momentum, four-momentum conservation is replaced by a weaker condition stating that the products of phase factors representing incoming and outgoing four-momenta are identical. This means that positive and negative energy states at opposite boundaries of CD would correspond to complex conjugate representations of the fusion algebra. In particular, the product of phase factors in the decomposition of the conformal field to a product of conformal fields should correspond to the original field value. This would give constraints on the trees physically possible in the operad formed by the fusion algebras. Quite generally, the phases expressible as products of phases $exp(in\pi/p)$, where $p \leq N$ is prime must be allowed in a given resolution and this

suggests that the hierarchy of p-adic primes is involved. At the limit of very large N exact momentum conservation should emerge.

3. Super-conformal invariance gives rise to mass shell conditions relating longitudinal and transversal momentum squared. The massivation of massless particles by Higgs mechanism and p-adic thermodynamics pose additional constraints to these phase factors.

6.4   What Does The Improvement Of Measurement Resolution Really Mean?

To proceed one must give a more precise meaning for the notion of measurement resolution. Two different views about the improvement of measurement resolution emerge. The first one relies on the replacement of braid strands with braids applies in SKM degrees of freedom and the homomorphism maps symplectic fields into their products. The homomorphism based on the averaging of symplectic fields over added points consistent with the extension of fusion algebra described in previous section is very natural in super-symplectic degrees of freedom. The directions of these two algebra homomorphisms are different. The question is whether both can be involved with both super-symplectic and SKM case. Since the end points of SKM braid strands correspond to both super-symplectic and SKM degrees of freedom, it seems that division of labor is the only reasonable option.

1. Quantum classical correspondence requires that measurement resolution has a purely geometric meaning. A purely geometric manner to interpret the increase of the measurement resolution is as a replacement of a braid strand with a braid in the improved resolution. If one assigns the phase factor assigned with the fusion algebra element with four-momentum, the conservation of the phase factor in the associated homomorphism is a natural constraint. The mapping of a fusion algebra element (strand) to a product of fusion algebra elements (braid) allows to realize this condition. Similar mapping of field value to a product of field values should hold true for conformal parts of the fields. There exists a large number equivalent geometric representations for a given symplectic field value so that one obtains automatically an averaging in conformal degrees of freedom. This interpretation for the improvement of measurement resolution looks especially natural for SKM degrees of freedom for which braids emerge naturally.

2. One can also consider the replacement of symplecto-conformal field with an average over the points becoming visible in the improved resolution. In super-symplectic degrees of freedom this looks especially natural since the assignment of a braid with light-cone boundary is not so natural as with light-like 3-surface. This map does not conserve the phase factor but this could be interpreted as reflecting the fact that the values of the light-like radial coordinate are different for points involved. The proposed extension of the symplectic algebra proposed in the previous section conforms with this interpretation.

3. In the super-symplectic case the improvement of measurement resolution means improvement of angular resolution at sphere S^2. In SKM sector it means improved resolution for the position at partonic 2-surface. This generalizes also to the 3-D symplectic triangulations. For SKM algebra the increase of the measurement resolution related to the braiding takes place inside light-like 3-surface. This operation corresponds naturally to an addition of sub-CD inside which braid strands are replaced with braids. This is like looking with a microscope a particular part of line of generalized Feynman graph inside CD and corresponds to a genuine physical process inside parton. In super-symplectic case the replacement of a braid strand with braid (at light-cone boundary) is induced by the replacement of the projection of a point of a partonic 2-surface to S^2 with a a collection of points coming from several partonic 2-surfaces. This replaces the point s of S^2 associated with CD with a set of points s_k of S^2 associated with sub-CD. Note that the solid angle spanned by these points can be rather larger so that zoom-up is in question.

4. The improved measurement resolution means that a point of S^2 (X^2) at boundary of CD is replaced with a point set of S^2 (X^2) assignable to sub-CD. The task is to map the point set

to a small disk around the point. Light-like geodesics along light-like X^3 defines this map naturally in both cases. In super-symplectic case this map means scaling down of the solid angle spanned by the points of S^2 associated with sub-CD.

6.5 How Do The Operads Formed By Generalized Feynman Diagrams And Symplecto-Conformal Fields Relate?

The discussion above leads to following overall view about the situation. The basic operation for both symplectic and Feynman graph operads corresponds to an improvement of measurement resolution. In the case of planar disk operad this means to a replacement of a white region of a map with smaller white regions. In the case of Feynman graph operad this means better space-time resolution leading to a replacement of generalized Feynman graph with a new one containing new sub-CD bringing new vertices into daylight. For braid operad the basic operation means looking a braid strand with a microscope so that it can resolve into a braid: braid becomes a braid of braids. The latter two views are equivalent if sub-CD contains the braid of braids.

The disks D^2 of the planar disk operad has natural counterparts in both super-symplectic and SKM sector.

1. For the geometric representations of the symplectic algebra the image points vary in continuous regions of S^2 (X^2) since the symplectic area of the symplectic triangle is a highly flexible constraint. Posing the condition that any point at the edges of symplectic triangle can be connected to any another edge excludes symplectic triangles with loopy sides so that constraint becomes non-trivial. In fact, since two different elements of the symplectic algebra cannot correspond to the same point for a given geometric representation, each element must correspond to a connected region of S^2 (X^2). This allows a huge number of representations related by the symplectic transformations S^2 in super-symplectic case and by the symplectic transformations of CP_2 in SKM case. In the case of planar disk operad different representations are related by isotopies of plane.

   This decomposition to disjoint regions naturally correspond to the decomposition of the disk to disjoint regions in the case of planar disk operad and Feynman graph operad (allowing zero energy insertions). Perhaps one might say that N-dimensional elementary symplectic algebra defines an N-coloring of S^2 (S^2) which is however not the same thing as the 2-coloring possible for the planar operad. TGD based view about Higgs mechanism leads to a decomposition of partonic 2-surface X^2 (its light-like orbit X^3) into conformal patches. Since also these decompositions correspond to effective discretizations of X^2 (X^3), these two decompositions would naturally correspond to each other.

2. In SKM sector disk D^2 of the planar disk operad is replaced with the partonic 2-surface X^2 and since measurement resolution is a local notion, the topology of X^2 does not matter. The improvement of measurement resolution corresponds to the replacement of braid strand with braid and homomorphism is to the direction of improved spatial resolution.

3. In super-symplectic case D^2 is replaced with the sphere S^2 of light-cone boundary. The improvement of measurement resolution corresponds to introducing points near the original point and the homomorphism maps field to its average. For the operad of generalized Feynman diagrams CD defined by future and past directed light-cones is the basic object. Given CD can be indeed mapped to sphere S^2 in a natural manner. The light-like boundaries of CDs are metrically spheres S^2. The points of light-cone boundaries can be projected to any sphere at light-cone boundary. Since the symplectic area of the sphere corresponds to solid angle, the choice of the representative for S^2 does not matter. The sphere defined by the intersection of future and past light-cones of CD however provides a natural identification of points associated with positive and negative energy parts of the state as points of the same sphere. The points of S^2 appearing in n-point function are replaced by point sets in a small disks around the n points.

4. In both super-symplectic and SKM sectors light-like geodesic along X^3 mediate the analog of the map gluing smaller disk to a hole of a disk in the case of planar disk operad defining the decomposition of planar tangles. In super-symplectic sector the set of points at the sphere

corresponding to a sub-CD is mapped by SKM braid to the larger CD and for a typical braid corresponds to a larger angular span at sub-CD. This corresponds to the gluing of D^2 along its boundaries to a hole in D^2 in disk operad. A scaling transformation allowed by the conformal invariance is in question. This scaling can have a non-trivial effect if the conformal fields have anomalous scaling dimensions.

5. Homomorphisms between the algebraic structures assignable to the basic structures of the operad (say tangles in the case of planar tangle operad) are an essential part of the power of the operad. These homomorphisms associated with super-symplectic and SKM sector code for two views about improvement of measurement resolution and might lead to a highly unique construction of M-matrix elements.

The operad picture gives good hopes of understanding how M-matrices corresponding to a hierarchy of measurement resolutions can be constructed using only discrete data.

1. In this process the n-point function defining M-matrix element is replaced with a superposition of n-point functions for which the number of points is larger: $n \to \sum_{k=1,...,m} n_k$. The numbers n_k vary in the superposition. The points are also obtained by downwards scaling from those of smaller S^2. Similar scaling accompanies the composition of tangles in the case of planar disk operad. Algebra homomorphism property gives constraints on the compositeness and should govern to a high degree how the improved measurement resolution affects the amplitude. In the lowest order approximation the M-matrix element is just an n-point function for conformal fields of positive and negative energy parts of the state at this sphere and one would obtain ordinary stringy amplitude in this approximation.

2. Zero energy ontology means also that each addition in principle brings in a new zero energy insertion as the resolution is improved. Zero energy insertions describe actual physical processes in shorter scales in principle affecting the outcome of the experiment in longer time scales. Since zero energy states can interact with positive (negative) energy particles, zero energy insertions are not completely analogous to vacuum bubbles and cannot be neglected. In an idealized experiment these zero energy states can be assumed to be absent. The homomorphism property must hold true also in the presence of the zero energy insertions. Note that the Feynman graph operad reduces to planar disk operad in absence of zero energy insertions.

7 Possible Other Applications Of Category Theory

It is not difficult to imagine also other applications of category theory in TGD framework.

7.1 Categorification And Finite Measurement Resolution

I read a very stimulating article by John Baez with title "Categorification" (see `http://tinyurl.com/ych6a8oa`) [A12] about the basic ideas behind a process called categorification. The process starts from sets consisting of elements. In the following I describe the basic ideas and propose how categorification could be applied to realize the notion of finite measurement resolution in TGD framework.

7.1.1 What categorification is?

In categorification sets are replaced with categories and elements of sets are replaced with objects. Equations between elements are replaced with isomorphisms between objects: the right and left hand sides of equations are not the same thing but only related by an isomorphism so that they are not tautologies anymore. Functions between sets are replaced with functors between categories taking objects to objects and morphisms to morphisms and respecting the composition of morphisms. Equations between functions are replaced with natural isomorphisms between functors, which must satisfy certain coherence laws representable in terms of commuting diagrams expressing conditions such as commutativity and associativity.

The isomorphism between objects represents equation between elements of set replaces identity. What about isomorphisms themselves? Should also these be defined only up to an isomorphism of isomorphism? And what about functors? Should one continue this replacement ad infinitum to obtain a hierarchy of what might be called n-categories, for which the process stops after n: th level. This rather fuzzy buisiness is what mathematicians like John Baez are actually doing.

7.1.2   Why categorification?

There are good motivations for the categofication. Consider the fact that natural numbers. Mathematically oriented person would think number "3" in terms of an abstract set theoretic axiomatization of natural numbers. One could also identify numbers as a series of digits. In the real life the representations of three-ness are more concrete involving many kinds of associations. For a child "3" could correspond to three fingers. For a mystic it could correspond to holy trinity. For a Christian "faith, hope, love". All these representations are isomorphic representation of threeness but as real life objects three sheeps and three cows are not identical.

We have however performed what might be called decategorification: that is forgitten that the isomorphic objects are not equal. Decatecorification was of course a stroke of mathematical genius with enormous practical implications: our information society represents all kinds of things in terms of numbers and simulates successfully the real world using only bit sequences. The dark side is that treating people as mere numbers can lead to a rather cold society.

Equally brilliant stroke of mathematical genius is the realization that isomorphic objects are not equal. Decategorization means a loss of information. Categorification brings back this information by bringing in consistency conditions known as coherence laws and finding these laws is the hard part of categorization meaning discovery of new mathematics. For instance, for braid groups commutativity modulo isomorphisms defines a highly non-trivial coherence law leading to an extremely powerful notion of quantum group having among other things applications in topological quantum computation.

The so called associahedrons (see `http://tinyurl.com/ng2fqro`) [A8] emerging in n-category theory could replace space-time and space as fundamental objects. Associahedrons are polygons used to represent geometrically associativity or its weaker form modulo isomorphism for the products of n objects bracketed in all possible ways. The polygon defines a hierarchy containing sub-polygons as its edges containing.... Associativity states the isomorphy of these polygons. Associahedrons and related geometric representations of category theoretical arrow complexes in terms or simplexes allow a beautiful geometric realization of the coherence laws. One could perhaps say that categories as discrete structures are not enough: only by introducing the continuum allowing geometric representations of the coherence laws things become simple.

No-one would have proposed categorification unless it were demanded by practical needs of mathematics. In many mathematical applications it is obvious that isomorphism does not mean identity. For instance, in homotopy theory all paths deformable to each other in continuous manner are homotopy equivalent but not identical. Isomorphism is now homotopy. These paths can be connected and form a groupoid. The outcome of the groupoid operation is determined up to homotopy. The deformations of closed path starting from a given point modulo homotopies form homotopy group and one can interpret the elements of homotopy group as copies of the point which are isomorphic. The replacement of the space with its universal covering makes this distinction explicit. One can form homotopies of homotopies and continue this process ad infinitum and obtain in this manner homotopy groups as characterizes of the topology of the space.

7.1.3   Cateforification as a way to describe finite measurement resolution?

In quantum physics gauge equivalence represents a standard example about equivalence modulo isomorphisms which are now gauge transformations. There is a practical strategy to treat the situation: perform a gauge choice by picking up one representative amongst infinitely many isomorphic objects. At the level of natural numbers a very convenient gauge fixing would correspond the representation of natural number as a sequence of decimal digits rather than image of three cows.

In TGD framework a excellent motivation for categorification is the need to find an elegant mathematical realization for the notion of finite measurement resolution. Finite measurement

resolutions (or cognitive resolutions) at various levels of information transfer hierarchy imply accumulation of uncertainties. Consider as a concrete example uncertainty in the determination of basic parameters of a mathematical model. This uncertainty is reflected to final outcome as via a long sequence of mathematical maps and additional uncertainties are produced by the approximations at each step of this process.

How could onbe describe the finite measurement resolution elegantly in TGD Universe? Categorification suggests a natural method. The points equivalent with measurement resolution are isomorphic with each other. A natural guess inspired by gauge theories is that one should perform a gauge choice as an analog of decategorification. This allows also to avoid continuum of objects connected by arrows not n spirit with the discreteness of category theoretical approach.

1. At space-time level gauge choice means discretization of partonic 2-surfaces replacing them with a discrete set points serving as representatives of equivalence classes of points equivalent under finite measurement resolution. An especially interesting choice of points is as rational points or algebraic numbers and emerges naturally in p-adicization process. One can also introduce what I have called symplectic triangulation of partonic 2-surfaces with the nodes of the triangulation representing the discretization and carrying quantum numbers of various kinds.

2. At the level of "world classical worlds" (WCW) this means the replacement of the sub-group if the symplectic group of $\delta M^4 \times CP_2$ -call it G - permuting the points of the symplectic triangulation with its discrete subgroup obtained as a factor group G/H, where H is the normal subgroup of G leaving the points of the symplectic triangulation fixed. One can also consider subgroups of the permutation group for the points of the triangulation. One can also consider flows with these properties to get braided variant of G/H. It would seem that one cannot regard the points of triangulation as isomorphic in the category theoretical sense. This because, one can have quantum superpositions of states located at these points and the factor group acts as the analog of isometry group. One can also have many-particle states with quantum numbers at several points. The possibility to assign quantum numbers to a given point becomes the physical counterpart for the axiom of choice.

   The finite measurement resolution leads to a replacement of the infinite-dimensional world of classical worlds with a discrete structure. Therefore operation like integration over entire "world of classical worlds" is replaced with a discrete sum.

3. What suggests itself strongly is a hierarchy of n-categories as a proper description for the finite measurement resolution. The increase of measurement resolution means increase for the number of braid points. One has also braids of braids of braids structure implied by the possibility to map infinite primes, integers, and rationals to rational functions of several variables and the conjecture possibility to represent the hierarchy of Galois groups involved as symplectic flows. If so the hierarchy of n-categories would correspond to the hierarchy of infinite primes having also interpretation in terms of repeated second quantization of an arithmetic SUSY such that many particle states of previous level become single particle states of the next level.

The finite measurement resolution has also a representation in terms of inclusions of hyperfinite factors of type II_1 defined by the Clifford algebra generated by the gamma matrices of WCW [K7]

1. The included algebra represents finite measurement resolution in the sense that its action generates states which are cannot be distinguished from each other within measurement resolution used. The natural conjecture is that this indistuinguishability corresponds to a gauge invariance for some gauge group and that TGD Universe is analogous to Turing machine in that almost any gauge group can be represented in terms of finite measurement resolution.

2. Second natural conjecture inspired by the fact that symplectic groups have enormous representabive power is that these gauge symmetries allow representation as subgroups of the symplectic group of $\delta M^4 \times CP_2$. A nice article about universality of symplectic groups is the article "The symplectification of science" (see `http://tinyurl.com/y8us9sgw`) by Mark. J. Gotay [A7].

3. An interesting question is whether there exists a finite-dimensional space, whose symplecto-morphisms would allow a representation of any gauge group (or of all possible Galois groups as factor groups) and whether $\delta M^4 \times CP_2$ could be a space of this kind with the smallest possible dimension.

7.2   Inclusions Of HFFs And Planar Tangles

Finite index inclusions of HFFs are characterized by non-branched planar algebras for which only an even number of lines can emanate from a given disk. This makes possible a consistent coloring of the k-tangle by black and white by painting the regions separated by a curve using opposite colors. For more general algebras, also for possibly existing branched tangle algebras, the minimum number of colors is four by four-color theorem. For the description of zero energy states the 2-color assumption is not needed so that the necessity to have general branched planar algebras is internally consistent. The idea about the inclusion of positive energy state space into the space of negative energy states might be consistent with branched planar algebras and the requirement of four colors since this inclusion involves also conjugation and is thus not direct.

In [A6] if was proposed that planar operads are associated with conformal field theories at sphere possessing defect lines separating regions with different color. In TGD framework and for branched planar algebras these defect lines would correspond to light-like 3-surfaces. For fermions one has single wormhole throat associated with topologically condensed CP_2 type extremal and the signature of the induced metric changes at the throat. Bosons correspond to pairs of wormhole throats associated with wormhole contacts connecting two space-time sheets modellable as a piece of CP_2 type extremal. Each boson thus corresponds to 2 lines within CP_2 radius so that in purely bosonic case the planar algebra can correspond to that associated with an inclusion of HFFs.

7.3   2-Plectic Structures And TGD

Chris Rogers and Alex Hoffnung have demonstrated [A15] that the notion of symplectic structure generalizes to n-plectic structure and in $n = 2$ case leads to a categorification of Lie algebra to 2-Lie-algebra. In this case the generalization replaces the closed symplectic 2-form with a closed 3-form ω and assigns to a subset of one-forms defining generalized Hamiltonians vector fields leaving the 3-form invariant.

There are two equivalent definitions of the Poisson bracket in the sense that these Poisson brackets differ only by a gradient, which does not affect the vector field assignable to the Hamiltonian one-form. The first bracket is simply the Lie-derivate of Hamiltonian one form G with respect to vector field assigned to F. Second bracket is contraction of Hamiltonian one-forms with the three-form ω. For the first variant Jacobi identities hold true but Poisson bracket is antisymmetric only modulo gradient. For the second variant Jacobi identities hold true only modulo gradient but Poisson bracket is antisymmetric. This modulo property is in accordance with category theoretic thinking in which commutativity, associativity, antisymmetry, ... hold true only up to isomorphism.

For 3-dimensional manifolds n=2-plectic structure has the very nice property that *all* one-forms give rise to Hamiltonian vector field. In this case any 3-form is automatically closed so that a large variety of 2-plectic structures exists. In TGD framework the natural choice for the 3-form ω is as Chern-Simons 3-form defined by the projection of the Kähler gauge potential to the light-like 3-surface. Despite the fact the induced metric is degenerate, one can deduce the Hamiltonian vector field associated with the one-form using the general defining conditions

$$ i_{v_F} \omega \quad = \quad dF \tag{7.1} $$

since the vanishing of the metric determinant appearing in the formal definition cancels out in the expression of the Hamiltonian vector field.

The explicit formula is obtained by writing ω as

$$ \omega = K \epsilon_{\alpha\beta\gamma} \times \epsilon^{\mu\nu\delta} A_\mu J_{\nu\delta} \sqrt{g} \quad = \epsilon_{\alpha\beta\gamma} \times C - S \ , $$
$$ C - S = K E^{\alpha\beta\gamma} A_\alpha J_{\beta\gamma} \ . \tag{7.2} $$

Here $E^{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma}$ holds true numerically and metric determinant, which vanishes for light-like 3-surfaces, has disappeared.

The Hamiltonian vector field is the curl of F divided by the Chern-Simons action density $C - S$:

$$v_F^\alpha = \tfrac{1}{2} \times \frac{\epsilon^{\alpha\beta\gamma}(\partial_\beta F_\gamma - \partial_\gamma F_\beta)\sqrt{g}}{C - S \sqrt{g}} = \tfrac{1}{2} \times \frac{E^{\alpha\beta\gamma}(\partial_\beta F_\gamma - \partial_\gamma F_\beta)}{C - S} \quad . \tag{7.3}$$

The Hamiltonian vector field multiplied by the dual of 3-form multiplied by the metric determinant has a vanishing divergence and is analogous to a vector field generating volume preserving flow. and the value of Chern Simons 3-form defines the analog of the metric determinant for light-like 3-surfaces. The generalized Poisson bracket for Hamiltonian 1-forms defined in terms of the action of Hamiltonian vector field on Hamiltonian as $J_1^\beta D_\beta F_2\alpha - J_2^\beta D_\beta H_2\alpha$ is Hamiltonian 1-form. Here J_i denotes the Hamiltonian vector field associated with F_i. The bracked unique apart from gradient. The corresponding vector field is the commutator of the Hamiltonian vector fields.

The objection is that gauge invariance is broken since the expression for the vector field assigned to the Hamiltonian one-form depends on gauge. In TGD framework there is no need to worry since Kähler gauge potential has unique natural expression and the $U(1)$ gauge transformations of Kähler gauge potential induced by symplectic transformations of CP_2 are not genuine gauge transformations but dynamical symmetries since the induced metric changes and space-time surface is deformed. Another important point is that Kähler gauge potential for a given CD has M^4 part which is "pure gauge" constant Lorentz invariant vector and proportional to the inverse of gravitational constant G. Its ratio to CP_2 radius squared is determined from electron mass by p-adic mass calculations and mathematically by quantum criticality fixing also the value of Kähler coupling strength.

7.4 TGD Variant For The Category Ncob

John Baez has suggested that quantum field theories could be formulated as functors from the category of n-cobordisms to the category of Hilbert spaces [A13, A9]. In TGD framework light-like 3-surfaces containing the number theoretical braids define the analogs of 3-cobordisms and surface property brings in new structure. The motion of topological condensed 3-surfaces along 4-D space-time sheets brings in non-trivial topology analogous to braiding and not present in category nCob.

Intuitively it seems possible to speak about one-dimensional orbits of wormhole throats and -contacts (fermions and bosons) in background space-time (homological dimension). In this case linking or knotting are not possible since knotting is co-dimension 2 phenomenon and only objects whose homological dimensions sum up to $D - 1$ can get linked in dimension D. String like objects could topologically condense along wormhole contact which is string like object. The orbits of closed string like objects are homologically co-dimension 2 objects and could get knotted if one does not allow space-time sheets describing un-knotting. The simplest examples are ordinary knots which are not allowed to evolve by forming self intersections. The orbits of point like wormhole contact and closed string like wormhole contact can get linked: a point particle moving through a closed string is basic dynamical example. There is no good reason preventing unknotting and unlinking in absolute sense.

7.5 Number Theoretical Universality And Category Theory

Category theory might be also a useful tool to formulate rigorously the idea of number theoretical universality and ideas about cognition. What comes into mind first are functors real to p-adic physics and vice versa. They would be obtained by composition of functors from real to rational physics and back to p-adic physics or vice versa. The functors from real to p-adic physics would provide cognitive representations and the reverse functors would correspond to the realization of intentional action. The functor mapping real 3-surface to p-adic 3-surfaces would be simple: interpret the equations of 3-surface in terms of rational functions with coefficients in some algebraic extension of rationals as equations in arbitrary number field. Whether this description applies or is needed for 4-D space-time surface is not clear.

At the Hilbert space level the realization of these functors would be quantum jump in which quantum state localized to p-adic sector tunnels to real sector or vice versa. In zero energy

ontology this process is allowed by conservation laws even in the case that one cannot assign classical conserved quantities to p-adic states (their definition as integrals of conserved currents does not make sense since definite integral is not a well-defined concept in p-adic physics). The interpretation would be in terms of generalized M-matrix applying to cognition and intentionality. This M-matrix would have values in the field of rationals or some algebraic extension of rationals. Again a generalization of Connes tensor product is suggestive.

7.6   Category Theory And Fermionic Parts Of Zero Energy States As Logical Deductions

Category theory has natural applications to quantum and classical logic and theory of computation [A9]. In TGD framework these applications are very closely related to quantum TGD itself since it is possible to identify the positive and negative energy pieces of fermionic part of the zero energy state as a pair of Boolean statements connected by a logical deduction, or rather- quantum superposition of them. An alternative interpretation is as rules for the behavior of the Universe coded by the quantum state of Universe itself. A further interpretation is as structures analogous to quantum computation programs with internal lines of Feynman diagram would represent communication and vertices computational steps and replication of classical information coded by number theoretical braids.

7.7   Category Theory And Hierarchy Of Planck Constants

Category theory might help to characterize more precisely the proposed geometric realization of the hierarchy of Planck constants explaining dark matter as phases with non-standard value of Planck constant. The situation is topologically very similar to that encountered for generalized Feynman diagrams. Singular coverings and factor spaces of M^4 and CP_2 are glued together along 2-D manifolds playing the role of object and space-time sheets at different vertices could be interpreted as arrows going through this object.

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