

Number theoretic vision, Hyper-finite Factors and S-matrix

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Abstract

During years the basic mathematical and conceptual building bricks of quantum TGD have become rather obvious. The basic goal is the construction of scattering amplitudes.

1. Zero Energy Ontology (ZEO) forces to generalize the notion of S-matrix by introducing M-matrix as a matrix characterizing the entanglement between pairs of states forming zero energy states.
2. Second building brick consists of various hierarchies and connections between them. There is the hierarchy of quantum criticalities for super-symplectic algebra and its Yangian extension acting as a spectrum generating algebra. This hierarchy is closely related to the hierarchy of Planck constants $h_{eff} = n \times h$. The hierarchies of criticalities correspond also to fractal hierarchies of breakings of super-symplectic gauge conformal symmetry: only the sub-algebra isomorphic to the original gauge algebra acts as gauge algebra after the breaking. At each step one criticality is reduced and the number of physical degrees of freedom increases.

There is a natural connection between these hierarchies with the hierarchies of hyperfinite factors of type II_1 (HFFs) and their inclusions providing a description for the notion of measurement resolution.

3. Number theoretic realized as adelic physics fusing real number based physics as a correlate of sensory experience and p-adic physics as correlate of cognition involves several elements: $M^8 - H$ duality, hierarchy of effective Planck constants $h_{eff} = nh_0$ with n identified as a dimension of extension of rationals, cognitive representations characterized by extensions of rationals, and p-adic length scale hypothesis.

The identification of the TGD counterpart of S-matrix is the key topic of this chapter. What this matrix actually means is far from obvious.

1. One can characterize zero energy state by a "square root" of density matrix which is product of hermitian matrix and unitary matrix: I have called this matrix *M*-matrix. The unitary matrix related to the *M*-matrix could relate closely to the *S*-matrix assigned with particle reactions.
2. One can assign the analog of unitary *S*-matrix to "small" state function reductions (SSFRs) defining the TGD counterparts of "weak" measurements. The states at the passive boundary PB are unaffected, which has interpretation as the TGD counterpart of Zeno effect. This *S*-matrix could relate to the evolution of self as a conscious entity and to its cognitive time evolution.
3. One can also assign an *S*-matrix like entity to "big" SFRs (BSFRs) in which the arrow of time changes. This *S*-matrix would be the counterpart of the ordinary *S*-matrix and should closely relate to the *M*-matrix.
4. I have also introduced the notion of *U*-matrix, which would be defined between zero energy states without fixing states at the passive boundary essential for fixing the arrow of time. This notion has remained somewhat misty and it seems that it is not needed since the matrices assigned SSFRs and BSFRs indeed are between zero energy states.

The construction of these matrices is discussed at the general level.

1 Introduction

Quantum criticality has been the key idea from beginning but its understanding has evolved rather slowly. Quantum criticality accompanies several hierarchies: hierarchy of p-adic length scales and hierarchy of space-time sheets glued to larger space-time sheets; hierarchy of Planck constants labelling phases of ordinary matter behaving like dark matter; hierarchy of breakings of super-symplectic symmetry represented as gauge symmetry; hierarchy of causal diamonds (CDs); hierarchies of inclusions of hyperfinite factors of type II_1 (HFFs); hierarchies of extensions of extensions of of rationals emerging in M^8 picture about TGD; hierarchies of conscious entities with lower level represented as mental images.

1.1 Zero energy ontology and the interpretation of light-like 3-surfaces as generalized Feynman diagrams

Zero energy ontology (ZEO) is discussed in [K30] but its role in the construction of scattering amplitudes deserves a brief discussion also here.

1. ZEO is the cornerstone of the construction. Zero energy states have vanishing net quantum numbers and consist of positive and negative energy parts, which can be thought of as being localized at the boundaries of light-like 3-surface X_l^3 connecting the light-like boundaries of a causal diamond CD identified as intersection of future and past directed light-cones. There is entire hierarchy of CDs, whose scales are suggested to come as powers of 2. A more general proposal is that prime powers of fundamental size scale are possible and would conform with the most general form of p-adic length scale hypothesis. The hierarchy of size scales assignable to CDs corresponds to a hierarchy of length scales and code for a hierarchy of radiative corrections to generalized Feynman diagrams.
2. Either space-like 3-surfaces at the boundaries of CDs or light-like 3-surfaces connecting the boundaries of CDs can be seen as the basic dynamical objects of quantum TGD and have interpretation as generalized Feynman diagrams having light-like 3-surfaces as lines glued together along their ends defining vertices as 2-surfaces. By effective 2-dimensionality (holography) of light-like 3-surfaces the interiors of light-like 3-surfaces are analogous to gauge degrees of freedom and partially parameterized by Kac-Moody group respecting the light-likeness of 3-surfaces. This picture differs dramatically from that of string models since light-like 3-surfaces replacing stringy diagrams are singular as manifolds whereas 2-surfaces representing vertices are not.
3. String world sheets and partonic 2-surfaces however appear also in TGD as carriers of spinor modes: this follows from the condition that em charge is well defined for the modes. The condition follows also from number theoretic arguments and is assumed quite generally. This has far reaching consequences for the understanding of gravitation in TGD framework and profound deviations from string models are predicted due to the hierarchy of Planck constants absolutely essential for the description of gravitational bound states in terms of strings connecting partonic 2-surfaces. Macroscopic quantum coherence in even astrophysical scales is predicted [?, K22].

1.2 About the identification of various TGD counterparts of S -matrix

The identification of the TGD counterpart of S -matrix is the key topic of this chapter. What this matrix actually means is far from obvious.

1. One can characterize zero energy state by a "square root" of density matrix which is product of hermitian matrix and unitary matrix: I have called this matrix M -matrix. The unitary matrix related to the M -matrix could relate closely to the S -matrix assigned with particle reactions.
2. One can assign the analog of unitary S -matrix to "small" state function reductions (SSFRs) defining the TGD counterparts of "weak" measurements. The states at the passive boundary PB are unaffected, which has interpretation as the TGD counterpart of Zeno effect. This S -matrix could relate to the evolution of self as a conscious entity and to its cognitive time evolution [L26, L30].
3. One can also assign an S -matrix like entity to "big" SFRs (BSFRs) in which the arrow of time changes. This S -matrix would be the counterpart of the ordinary S -matrix and should closely relate to the M -matrix.
4. I have also introduced the notion of U -matrix, which would be defined between zero energy states without fixing states at the passive boundary essential for fixing the arrow of time. This notion has remained somewhat misty and it seems that this notion is not needed since the matrices assigned SSFRs and BSFRs indeed are between zero energy states.

The TGD counterpart of S -matrix - call it M -matrix- defines time-like entanglement coefficients between positive and negative energy parts of zero energy state located at the light-like boundaries of CD.

1. M -matrix need not be unitary unlike the U -matrix characterizing the unitary process forming part of quantum jump. There are several arguments suggesting that M -matrix cannot be unitary but can be regarded as thermal S -matrix so that thermodynamics would become an essential part of quantum theory. In fact, M -matrix can be decomposed to a product of positive diagonal matrix identifiable as square root of density matrix and unitary matrix so that quantum theory would be kind of square root of thermodynamics. Path integral formalism is given up although functional integral over the 3-surfaces is present.
2. In the general case only thermal M -matrix defines a normalizable zero energy state so that thermodynamics or at least formalism resembling thermodynamics becomes part of quantum theory. One can assign to M -matrix a complex parameter whose real part has interpretation as interaction time and imaginary part as the inverse temperature.

In the chapter “*Zero energy ontology, hierarchy of Planck constants, and Kähler metric replacing unitary S -matrix: three pillars of new quantum theory*” [K31] of this book, the idea that scattering amplitudes could allow a geometrization in terms of the Kähler metric of WCW is considered. The role of $M^8 - H$ duality in the construction of scattering amplitudes as M -matrix is discussed in chapter “*Breakthrough in understanding of $M^8 - H$ duality*” [K5] of this book. The idea would be that the descriptions at the level of M^8 and H provide momentum-space and space-time descriptions of the scattering amplitudes.

1.3 Topics of the chapter

The first sections provide conceptual background for the attempts to identify scattering amplitudes in TGD framework. The other chapters discuss more detailed attempts.

1.3.1 $M^8 - H$ duality, hierarchy of Planck constants, and p-adic length scale hypothesis

$M^8 - H$ duality provides a cornerstone of TGD and one can consider the construction of scattering amplitudes both at M - and H -level. This motivates the discussion of in the section “*About relationship between $M^8 - H$ duality, hierarchy of Planck constants, and p-adic length scale hypothesis*”.

The meaning of $M^8 - H$ duality in fermionic sector is considered in the section “*Fermionic variant of $M^8 - H$ duality*”. The role of second quantized spinors in H is well-understood but in M^8 the situation is different. The basic guideline is that also fermionic dynamics at the level of M^8 should be algebraic and number theoretical.

1. Spinors should be octonionic. I have already earlier considered their possible physical interpretation [L1].
2. Dirac equation as linear partial differential equation should be replaced with a linear algebraic equation for octonionic spinors which are complexified octonions. The momentum space variant of the ordinary Dirac equation is an algebraic equation and the proposal is obvious: $P\Psi = 0$, where P is the octonionic continuation of the polynomial defining the space-time surface and multiplication is in octonionic sense. The conjugation in O_c is induced by the conjugation of the commuting imaginary unit i . The square of the Dirac equation is real if the space-time surface corresponds to a projection $O_c \rightarrow M^8 \rightarrow M^4$ with real time coordinate and imaginary spatial coordinates so that the metric defined by the octonionic norm is real and has Minkowskian signature. Hence the notion of Minkowski metric reduces to octonionic norm for O_c - a purely number theoretic notion.

The masslessness condition restricts the solutions to light-like 3-surfaces $m_{kl}P^kP^l = 0$ in Minkowskian sector analogous to mass shells in momentum space - just as in the case of ordinary massless Dirac equation. $P(o)$ rather than octonionic coordinate o would define momentum. These mass shells should be mapped to light-like partonic orbits in H .

3. This picture leads to the earlier phenomenological picture about induced spinors in H . Twistor Grassmann approach suggests the localization of the induced spinor fields at light-like partonic orbits in H . If the induced spinor field allows a continuation from 3-D partonic orbits to the interior of X^4 , it would serve as a counterpart of virtual particle in accordance with quantum field theoretical picture.

1.3.2 Hyper-finite factors and M -matrix

The notion of hyper-finite factor is expected to play central role in the mathematical description of finite measurement resolution, in the realization of the hierarchy of Planck constants [K11, ?], the hierarchy quantum criticalities, and the hierarchy of gauge symmetry breakings for the super-symplectic algebra. This motivates the discussion of the basic results and ideas are about HFFs. The views about M -matrix as a characterizer of time-like entanglement and M -matrix as a functor are analyzed. The role of hyper-finite factors in the construction of M -matrix is considered. One section is devoted to the possibility that Connes tensor product could define fundamental vertices. A more detailed discussion can be found in the book [K13], in particular in chapter [K28].

I do not pretend of having handle about the huge technical complexities and can only recommend the works of von Neumann [A18, A22, A20, A15]. Tomita [A17]. [B2, B1, B3]. the work of Powers and Araki and Woods which served as starting point for the work of Connes [A5, A4]. The work of Jones [A11], and other leading figures in the field. What is may main contribution is fresh physical interpretation of this mathematics which also helps to make mathematical conjectures. The book of Connes [A5] available in web provides an excellent overall view about von Neumann algebras and non-commutative geometry.

The role of HFFs in the construction of M -matrix is considered in the section “*A vision about the role of HFFs in TGD*”.

1.3.3 Number theoretic approach to the S -matrix associated with SFRs

Adelic physics, $M^8 - H$ duality, and ZEO to a proposal that the dynamics involved with “small” state function reductions (SSFRs) as counterparts of ”weak” measurements could be basically number theoretical dynamics with SSFRs identified as SFR cascades leading to completely un-entangled state in the space of wave functions in Galois group of extension of rationals identifiable as wave functions in the space of cognitive representations. This is discussed in the section “it The dynamics of SSFRs as quantum measurement cascades in the group algebra of Galois group” [L26].

As a side product a prime factorization of the order of Galois group is obtained. The result looks even more fascinating if the cognitive dynamics is a representation for the dynamics in real degrees of freedom in finite resolution characterized by the extension of rationals. If cognitive representations represent reality approximately, this indeed looks very natural and would provide an analog for adèle formula expressing the norm of a rational as the inverse of the product of its p-adic norms. The results can be applied to the TGD inspired model of genetic code.

The last section “it The relation between U -Matrix and M -matrices” includes some old and perhaps obsolete speculations about the admittedly misty U -matrix. The negative and positive energy parts of zero energy state can contain zero energy parts in shorter scales - quantum field theorist might talk about quantum fluctuations. One can have also U -matrix and M -matrix elements between this kind of states and even between zero energy states and a hierarchy suggests itself. Since fermions could be seen as correlates of Boolean cognition and zero energy states in fermion sectors as quantal Boolean statements, one can ask whether these matrices could define Boolean hierarchies: statements about statements about...

2 About $M^8 - H$ -duality, p-adic length scale hypothesis and dark matter hierarchy

$M^8 - H$ duality, p-adic length scale hypothesis and dark matter hierarchy as phases of ordinary matter with effective Planck constant $h_{eff} = nh_0$ are basic assumptions of TGD, which all reduce to number theoretic vision. In the sequel $M^8 - H$ duality, p-adic length scale hypothesis and dark matter hierarchy are discussed from number theoretic perspective.

Several new results emerge. Strong form of holography (SH) allows to weaken strong form of $M^8 - H$ duality mapping space-time surfaces $X^4 \subset M^8$ to $H = M^4 \times CP_2$ that it allows to map only certain complex 2-D sub-manifolds of quaternionic space-time surface to H : SH allows to determine $X^4 \subset H$ from this 2-D data. Complex sub-manifolds are determined by conditions completely analogous to those determined space-time surface as quaternionic sub-manifold and only discrete set of them is obtained.

M^8 duality allows to relate p-adic length scales L_p to differences for the roots of the polynomial defining the extension defining “special moments in the life of self” assignable causal diamond (CD) central in zero energy ontology (ZEO). Hence p-adic length scale hypothesis emerges both from p-adic mass calculations and $M^8 - H$ duality. It is proposed that the size scale of CD correspond to the largest dark scale nL_p for the extension and that the sub-extensions of extensions could define hierarchy of sub-CDs. Skyrmions are an important notion if nuclear and hadron physics, $M^8 - H$ duality suggests an interpretation of skyrmion number as winding number as that for a map defined by complex polynomial.

2.1 Some background

A summary of the basic notions and ideas involved is in order.

2.1.1 p-Adic length scale hypothesis

In p-adic mass calculations [K16] real mass squared is obtained by so called canonical identification from p-adic valued mass squared identified as analog of thermodynamical mass squared using p-adic generalization of thermodynamics assuming super-conformal invariance and Kac-Moody algebras assignable to isometries and holonomies of $H = M^4 \times CP_2$. This implies that the mass squared is essentially the expectation value of sum of scaling generators associated with various tensor factors of the representations for the direct sum of super-conformal algebras and if the number of factors is 5 one obtains rather predictive scenario since the p-adic temperature T_p must be inverse integer in order that the analogs of Boltzmann factors identified essentially as p^{L_0/T_p} .

The p-adic mass squared is of form $Xp + O(p^2)$ and mapped to $X/p + O(1/p^2)$. For the p-adic primes assignable to elementary particles ($M_{127} = 2^{127} - 1$ for electron) the higher order corrections are in general extremely small unless the coefficient of second order contribution is larger integer of order p so that calculations are practically exact.

Elementary particles seem to correspond to p-adic primes near powers 2^k . Corresponding p-adic length - and time scales would come as half-octaves of basic scale if all integers k are allowed. For odd values of k one would have octaves as analog for period doubling. In chaotic systems also the generalization of period doubling in which prime $p = 2$ is replaced by some other small prime appear and there is indeed evidence for powers of $p = 3$ (period tripling as approach to chaos). Many elementary particles and also hadron physics and electroweak physics seem to correspond to Mersenne primes and Gaussian Mersennes which are maximally near to powers of 2.

For given prime p also higher powers of p define p-adic length scales: for instance, for electron the secondary p-adic time scale is .1 seconds characterizing fundamental bio-rhythm. Quite generally, elementary particles would be accompanied by macroscopic length and time scales perhaps assignable to their magnetic bodies or causal diamonds (CDs) accompanying them.

This inspired p-adic length scale hypothesis stating the size scales of space-time surface correspond to primes near half-octaves of 2. The predictions of p-adic are exponentially sensitive to the value of k and their success gives strong support for p-adic length scale hypothesis. This hypothesis applied not only to elementary particle physics but also to biology and even astrophysics and cosmology. TGD Universe could be p-adic fractal.

2.1.2 Dark matter as phases of ordinary matter with $h_{eff} = nh_0$

The identification of dark matter as phases of ordinary matter with effective Planck constant $h_{eff} = nh_0$ is second key hypothesis of TGD. To be precise, these phases behave like dark matter and galactic dark matter could correspond to dark energy in TGD sense assignable to cosmic strings thickened to magnetic flux tubes.

There are good arguments in favor of the identification $h = 6h_0$ [L2, L11]. “Effective” means that the actual value of Planck constant is h_0 but in many-sheeted space-time n counts the number of symmetry related space-time sheets defining space-time surface as a covering. Each sheet gives identical contribution to action and this implies that effective value of Planck constant is nh_0 .

2.1.3 $M^8 - H$ duality

$M^8 - H$ duality ($H = M^4 \times CP_2$) [L16] has taken a central role in TGD framework. $M^8 - H$ duality allows to identify space-time regions as ”roots” of octonionic polynomials P in complexified $M^8 - M_c^8$ - or as minimal surfaces in $H = M^4 \times CP_2$ having 2-D singularities.

Remark: O_c, H_c, C_c, R_c will be used in the sequel for complexifications of octonions, quaternions, etc.. number fields using commuting imaginary unit i appearing naturally via the roots of real polynomials.

The precise form of $M^8 - H$ duality has however remained unclear. Two assumptions are involved.

1. Associativity stating that the tangent or normal space of at the point of the space-time space-time surface M^8 is associative - that is quaternionic. There are good reasons to believe that this is true for the polynomial ansatz everywhere but there is no rigorous proof.
2. The tangent space of the point of space-time surface at points mappable from M^8 to H must contain fixed $M^2 \subset M^4 \subset M^8$ or an integrable distribution of $M^2(x)$ so that the 2-surface of M^4 determined by it belongs to space-time surface.

The strongest, global form of $M^8 - H$ duality states that $M^2(x)$ is contained to tangent spaces of X^4 at all points x . Strong form of holography (SH) states allows also the option for which this holds true only for 2-D surfaces - string world sheets and partonic 2-surfaces - therefore mappable to H and that SH allows to determined $X^4 \subset H$ from this data. In the following a realization of this weaker form of $M^8 - H$ duality is found. Note however that one cannot exclude the possibility that also associativity is true only at these surfaces for the polynomial ansatz.

2.1.4 Number theoretic origin of p-adic primes and dark matter

There are several questions to be answered. How to fuse real number based physics with various p-adic physics? How p-adic length scale hypothesis and dark matter hypothesis emerge from TGD?

The properties of p-adic number fields and the strange failure of complete non-determinism for p-adic differential equations led to the proposal that p-adic physics might serve as a correlate for cognition, imagination, and intention. This led to a development of number theoretic vision which I call adelic physics. A given adèle corresponds to a fusion of reals and extensions of various p-adic number fields induced by a given extension of rationals.

The notion of space-time generalizes to a book like structure having real space-time surfaces and their p-adic counterparts as pages. The common points of pages defining is back correspond to points with coordinates in the extension of rationals considered. This discretization of space-time surface is in general finite and unique and is identified as what I call cognitive representation. The Galois group of extension becomes symmetry group in cognitive degrees of freedom. The ramified primes of extension are exceptionally interesting and are identified as preferred p-adic primes for the extension considered.

The basic challenge is to identify dark scale. There are some reasons to expect correlation between p-adic and dark scales which would mean that the dark scale would depend on ramified primes, which characterize roots of the polynomial defining the extensions and are thus not defined completely by extension alone. Same extension can be defined by many polynomials. The naïve guess is that the scale is proportional to the dimension n of extension serving as a measure for algebraic complexity (there are also other measures). p-Adic length scales L_p would be proportional nL_p , p ramified prime of extension? The motivation would be that quantum scales are typically proportional to Planck constant. It turns out that the identification of CD scale as dark scale is rather natural.

2.2 New results about $M^8 - H$ duality

In the sequel some new results about $M^8 - H$ duality are deduced. Strong form of holography (SH) allows to weaken the assumptions making possible $M^8 - H$ duality. It would be enough to map only certain complex 2-D sub-manifolds of quaternionic space-time surface in M^8 to H : SH would allow to determine $X^4 \subset H$ from this 2-D data. Complex sub-manifolds would be determined by conditions completely analogous to those determined space-time surface as quaternionic sub-manifold and they form a discrete set.

2.2.1 Strong form of holography (SH)

Ordinary 3-D holography is forced by general coordinate invariance (GCI) and loosely states that the data at 3-D surfaces allows to determine space-time surface $X^4 \subset H$. In ZEO 3-surfaces correspond to pairs of 3-surfaces with members at the opposite light-like boundaries of causal diamond (CD) and are analogous to initial and final states of deterministic time evolution as Bohr orbit.

This poses additional strong conditions on the space-time surface.

1. The conjecture is that these conditions state the vanishing of super-symplectic Noether charges for a sub-algebra of super-symplectic algebra SC_n with radial conformal weights coming as n -multiples of those for the entire algebra SC and its commutator $[SC_n, SC]$ with the entire algebra: these conditions generalize super conformal conditions and one obtains a hierarchy of realizations.

This hierarchy of minimal surfaces would naturally correspond to the hierarchy of extensions of rationals with n identifiable as dimension of the extension giving rise to effective Planck constant. At the level of Hilbert spaces the inclusion hierarchies for extensions could also correspond to the inclusion hierarchies of hyper-finite factors of type I_1 [K28] so that $M^8 - H$ duality would imply beautiful connections between key ideas of TGD.

2. Second conjecture is that the preferred extremals (PEs) are extremals of both the volume term and Kähler action term of the action resulting by dimensional reduction making possible the induction of twistor structure from the product of twistor spaces of M^4 and CP_2 to 6-D S^2 bundle over X^4 defining the analog of twistor space. These twistor spaces must have Kähler structure since action for 6-D surfaces is Kähler action - it exists only in these two cases [A13] so that TGD is unique.

Strong form of holography (SH) is a strengthening of 3-D holography. Strong form of GCI requires that one can use either the data associated either with

- light-like 3-surfaces defining partonic orbits as surfaces at which signature of the induced metric changes from Euclidian to Minkowskian or
- the space-like 3-surfaces at the ends of CD to determine space-time surface as PE (in case that it exists).

This suggests that the data at the intersections of these 2-surfaces defined by partonic 2-surfaces might be enough for holography. A slightly weaker form of SH is that also string world sheets intersecting partonic orbits along their 1-D boundaries is needed and this form seems more realistic.

SH allows to weaken strong form of $M^8 - H$ duality mapping space-time surfaces $X^4 \subset M^8$ to $H = M^4 \times CP_2$ that it allows to map only certain complex 2-D sub-manifolds of quaternionic space-time surface to H : SH allows to determine $X^4 \subset H$ from this 2-D data. Complex sub-manifolds are determined by conditions completely analogous to those determined space-time surface as quaternionic sub-manifold and only discrete set of them is obtained.

2.2.2 Space-time as algebraic surface in M_c^8 regarded complexified octonions

The octonionic polynomial giving rise to space-time surface as its “root” is obtained from ordinary real polynomial P with rational coefficients by algebraic continuation. The conjecture is that the identification in terms of roots of polynomials of even real analytic functions guarantees

associativity and one can formulate this as rather convincing argument [?] Space-time surface X_c^4 is identified as a 4-D root for a H_c -valued “imaginary” or “real” part of O_c valued polynomial obtained as an O_c continuation of a real polynomial P with rational coefficients, which can be chosen to be integers. These options correspond to complexified-quaternionic tangent- or normal spaces. For $P(x) = x^n + \dots$ ordinary roots are algebraic integers. The real 4-D space-time surface is projection of this surface from M_c^8 to M^8 . One could drop the subscripts “ c ” but in the sequel they will be kept.

M_c^4 appears as a special solution for any polynomial P . M_c^4 seems to be like a universal reference solution with which to compare other solutions.

One obtains also brane-like 6-surfaces as 6-spheres as universal solutions. They have M^4 projection, which is a piece of hyper-surface for which Minkowski time as time coordinate of CD corresponds to a root $t = r_n$ of P . For monic polynomials these time values are algebraic integers and Galois group permutes them.

One cannot exclude rational functions or even real analytic functions in the sense that Taylor coefficients are octonionically real (proportional to octonionic real unit). Number theoretical vision - adelic physics [?, ?] suggests that polynomial coefficients are rational or perhaps in extensions of rationals. The real coefficients could in principle be replaced with complex numbers $a + ib$, where i commutes with the octonionic units and defines complexification of octonions. i appears also in the roots defining complex extensions of rationals.

2.2.3 How do the solutions assignable to the opposite boundaries of CD relate to each other?

CD has two boundaries. The polynomials associated with them could be different in the general formulation discussed in [L24, L26] but they could be also same. How are the solutions associated with opposite boundaries of CD glued together in a continuous manner?

1. The polynomials assignable to the opposite boundaries of CD are allowed to be polynomials of o resp. $(o - T)$: here T is the distance between the tips of CD.
2. CD brings in mind the realization of conformal invariance at sphere: the two hemispheres correspond to powers of z and $1/z$: the condition $z = \bar{1}/z$ at unit circle is essential and there is no real conjugation. How the sphere is replaced with 8-D CD which is also complexified. The absence of conjugation looks natural also now: could CD contain a 3-surface analogous to the unit circle of sphere at which the analog of $z = \bar{1}/z$ holds true? If so, one has $P(o, z) = P(1/o, z)$ and the solutions representing roots of $P(o, z)$ and $P(1/o, z)$ can be glued together.

Note that $1/o$ can be expressed as $\bar{o}/o\bar{o}$ when the Minkowskian norm squared $\bar{o}o$ is non-vanishing and one has polynomial equation also now. This condition is true outside the boundary of 8-D light-cone, in particular near the upper boundary of CD.

The counter part for the length squared of octonion in Minkowskian signature is light-one proper time coordinate $a^2 = t^2 - r^2$ for M_+^8 . Replacing o which scaled dimensionless variable $o_1 = o/(T/2)$ the gluing take place along $a = T/2$ hyperboloid.

One has algebraic holomorphy with respect to o but also anti-holomorphy is possible. What could these two options correspond to? Could the space-time surfaces assignable to self and its time-reversal relate by octonionic conjugation $o \rightarrow \bar{o}$ relating two Fock vacuums annihilated by fermionic annihilation resp. creation operators?

In [L24, L26] the possibility that the sequence of SSFRs or BSFRs could involve iteration of the polynomial defining space-time surface - actually different polynomials were allowed for two boundaries. There are 3 options: each SSFR would involve the replacement $Q = P \circ \dots \circ P \rightarrow P \circ Q$, the replacement occurs only when new “special moments in the life of self” defined by the roots of P as $t = r_n$ balls of cd, or the replacement can occur in BSFR when the metabolic resources do not allow to continue the iteration (the increase of h_{eff} during iteration increases the needed metabolic feed).

The iteration is compatible with the proposed picture. The assumption $P(0) = 0$ implies that iterates of P contain also the roots of P as roots - they are like conserved genes. Also the 8-D

light-cone boundary remains invariant under iteration. Even more general function decompositions $P \rightarrow Q \rightarrow P$ are consistent with the proposed picture.

2.2.4 Brane-like solutions

One obtains also 6-D brane-like solutions to the equations.

1. In general the zero loci for imaginary or real part are 4-D but the 7-D light-cone δM_+^8 of M^8 with tip at the origin of coordinates is an exception [L4, L5, L6]. At δM_+^8 the octonionic coordinate o is light-like and one can write $o = re$, where 8-D time coordinate and radial coordinate are related by $t = r$ and one has $e = (1 + e_r)/\sqrt{2}$ such that one as $e^2 = e$.

Polynomial $P(o)$ can be written at δM_+^8 as $P(o) = P(r)e$ and its roots correspond to 6-spheres S^6 represented as surfaces $t_M = t = r_N$, $r_M = \sqrt{r_N^2 - r_E^2} \leq r_N$, $r_E \leq r_N$, where the value of Minkowski time $t = r = r_N$ is a root of $P(r)$ and r_M denotes radial Minkowski coordinate. The points with distance r_M from origin of $t = r_N$ ball of M^4 has as fiber 3-sphere with radius $r = \sqrt{r_N^2 - r_E^2}$. At the boundary of S^3 contracts to a point.

2. These 6-spheres are analogous to 6-D branes in that the 4-D solutions would intersect them in the generic case along 2-D surfaces X^2 . The boundaries $r_M = r_N$ of balls belong to the boundary of M^4 light-cone. In this case the intersection would be that of 4-D and 3-D surface, and empty in the generic case (it is however quite not clear whether topological notion of “genericity” applies to octonionic polynomials with very special symmetry properties).
3. The 6-spheres $t_M = r_N$ would be very special. At these 6-spheres the 4-D space-time surfaces X^4 as usual roots of $P(o)$ could meet. Brane picture suggests that the 4-D solutions connect the 6-D branes with different values of r_n .

The basic assumption has been that particle vertices are 2-D partonic 2-surfaces and light-like 3-D surfaces - partonic orbits identified as boundaries between Minkowskian and Euclidian regions of space-time surface in the induced metric (at least at H level) - meet along their 2-D ends X^2 at these partonic 2-surfaces. This would generalize the vertices of ordinary Feynman diagrams. Obviously this would make the definition of the generalized vertices mathematically elegant and simple.

Note that this does not require that space-time surfaces X^4 meet along 3-D surfaces at S^6 . The interpretation of the times t_n as moments of phase transition like phenomena is suggestive. ZEO based theory of consciousness suggests interpretation as moments for state function reductions analogous to weak measurements and giving rise to the flow of experienced time.

4. One could perhaps interpret the free selection of 2-D partonic surfaces at the 6-D roots as initial data fixing the 4-D roots of polynomials. This would give precise content to strong form of holography (SH), which is one of the central ideas of TGD and strengthens the 3-D holography coded by ZEO alone in the sense that pairs of 3-surfaces at boundaries of CD define unique preferred extremals. The reduction to 2-D holography would be due to preferred extremal property realizing the huge symplectic symmetries and making $M^8 - H$ duality possible as also classical twistor lift.

I have also considered the possibility that 2-D string world sheets in M^8 could correspond to intersections $X^4 \cap S^6$? This is not possible since time coordinate t_M constant at the roots and varies at string world sheets.

Note that the complexification of M^8 (or equivalently octonionic E^8) allows to consider also different variants for the signature of the 6-D roots and hyperbolic spaces would appear for $(\epsilon_1, \epsilon_i, \dots, \epsilon_8)$, $\epsilonpsilon_i = \pm 1$ signatures. Their physical interpretation - if any - remains open at this moment.

5. The universal 6-D brane-like solutions S_c^6 have also lower-D counterparts. The condition determining X^2 states that the C_c -valued “real” or “imaginary” for the non-vanishing Q_c -valued “real” or “imaginary” for P vanishes. This condition allows universal brane-like solution as a restriction of O_c to M_c^4 (that is CD_c) and corresponds to the complexified

time=constant hyperplanes defined by the roots $t = r_n$ of P defining “special moments in the life of self” assignable to CD. The condition for reality in R_c sense in turn gives roots of $t = r_n$ a hyper-surfaces in M_c^2 .

2.2.5 Explicit realization of $M^8 - H$ duality

$M^8 - H$ duality allows to map space-time surfaces in M^8 to H so that one has two equivalent descriptions for the space-time surfaces as algebraic surfaces in M^8 and as minimal surfaces with 2-D singularities in H satisfying an infinite number of additional conditions stating vanishing of Noether charges for super-symplectic algebra acting as isometries for the “world of classical worlds” (WCW). Twistor lift allows variants of this duality. M_H^8 duality predicts that space-time surfaces form a hierarchy induced by the hierarchy of extensions of rationals defining an evolutionary hierarchy. This forms the basis for the number theoretical vision about TGD.

$M^8 - H$ duality makes sense under 2 additional assumptions to be considered in the following more explicitly than in earlier discussions.

1. Associativity condition for tangent-/normal space is the first essential condition for the existence of $M^8 - H$ duality and means that tangent - or normal space is quaternionic.
2. The tangent space of space-time surface and thus space-time surface itself must contain a preferred $M_c^2 \subset M_c^4$ or more generally, an integrable distribution of tangent spaces $M_c^2(x)$ and similar distribution of their complements $E_c^2(x)$. The string world sheet like entity defined by this distribution is 2-D surface $X_c^2 \subset X_c^4$ in R_c sense. $E_c^2(x)$ would correspond to partonic 2-surface.

One can imagine two realizations for this condition.

Option I: Global option states that the distributions $M_c^2(x)$ and $E_c^2(x)$ define slicing of X_c^4 .

Option II: Only discrete set of 2-surfaces satisfying the conditions exist, they are mapped to H , and strong form of holography (SH) applied in H allows to deduce space-time surfaces in H . This would be the minimal option.

That the selection between these options is not trivial is suggested by following.

1. For massless extremals (MEs, topological light rays) parameterized by light-like vector k defining $M^2 \subset M^2 \times E^2 \subset M^4$ at each point and by space-like polarization vector ϵ depending on single transversal coordinate of E^2 [K3].
2. CP_2 coordinates have an arbitrary dependence on both $u = k \cdot m$ and $w = \epsilon \cdot m$ and can be also multivalued functions of u and w . Single light-like vector k is enough to identify M^2 . CP_2 type extremals having metric and Kähler form of CP_2 have light-like geodesic as M^4 projection defining M^2 and its complement E^2 in the normal space.
3. String like objects $X^2 \times Y^2 \subset M^4 \times CP_2$ are minimal surfaces and X^2 defines the distribution of $M^2(x) \subset M^4$. Y^2 defines the complement of this distribution.

Option I is realized in all 3 cases. It is not clear whether M^2 can depend on position in the first 2 cases and also CP_2 point in the third case. It could be that only a discrete set of these string world sheets assignable to wormhole contacts representing massless particles is possible (**Option II**).

How these conditions would be realized?

1. The basic observation is that X^2c can be fixed by posing to the non-vanishing H_c -valued part of octonionic polynomial P condition that the C_c valued “real” or “imaginary” part in C_c sense for P vanishes. M_c^2 would be the simplest solution but also more general complex sub-manifolds $X_c^2 \subset M_c^4$ are possible. This condition allows only a discrete set of 2-surfaces as its solutions so that it works only for **Option II**.

These surfaces would be like the families of curves in complex plane defined by $u = 0$ and $v = 0$ curves of analytic function $f(z) = u + iv$. One should have family of polynomials differing by a constant term, which should be real so that $v = 0$ surfaces would form a discrete set.

2. As found, there are also classes special global solutions for which the choice of M_c^2 is global and does not depend on space-time point. The interpretation would be in terms of modes of classical massless fields characterized by polarization and momentum. If the identification of M_c^2 is correct, these surfaces are however unstable against perturbations generating discrete string world sheets and orbits of partonic 2-surfaces having interpretation space-time counterparts of quanta. That fields are detected via their quanta was the revolutionary observation that led to quantum theory. Could quantum measurement induce the instability decomposing the field to quanta at the level of space-time topology?
3. One can generalize this condition so that it selects 1-D surface in X_c^2 . By assuming that R_c -valued “real” or “imaginary” part of quaternionic part of P at this 2-surface vanishes. one obtains preferred M_c^1 or E_c^1 containing octonionic real and preferred imaginary unit or distribution of the imaginary unit having interpretation as complexified string. Together these kind 1-D surfaces in R_c sense would define local quantization axis of energy and spin. The outcome would be a realization of the hierarchy $R_c \rightarrow C_c \rightarrow H_c \rightarrow O_c$ realized as surfaces.

This option could be made possible by SH. SH states that preferred extremals are determined by data at 2-D surfaces of X^4 . Even if the conditions defining X_c^2 have only a discrete set of solutions, SH at the level of H could allow to deduce the preferred extremals from the data provided by the images of these 2-surfaces under $M^8 - H$ duality. Associativity and existence of $M^2(x)$ would be required only at the 2-D surfaces.

4. I have proposed that physical string world sheets and partonic 2-surfaces appear as singularities and correspond to 2-D folds of space-time surfaces at which the dimension of the quaternionic tangent space degenerates from 4 to 2 [L15] [K3]. This interpretation is consistent with a book like structure with 2-pages. Also 1-D real and imaginary manifolds could be interpreted as folds or equivalently books with 2 pages.

For the singular surfaces the dimension quaternionic tangent or normal space would reduce from 4 to 2 and it is not possible to assign CP_2 point to the tangent space. This does not of course preclude the singular surfaces and they could be analogous to poles of analytic function. Light-like orbits of partonic 2-surfaces would in turn correspond to cuts.

5. What could the normal space singularity mean at the level of H ? Second fundamental form defining vector basis in normal space is expected to vanish. This would be the case for minimal surfaces.
 - (a) String world sheets with Minkowskian signature (in M^4 actually) are expected to be minimal surfaces. In this case T matters and string world sheets could be mapped to H by $M^8 - H$ duality and SH would work for them.
 - (b) The light-like orbits of partonic 2-surfaces with Euclidian signature in H would serve as analogs of cuts. N is expected to matter and partonic 2-surfaces should be minimal surfaces. Their branching of partonic 2-surfaces is thus possible and would make possible (note the analogy with the branching of soap films) for them to appear as 2-D vertices in H .

The problem is to identify the pre-images of partonic 2-surfaces in M^8 . The light-likeness of the orbits of partonic 2-surfaces (induced 4-metric changes its signature and degenerates to 3-D) should be important. Could light-likeness in this sense define the pre-images partonic orbits in M^8 ?

Remark: It must be emphasized that SH makes possible $M^8 - H$ correspondence assuming that also associativity conditions hold true only at partonic 2-surfaces and string world sheets. Thus one could give up the conjecture that the polynomial ansatz implies that tangent or normal spaces are associative. Proving that this is the case for the tangent/normal spaces of these 2-surfaces should be easier.

2.2.6 Does $M^8 - H$ duality relate hadron physics at high and low energies?

During the writing of this article I realized that $M^8 - H$ duality has very nice interpretation in terms of symmetries. For $H = M^4 \times CP_2$ the isometries correspond to Poincare symmetries and color $SU(3)$ plus electroweak symmetries as holonomies of CP_2 . For octonionic M^8 the subgroup $SU(3) \subset G_2$ is the sub-group of octonionic automorphisms leaving fixed octonionic imaginary unit invariant - this is essential for $M^8 - H$ duality. $SU(3)$ is also subgroup of $SO(6) \equiv SU(4)$ acting as rotation on $M^8 = M^2 \times E^6$. The subgroup of the holonomy group of $SO(4)$ for E^4 factor of $M^8 = M^4 \times E^4$ is $SU(2) \times U(1)$ and corresponds to electroweak symmetries. One can say that at the level of M^8 one has symmetry breaking from $SO(6)$ to $SU(3)$ and from $SO(4) = SU(2) \times SO(3)$ to $U(2)$.

This interpretation gives a justification for the earlier proposal that the descriptions provided by the old-fashioned low energy hadron physics assuming $SU(2)_L \times SU(2)_R$ and acting acting as covering group for isometries $SO(4)$ of E^4 and by high energy hadron physics relying on color group $SU(3)$ are dual to each other.

2.2.7 Skyrmions and $M^8 - H$ duality

I received a link (<https://tinyurl.com/ycathr3u>) to an article telling about research (<https://tinyurl.com/yddwhr2o>) carried out for skyrmions, which are very general condensed matter quasiparticles. They were found to replicate like DNA and cells. I realized that I have not clarified myself the possibility of skyrmions on TGD world and decided to clarify my thoughts.

1. What skyrmions are?

Consider first what skyrmions are.

1. Skyrmions are topological entities. One has some order parameter having values in some compact space S . This parameter is defined in say 3-ball such that the parameter is constant at the boundary meaning that one has effectively 3-sphere. If the 3rd homotopy group of S characterizing topology equivalence classes of maps from 3-sphere to S is non-trivial, you get soliton-like entities, stable field configurations not deformable to trivial ones (constant value). Skyrmions can be assigned to space S which is coset space $SU(2)_L \times SU(2)_R / SU(2)_V$, essentially S^3 and are labelled by conserved integer-valued topological quantum number.
2. One can imagine variants of this. For instance, one can replace 3-ball with disk. $SO(3) = S^3$ with 2-sphere S^2 . The example considered in the article corresponds to discretized situation in which one has magnetic dipoles/spins at points of say discretized disk such that spins have same direction about boundary circle. The distribution of directions of spin can give rise to skyrmion-like entity. Second option is distribution of molecules which do not have symmetry axis so that as rigid bodies the space of their orientations is discretized version of $SO(3)$. The field would be the orientation of a molecule of lattice and one has also now discrete analogs of skyrmions.
3. More generally, skyrmions emerge naturally in old-fashioned hadron physics, where $SU(2)_L \times SU(2)_R / SU(2)_V$ involves left-handed, right-handed and vectorial subgroups of $SO(4) = SU(2)_L \times SU(2)_R$. The realization would be in terms of 4-component field (π, σ) , where π is charged pion with 3 components - axial vector - and σ which is scalar. The additional constraint $\pi \cdot \pi + \sigma^2 = \text{constant}$ defines 3-sphere so that one has field with values in S^3 . There are models assigning this kind of skyrmion with nucleon, atomic nuclei, and also in the bag model of hadrons bag can be thought of as a hole inside skyrmion. These models seem to have something to do with reality so that a natural question is whether skyrmions might appear in TGD.

2. Skyrmion number as winding number

In TGD framework one can regard space-time as 4-surface in either octonionic M_c^8 , c refers here to complexification by an imaginary unit i commuting with octonions, or in $M^4 \times CP_2$. For the solution surfaces M^8 has natural decomposition $M^8 = M^2 \times E^6$ and E^6 has $SO(6)$ as isometry group containing subgroup $SU(3)$ having automorphisms of octonions as subgroup leaving M^2

invariant. $SO(6) = SU(4)$ contains $SU(3)$ as subgroup, which has interpretation as isometries of CP_2 and counterpart of color gauge group. This supports $M^8 - H$ duality, whose most recent form is discussed in [L22].

The map $S^3 \rightarrow S^3$ defining skyrmion could be taken as a phenomenological consequence of $M^8 - H$ duality implying the old-fashioned description of hadrons involving broken $SO(4)$ symmetry (PCAC) and unbroken symmetry for diagonal group $SO(3)_V$ (CCV). The analog of (π, σ) field could correspond to a B-E condensate of pions (π, σ) .

The obvious question is whether the map $S^3 \rightarrow S^3$ defining skyrmion could have a deeper interpretation in TGD framework. I failed to find any elegant formulation. One could however generalize and ask whether skyrmion like entities characterize by winding number are predicted by basic TGD.

1. In the models of nucleon and nuclei the interpretation of conserved topological skyrmion number is as baryon number. This number should correspond to the homotopy class of the map in question, essentially winding number. For polynomials of complex number degree corresponds to winding number. Could the degree $n = h_{eff}/h_0$ of polynomial P having interpretation as effective Planck constant and measure of complexity - kind of number theoretic IQ - be identifiable as skyrmion number? Could it be interpreted as baryon number too?
2. For leptons regarded as local 3 anti-quark composites in TGD based view about SUSY [L18] the same interpretation would make sense. It seems however that the winding number must have both signs. Degree is n is however non-negative.

Here complexification of M^8 to M_c^8 is essential. One can allow both holomorphic and anti-holomorphic continuations of real polynomials P (with rational coefficients) using complexification defined by commutative imaginary unit i in M_c^8 so that one has polynomials $P(z)$ resp. $P(\bar{z})$ in turn algebraically continued to complexified octonionic polynomials $P(z, o)$ resp. $P(\bar{z}, o)$.

Particles resp. antiparticles would correspond to the roots of octonionic polynomial $P(z, o)$ resp. $P(\bar{z}, o)$ meaning space-time geometrization of the particle-antiparticle dichotomy and would be conjugates of each other. This could give a nice physical interpretation to the somewhat mysterious complex roots of P .

3. More detailed formulation

To make this formulation more detailed one must ask how 4-D space-time surfaces correspond to 8-D "roots" for the "imaginary" ("real") part of complexified octonionic polynomial as surfaces in M_c^8 .

1. Equations state the simultaneous vanishing of the 4 components of complexified quaternion valued polynomial having degree n and with coefficients depending on the components of O_c , which are regarded as complex numbers $x + iy$, where i commutes with octonionic units. The coefficients of polynomials depend on complex coordinates associated with non-vanishing "real" ("imaginary") part of the O_c valued polynomial.
2. To get perspective, one can compare the situation with that in catastrophe theory in which one considers roots for the gradient of potential function of behavior variables x^i . Potential function is polynomial having control variables as parameters. Now behavior variable correspond "imaginary" ("real") part and control variables to "real" ("imaginary") of octonionic polynomial.

For a polynomial with real coefficients the solution divides to regions in which some roots are real and some roots are complex. In the case of cusp catastrophe one has cusp region with 3-D region of the parameter defined by behavior variable x and 2 control parameters with 3 real roots, the region in which one has one real root. The boundaries for the projection of 3-sheeted cusp to the plane defined by control variables correspond to degeneration of two complex roots to one real root.

In the recent case it is not clear whether one cannot require the M_c^8 coordinates for space-time surface to be real but to be in $M^8 = M^1 + iE^7$.

3. Allowing complex roots gives 8-D space-time surfaces. How to obtain real 4-D space-time surfaces?
 - (a) One could project space-time surfaces to real M^8 to obtain 4-D real space-time surfaces. For M^8 this would mean projection to $M^1 + iE^7$ and in time direction the real part of root is accepted and is same for the root and its conjugate. For E^7 this would mean that imaginary part is accepted and means that conjugate roots correspond to different space-time surfaces and the notion of baryon number is realized at space-time level.
 - (b) If one allows only real roots, the complex conjugation proposed to relate fermions and anti-fermions would be lost.
4. One can select for 4 complex M_c^8 coordinates X^k of the surface and the remaining 4 coordinates Y^k can be formally solved as roots of n :th degree polynomial with dynamical coefficients depending on X^k and the remaining Y^k . This is expected to give rise to preferred extremals with varying dimension of M^4 and CP_2 projections.
5. It seems that all roots must be complex.
 - (a) The holomorphy of the polynomials with respect to the complex M_c^8 coordinates implies that the coefficients are complex in the generic point M_c^8 . If so, all 4 roots are in general complex but do not appear as conjugate pairs. The naïve guess is that the maximal number of solutions would be n^4 for a given choice of M^8 coordinates solved as roots. An open question is whether one can select subset of roots and what happens at $t = r_n$ surfaces: could different solutions be glued together at them.
 - (b) Just for completeness one can consider also the case that the dynamical coefficients are real - this is true in the E^8 sector and whether it has physical meaning is not clear. In this case the roots come as real roots and pairs formed by complex root and its conjugate. The solution surface can be divided into regions depending on the character of 4 roots. The n roots consist of complex root pairs and real roots. The members or complex root pairs are mapped to same point in E^8 .

4. Could skyrmions in TGD sense replicate?

What about the observation that condensed matter skyrmions replicate? Could this have analog at fundamental level?

1. The assignment of conserved topological quantum number to the skyrmion is not consistent with replication unless the skyrmion numbers of outgoing states sum up to that of the initial state. If the system is open one can circumvent this objection. The replication would be like replication of DNA in which nucleotides of new DNA strands are brought to the system to form new strands.
2. It would be fascinating if all skyrmions would correspond to space-time surfaces at fundamental M^8 level. If so, skyrmion property also in magnetic sense could be induced by from a deeper geometric skyrmion property of the MB of the system. The openness of the system would be essential to guarantee conservation of baryon number. Here the fact that leptons and baryons have opposite baryon numbers helps in TGD framework. Note also ordinary DNA replication could correspond to replication of MB and thus of skyrmion sequences.

2.3 About p-adic length scale hypothesis and dark matter hierarchy

It is good to introduce first some background related to p-adic length scale hypothesis discussed in chapters of [K19] and dark matter hierarchy discussed in chapters [K14, K15], in particular in chatper [?].

2.3.1 General form of p-adic length scale hypothesis

The most general form of p-adic length scale hypothesis does not pose conditions on allowed p-adic primes and emerges from p-adic mass calculations [K6, K16, K20]. It has two forms corresponding to massive particles and massless particles.

1. For massive particles the preferred p-adic mass calculations based on p-adic thermodynamics predicts the p-adic mass squared m^2 to be proportional to p or its power- the real counterpart of m^2 is proportional to $1/p$ or its power. In the simplest case one has

$$m^2 = \frac{X}{p} \frac{\hbar}{L_0} ,$$

where L_0 is apart from numerical constant the length R of CP_2 geodesic circle. X is a numerical constant not far from unity. $X \geq 1$ is small integer in good approximation. For instance for electron one has $x = 5$.

By Uncertainty Principle the Compton length of particle is characterizing the size of 3-surfaces assignable to particle are proportional to \sqrt{p} :

$$L_c(m) = \frac{\hbar}{m} = \sqrt{\frac{1}{X}} L_p , \quad L_p = \sqrt{p} L_0 = .$$

Here L_p is p-adic length scale and corresponds to minimal mass for given p-adic prime. p-Adic length scale would be would characterize the size of the 3-surface assignable to the particle and would correspond to Compton length.

2. For massless particles mass vanishes and the above picture is not possible unless there is very small mass coming from p-adic thermodynamics and determined by the size scale of CD - this is quite possible. The preferred time/spatial scales p-adic energy- equivalently 3-momentum are proportional to p-adic prime p or its power. The real energy is proportional to $1/p$. At the embedding space level the size of scale causal diamond (CD) [L17] would be proportional to p : $L = T = pL_0$, $L_0 = T_0$ for $c = 1$. The interpretation in terms of Uncertainty Principle is possible.

There would be therefore two levels: space-time level and embedding space level . At the space-time level the primary p-adic length scale would be proportional to \sqrt{p} whereas the p-adic length scale at embedding space-time would correspond to secondary p-adic length scale proportional to p . The secondary p-adic length scales would assign to elementary new physics in macroscopic scales. For electron the size scale of CD would be about .1 seconds, the time scale associated with the fundamental bio-rhythm of about 10 Hz.

3. A third piece in the picture is adelic physics [L8, L9] inspiring the hypothesis that effective Planck constant h_{eff} given by $h_{eff}/h_0 = n$, $h = 6h_0$, labels the phases of ordinary matter identified as dark matter. n would correspond to the dimension of extension of rationals.

The connection between preferred primes and the value of $n = h_{eff}/h_0$ is interesting. One proposal is that preferred primes p in p-adic length scale hypothesis determining the mass scale of particle correspond to so called ramified primes, which characterize the extensions. The p-adic variant of the polynomial defining space-time surfaces in M^8 picture would have vanishing discriminant in order $O(p)$. Since discriminant is proportional to the product of differences of different roots of the polynomial, two roots would be very near to each other p-adically. This would be mathematical correlate for criticality in p-adic sense.

$M^8 - H$ duality [L16, L14] leads to the prediction that the roots r_n of polynomial defining the space-time region in M^8 correspond to preferred time values $t = t_n = \propto r_n$ - I have called $t = t_n$ "special moments in the life of self". Since the squares for the differences for the roots are proportional to ramified primes, these time differences would code for ramified primes assignable to the space-time surface. There would be several p-adic time scales involved and they would be coded by $t_{ij} = r_i - r_j$, whose moduli squared are divided by so called ramified primes defining excellent candidates for preferred p-adic primes. p-Adic physics would make itself visible at the level of space-time surface in terms of "special moments in the life of self".

4. p-Adic length scales emerge naturally from $M^8 - H$ duality [L16, L14]. Ramified primes would in M^8 picture appear as factors of time differences associated with “special moments in the life of self” associated with CD [L14]. One has $|t_i - t_j| \propto \sqrt{p_{ij}}$, p_{ij} ramified prime. It is essential that square root of ramified prime appears here.

This suggests strongly that p-adic length scale hypothesis is realized at the level of space-time surface and there are several p-adic length scales present coded to the time differences. Knowing of the polynomial would give information about p-adic physics involved. If dark scales correlate with p-adic length scales as proposed, the definition of dark scale should assume the dependence of ramified primes quite generally rather than as a result of number theoretic survival of fittest as one might also think.

The factors $t_i - t_j$ are proportional - not only to the typically very large p-adic prime p_{max} charactering the system - but also smaller primes or their powers. Could the scales in question be of form $l_p = \sqrt{X} \sqrt{p_{max}} L_0$ rather than p-adic length scales $L_{p_{ram}}$ defined by various ramified primes. Here X would be integer consisting of small ramified primes.

p-Adic mass calculations predict in an excellent approximation the mass of the particle is given by $m = (\sqrt{X}/\sqrt{p})m_0$, X small integer and $m_0 = 1/L_0$. Compton length would be given by $L_c(p) = \sqrt{p}/\sqrt{X} L_0$. The identification $l_p = L_c(p)$ would be attractive but is not possible unless one has $X = 1$. In this case one would be considering p-adic length scale L_p . the interpretation in terms of multi-p-adicity seems to be the realistic option.

2.3.2 About more detailed form of p-adic length scale hypothesis

More specific form of p-adic length scale hypothesis poses conditions on physically preferred p-adic primes. There are several guesses for preferred primes. They could be primes near to integer powers 2^k , where k could be positive integer, which could satisfy additional conditions such as being odd, prime or be associated with Mersenne prime or Gaussian Mersenne. One can consider also powers of other small primes such as $p = 2, 3, 5$. p-Adic length scale hypothesis in its basic form would generalize the notion of period doubling. For odd values of k one would indeed obtain period doubling, tripling, etc... suggesting strongly chaos theoretic origin.

1. p-Adic length scale hypothesis in its basic form

Consider first p-adic length scale hypothesis in its basic form.

1. In its basic form states that primes $p \simeq 2^k$ are preferred p-adic primes and correspond by p-adic mass calculations p-adic length scales $L_p \equiv L(k) \propto \sqrt{p} = 2^{k/2}$. Mersenne primes and primes associated with Gaussian Mersennes as especially favored primes and charged leptons ($k \in \{127, 113, 107\}$) and Higgs boson ($k = 89$) correspond to them. Also hadron physics ($k = 107$) and nuclear physics ($k = 113$) correspond to these scales. One can assign also to hadron physics Mersenne prime and the conjecture is that Mersennes and Gaussian Mersennes define scaled variants of hadron physics and electroweak physics. In the length scale between cell membrane thickness fo 10 nm and nuclear size about $2.5 \mu\text{m}$ there are as many as 4 Gaussian Mersennes corresponding to $k \in \{151, 157, 163, 167\}$.

Mersenne primes correspond to prime values of k and I have proposed that k is prime for fundamental p-adic length scales quite generally. There are also however also other p-adic length scales - for instance, for quarks k need not be prime - and it has remained unclear what criterion could select the preferred exponents k . One can consider also the option that odd values of k defined fundamental p-adic length scales.

2. What makes p-adic length scale hypothesis powerful is that masses of say scaled up variant of hadron physics can be estimated by simple scaling arguments. It is convenient to use electron's p-adic length scale and calculate other p-adic length scales by scaling $L(k) = 2^{(k-127)/2} L(127)$.

Here one must make clear that there has been a confusion in the definitions, which was originally due to a calculational error.

1. I identified the p-adic length scale $L(151)$ mistakenly as $L(151) = 2^{(k-127)/2}L_e(127)$ by using instead of $L(127)$ electron Compton length $L_e \simeq L(127)/\sqrt{5}$. The notation for these scales would be therefore $L_e(k)$ identified as $L_e(k) = 2^{(k-127)/2}L_e(127)$ and I have tried to use it systematically but failed to use the wrong notation in informal discussions.
2. This mistake might reflect highly non-trivial physics. It is scaled up variants of L_e which seem to appear in physics. For instance, $L_e(151) \simeq 10$ nm corresponds to basic scale in living matter. Why the biological important scales should correspond to scaled up Compton lengths for electron? Could dark electrons with scaled up Compton scales equal to $L_e(k)$ be important in these scales? And what about the real p-adic length scales relate to these scales by a scaling factor $\sqrt{5} \simeq 2.23$?

2. Possible modifications of the p-adic length scale hypothesis

One can consider also possible modifications of the p-adic length scale hypothesis. In an attempt to understand the scales associated with INW structures in terms of p-adic length scale hypothesis it occurred to me that the scales which do not correspond to Mersenne primes or Gaussian Mersennes might be generated somehow from the these scales.

1. Geometric mean $L = \sqrt{L(k_1)L(k_2)}$ would length scale which would correspond to L_p with $p \simeq 2^{(k_1+k_2)/2}$. This is of the required form only if $k = k_1 + k_2$ is even so that k_1 and k_2 are both even or odd. If one starts from Mersennes and Gaussian Mersennes the condition is satisfied. The value of $k = (k_1 + k_2)/2$ can be also even.

Remark: The geometric mean $(127 + 107)/2 = 117$ of electronic and hadronic Mersennes corresponding to mass 16 MeV rather near to the mass of so called X boson [L3] (<https://tinyurl.com/ya3yuzeb>).

2. One can also consider the formula $L = (L(k_1)L(k_2)..L(k_n))^{1/n}$ but in this case the scale would correspond to prime $p \simeq 2^{(k_1+..k_n)/n}$. Since $(k_1 + ..k_n)/n$ is integer only if $k_1 + ..k_n$ is proportional to n .

What about the allowed values of fundamental integers k ? It seems that one must allow all odd integers.

1. If only prime values of k are allowed, one can obtain obtain for twin prime pair $(k-1, k+1)$ even integer k as geometric mean \sqrt{k} if k is square. If prime k is not a member of this kind of pair, it is not possible to get integers $k-1$ and $k+1$. If only prime values of k are fundamental, one could assign to $k = 89$ characterizing Higgs boson weak bosons $k = 90$ possibly characterizing weak bosons. Therefore it seems that one must allow all odd integers with the additional condition already explained.
2. Just for fun one can check whether $k = 161$ forced by the argument related to electroweak scale and h_{eff} corresponds to a geometric mean of two Gaussian Mersennes. One has $k(k_1, k_2) = (k_2 + k_2)/2$ giving the list $k(151, 157) = 154$, $k(151, 163) = 157$ Gaussian Mersenne itself, $k(151, 167) = 159$, $k(157, 163) = 160$, $k(157, 167) = 162$, $k(163, 167) = 165$. Unfortunately, $k = 161$ does not belong to this set. If one allows all odd values of k as fundamental, the problem disappears.

One can also consider refinements of p-adic length scale hypothesis in its basic form.

1. One can consider also a generalization of p-adic length scale hypothesis to allow length scales coming as powers of small primes. The small primes $p = 2, 3, 5$ assignable to Platonic solids would be especially interesting. $p = 2, 3, 5$ and also Fermat primes and Mersenne primes are maximally near to powers of two and their powers would define secondary and higher p-adic length scales. In this sense the extension would not actually bring anything new.

There is evidence for the occurrence of long p-adic time scales coming as powers of 3 [?, ?] (<http://tinyurl.com/ycesc5mq>) and [K21] (<https://tinyurl.com/y8camqlt>). Furthermore, prime 5 and Golden Mean are related closely to DNA helical structure. Portion of DNA with $L(151)$ contains 10 DNA codons and is the minimal length containing an integer number of codons.

2. The presence of length scales associated with 1 nm and 2 nm thick structures encourage to consider the possibility of p-adic primes near integers $2^k 3^l 5^m$ defining generators of multiplicative ideals of integers. They do not satisfy the maximal nearness criterion anymore but would be near to integers representable as products of powers of primes maximally near to powers of two.

What could be the interpretation of the integer k appearing in $p \simeq 2^k$? Elementary particle quantum numbers would be associated with wormhole contacts with size scale of CP_2 whereas elementary particles correspond to p-adic size scale about Compton length. What could determine the size scale of wormhole contact? I have proposed that to p-adic length scale there is associated a scale characterizing wormhole contact and depending logarithmically on it and corresponds to $L_k = (1/2)\log(p)L_0 = (k/2)\log(2)L_0$. The generalization of this hypothesis to the case of $p \simeq 2^k 3^l 5^m \dots$ be straightforward and be $L_{k,l,m} = (1/2)(k\log(2) + l\log(3) + m\log(5) + \dots)$.

2.3.3 Dark scales and scales of CDs and their relation to p-adic length scale hierarchy

There are two length scale hierarchies. p-Adic length scale hierarchy assignable to space-time surfaces and the dark hierarchy assignable to CDs. One should find an identification of dark scales and understand their relationship to p-adic length scales.

1. Identification of dark scales

The dimension n of the extension provides the roughest measure for its complexity via the formula $h_{eff}/h_0 = n$. The basic - rather ad hoc - assumption has been that n as dimension of extension defines not only h_{eff} but also the size scale of CD via $L = nL_0$.

This assumption need not be true generally and already the attempt to understand gravitational constant [L23] as a prediction of TGD led to the proposal that gravitational Planck constant $h_{gr} = n_{gr}h_0 = GMm/v_0$ [?] could be coded by the data relating to a normal subgroup of Galois group appearing as a factor of n .

The most general option is that dark scale is coded by a data related to extension of its sub-extension and this data involves ramified primes. Ramified primes depend on the polynomial defining the extension and there is large number polynomials defining the same extension. Therefore ramified ramifies code information also about polynomial and dynamics of space-time surface.

First some observations.

1. For Galois extension the order n has a natural decomposition to a product of orders n_i of its normal subgroups serving also as dimensions of corresponding extensions: $n = \prod_i n_i$. This implies a decomposition of the group algebra of Galois group to a tensor product of state spaces with dimensions n_i [L26].
2. Could one actually identify several dark scales as the proposed identifications of gravitational, electromagnetic, etc variants of h_{eff} suggest? The hierarchy of normal subgroups of Galois group of rationals corresponds to sub-groups with orders given by $N(i, 1) = n_i n_{i-1} \dots n_{i-1}$ of n define orders for the normal subgroups of Galois group. For extensions of $k - 1$:th extension of rationals one has $N(i, k) = n_i n_{i-1} \dots n_{i-k}$. The most general option is that these normal subgroups provide only the data allowing to associate dark scales to each of them. The spectrum of h_{eff} could correspond to the $\{N_{i,k}\}$ or at least the set $\{N_{i,1}\}$.
3. The extensions with prime dimension $n = p$ have no non-trivial normal subgroups and $n = p$ would hold for them. For these extensions the state space of group algebra is prime as Hilbert space and does not decompose to tensor product so that it would represent fundamental system. Could these extensions be of special interest physically? SSFRs would naturally involve state function reduction cascades proceeding downwards along hierarchy of normal subgroups and would represent cognitive measurements [L26].

The original guess was that dark scale $L_D = nL_p$, where n is the order n for the extensions and p is a ramified prime for the extension. A generalized form would allow $L_D = N(i, 1)L_{p_k}$ for the sub-extension such that p_k is ramified prime for the sub-extension.

2. *Can one identify the size scale of CD as dark scale?*

It would be natural if the scale of CD would be determined by the extension of rationals. Or more generally, the scales of CD and hierarchy of sub-CDs associated with the extension would be determined by the inclusion hierarchy of extensions and thus correspond to the hierarchy of normal sub-groups of Galois group.

The simplest option would be $L_{CD} = L_D$ so that the size scales of sub-CD would correspond dark scales for sub-extension given by $L_{CD,i} = N(i, 1)L_{p_k}$, p_k ramified prime of sub-extension.

1. The differences $|r_i - r_j|$ would correspond to differences for Minkowski time of CD. CD need not contain all values of hyperplanes $t = r_i$ and the evolution by SSFR would gradually bring in day-light all roots r_n of the polynomial P defining space-time surface as “very special moments in the life of self”. If the size scale of CD is so large that also the largest value of $|r_i|$ is inside the upper or lower half of CD, the size scale of CD would correspond roughly to the largest p-adic length scale.

CD contains sub-CDs and these could correspond to normal subgroups of Galois extension as extension of extension of

2. One can ask what happens when all special moments $t = r_n$ have been experienced? Does BSFR meaning death of conscious entity take place or is there some other option? In [L24] I considered a proposal for how chaos could emerge via iterations of P during the sequence of SSFRs.

One could argue that when CD has reached by SSFRs following unitary evolutions a size for which all roots r_n have become visible, the evolution could continue by the replacement of P with $P \circ P$, and so on. This would give rise to iteration and space-time analog for the approach to chaos.

3. Eventually the evolution by SSFRs must stop. Biological arguments suggests that metabolic limitations cause the death of self since the metabolic energy feed is not enough to preserve the distribution of values of h_{eff} (energies increase with $h_{eff} \propto Nn$, for N :th iteration and h_{eff} is reduced spontaneously) [L27].

3 Fermionic variant of $M^8 - H$ duality

The topics of this section is $M^8 - H$ duality for fermions. Consider first the bosonic counterpart of $M^8 - H$ duality.

1. The octonionic polynomial giving rise to space-time surface X^4 as its “root” is obtained from ordinary real polynomial P with rational coefficients by algebraic continuation. The conjecture is that the identification in terms of roots of polynomials of even real analytic functions guarantees associativity and one can formulate this as rather convincing argument [L4, L5, L6]. Space-time surface X_c^4 is identified as a 4-D root for a H_c -valued “imaginary” or “real” part of O_c valued polynomial obtained as an O_c continuation of a real polynomial P with rational coefficients, which can be chosen to be integers. These options correspond to complexified-quaternionic tangent- or normal spaces. For $P(x) = x^n + ..$ ordinary roots are algebraic integers. The real 4-D space-time surface is projection of this surface from M_c^8 to M^8 . One could drop the subscripts “ c ” but in the sequel they will be kept.

M_c^4 appears as a special solution for any polynomial P . M_c^4 seems to be like a universal reference solution with which to compare other solutions.

One obtains also brane-like 6-surfaces as 6-spheres as universal solutions. They have M^4 projection, which is a piece of hyper-surface for which Minkowski time as time coordinate of CD corresponds to a root $t = r_n$ of P . For monic polynomials these time values are algebraic integers and Galois group permutes them.

2. One cannot exclude rational functions or even real analytic functions in the sense that Taylor coefficients are octonionically real (proportional to octonionic real unit). Number theoretical vision - adelic physics [L8], suggests that polynomial coefficients are rational or perhaps in

extensions of rationals. The real coefficients could in principle be replaced with complex numbers $a + ib$, where i commutes with the octonionic units and defines complexification of octonions. i appears also in the roots defining complex extensions of rationals.

The generalization of the relationship between reals, extensions of p-adic number fields, and algebraic numbers in their intersection is suggestive. The “world of classical worlds” (WCW) would contain the space-time surfaces defined by polynomials with general real coefficients. Real WCW would be continuous space in real topology. The surfaces defined by rational or perhaps even algebraic coefficients for given extension would represent the intersection of real WCW with the p-adic variants of WCW labelled by the extension.

3. $M^8 - H$ duality requires additional condition realized as condition that also space-time surface itself contains 2-surfaces having commutative (complex) tangent or normal space. These surfaces can be 2-D also in metric sense that is light-like 3-D surfaces. The number of these surfaces is finite in generic case and they do not define a slicing of X^4 as was the first expectation. Strong form of holography (SH) makes it possible to map these surfaces and their tangent/normal spaces to 2-D surfaces $M^4 \times CP_2$ and to serve as boundary values for the partial differential equations for variational principle defined by twistor lift. Space-time surfaces in H would be minimal surface apart from singularities.

Concerning $M^8 - H$ duality for fermions, there are strong guidelines: also fermionic dynamics should be algebraic and number theoretical.

1. Spinors should be octonionic. I have already earlier considered their possible physical interpretation. [L1].
2. Dirac equation as linear partial differential equation should be replaced with a linear algebraic equation for octonionic spinors which are complexified octonions. The momentum space variant of the ordinary Dirac equation is an algebraic equation and the proposal is obvious: $P\Psi = 0$, where P is the octonionic continuation of the polynomial defining the space-time surface and multiplication is in octonionic sense. The conjugation in O_c is induced by the conjugation of the commuting imaginary unit i . The square of the Dirac operator is real if the space-time surface corresponds to the projection $O_c \rightarrow M^8 \rightarrow M^4$ with real time coordinate and imaginary spatial coordinates so that the metric defined by the octonionic norm is real and has Minkowskian signature. Hence the notion of Minkowski metric reduces to octonionic norm for O_c - a purely number theoretic notion.

The masslessness condition restricts the solutions to light-like 3-surfaces $m_{kl}P^kP^l = 0$ in Minkowskian sector analogous to mass shells in momentum space - just as in the case of ordinary massless Dirac equation. $P(o)$ rather than octonionic coordinate o would define momentum. These mass shells should be mapped to light-like partonic orbits in H .

3. This picture leads to the earlier phenomenological picture about induced spinors in H . Twistor Grassmann approach suggests the localization of the induced spinor fields at light-like partonic orbits in H . If the induced spinor field allows a continuation from 3-D partonic orbits to the interior of X^4 , it would serve as a counterpart of virtual particle in accordance with quantum field theoretical picture.

3.1 $M^8 - H$ duality for space-time surfaces

It is good to explain $M^8 - H$ duality for space-time surfaces before discussing it in fermionic sector.

3.1.1 Space-time as 4-surface in $M_c^8 = O_c$

One can regard real space-time surface $X^4 \subset M^8$ as a M^8 -projection of $X_c^4 \subset M_c^8 = O_c$. M_c^4 is identified as complexified quaternions H_c [L16, L22]. The dynamics is purely algebraic and therefore local an associativity is the basic dynamical principle.

1. The basic condition is associativity of $X^4 \subset M^8$ in the sense that either the tangent space or normal space is associative - that is quaternionic. This would be realized if X_c^4 as a root for

the quaternion-valued “real” or “imaginary part” for the O_c algebraic continuation of real analytic function $P(x)$ in octonionic sense. Number theoretical universality requires that the Taylor coefficients are rational numbers and that only polynomials are considered.

The 4-surfaces with associative normal space could correspond to elementary particle like entities with Euclidian signature (CP_2 type extremals) and those with associative tangent space to their interaction regions with Minkowskian signature. These two kinds space-time surfaces could meet along these 6-branes suggesting that interaction vertices are located at these branes.

2. The conditions allow also exceptional solutions for any polynomial for which both “real” and “imaginary” parts of the octonionic polynomial vanish. Brane-like solutions correspond to 6-spheres S^6 having $t = r_n$ 3-ball B^3 of light-cone as M^4 projection: here r_n is a root of the real polynomial with rational coefficients and can be also complex - one reason for complexification by commuting imaginary unit i . For scattering amplitudes the topological vertices as 2-surfaces would be located at the intersections of X_c^4 with 6-brane. Also Minkowski space M^4 is a universal solution appearing for any polynomial and would provide a universal reference space-time surface.
3. Polynomials with rational coefficients define EQs and these extensions form a hierarchy realized at the level of physics as evolutionary hierarchy. Given extension induces extensions of p-adic number fields and adeles and one obtains a hierarchy of adelic physics. The dimension n of extension allows interpretation in terms of effective Planck constant $h_{eff} = n \times h_0$. The phases of ordinary matter with effective Planck constant $h_{eff} = nh_0$ behave like dark matter and galactic dark matter could correspond to classical energy in TGD sense assignable to cosmic strings thickened to magnetic flux tubes. It is not completely clear whether number galactic dark matter must have $h_{eff} > h$. Dark energy in would correspond to the volume part of the energy of the flux tubes.

There are good arguments in favor of the identification $h = 6h_0$ [L10]. “Effective” means that the actual value of Planck constant is h_0 but in many-sheeted space-time n counts the number of symmetry related space-time sheets defining X^4 as a covering space locally. Each sheet gives identical contribution to action and this implies that effective value of Planck constant is nh_0 .

The ramified primes of extension in turn are identified as preferred p-adic primes. The moduli for the time differences $|t_r - t_s|$ have identification as p-adic time scales assignable to ramified primes [L22]. For ramified primes the p-adic variants of polynomials have degenerate zeros in $O(p) = 0$ approximation having interpretation in terms of quantum criticality central in TGD inspired biology.

4. During the preparation of this article I made a trivial but overall important observation. Standard Minkowski signature emerges as a prediction if conjugation in O_c corresponds to the conjugation with respect to commuting imaginary unit i rather than octonionic imaginary units as though earlier. If the space-time surface corresponds to the projection $O_c \rightarrow M^8 \rightarrow M^4$ with real time coordinate and imaginary spatial coordinates the metric defined by the octonionic norm is real and has Minkowskian signature. Hence the notion of Minkowski metric reduces to octonionic norm for O_c - a purely number theoretic notion.

3.1.2 Realization of $M^8 - H$ duality

$M^8 - H$ duality allows to $X^4 \subset M^8$ to $X^4 \subset H$ so that one has two equivalent descriptions for the space-time surfaces as algebraic surfaces in M^8 and as minimal surfaces with 2-D preferred 2-surfaces defining holography making possible $M^8 - H$ duality and possibly appearing as singularities in H . The dynamics of minimal surfaces, which are also extremals of Kähler action, reduces for known extremals to purely algebraic conditions analogous to holomorphy conditions in string models and thus involving only gradients of coordinates. This condition should hold generally and should induce the required huge reduction of degrees of freedom proposed to be realized also in terms of the vanishing of super-symplectic Noether charges already mentioned [K23].

Twistor lift allows several variants of this basic duality [L19]. M^8_H duality predicts that space-time surfaces form a hierarchy induced by the hierarchy of EQs defining an evolutionary hierarchy. This forms the basics for the number theoretical vision about TGD.

As already noticed, $X^4 \subset M^8$ would satisfy an infinite number of additional conditions stating vanishing of Noether charges for a sub-algebra $SSA_n \subset SSA$ of super-symplectic algebra SSA acting as isometries of WCW.

$M^8 - H$ duality makes sense under 2 additional assumptions to be considered in the following more explicitly than in earlier discussions [L16].

1. Associativity condition for tangent-/normal spaces is the first essential condition for the existence of $M^8 - H$ duality and means that tangent - or normal space is associative/quaternionic.
2. Each tangent space of X^4 at x must contain a preferred $M_c^2(x) \subset M_c^4$ such that $M_c^2(x)$ define an integrable distribution and therefore complexified string world sheet in M_c^4 . This gives similar distribution for their orthogonal complements $E_c^2(x)$. The string world sheet like entity defined by this distribution is 2-D surface $X_c^2 \subset X_c^4$ in R_c sense. $E_c^2(x)$ would correspond to partonic 2-surface. This condition generalizes for X^4 with quaternionic normal space. A possible interpretation is as a space-time correlate for the selection of quantization axes for energy (rest system) and spin.

One can imagine two realizations for the additional condition.

Option I: Global option states that the distributions $M_c^2(x)$ and $E_c^2(x)$ define a slicing of X_c^4 .

Option II: Only a discrete set of 2-surfaces satisfying the conditions exist, they are mapped to H , and strong form of holography (SH) applied in H allows to deduce $X^4 \subset H$. This would be the minimal option.

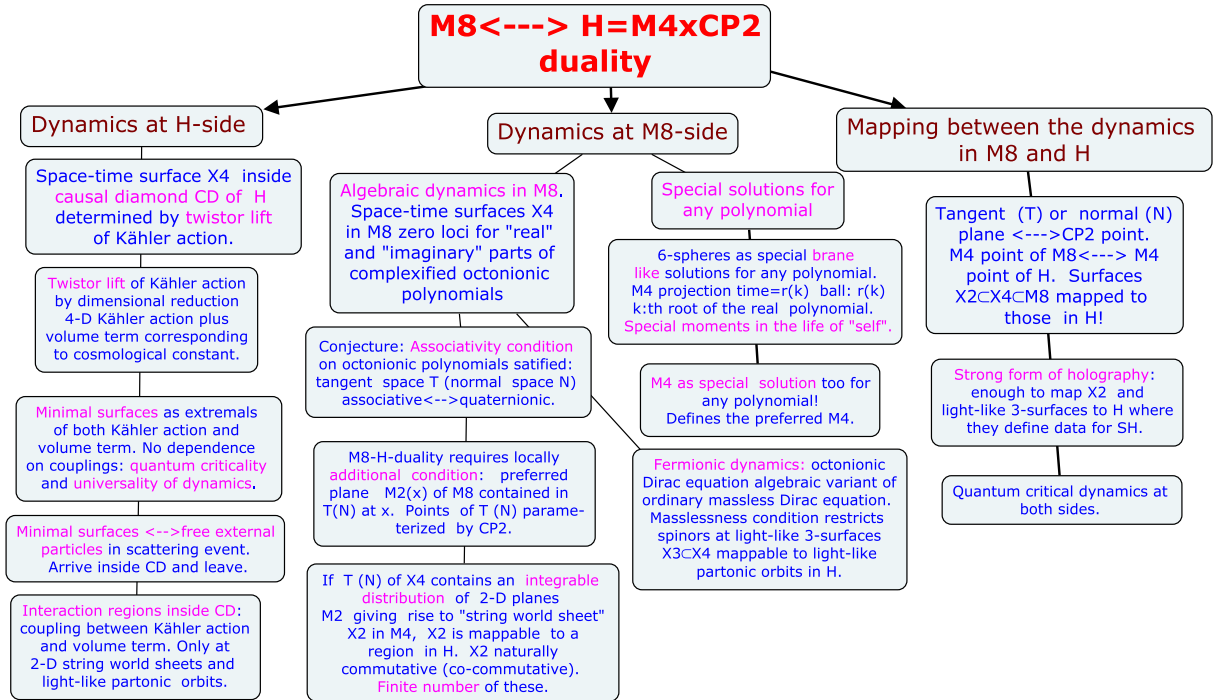
It seems that only **Option II** can be realized.

1. The basic observation is that X_c^2 can be fixed by posing to the non-vanishing H_c -valued part of octonionic polynomial P condition that the C_c -valued “real” or “imaginary” part in C_c sense for P vanishes. M_c^2 would be the simplest solution but also more general complex sub-manifolds $X_c^2 \subset M_c^4$ are possible. This condition allows only a discrete set of 2-surfaces as its solutions so that it works only for **Option II**.

These surfaces would be like the families of curves in complex plane defined by $u = 0$ and $v = 0$ curves of analytic function $f(z) = u + iv$. One should have family of polynomials differing by a constant term, which should be real so that $v = 0$ surfaces would form a discrete set.

2. SH makes possible $M^8 - H$ duality assuming that associativity conditions hold true only at 2-surfaces including partonic 2-surfaces or string world sheets or perhaps both. Thus one can give up the conjecture that the polynomial ansatz implies the additional condition globally. SH indeed states that PEs are determined by data at 2-D surfaces of X^4 . Even if the conditions defining X_c^2 have only a discrete set of solutions, SH at the level of H could allow to deduce the PEs from the data provided by the images of these 2-surfaces under $M^8 - H$ duality. The existence of $M^2(x)$ would be required only at the 2-D surfaces.
3. There is however a delicacy involved: X^2 might be 2-D only metrically but not topologically! The 3-D light-like surfaces X_L^3 indeed have metric dimension $D = 2$ since the induced 4-metric degenerates to 2-D metric at them. Therefore their pre-images in M^8 would be natural candidates for the singularities at which the dimension of the quaternionic tangent or normal space reduces to $D = 2$ [L15] [K3]. If this happens, SH would not be quite so strong as expected. The study of fermionic variant of $M^8 - H$ -duality supports this conclusion.

One can generalize the condition selecting X_c^2 so that it selects 1-D surface inside X_c^2 . By assuming that R_c -valued “real” or “imaginary” part of complex part of P sense at this 2-surface vanishes. One obtains preferred M_c^1 or E_c^1 containing octonionic real and preferred imaginary unit or distribution of the imaginary unit having interpretation as a complexified string. Together these kind 1-D surfaces in R_c sense would define local quantization axis of energy and spin. The outcome would be a realization of the hierarchy $R_c \rightarrow C_c \rightarrow H_c \rightarrow O_c$ realized as surfaces.

Figure 1: $M^8 - H$ duality.

3.2 What about $M^8 - H$ duality in the fermionic sector?

During the preparation of this article I become aware of the fact that the realization $M^8 - H$ duality in the fermionic sector has remained poorly understood. This led to a considerable integration of the ideas about $M^8 - H$ duality also in the bosonic sector and the existing phenomenological picture follows now from $M^8 - H$ duality. There are powerful mathematical guidelines available.

3.2.1 Octonionic spinors

By supersymmetry, octonionicity should have also fermionic counterpart.

1. The interpretation of M_c^8 as complexified octonions suggests that one should use complexified octonionic spinors in M_c^8 . This is also suggested by $SO(1,7)$ triality unique for dimension $d = 8$ and stating that the dimensions of vector representation, spinor representation and its conjugate are same and equal to $D = 8$. I have already earlier considered the possibility to interpret M^8 spinors as octonionic [L1]. Both octonionic gamma matrices and spinors have interpretation as octonions and gamma matrices satisfy the usual anti-commutation rules. The product for gamma matrices and gamma matrices and spinors is replaced with non-associative octonionic product.
2. Octonionic spinors allow only one M^8 -chirality, which conforms with the assumption of TGD inspired SUSY that only quarks are fundamental fermions and leptons are their local composites [L18].

3. The decomposition of $X^2 \subset X^4 \subset M^8$ corresponding to $R \subset C \subset Q \subset O$ should have analog for the O_c spinors as a tensor product decomposition. The special feature of dimension $D = 8$ is that the dimensions of spinor spaces associated with these factors are indeed 1, 2, 4, and 8 and correspond to dimensions for the surfaces!

One can define for octonionic spinors associative/co-associative sub-spaces as quaternionic/co-quaternionic spinors by posing chirality conditions. For $X^4 \subset M_c^8$ one could define the analogs of projection operators $P_{\pm} = (1 \pm \gamma_5)/2$ as projection operators to either factor of the spinor space as tensor product of spinor space associated with the tangent and normal spaces of X^4 : the analog of γ_5 would correspond to tangent or normal space depending on whether tangent or normal space is associative. For the spinors with definite chirality there would be no entanglement between the tensor factors. The condition would generalize the chirality condition for massless M^4 spinors to a condition holding for the local M^4 appearing as tangent/normal space of X^4 .

4. The chirality condition makes sense also for $X^2 \subset X^4$ identified as complex/co-complex surface of X^4 . Now γ_5 is replaced with γ_3 and states that the spinor has well-defined spin in the direction of axis defined by the decomposition of X^2 tangent space to $M^1 \times E^1$ with M^1 defining real octonion axis and selecting rest frame. Interpretation in terms of quantum measurement theory is suggestive.

What about tangent space quantum numbers in M^8 picture. In H -picture they correspond to spin and electroweak quantum numbers. In M^8 picture the geometric tangent space group for a rest system is product $SU(2) \times SU(2)$ with possible modifications due to octonionicity reducing tangent space group to those respecting octonionic automorphisms.

What about the sigma matrices for the octonionic gamma matrices? The surprise is that the commutators of M^4 sigma matrices and those of E^4 sigma matrices close to the same $SO(3)$ algebra allowing interpretation as representation for quaternionic automorphisms. Lorentz boosts are represented trivially, which conforms with the fact that octonion structure fixes unique rest system. Analogous result holds in E^4 degrees of freedom. Besides this one has unit matrix assignable to the generalize spinor structure of CP_2 so that also electroweak $U(1)$ factor is obtained.

One can understand this result by noticing that octonionic spinors correspond to 2 copies of a tensor products of the spinor doublets associated with spin and weak isospin. One has $2 \otimes 2 = 3 \oplus 1$ so that one must have $1 \oplus 3 \oplus 1 \oplus 3$. The octonionic spinors indeed decompose like $1 + 1 + 3 + \bar{3}$ under $SU(3)$ representing automorphisms of the octonions. $SO(3)$ could be interpreted as $SO(3) \subset SU(3)$. $SU(3)$ would be represented as tangent space rotations.

3.2.2 Dirac equation as partial differential equation must be replaced by an algebraic equation

Algebraization of dynamics should be also supersymmetric. The modified Dirac equation in H is linear partial differential equation and should correspond to a linear algebraic equation in M^8 .

1. The key observation is that for the ordinary Dirac equation the momentum space variant of Dirac equation for momentum eigenstates is algebraic! Could the interpretation for $M^8 - H$ duality as an analog of momentum-position duality of wave mechanics considered already earlier make sense! This could also have something to do with the dual descriptions of twistorial scattering amplitudes in terms of either twistor and momentum twistors. Already the earlier work excludes the interpretation of the octonionic coordinate o as 8-momentum. Rather, $P(o)$ has this interpretation and o corresponds to embedding space coordinate.
2. The first guess for the counterpart of the modified Dirac equation at the level of $X^4 \subset M^8$ is $P\Psi = 0$, where Ψ is octonionic spinor and the octonionic polynomial P defining the space-time surface can be seen as a generalization of momentum space Dirac operator with octonion units representing gamma matrices. If associativity/co-associativity holds true, the equation becomes quaternionic/co-quaternionic and reduces to the 4-D analog of massless Dirac equation and of modified Dirac equation in H . Associativity holds true if also Ψ satisfies associativity/co-associativity condition as proposed above.

3. What about the square of the Dirac operator? There are 3 conjugations involved: quaternionic conjugation assumed in the earlier work, conjugation with respect to i , and their combination. The analog of octonionic norm squared defined as the product $o_c o_c^*$ with conjugation with respect to i only, gives Minkowskian metric $m_{kl} o^k \bar{o}^l$ as its real part. The imaginary part of the norm squared is vanishing for the projection $O_c \rightarrow M^8 \rightarrow M^4$ so that time coordinate is real and spatial coordinates imaginary. Therefore Dirac equation allows solutions only for the M^4 projection X^4 and M^4 (M^8) signature of the metric can be said to be an outcome of quaternionicity (octonionicity) alone in accordance with the duality between metric and algebraic pictures.

Both $P^\dagger P$ and PP should annihilate Ψ . $P^\dagger P \Psi = 0$ gives $m_{kl} P^k \bar{P}^l = 0$ as the analog of vanishing mass squared in M^4 signature in both associative and co-associative cases. $PP \Psi = 0$ reduces to $P \Psi = 0$ by masslessness condition. One could perhaps interpret the projection $X_c^4 \rightarrow M^8 \rightarrow M^4$ in terms of Uncertainty Principle.

There is a $U(1)$ symmetry involved: instead of the plane M^8 one can choose any plane obtained by a rotation $exp(i\phi)$ from it. Could it realize quark number conservation in M^8 picture?

For $P = o$ having only $o = 0$ as root $Po = 0$ reduces to $o^\dagger o = 0$ and o takes the role of momentum, which is however vanishing. 6-D brane like solutions S^6 having $t = r_n$ balls $B^3 \subset CD_4$ as M^4 projections one has $P = 0$ so that the Dirac equation trivializes and does not pose conditions on Ψ . o would have interpretation as space-time coordinates and $P(o)$ as position dependent momentum components P^k .

The variation of P at mass shell of M_c^8 (to be precise) could be interpreted in terms of the width of the wave packet representing particle. Since the light-like curve at partonic 2-surface for fermion at X_L^3 is not a geodesic, mass squared in M^4 sense is not vanishing. Could one understand mass squared and the decay width of the particle geometrically? Note that mass squared is predicted also by p-adic thermodynamics [K16].

4. The masslessness condition restricts the spinors at 3-D light-cone boundary in $P(M^8)$. $M^8 - H$ duality [L16] suggests that this boundary is mapped to $X_L^3 \subset H$ defining the light-like orbit of the partonic 2-surface in H . The identification of the images of $P_k P^k = 0$ surfaces as X_L^3 gives a very powerful constraint on SH and $M^8 - H$ duality.
5. Also at 2-surfaces $X^2 \subset X^4$ an the variant Dirac equation would hold true and should commute with the corresponding chirality condition. Now $D^\dagger D \Psi = 0$ gives 2-D variant of masslessness condition with 2-momentum components represented by those of P . 2-D masslessness locates the spinor to a 1-D curve X_L^1 . Its H -image would naturally contain the boundary of the string world sheet at X_L^3 assumed to carry fermion quantum numbers and also the boundary of string world sheet at the light-like boundary of CD_4 . The interior of string world sheet in H would not carry induced spinor field.
6. The general solution for both 4-D and 2-D cases can be written as $\Psi = P \Psi_0$, Ψ_0 a constant spinor - this in a complete analogy with the solution of modified Dirac equation in H . P depends on position: the WKB approximation using plane waves with position dependent momentum seems to be nearer to reality than one might expect.

3.2.3 The phenomenological picture at H -level follows from the M^8 -picture

Remarkably, the partly phenomenological picture developed at the level of H is reproduced at the level of M^8 . Whether the induced spinor fields in the interior of X^4 are present or not, has been long standing question since they do not seem to have any role in the physical picture. The proposed picture answers this question.

Consider now the explicit realization of $M^8 - H$ -duality for fermions.

1. SH and the expected analogy with the bosonic variant of $M^8 - H$ duality lead to the first guess. The spinor modes in $X^4 \subset M^8$ restricted to X^2 can be mapped by $M^8 - H$ -duality to those at their images $X^2 \subset H$, and define boundary conditions allowing to deduce the

solution of the modified Dirac equation at $X^4 \subset H$. X^2 would correspond to string world sheets having boundaries X_L^1 at X_L^3 .

The guess is not quite correct. Algebraic Dirac equation requires that the solutions are restricted to the 3-D and 1-D mass shells $P_k P^k = 0$ in M^8 . This should remain true also in H and X_L^3 and their 1-D intersections X_L^1 with string world sheets remain. Fermions would live at boundaries. This is just the picture proposed for the TGD counterparts of the twistor amplitudes and corresponds to that used in twistor Grassmann approach!

For 2-D case constant octonionic spinors Ψ_0 and gamma matrix algebra are equivalent with the ordinary Weyl spinors and gamma matrix algebra and can be mapped as such to H . This gives one additional reason for why SH must be involved.

2. At the level of H the first guess is that the modified Dirac equation $D\Psi = 0$ is true for D based on the modified gamma matrices associated with both volume action and Kähler action. This would select preferred solutions of modified Dirac equation and conform with the vanishing of super-symplectic Noether charges for SSA_n for the spinor modes. The guess is not quite correct. The restriction of the induced spinors to X_L^3 requires that Chern-Simons action at X_L^3 defines the modified Dirac action.
3. The question has been whether the 2-D modified Dirac action emerges as a singular part of 4-D modified Dirac action assignable to singular 2-surface or can one assign an independent 2-D Dirac action assignable to 2-surfaces selected by some other criterion. For singular surfaces $M^8 - H$ duality fails since tangent space would reduce to 2-D space so that only their images can appear in SH at the level of H .

This supports the view that singular surfaces are actually 3-D mass shells M^8 mapped to X_L^3 for which 4-D tangent space is 2-D by the vanishing of $\sqrt{g_4}$ and light-likeness. String world sheets would correspond to non-singular $X^2 \subset M^8$ mapped to H and defining data for SH and their boundaries $X_L^1 \subset X_L^3$ and $X_L^1 \subset CD_4$ would define fermionic variant of SH.

What about the modified Dirac operator D in H ?

1. For X_L^3 modified Dirac equation $D\Psi = 0$ based on 4-D action S containing volume and Kähler term is problematic since the induced metric fails to have inverse at X_L^3 . The only possible action is Chern-Simons action S_{CS} used in topological quantum field theories and now defined as sum of C-S terms for Kähler actions in M^4 and CP_2 degrees of freedom. The presence of M^4 part of Kähler form of M^8 is forced by the twistor lift, and would give rise to small CP breaking effects explaining matter antimatter asymmetry [L18]. S_{C-S} could emerge as a limit of 4-D action.

The modified Dirac operator D_{C-S} uses modified gamma matrices identified as contractions $\Gamma_{CS}^\alpha = T^{\alpha k} \gamma_k$, where $T^{\alpha k} = \partial L_{CS} / \partial (\partial_\alpha h^k)$ are canonical momentum currents for S_{C-S} defined by a standard formula.

2. CP_2 part would give conserved Noether currents for color in and M^4 part Poincare quantum numbers: the apparently small CP breaking term would give masses for quarks and leptons! The bosonic Noether current $J_{B,A}$ for Killing vector j_A^k would be proportional to $J_{B,A}^\alpha = T_k^\alpha j_A^k$ and given by $J_{B,A} = \epsilon^{\alpha\beta\gamma} [J_{\beta\gamma} A_k + A_\beta J_{\gamma k}] j_A^k$.

Fermionic Noether current would be $J_{F,A} = \bar{\Psi} J^\alpha \Psi$ 3-D Riemann spaces allow coordinates in which the metric tensor is a direct sum of 1-D and 2-D contributions and are analogous to expectation values of bosonic Noether currents. One can also identify also finite number of Noether super currents by replacing $\bar{\Psi}$ or Ψ by its modes.

3. In the case of X_L^3 the 1-D part light-like part would vanish. If also induced Kähler form is non-vanishing only in 2-D degrees of freedom, the Noether charge densities J^t reduce to $J^t = J A_k j_A^k$, $J = \epsilon^{\alpha\beta\gamma} J_{\beta\gamma}$ defining magnetic flux. Modified Dirac operator would reduce to $D = J A_k \gamma^k D_t$ and 3-D solutions would be covariantly constant spinors along the light-like geodesics parameterized by the points 2-D cross section. One could say that the number of solutions is finite and corresponds to covariantly constant modes continued from X_L^1 to X_L^3 . This picture is just what twistor Grassmannian approach led to [L12].

3.2.4 A comment inspired by the ZEO based quantum measurement theory

I cannot resist the temptation to make a comment relating to quantum measurement theory inspired by zero energy ontology (ZEO) extending to a theory of consciousness [L17, L26, L27].

I have proposed [L22, L24] that the time evolution by “big” state function reductions (BSFRs) could be induced by iteration of real polynomial P - at least in some special cases. The roots of the real polynomial P would define a fractal at the limit of larger number of iterations. The roots of n -fold iterate $\circ^n P$ would contain the inverse images under $\circ^{-n+1} P$ of roots of P and for $P(0) = 0$ the inverse image $\circ^n P$ would consist of inverse images under $\circ^{-k} P$, $k = 0, \dots, n-1$, of roots of P .

Also the mass shells for $\circ^n P$ would be unions of inverse images under $\circ^{-k} P$, $k = 0, \dots, n-1$, of roots of P . This gives rather concrete view about evolution of M^4 projections of the partonic orbits. A rough approximate expression for the largest root of real P approximated as $P(x) \simeq a_n x^n + a_{n-1} x^{n-1}$ for large x is $x_{max} \sim a_n/a_{n-1}$. For $\circ^n P$ one obtains the same estimate. This suggests that the size scales of the partonic orbits are same for the iterates. The mass shells would not differ dramatically: could they have an interpretation in terms of mass splitting?

The evolution by iteration would add new partonic orbits and preserve the existing ones: this brings in mind conservation of genes in biological evolution. This is true also for a more general evolution allowing general functional decomposition $Q \rightarrow Q \circ P$ to occur in BSFR.

3.2.5 What next in TGD?

The construction of scattering amplitudes has been the dream impossible that has driven me for decades. Maybe the understanding of fermionic $M^8 - H$ duality provides the needed additional conceptual tools. The key observation is utterly trivial but far reaching: there are 3 possible conjugations for octonions corresponding to the conjugation of commutative imaginary unit or of octonionic imaginary units or both of them. 1st norm gives a real valued norm squared in Minkowski signature natural at M^8 level! Second one gives a complex valued norm squared in Euclidian signature. 1st and 2nd norms are equivalent for octonions light-like with respect to the first norm. The 3rd conjugation gives a real-valued Euclidian norm natural at the level of Hilbert space.

1. M^8 picture looks simple. Space-time surfaces in M^8 can be constructed from real polynomials with real (rational) coefficients, actually knowledge of their roots is enough. Discrete data - roots of the polynomial! - determine space-time surface as associative or co-associative region! Besides this one must pose additional condition selecting 2-D string world sheets and 3-D light-like surfaces as orbits of partonic 2-surfaces. These would define strong form of holography (SH) allowing to map space-time surfaces in M^8 to $M^4 \times CP_2$.
2. Could SH generalize to the level of scattering amplitudes expressible in terms of n-point functions of CFT?! Could the n points correspond to the roots of the polynomial defining space-time region!

Algebraic continuation to quaternion valued scattering amplitudes analogous to that giving space-time sheets from the data coded SH should be the key idea. Their moduli squared are real - this led to the emergence of Minkowski metric for complexified octonions/quaternions) would give the real scattering rates: this is enough! This would mean a number theoretic generalization of quantum theory.

3. One can start from complex numbers and string world sheets/partonic 2-surfaces. Conformal field theories (CFTs) in 2-D play fundamental role in the construction of scattering string theories and in modelling 2-D statistical systems. In TGD 2-D surfaces (2-D at least metrically) code for information about space-time surface by strong holography (SH) .

Are CFTs at partonic 2-surfaces and string world sheets the basic building bricks? Could 2-D conformal invariance dictate the data needed to construct the scattering amplitudes for given space-time region defined by causal diamond (CD) taking the role of sphere S^2 in CFTs. Could the generalization for metrically 2-D light-like 3-surfaces be needed at the level of “world of classical worlds” (WCW) when states are superpositions of space-time surfaces, preferred extremals?

The challenge is to develop a concrete number theoretic hierarchy for scattering amplitudes: $R \rightarrow C \rightarrow H \rightarrow O$ - actually their complexifications.

1. In the case of fermions one can start from 1-D data at light-like boundaries LB of string world sheets at light-like orbits of partonic 2-surfaces. Fermionic propagators assignable to LB would be coded by 2-D Minkowskian QFT in manner analogous to that in twistor Grassmann approach. n-point vertices would be expressible in terms of Euclidian n-point functions for partonic 2-surfaces: the latter element would be new as compared to QFTs since point-like vertex is replaced with partonic 2-surface.
2. The fusion (product?) of these Minkowskian and Euclidian CFT entities corresponding to different realization of complex numbers as sub-field of quaternions would give rise to 4-D quaternionic valued scattering amplitudes for given space-time sheet. Most importantly: there moduli squared are real for both norms.

It is not quite clear whether one must use the 1st Minkowskian norm requiring “time-like” scattering amplitudes to achieve non-negative probabilities or use the 3rd norm to get the ordinary positive-definite Hilbert space norm. A generalization of quantum theory (CFT) from complex numbers to quaternions (quaternionic ”CFT”) would be in question.

3. What about several space-time sheets? Could one allow fusion of different quaternionic scattering amplitudes corresponding to different quaternionic sub-spaces of complexified octonions to get octonion-valued non-associative scattering amplitudes. Again scattering rates would be real. This would be a further generalization of quantum theory.

There is also the challenge to relate M^8 - and H -pictures at the level of WCW. The formulation of physics in terms of WCW geometry [K23, ?] leads to the hypothesis that WCW Kähler geometry is determined by Kähler function identified as the 4-D action resulting by dimensional reduction of 6-D surfaces in the product of twistor spaces of M^4 and CP_2 to twistor bundles having S^2 as fiber and space-time surface $X^4 \subset H$ as base. The 6-D Kähler action reduces to the sum of 4-D Kähler action and volume term having interpretation in terms of cosmological constant.

The question is whether the Kähler function - an essentially geometric notion - can have a counterpart at the level of M^8 .

1. SH suggests that the Kähler function identified in the proposed manner can be expressed by using 2-D data or at least metrically 2-D data (light-like partonic orbits and light-like boundaries of CD). Note that each WCW would correspond to a particular CD.
2. Since 2-D conformal symmetry is involved, one expects also modular invariance meaning that WCW Kähler function is modular invariant, so that they have the same value for $X^4 \subset H$ for which partonic 2-surfaces have induced metric in the same conformal equivalence class.
3. Also the analogs of Kac-Moody type symmetries would be realized as symmetries of Kähler function. The algebra of super-symplectic symmetries of the light-cone boundary can be regarded as an analog of Kac-Moody algebra. Light-cone boundary has topology $S^2 \times R_+$ where R_+ corresponds to radial light-like ray parameterized by radial light-like coordinate r . Super symplectic transformations of $S^2 \times CP_2$ depend on the light-like radial coordinate r , which is analogous to the complex coordinate z for the Kac-Moody algebras.

The infinitesimal super-symplectic transformations form algebra SSA with generators proportional to powers r^n . The Kac-Moody invariance for physical states generalizes to a hierarchy of similar invariances. There is infinite fractal hierarchy of sub-algebras $SSA_n \subset SSA$ with conformal weights coming as n -multiples of those for SSA. For physical states SSA_n and $[SSA_n, SSA]$ would act as gauge symmetries. They would leave invariant also Kähler function in the sector WCW_n defined by n . This would define a hierarchy of sub- WCWs of the WCW assignable to given CD.

The sector WCW_n could correspond to extensions of rationals with dimension n , and one would have inclusion hierarchies consisting of sequences of n_i with n_i dividing n_{i+1} . These inclusion hierarchies would naturally correspond to those for hyper-finite factors of type II_1 [K28].

4 A vision about the role of HFFs in TGD

It is clear that at least the hyper-finite factors of type II_1 assignable to WCW spinors must have a profound role in TGD. Whether also HFFs of type III_1 appearing also in relativistic quantum field theories emerge when WCW spinors are replaced with spinor fields is not completely clear. I have proposed several ideas about the role of hyper-finite factors in TGD framework. In particular, Connes tensor product is an excellent candidate for defining the notion of measurement resolution.

In the following this topic is discussed from the perspective made possible by zero energy ontology and the recent advances in the understanding of M-matrix using the notion of bosonic emergence. The conclusion is that the notion of state as it appears in the theory of factors is not enough for the purposes of quantum TGD. The reason is that state in this sense is essentially the counterpart of thermodynamical state. The construction of M-matrix might be understood in the framework of factors if one replaces state with its “complex square root” natural if quantum theory is regarded as a “complex square root” of thermodynamics. It is also found that the idea that Connes tensor product could fix M-matrix is too optimistic but an elegant formulation in terms of partial trace for the notion of M-matrix modulo measurement resolution exists and Connes tensor product allows interpretation as entanglement between sub-spaces consisting of states not distinguishable in the measurement resolution used. The partial trace also gives rise to non-pure states naturally.

The newest element in the vision is the proposal that quantum criticality of TGD Universe is realized as hierarchies of inclusions of super-conformal algebras with conformal weights coming as multiples of integer n , where n varies. If n_1 divides n_2 then various super-conformal algebras C_{n_2} are contained in C_{n_1} . This would define naturally the inclusion.

4.1 Basic facts about factors

In this section basic facts about factors are discussed. My hope that the discussion is more mature than or at least complementary to the summary that I could afford when I started the work with factors for more than half decade ago. I of course admit that this just a humble attempt of a physicist to express physical vision in terms of only superficially understood mathematical notions.

4.1.1 Basic notions

First some standard notations. Let $\mathcal{B}(\mathcal{H})$ denote the algebra of linear operators of Hilbert space \mathcal{H} bounded in the norm topology with norm defined by the supremum for the length of the image of a point of unit sphere \mathcal{H} . This algebra has a lot of common with complex numbers in that the counterparts of complex conjugation, order structure and metric structure determined by the algebraic structure exist. This means the existence involution -that is *- algebra property. The order structure determined by algebraic structure means following: $A \geq 0$ defined as the condition $(A\xi, \xi) \geq 0$ is equivalent with $A = B^*B$. The algebra has also metric structure $\|AB\| \leq \|A\|\|B\|$ (Banach algebra property) determined by the algebraic structure. The algebra is also C^* algebra: $\|A^*A\| = \|A\|^2$ meaning that the norm is algebraically like that for complex numbers.

A von Neumann algebra \mathcal{M} [A3] is defined as a weakly closed non-degenerate *-subalgebra of $\mathcal{B}(\mathcal{H})$ and has therefore all the above mentioned properties. From the point of view of physicist it is important that a sub-algebra is in question.

In order to define factors one must introduce additional structure.

- (a) Let \mathcal{M} be subalgebra of $\mathcal{B}(\mathcal{H})$ and denote by \mathcal{M}' its commutant (\mathcal{H}) commuting with it and allowing to express $\mathcal{B}(\mathcal{H})$ as $\mathcal{B}(\mathcal{H}) = \mathcal{M} \vee \mathcal{M}'$.
- (b) A factor is defined as a von Neumann algebra satisfying $\mathcal{M}'' = \mathcal{M}$ \mathcal{M} is called factor. The equality of double commutant with the original algebra is thus the defining condition

so that also the commutant is a factor. An equivalent definition for factor is as the condition that the intersection of the algebra and its commutant reduces to a complex line spanned by a unit operator. The condition that the only operator commuting with all operators of the factor is unit operator corresponds to irreducibility in representation theory.

- (c) Some further basic definitions are needed. $\Omega \in \mathcal{H}$ is cyclic if the closure of $\mathcal{M}\Omega$ is \mathcal{H} and separating if the only element of \mathcal{M} annihilating Ω is zero. Ω is cyclic for \mathcal{M} if and only if it is separating for its commutant. In so called standard representation Ω is both cyclic and separating.
- (d) For hyperfinite factors an inclusion hierarchy of finite-dimensional algebras whose union is dense in the factor exists. This roughly means that one can approximate the algebra in arbitrary accuracy with a finite-dimensional sub-algebra.

The definition of the factor might look somewhat artificial unless one is aware of the underlying physical motivations. The motivating question is what the decomposition of a physical system to non-interacting sub-systems could mean. The decomposition of $\mathcal{B}(\mathcal{H})$ to \vee product realizes this decomposition.

- (a) Tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is the decomposition according to the standard quantum measurement theory and means the decomposition of operators in $\mathcal{B}(\mathcal{H})$ to tensor products of mutually commuting operators in $\mathcal{M} = \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{M}' = \mathcal{B}(\mathcal{H}_2)$. The information about \mathcal{M} can be coded in terms of projection operators. In this case projection operators projecting to a complex ray of Hilbert space exist and arbitrary compact operator can be expressed as a sum of these projectors. For factors of type I minimal projectors exist. Factors of type I_n correspond to sub-algebras of $\mathcal{B}(\mathcal{H})$ associated with infinite-dimensional Hilbert space and I_∞ to $\mathcal{B}(\mathcal{H})$ itself. These factors appear in the standard quantum measurement theory where state function reduction can lead to a ray of Hilbert space.
- (b) For factors of type II no minimal projectors exist whereas finite projectors exist. For factors of type II_1 all projectors have trace not larger than one and the trace varies in the range $(0, 1]$. In this case cyclic vectors Ω exist. State function reduction can lead only to an infinite-dimensional subspace characterized by a projector with trace smaller than 1 but larger than zero. The natural interpretation would be in terms of finite measurement resolution. The tensor product of II_1 factor and I_∞ is II_∞ factor for which the trace for a projector can have arbitrarily large values. II_1 factor has a unique finite tracial state and the set of traces of projections spans unit interval. There is uncountable number of factors of type II but hyper-finite factors of type II_1 are the exceptional ones and physically most interesting.
- (c) Factors of type III correspond to an extreme situation. In this case the projection operators E spanning the factor have either infinite or vanishing trace and there exists an isometry mapping $E\mathcal{H}$ to \mathcal{H} meaning that the projection operator spans almost all of \mathcal{H} . All projectors are also related to each other by isometry. Factors of type III are smallest if the factors are regarded as sub-algebras of a fixed $\mathcal{B}(\mathcal{H})$ where \mathcal{H} corresponds to isomorphism class of Hilbert spaces. Situation changes when one speaks about concrete representations. Also now hyper-finite factors are exceptional.
- (d) Von Neumann algebras define a non-commutative measure theory. Commutative von Neumann algebras indeed reduce to $L^\infty(X)$ for some measure space (X, μ) and vice versa.

4.1.2 Weights, states and traces

The notions of weight, state, and trace are standard notions in the theory of von Neumann algebras.

- (a) A weight of von Neumann algebra is a linear map from the set of positive elements (those of form a^*a) to non-negative reals.

- (b) A positive linear functional is weight with $\omega(1)$ finite.
- (c) A state is a weight with $\omega(1) = 1$.
- (d) A trace is a weight with $\omega(aa^*) = \omega(a^*a)$ for all a .
- (e) A tracial state is a weight with $\omega(1) = 1$.

A factor has a trace such that the trace of a non-zero projector is non-zero and the trace of projection is infinite only if the projection is infinite. The trace is unique up to a rescaling. For factors that are separable or finite, two projections are equivalent if and only if they have the same trace. Factors of type I_n the values of trace are equal to multiples of $1/n$. For a factor of type I_∞ the value of trace are $0, 1, 2, \dots$. For factors of type II_1 the values span the range $[0, 1]$ and for factors of type II_∞ in the range $[0, \infty)$. For factors of type III the values of the trace are 0 , and ∞ .

4.1.3 Tomita-Takesaki theory

Tomita-Takesaki theory is a vital part of the theory of factors. First some definitions.

- (a) Let $\omega(x)$ be a faithful state of von Neumann algebra so that one has $\omega(xx^*) > 0$ for $x > 0$. Assume by Riesz lemma the representation of ω as a vacuum expectation value: $\omega = (\cdot\Omega, \Omega)$, where Ω is cyclic and separating state.
- (b) Let

$$L^\infty(\mathcal{M}) \equiv \mathcal{M} \ , \quad L^2(\mathcal{M}) = \mathcal{H} \ , \quad L^1(\mathcal{M}) = \mathcal{M}_* \ , \quad (4.1)$$

where \mathcal{M}_* is the pre-dual of \mathcal{M} defined by linear functionals in \mathcal{M} . One has $\mathcal{M}_*^* = \mathcal{M}$.

- (c) The conjugation $x \rightarrow x^*$ is isometric in \mathcal{M} and defines a map $\mathcal{M} \rightarrow L^2(\mathcal{M})$ via $x \rightarrow x\Omega$. The map $S_0; x\Omega \rightarrow x^*\Omega$ is however non-isometric.
- (d) Denote by S the closure of the anti-linear operator S_0 and by $S = J\Delta^{1/2}$ its polar decomposition analogous that for complex number and generalizing polar decomposition of linear operators by replacing (almost) unitary operator with anti-unitary J . Therefore $\Delta = S^*S > 0$ is positive self-adjoint and J an anti-unitary involution. The non-triviality of Δ reflects the fact that the state is not trace so that hermitian conjugation represented by S in the state space brings in additional factor $\Delta^{1/2}$.
- (e) What x can be is puzzling to physicists. The restriction fermionic Fock space and thus to creation operators would imply that Δ would act non-trivially only vacuum state so that $\Delta > 0$ condition would not hold true. The resolution of puzzle is the allowance of tensor product of Fock spaces for which vacua are conjugates: only this gives cyclic and separating state. This is natural in ZEO.

The basic results of Tomita-Takesaki theory are following.

- (a) The basic result can be summarized through the following formulas

$$\Delta^{it} M \Delta^{-it} = \mathcal{M} \ , \ J \mathcal{M} J = \mathcal{M}' \ .$$

- (b) The latter formula implies that \mathcal{M} and \mathcal{M}' are isomorphic algebras. The first formula implies that a one parameter group of modular automorphisms characterizes partially the factor. The physical meaning of modular automorphisms is discussed in [A9, A16] Δ is Hermitian and positive definite so that the eigenvalues of $\log(\Delta)$ are real but can be negative. Δ^{it} is however not unitary for factors of type II and III. Physically the non-unitarity must relate to the fact that the flow is contracting so that hermiticity as a local condition is not enough to guarantee unitarity.

- (c) $\omega \rightarrow \sigma_t^\omega = Ad\Delta^{it}$ defines a canonical evolution -modular automorphism- associated with ω and depending on it. The Δ :s associated with different ω :s are related by a unitary inner automorphism so that their equivalence classes define an invariant of the factor.

Tomita-Takesaki theory gives rise to a non-commutative measure theory which is highly non-trivial. In particular the spectrum of Δ can be used to classify the factors of type II and III.

4.1.4 Modular automorphisms

Modular automorphisms of factors are central for their classification.

- (a) One can divide the automorphisms to inner and outer ones. Inner automorphisms correspond to unitary operators obtained by exponentiating Hermitian Hamiltonian belonging to the factor and connected to identity by a flow. Outer automorphisms do not allow a representation as a unitary transformations although $\log(\Delta)$ is formally a Hermitian operator.
- (b) The fundamental group of the type II_1 factor defined as fundamental group of corresponding II_∞ factor characterizes partially a factor of type II_1 . This group consists real numbers λ such that there is an automorphism scaling the trace by λ . Fundamental group typically contains all reals but it can be also discrete and even trivial.
- (c) Factors of type III allow a one-parameter group of modular automorphisms, which can be used to achieve a partial classification of these factors. These automorphisms define a flow in the center of the factor known as flow of weights. The set of parameter values λ for which ω is mapped to itself and the center of the factor defined by the identity operator (projector to the factor as a sub-algebra of $\mathcal{B}(\mathcal{H})$) is mapped to itself in the modular automorphism defines the Connes spectrum of the factor. For factors of type III_λ this set consists of powers of $\lambda < 1$. For factors of type III_0 this set contains only identity automorphism so that there is no periodicity. For factors of type III_1 Connes spectrum contains all real numbers so that the automorphisms do not affect the identity operator of the factor at all.

The modules over a factor correspond to separable Hilbert spaces that the factor acts on. These modules can be characterized by M-dimension. The idea is roughly that complex rays are replaced by the sub-spaces defined by the action of \mathcal{M} as basic units. M-dimension is not integer valued in general. The so called standard module has a cyclic separating vector and each factor has a standard representation possessing antilinear involution J such that $\mathcal{M}' = J\mathcal{M}J$ holds true (note that J changes the order of the operators in conjugation). The inclusions of factors define modules having interpretation in terms of a finite measurement resolution defined by \mathcal{M} .

4.1.5 Crossed product as a way to construct factors of type III

By using so called crossed product crossedproduct for a group G acting in algebra A one can obtain new von Neumann algebras. One ends up with crossed product by a two-step generalization by starting from the semidirect product $G \triangleleft H$ for groups defined as $(g_1, h_1)(g_2, h_2) = (g_1 h_1(g_2), h_1 h_2)$ (note that Poincare group has interpretation as a semidirect product $M^4 \triangleleft SO(3, 1)$ of Lorentz and translation groups). At the first step one replaces the group H with its group algebra. At the second step the the group algebra is replaced with a more general algebra. What is formed is the semidirect product $A \triangleleft G$ which is sum of algebras Ag . The product is given by $(a_1, g_1)(a_2, g_2) = (a_1 g_1(a_2), g_1 g_2)$. This construction works for both locally compact groups and quantum groups. A not too highly educated guess is that the construction in the case of quantum groups gives the factor \mathcal{M} as a crossed product of the included factor \mathcal{N} and quantum group defined by the factor space \mathcal{M}/\mathcal{N} .

The construction allows to express factors of type III as crossed products of factors of type II_∞ and the 1-parameter group G of modular automorphisms assignable to any vector which

is cyclic for both factor and its commutant. The ergodic flow θ_λ scales the trace of projector in II_∞ factor by $\lambda > 0$. The dual flow defined by G restricted to the center of II_∞ factor does not depend on the choice of cyclic vector.

The Connes spectrum - a closed subgroup of positive reals - is obtained as the exponent of the kernel of the dual flow defined as set of values of flow parameter λ for which the flow in the center is trivial. Kernel equals to $\{0\}$ for III_0 , contains numbers of form $\log(\lambda)Z$ for factors of type III_λ and contains all real numbers for factors of type III_1 meaning that the flow does not affect the center.

4.1.6 Inclusions and Connes tensor product

Inclusions $\mathcal{N} \subset \mathcal{M}$ of von Neumann algebras have physical interpretation as a mathematical description for sub-system-system relation. In [K28] there is more extensive TGD colored description of inclusions and their role in TGD. Here only basic facts are listed and the Connes tensor product is explained.

For type I algebras the inclusions are trivial and tensor product description applies as such. For factors of II_1 and III the inclusions are highly non-trivial. The inclusion of type II_1 factors were understood by Vaughan Jones [A1] and those of factors of type III by Alain Connes [A4] .

Formally sub-factor \mathcal{N} of \mathcal{M} is defined as a closed $*$ -stable C -subalgebra of \mathcal{M} . Let \mathcal{N} be a sub-factor of type II_1 factor \mathcal{M} . Jones index $\mathcal{M} : \mathcal{N}$ for the inclusion $\mathcal{N} \subset \mathcal{M}$ can be defined as $\mathcal{M} : \mathcal{N} = \dim_{\mathcal{N}}(L^2(\mathcal{M})) = \text{Tr}_{\mathcal{N}'}(\text{id}_{L^2(\mathcal{M})})$. One can say that the dimension of completion of \mathcal{M} as \mathcal{N} module is in question.

4.1.7 Basic findings about inclusions

What makes the inclusions non-trivial is that the position of \mathcal{N} in \mathcal{M} matters. This position is characterized in case of hyper-finite II_1 factors by index $\mathcal{M} : \mathcal{N}$ which can be said to the dimension of \mathcal{M} as \mathcal{N} module and also as the inverse of the dimension defined by the trace of the projector from \mathcal{M} to \mathcal{N} . It is important to notice that $\mathcal{M} : \mathcal{N}$ does not characterize either \mathcal{M} or \mathcal{N} , only the embedding.

The basic facts proved by Jones are following [A1] .

- (a) For pairs $\mathcal{N} \subset \mathcal{M}$ with a finite principal graph the values of $\mathcal{M} : \mathcal{N}$ are given by

$$\begin{aligned} a) \quad \mathcal{M} : \mathcal{N} &= 4\cos^2(\pi/h) \quad , \quad h \geq 3 \quad , \\ b) \quad \mathcal{M} : \mathcal{N} &\geq 4 \quad . \end{aligned} \tag{4.2}$$

the numbers at right hand side are known as Beraha numbers [A12] . The comments below give a rough idea about what finiteness of principal graph means.

- (b) As explained in [B4] , for $\mathcal{M} : \mathcal{N} < 4$ one can assign to the inclusion Dynkin graph of ADE type Lie-algebra g with h equal to the Coxeter number h of the Lie algebra given in terms of its dimension and dimension r of Cartan algebra r as $h = (\dim g(g) - r)/r$. For $\mathcal{M} : \mathcal{N} < 4$ ordinary Dynkin graphs of D_{2n} and E_6, E_8 are allowed. The Dynkin graphs of Lie algebras of $SU(n)$, E_7 and D_{2n+1} are however not allowed. $E_6, E_7, \text{ and } E_8$ correspond to symmetry groups of tetrahedron, octahedron/cube, and icosahedron/dodecahedron. The group for octahedron/cube is missing: what could this mean?

For $\mathcal{M} : \mathcal{N} = 4$ one can assign to the inclusion an extended Dynkin graph of type ADE characterizing Kac Moody algebra. Extended ADE diagrams characterize also the subgroups of $SU(2)$ and the interpretation proposed in [A21] is following-

The ADE diagrams are associated with the $n = \infty$ case having $\mathcal{M} : \mathcal{N} \geq 4$. There are diagrams corresponding to infinite subgroups: A_∞ corresponding to $SU(2)$ itself,

$A_{-\infty, \infty}$ corresponding to circle group $U(1)$, and infinite dihedral groups (generated by a rotation by a non-rational angle and reflection).

One can construct also inclusions for which the diagrams corresponding to finite subgroups $G \subset SU(2)$ are extension of A_n for cyclic groups, of D_n dihedral groups, and of E_n with $n = 6, 7, 8$ for tetrahedron, cube, dodecahedron. These extensions correspond to ADE type Kac-Moody algebras.

The extension is constructed by constructing first factor R as infinite tensor power of $M_2(C)$ (complexified quaternions). Sub-factor R_0 consists elements of R of form $Id \otimes x$. $SU(2)$ preserves R_0 and for any subgroup G of $SU(2)$ one can identify the inclusion $N \subset M$ in terms of $N = R_0^G$ and $M = R^G$, where $N = R_0^G$ and $M = R^G$ consists of fixed points of R_0 and R under the action of G . The principal graph for $N \subset M$ is the extended Coxeter-Dynk graph for the subgroup G .

Physicist might try to interpret this by saying that one considers only sub-algebras R_0^G and R^G of observables invariant under G and obtains extended Dynkin diagram of G defining an ADE type Kac-Moody algebra. Could the condition that Kac-Moody algebra elements with non-vanishing conformal weight annihilate the physical states state that the state is invariant under R_0 defining measurement resolution. Besides this the states are also invariant under finite group G ? Could R_0^G and R^G correspond just to states which are also invariant under finite group G .

4.1.8 Connes tensor product

The basic idea of Connes tensor product is that a sub-space generated sub-factor \mathcal{N} takes the role of the complex ray of Hilbert space. The physical interpretation is in terms of finite measurement resolution: it is not possible to distinguish between states obtained by applying elements of \mathcal{N} .

Intuitively it is clear that it should be possible to decompose \mathcal{M} to a tensor product of factor space \mathcal{M}/\mathcal{N} and \mathcal{N} :

$$\mathcal{M} = \mathcal{M}/\mathcal{N} \otimes \mathcal{N} . \quad (4.3)$$

One could regard the factor space \mathcal{M}/\mathcal{N} as a non-commutative space in which each point corresponds to a particular representative in the equivalence class of points defined by \mathcal{N} . The connections between quantum groups and Jones inclusions suggest that this space closely relates to quantum groups. An alternative interpretation is as an ordinary linear space obtained by mapping \mathcal{N} rays to ordinary complex rays. These spaces appear in the representations of quantum groups. Similar procedure makes sense also for the Hilbert spaces in which \mathcal{M} acts.

Connes tensor product can be defined in the space $\mathcal{M} \otimes \mathcal{M}$ as entanglement which effectively reduces to entanglement between \mathcal{N} sub-spaces. This is achieved if \mathcal{N} multiplication from right is equivalent with \mathcal{N} multiplication from left so that \mathcal{N} acts like complex numbers on states. One can imagine variants of the Connes tensor product and in TGD framework one particular variant appears naturally as will be found.

In the finite-dimensional case Connes tensor product of Hilbert spaces has a rather simple representation. If the matrix algebra N of $n \times n$ matrices acts on V from right, V can be regarded as a space formed by $m \times n$ matrices for some value of m . If N acts from left on W , W can be regarded as space of $n \times r$ matrices.

- (a) In the first representation the Connes tensor product of spaces V and W consists of $m \times r$ matrices and Connes tensor product is represented as the product VW of matrices as $(VW)_{mr} e^{mr}$. In this representation the information about N disappears completely as the interpretation in terms of measurement resolution suggests. The sum over intermediate states defined by N brings in mind path integral.

- (b) An alternative and more physical representation is as a state

$$\sum_n V_{mn} W_{nr} e^{mn} \otimes e^{nr}$$

in the tensor product $V \otimes W$.

- (c) One can also consider two spaces V and W in which N acts from right and define Connes tensor product for $A^\dagger \otimes_N B$ or its tensor product counterpart. This case corresponds to the modification of the Connes tensor product of positive and negative energy states. Since Hermitian conjugation is involved, matrix product does not define the Connes tensor product now. For $m = r$ case entanglement coefficients should define a unitary matrix commuting with the action of the Hermitian matrices of N and interpretation would be in terms of symmetry. HFF property would encourage to think that this representation has an analog in the case of HFFs of type II_1 .
- (d) Also type I_n factors are possible and for them Connes tensor product makes sense if one can assign the inclusion of finite-D matrix algebras to a measurement resolution.

4.1.9 Factors in quantum field theory and thermodynamics

Factors arise in thermodynamics and in quantum field theories [A19, A9, A16]. There are good arguments showing that in HFFs of III_1 appear are relativistic quantum field theories. In non-relativistic QFTs the factors of type I appear so that the non-compactness of Lorentz group is essential. Factors of type III_1 and III_λ appear also in relativistic thermodynamics.

The geometric picture about factors is based on open subsets of Minkowski space. The basic intuitive view is that for two subsets of M^4 , which cannot be connected by a classical signal moving with at most light velocity, the von Neumann algebras commute with each other so that \vee product should make sense.

Some basic mathematical results of algebraic quantum field theory [A16] deserve to be listed since they are suggestive also from the point of view of TGD.

- (a) Let \mathcal{O} be a bounded region of R^4 and define the region of M^4 as a union $\cup_{|x| < \epsilon} (\mathcal{O} + x)$ where $(\mathcal{O} + x)$ is the translate of \mathcal{O} and $|x|$ denotes Minkowski norm. Then every projection $E \in \mathcal{M}(\mathcal{O})$ can be written as WW^* with $W \in \mathcal{M}(\mathcal{O}_\epsilon)$ and $W^*W = 1$. Note that the union is not a bounded set of M^4 . This almost establishes the type III property.
- (b) Both the complement of light-cone and double light-cone define HFF of type III_1 . Lorentz boosts induce modular automorphisms.
- (c) The so called split property suggested by the description of two systems of this kind as a tensor product in relativistic QFTs is believed to hold true. This means that the HFFs of type III_1 associated with causally disjoint regions are sub-factors of factor of type I_∞ . This means

$$\mathcal{M}_1 \subset \mathcal{B}(\mathcal{H}_1) \times 1 \quad , \quad \mathcal{M}_2 \subset 1 \otimes \mathcal{B}(\mathcal{H}_2) \quad .$$

An infinite hierarchy of inclusions of HFFs of type III_1 s is induced by set theoretic inclusions.

4.1.10 Factors in quantum field theory and thermodynamics

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- (b) Both the complement of light-cone and double light-cone define HFF of type III₁. Lorentz boosts induce modular automorphisms.
- (c) The so called split property suggested by the description of two systems of this kind as a tensor product in relativistic QFTs is believed to hold true. This means that the HFFs of type III₁ associated with causally disjoint regions are sub-factors of factor of type I_∞ . This means

$$\mathcal{M}_1 \subset \mathcal{B}(\mathcal{H}_1) \times 1 \quad , \quad \mathcal{M}_2 \subset 1 \otimes \mathcal{B}(\mathcal{H}_2) \quad .$$

An infinite hierarchy of inclusions of HFFs of type III₁s is induced by set theoretic inclusions.

4.2 TGD and factors

The following vision about TGD and factors relies heavily on zero energy ontology, TGD inspired quantum measurement theory, basic vision about quantum TGD, and bosonic emergence.

4.2.1 The problems

Concerning the role of factors in TGD framework there are several problems of both conceptual and technical character.

1. Conceptual problems

It is safest to start from the conceptual problems and take a role of skeptic.

- (a) Under what conditions the assumptions of Tomita-Takesaki formula stating the existence of modular automorphism and isomorphy of the factor and its commutant hold true? What is the physical interpretation of the formula $\mathcal{M}' = J\mathcal{M}J$ relating factor and its commutant in TGD framework?
- (b) Is the identification $M = \Delta^{it}$ sensible in quantum TGD and ZEO, where M-matrix is “complex square root” of exponent of Hamiltonian defining thermodynamical state and the notion of unitary time evolution is given up? The notion of state ω leading to Δ is essentially thermodynamical and one can wonder whether one should take also a “complex square root” of ω to get M-matrix giving rise to a genuine quantum theory.
- (c) TGD based quantum measurement theory involves both quantum fluctuating degrees of freedom assignable to light-like 3-surfaces and zero modes identifiable as classical degrees of freedom assignable to interior of the space-time sheet. Zero modes have also fermionic counterparts. State preparation should generate entanglement between the quantal and classical states. What this means at the level of von Neumann algebras?
- (d) What is the TGD counterpart for causal disjointness. At space-time level different space-time sheets could correspond to such regions whereas at embedding space level causally disjoint CDs would represent such regions.

2. *Technical problems*

There are also more technical questions.

- (a) What is the von Neumann algebra needed in TGD framework? Does one have a direct integral over factors? Which factors appear in it? Can one construct the factor as a crossed product of some group G with direct physical interpretation and of naturally appearing factor A ? Is A a HFF of type II_∞ ? assignable to a fixed CD? What is the natural Hilbert space \mathcal{H} in which A acts?
- (b) What are the geometric transformations inducing modular automorphisms of II_∞ inducing the scaling down of the trace? Is the action of G induced by the boosts in Lorentz group. Could also translations and scalings induce the action? What is the factor associated with the union of Poincare transforms of CD? $\log(\Delta)$ is Hermitian algebraically: what does the non-unitarity of $\exp(\log(\Delta)it)$ mean physically?
- (c) Could Ω correspond to a vacuum which in conformal degrees of freedom depends on the choice of the sphere S^2 defining the radial coordinate playing the role of complex variable in the case of the radial conformal algebra. Does *-operation in \mathcal{M} correspond to Hermitian conjugation for fermionic oscillator operators and change of sign of super conformal weights?

The exponent of the Kähler-Dirac action gives rise to the exponent of Kähler function as Dirac determinant and fermionic inner product defined by fermionic Feynman rules. It is implausible that this exponent could as such correspond to ω or Δ^{it} having conceptual roots in thermodynamics rather than QFT. If one assumes that the exponent of the Kähler-Dirac action defines a “complex square root” of ω the situation changes. This raises technical questions relating to the notion of square root of ω .

- (a) Does the complex square root of ω have a polar decomposition to a product of positive definite matrix (square root of the density matrix) and unitary matrix and does $\omega^{1/2}$ correspond to the modulus in the decomposition? Does the square root of Δ have similar decomposition with modulus equal equal to $\Delta^{1/2}$ in standard picture so that modular automorphism, which is inherent property of von Neumann algebra, would not be affected?
- (b) Δ^{it} or rather its generalization is defined modulo a unitary operator defined by some Hamiltonian and is therefore highly non-unique as such. This non-uniqueness applies also to $|\Delta|$. Could this non-uniqueness correspond to the thermodynamical degrees of freedom?

4.2.2 ZEO and factors

The first question concerns the identification of the Hilbert space associated with the factors in ZEO. As the positive or negative energy part of the zero energy state space or as the entire space of zero energy states? The latter option would look more natural physically and is forced by the condition that the vacuum state is cyclic and separating.

- (a) The commutant of HFF given as $\mathcal{M}' = J\mathcal{M}J$, where J is involution transforming fermionic oscillator operators and bosonic vector fields to their Hermitian conjugates. Also conformal weights would change sign in the map which conforms with the view that the light-like boundaries of CD are analogous to upper and lower hemispheres of S^2 in conformal field theory. The presence of J representing essentially Hermitian conjugation would suggest that positive and zero energy parts of zero energy states are related by this formula so that state space decomposes to a tensor product of positive and negative energy states and M -matrix can be regarded as a map between these two sub-spaces.
- (b) The fact that HFF of type II_1 has the algebra of fermionic oscillator operators as a canonical representation makes the situation puzzling for a novice. The assumption

that the vacuum is cyclic and separating means that neither creation nor annihilation operators can annihilate it. Therefore Fermionic Fock space cannot appear as the Hilbert space in the Tomita-Takesaki theorem. The paradox is circumvented if the action of $*$ transforms creation operators acting on the positive energy part of the state to annihilation operators acting on negative energy part of the state. If J permutes the two Fock vacuums in their tensor product, the action of S indeed maps permutes the tensor factors associated with \mathcal{M} and \mathcal{M}' .

It is far from obvious whether the identification $M = \Delta^{it}$ makes sense in ZEO.

- (a) In ZEO M -matrix defines time-like entanglement coefficients between positive and negative energy parts of the state. M -matrix is essentially “complex square root” of the density matrix and quantum theory similar square root of thermodynamics. The notion of state as it appears in the theory of HFFs is however essentially thermodynamical. Therefore it is good to ask whether the “complex square root of state” could make sense in the theory of factors.
- (b) Quantum field theory suggests an obvious proposal concerning the meaning of the square root: one replaces exponent of Hamiltonian with imaginary exponential of action at $T \rightarrow 0$ limit. In quantum TGD the exponent of Kähler-Dirac action giving exponent of Kähler function as real exponent could be the manner to take this complex square root. Kähler-Dirac action can therefore be regarded as a “square root” of Kähler action.
- (c) The identification $M = \Delta^{it}$ relies on the idea of unitary time evolution which is given up in ZEO based on CDs? Is the reduction of the quantum dynamics to a flow a realistic idea? As will be found this automorphism could correspond to a time translation or scaling for either upper or lower light-cone defining CD and can ask whether Δ^{it} corresponds to the exponent of scaling operator L_0 defining single particle propagator as one integrates over t . Its complex square root would correspond to fermionic propagator.
- (d) In this framework $J\Delta^{it}$ would map the positive energy and negative energy sectors to each other. If the positive and negative energy state spaces can be identified by isometry then $M = J\Delta^{it}$ identification can be considered but seems unrealistic. $S = J\Delta^{1/2}$ maps positive and negative energy states to each other: could S or its generalization appear in M -matrix as a part which gives thermodynamics? The exponent of the Kähler-Dirac action does not seem to provide thermodynamical aspect and p-adic thermodynamics suggests strongly the presence exponent of $\exp(-L_0/T_p)$ with T_p chosen in such manner that consistency with p-adic thermodynamics is obtained. Could the generalization of $J\Delta^{n/2}$ with Δ replaced with its “square root” give rise to p-adic thermodynamics and also ordinary thermodynamics at the level of density matrix? The minimal option would be that power of Δ^{it} which imaginary value of t is responsible for thermodynamical degrees of freedom whereas everything else is dictated by the unitary S -matrix appearing as phase of the “square root” of ω .

4.2.3 Zero modes and factors

The presence of zero modes justifies quantum measurement theory in TGD framework and the relationship between zero modes and HFFs involves further conceptual problems.

- (a) The presence of zero modes means that one has a direct integral over HFFs labeled by zero modes which by definition do not contribute to WCW line element. The realization of quantum criticality in terms of Kähler-Dirac action [K29] suggests that also fermionic zero mode degrees of freedom are present and correspond to conserved charges assignable to the critical deformations of the space-time sheets. Induced Kähler form characterizes the values of zero modes for a given space-time sheet and the symplectic group of light-cone boundary characterizes the quantum fluctuating degrees of freedom. The entanglement between zero modes and quantum fluctuating degrees of freedom is essential for quantum measurement theory. One should understand this entanglement.

- (b) Physical intuition suggests that classical observables should correspond to longer length scale than quantal ones. Hence it would seem that the interior degrees of freedom outside CD should correspond to classical degrees of freedom correlating with quantum fluctuating degrees of freedom of CD.
- (c) Quantum criticality means that Kähler-Dirac action allows an infinite number of conserved charges which correspond to deformations leaving metric invariant and therefore act on zero modes. Does this super-conformal algebra commute with the super-conformal algebra associated with quantum fluctuating degrees of freedom? Could the restriction of elements of quantum fluctuating currents to 3-D light-like 3-surfaces actually imply this commutativity. Quantum holography would suggest a duality between these algebras. Quantum measurement theory suggests even 1-1 correspondence between the elements of the two super-conformal algebras. The entanglement between classical and quantum degrees of freedom would mean that prepared quantum states are created by operators for which the operators in the two algebras are entangled in diagonal manner.
- (d) The notion of finite measurement resolution has become key element of quantum TGD and one should understand how finite measurement resolution is realized in terms of inclusions of hyper-finite factors for which sub-factor defines the resolution in the sense that its action creates states not distinguishable from each other in the resolution used. The notion of finite measurement resolution suggests that one should speak about entanglement between sub-factors and corresponding sub-spaces rather than between states. Connes tensor product would code for the idea that the action of sub-factors is analogous to that of complex numbers and tracing over sub-factor realizes this idea.
- (e) Just for fun one can ask whether the duality between zero modes and quantum fluctuating degrees of freedom representing quantum holography could correspond to $\mathcal{M}' = J\mathcal{M}J$? This interpretation must be consistent with the interpretation forced by zero energy ontology. If this crazy guess is correct (very probably not!), both positive and negative energy states would be observed in quantum measurement but in totally different manner. Since this identity would simplify enormously the structure of the theory, it deserves therefore to be shown wrong.

4.2.4 Crossed product construction in TGD framework

The identification of the von Neumann algebra by crossed product construction is the basic challenge. Consider first the question how HFFs of type II_∞ emerge, how modular automorphisms act on them, and how one can understand the non-unitary character of the Δ^{it} in an apparent conflict with the hermiticity and positivity of Δ .

- (a) The Clifford algebra at a given point of WCW(CD) (light-like 3-surfaces with ends at the boundaries of CD) defines HFF of type II_1 or possibly a direct integral of them. For a given CD having compact isotropy group $SO(3)$ leaving the rest frame defined by the tips of CD invariant the factor defined by Clifford algebra valued fields in WCW(CD) is most naturally HFF of type II_∞ . The Hilbert space in which this Clifford algebra acts, consists of spinor fields in WCW(CD). Also the symplectic transformations of light-cone boundary leaving light-like 3-surfaces inside CD can be included to G . In fact all conformal algebras leaving CD invariant could be included in CD.
- (b) The downwards scalings of the radial coordinate r_M of the light-cone boundary applied to the basis of WCW (CD) spinor fields could induce modular automorphism. These scalings reduce the size of the portion of light-cone in which the WCW spinor fields are non-vanishing and effectively scale down the size of CD. $\exp(iL_0)$ as algebraic operator acts as a phase multiplication on eigen states of conformal weight and therefore as apparently unitary operator. The geometric flow however contracts the CD so that the interpretation of $\exp(itL_0)$ as a unitary modular automorphism is not possible. The scaling down of CD reduces the value of the trace if it involves integral over the boundary of CD. A similar reduction is implied by the downward shift of the upper boundary of

CD so that also time translations would induce modular automorphism. These shifts seem to be necessary to define rest energies of positive and negative energy parts of the zero energy state.

- (c) The non-triviality of the modular automorphisms of II_∞ factor reflects different choices of ω . The degeneracy of ω could be due to the non-uniqueness of conformal vacuum which is part of the definition of ω . The radial Virasoro algebra of light-cone boundary is generated by $L_n = L_{-n}^*$, $n \neq 0$ and $L_0 = L_0^*$ and negative and positive frequencies are in asymmetric position. The conformal gauge is fixed by the choice of $SO(3)$ subgroup of Lorentz group defining the slicing of light-cone boundary by spheres and the tips of CD fix $SO(3)$ uniquely. One can however consider also alternative choices of $SO(3)$ and each corresponds to a slicing of the light-cone boundary by spheres but in general the sphere defining the intersection of the two light-cone does not belong to the slicing. Hence the action of Lorentz transformation inducing different choice of $SO(3)$ can lead out from the preferred state space so that its representation must be non-unitary unless Virasoro generators annihilate the physical states. The non-vanishing of the conformal central charge c and vacuum weight h seems to be necessary and indeed can take place for super-symplectic algebra and Super Kac-Moody algebra since only the differences of the algebra elements are assumed to annihilate physical states.

Modular automorphism of HFFs type III_1 can be induced by several geometric transformations for HFFs of type III_1 obtained using the crossed product construction from II_∞ factor by extending CD to a union of its Lorentz transforms.

- (a) The crossed product would correspond to an extension of II_∞ by allowing a union of some geometric transforms of CD. If one assumes that only CDs for which the distance between tips is quantized in powers of 2, then scalings of either upper or lower boundary of CD cannot correspond to these transformations. Same applies to time translations acting on either boundary but not to ordinary translations. As found, the modular automorphisms reducing the size of CD could act in HFF of type II_∞ .
- (b) The geometric counterparts of the modular transformations would most naturally correspond to any non-compact one parameter sub-group of Lorentz group as also QFT suggests. The Lorentz boosts would replace the radial coordinate r_M of the light-cone boundary associated with the radial Virasoro algebra with a new one so that the slicing of light-cone boundary with spheres would be affected and one could speak of a new conformal gauge. The temporal distance between tips of CD in the rest frame would not be affected. The effect would seem to be however unitary because the transformation does not only modify the states but also transforms CD.
- (c) Since Lorentz boosts affect the isotropy group $SO(3)$ of CD and thus also the conformal gauge defining the radial coordinate of the light-cone boundary, they affect also the definition of the conformal vacuum so that also ω is affected so that the interpretation as a modular automorphism makes sense. The simplistic intuition of the novice suggests that if one allows wave functions in the space of Lorentz transforms of CD, unitarity of Δ^{it} is possible. Note that the hierarchy of Planck constants assigns to CD preferred M^2 and thus direction of quantization axes of angular momentum and boosts in this direction would be in preferred role.
- (d) One can also consider the HFF of type III_λ if the radial scalings by negative powers of 2 correspond to the automorphism group of II_∞ factor as the vision about allowed CDs suggests. $\lambda = 1/2$ would naturally hold true for the factor obtained by allowing only the radial scalings. Lorentz boosts would expand the factor to HFF of type III_1 . Why scalings by powers of 2 would give rise to periodicity should be understood.

The identification of M -matrix as modular automorphism Δ^{it} , where t is complex number having as its real part the temporal distance between tips of CD quantized as 2^n and temperature as imaginary part, looks at first highly attractive, since it would mean that M -matrix indeed exists mathematically. The proposed interpretations of modular automorphisms do not support the idea that they could define the S-matrix of the theory. In any case, the

identification as modular automorphism would not lead to a magic universal formula since arbitrary unitary transformation is involved.

4.2.5 Quantum criticality and inclusions of factors

Quantum criticality fixes the value of Kähler coupling strength but is expected to have also an interpretation in terms of a hierarchies of broken conformal gauge symmetries suggesting hierarchies of inclusions.

- (a) In ZEO 3-surfaces are unions of space-like 3-surfaces at the ends of causal diamond (CD). Space-time surfaces connect 3-surfaces at the boundaries of CD. The non-determinism of Kähler action allows the possibility of having several space-time sheets connecting the ends of space-time surface but the conditions that classical charges are same for them reduces this number so that it could be finite. Quantum criticality in this sense implies non-determinism analogous to that of critical systems since preferred extremals can co-incide and suffer this kind of bifurcation in the interior of CD. This quantum criticality can be assigned to the hierarchy of Planck constants and the integer n in $h_{eff} = n \times h$ [K11] corresponds to the number of degenerate space-time sheets with same Kähler action and conserved classical charges.
- (b) Also now one expects a hierarchy of criticalities and since criticality and conformal invariance are closely related, a natural conjecture is that the fractal hierarchy of sub-algebras of conformal algebra isomorphic to conformal algebra itself and having conformal weights coming as multiples of n corresponds to the hierarchy of Planck constants. This hierarchy would define a hierarchy of symmetry breakings in the sense that only the sub-algebra would act as gauge symmetries.
- (c) The assignment of this hierarchy with super-symplectic algebra having conformal structure with respect to the light-like radial coordinate of light-cone boundary looks very attractive. An interesting question is what is the role of the super-conformal algebra associated with the isometries of light-cone boundary $R_+ \times S^2$ which are conformal transformations of sphere S^2 with a scaling of radial coordinate compensating the scaling induced by the conformal transformation. Does it act as dynamical or gauge symmetries?
- (d) The natural proposal is that the inclusions of various superconformal algebras in the hierarchy define inclusions of hyper-finite factors which would be thus labelled by integers. Any sequences of integers for which n_i divides n_{i+1} would define a hierarchy of inclusions proceeding in reverse direction. Physically inclusion hierarchy would correspond to an infinite hierarchy of criticalities within criticalities.

4.3 Can one identify M -matrix from physical arguments?

Consider next the identification of M -matrix from physical arguments from the point of view of factors.

4.3.1 A proposal for M -matrix

The proposed general picture reduces the core of U -matrix to the construction of S -matrix possibly having the real square roots of density matrices as symmetry algebra. This structure can be taken as a template as one tries to imagine how the construction of M -matrix could proceed in quantum TGD proper.

- (a) At the bosonic sector one would have converging functional integral over WCW. This is analogous to the path integral over bosonic fields in QFTs. The presence of Kähler function would make this integral well-defined and would not encounter the difficulties met in the case of path integrals.

- (b) In fermionic sector 1-D Dirac action and its bosonic counterpart imply that spinors modes localized at string world sheets are eigenstates of induced Dirac operator with generalized eigenvalue $p^k \gamma_k$ defining light-like 8-D momentum so that one would obtain fermionic propagators massless in 8-D sense at light-light geodesics of embedding space. The 8-D generalization of twistor Grassmann approach is suggestive and would mean that the residue integral over fermionic virtual momenta gives only integral over massless momenta and virtual fermions differ from real fermions only in that they have non-physical polarizations so that massless Dirac operator replacing the propagator does not annihilate the spinors at the other end of the line.
- (c) Fundamental bosons (not elementary particles) correspond to wormhole contacts having fermion and antifermion at opposite throats and bosonic propagators are composite of massless fermion propagators. The directions of virtual momenta are obviously strongly correlated so that the approximation as a gauge theory with gauge symmetry breaking in almost massless sector is natural. Massivation follows necessary from the fact that also elementary particles are bound states of two wormhole contacts.
- (d) Physical fermions and bosons correspond to pairs of wormhole contacts with throats carrying Kähler magnetic charge equal to Kähler electric charge (dyon). The absence of Dirac monopoles (as opposed to homological magnetic monopoles due to CP_2 topology) implies that wormhole contacts must appear as pairs (also large numbers of them are possible and 3 valence quarks inside baryons could form Kähler magnetic tripole). Hence elementary particles would correspond to pairs of monopoles and are accompanied by Kähler magnetic flux loop running along the two space-time sheets involved as well as fermionic strings connecting the monopole throats.

There seems to be no specific need to assign string to the wormhole contact and if is a piece of deformed CP_2 type vacuum extremal this might not be even possible: the Kähler-Dirac gamma matrices would not span 2-D space in this case since the CP_2 projection is 4-D. Hence massless fermion propagators would be assigned only with the boundaries of string world sheets at Minkowskian regions of space-time surface. One could say that physical particles are bound states of massless fundamental fermions and the non-collinearity of their four-momenta can make them massive. Therefore the breaking of conformal invariance would be due to the bound state formation and this would also resolve the infrared divergence problems plaguing Grassmann twistor approach by introducing natural length scale assignable to the size of particles defined by the string like flux tube connecting the wormhole contacts. This point is discussed in more detail in [K26].

The bound states would form representations of super-conformal algebras so that stringy mass formula would emerge naturally. p-Adic mass calculations indeed assume conformal invariance in CP_2 length scale assignable to wormhole contacts. Also the long flux tube strings contribute to the particle masses and would explain gauge boson masses.

- (e) The interaction vertices would correspond topologically to decays of 3-surface by splitting in complete analogy with ordinary Feynman diagrams. At the level of orbits of partonic 2-surface the vertices would be represented by partonic 2-surfaces. In [K26] the interpretation of scattering amplitudes as sequences of algebraic operations for the Yangian of super-symplectic algebra is proposed: product and co-product would define time 3-vertex and its time reversal. At the level of fermions the diagrams reduce to braid diagrams since fermions are “free”. At vertices fermions can however reflect in time direction so that fermion-antifermion annihilations in classical fields can be said to appear in the vertices.

The Yangian is generated by super-symplectic fermionic Noether charges assignable to the strings connecting partonic 2-surfaces. The interpretation of vertices as algebraic operations implies that all sequences of operations connecting given collections of elements of Yangian at the opposite boundaries of CD give rise to the same amplitude. This means a huge generalization of the duality symmetry of hadronic string models that I have proposed already earlier: the chapter [K4] is a remnant of an “idea that came too early”. The propagators are associated with the fermionic lines identifiable as boundaries of string world sheets. These lines are light-like geodesics of H and fermion

lines correspond to partial wave in the space S^3 of light like 8-momenta with fixed M^4 momentum. For external lines M^8 momentum corresponds to the $M^4 \times CP_2$ quantum numbers of a spinor harmonic.

The amplitudes can be formulated using only partonic 2-surfaces and string world sheets and the algebraic continuation to achieve number theoretic Universality should be rather straightforward: the parameters characterizing 2-surfaces - by conformal invariance various conformal moduli - in the algebraic extension of rationals are replaced with real and various p-adic numbers.

- (f) Wormhole contacts represent fundamental interaction vertex pairs and propagators between them and one has stringy super-conformal invariance. Therefore there are excellent reasons to expect that the perturbation theory is free of divergences. Without stringy contributions for massive conformal excitations of wormhole contacts one would obtain the usual logarithmic UV divergences of massless gauge theories. The fact that physical particles are bound states of massless particles, gives good hopes of avoiding IR divergences of massless theories.

The figures ??, ?? (<http://tgdtheory.fi/appfigures/elparticletgd.jpg> <http://tgdtheory.fi/appfigures/tgdgrpahs.jpg>) in the appendix of this book illustrate the relationship between TGD diagrammatics, QFT diagrammatics and stringy diagrammatics. In [K26] a more detailed construction based on the generalization of twistor approach and the idea that scattering amplitudes represent sequences of algebraic operation in the Yangian of super-symplectic algebra, is considered.

4.3.2 Quantum TGD as square root of thermodynamics

ZEO (ZEO) suggests strongly that quantum TGD corresponds to what might be called square root of thermodynamics. Since fermionic sector of TGD corresponds naturally to a hyper-finite factor of type II_1 , and super-conformal sector relates fermionic and bosonic sectors (WCW degrees of freedom), there is a temptation to suggest that the mathematics of von Neumann algebras generalizes: in other worlds it is possible to speak about the complex square root of ω defining a state of von Neumann algebra [A19] [K28]. This square root would bring in also the fermionic sector and realized super-conformal symmetry. The reduction of determinant with WCW vacuum functional would be one manifestation of this supersymmetry.

The exponent of Kähler function identified as real part of Kähler action for preferred extremals coming from Euclidian space-time regions defines the modulus of the bosonic vacuum functional appearing in the functional integral over WCW. The imaginary part of Kähler action coming from the Minkowskian regions is analogous to action of quantum field theories and would give rise to interference effects distinguishing thermodynamics from quantum theory. This would be something new from the point of view of the canonical theory of von Neumann algebra. The saddle points of the imaginary part appear in stationary phase approximation and the imaginary part serves the role of Morse function for WCW.

The exponent of Kähler function depends on the real part of t identified as Minkowski distance between the tips of CD. This dependence is not consistent with the dependence of the canonical unitary automorphism Δ^{it} of von Neumann algebra on t [A19], [K28] and the natural interpretation is that the vacuum functional can be included in the definition of the inner product for spinors fields of WCW. More formally, the exponent of Kähler function would define ω in bosonic degrees of freedom.

Note that the imaginary exponent is more natural for the imaginary part of Kähler action coming from Minkowskian region. In any case, one has combination of thermodynamics and QFT and the presence of thermodynamics makes the functional integral mathematically well-defined.

Number theoretic vision requiring number theoretical universality suggests that the value of CD size scales as defined by the distance between the tips is expected to come as integer multiples of CP_2 length scale - at least in the intersection of real and p-adic worlds. If

this is the case the continuous family of modular automorphisms would be replaced with a discrete family.

4.3.3 Quantum criticality and hierarchy of inclusions

Quantum criticality and related fractal hierarchies of breakings of conformal symmetry could allow to understand the inclusion hierarchies for hyper-finite factors. Quantum criticality - implied by the condition that the Kähler-Dirac action gives rise to conserved currents assignable to the deformations of the space-time surface - means the vanishing of the second variation of Kähler action for these deformations. Preferred extremals correspond to these 4-surfaces and $M^8 - M^4 \times CP_2$ duality would allow to identify them also as associative (co-associative) space-time surfaces.

Quantum criticality is basically due to the failure of strict determinism for Kähler action and leads to the hierarchy of dark matter phases labelled by the effective value of Planck constant $h_{eff} = n \times h$. These phases correspond to space-time surfaces connecting 3-surfaces at the ends of CD which are multi-sheeted having n conformal equivalence classes.

Conformal invariance indeed relates naturally to quantum criticality. This brings in n discrete degrees of freedom and one can technically describe the situation by using n -fold singular covering of the embedding space [K11]. One can say that there is hierarchy of broken conformal symmetries in the sense that for $h_{eff} = n \times h$ the sub-algebra of conformal algebras with conformal weights coming as multiples of n act as gauge symmetries. This implies that classical symplectic Noether charges vanish for this sub-algebra. The quantal conformal charges associated with induced spinor fields annihilate the physical states. Therefore it seems that the measured quantities are the symplectic charges and there is not need to introduce any measurement interaction term and the formalism simplifies dramatically.

The resolution increases with $h_{eff}/h = n$. Also the number of strings connecting partonic 2-surfaces (in practice elementary particles and their dark counterparts plus bound states generated by connecting dark strings) characterizes physically the finite measurement resolution. Their presence is also visible in the geometry of the space-time surfaces through the conditions that induced W fields vanish at them (well-definedness of em charge), and by the condition that the canonical momentum currents for Kähler action define an integrable distribution of planes parallel to the string world sheet. In spirit with holography, preferred extremal is constructed by fixing string world sheets and partonic 2-surfaces and possibly also their light-like orbits (should one fix wormhole contacts is not quite clear). If the analog of AdS/CFT correspondence holds true, the value of Kähler function is expressible as the energy of string defined by area in the effective metric defined by the anti-commutators of K-D gamma matrices.

Super-symplectic algebra, whose charges are represented by Noether charges associated with strings connecting partonic 2-surfaces extends to a Yangian algebra with multi-stringy generators [K26]. The better the measurement resolution, the larger the maximal number of strings associated with the multilocal generator.

Kac-Moody type transformations preserving light-likeness of partonic orbits and possibly also the light-like character of the boundaries of string world sheets carrying modes of induced spinor field underlie the conformal gauge symmetry. The minimal option is that only the light-likeness of the string end world line is preserved by the conformal symmetries. In fact, conformal symmetries was originally deduced from the light-likeness condition for the M^4 projection of CP_2 type vacuum extremals.

The inclusions of super-symplectic Yangians form a hierarchy and would naturally correspond to inclusions of hyperfinite factors of type II_1 . Conformal symmetries acting as gauge transformations would naturally correspond to degrees of freedom below measurement resolution and would correspond to included subalgebra. As h_{eff} increases, infinite number of these gauge degrees of freedom become dynamical and measurement resolution is increased. This picture is definitely in conflict with the original view but the reduction of criticality in the increase of h_{eff} forces it.

4.3.4 Summary

On basis of above considerations it seems that the idea about “complex square root” of the state ω of von Neumann algebras might make sense in quantum TGD. Also the discretized versions of modular automorphism assignable to the hierarchy of CDs would make sense and because of its non-uniqueness the generator Δ of the canonical automorphism could bring in the flexibility needed one wants thermodynamics. Stringy picture forces to ask whether Δ could in some situation be proportional $\exp(L_0)$, where L_0 represents as the infinitesimal scaling generator of either super-symplectic algebra or super Kac-Moody algebra (the choice does not matter since the differences of the generators annihilate physical states in coset construction). This would allow to reproduce real thermodynamics consistent with p-adic thermodynamics. Note that also p-adic thermodynamics would be replaced by its square root in ZEO.

4.4 Finite measurement resolution and HFFs

The finite resolution of quantum measurement leads in TGD framework naturally to the notion of quantum M -matrix for which elements have values in sub-factor \mathcal{N} of HFF rather than being complex numbers. M -matrix in the factor space \mathcal{M}/\mathcal{N} is obtained by tracing over \mathcal{N} . The condition that \mathcal{N} acts like complex numbers in the tracing implies that M -matrix elements are proportional to maximal projectors to \mathcal{N} so that M -matrix is effectively a matrix in \mathcal{M}/\mathcal{N} and situation becomes finite-dimensional. It is still possible to satisfy generalized unitarity conditions but in general case tracing gives a weighted sum of unitary M -matrices defining what can be regarded as a square root of density matrix.

4.4.1 About the notion of observable in ZEO

Some clarifications concerning the notion of observable in zero energy ontology are in order.

- (a) As in standard quantum theory observables correspond to hermitian operators acting on either positive or negative energy part of the state. One can indeed define hermitian conjugation for positive and negative energy parts of the states in standard manner.
- (b) Also the conjugation $A \rightarrow JAJ$ is analogous to hermitian conjugation. It exchanges the positive and negative energy parts of the states also maps the light-like 3-surfaces at the upper boundary of CD to the lower boundary and vice versa. The map is induced by time reflection in the rest frame of CD with respect to the origin at the center of CD and has a well defined action on light-like 3-surfaces and space-time surfaces. This operation cannot correspond to the sought for hermitian conjugation since JAJ and A commute.
- (c) In order to obtain non-trivial fermion propagator one must add to Dirac action 1-D Dirac action in induced metric with the boundaries of string world sheets at the light-like parton orbits. Its bosonic counterpart is line-length in induced metric. Field equations imply that the boundaries are light-like geodesics and fermion has light-like 8-momentum. This suggests strongly a connection with quantum field theory and an 8-D generalization of twistor Grassmannian approach. By field equations the bosonic part of this action does not contribute to the Kähler action. Chern-Simons Dirac terms to which Kähler action reduces could be responsible for the breaking of CP and T symmetries as they appear in CKM matrix.
- (d) ZEO gives Cartan sub-algebra of the Lie algebra of symmetries a special status. Only Cartan algebra acting on either positive or negative states respects the zero energy property but this is enough to define quantum numbers of the state. In absence of symmetry breaking positive and negative energy parts of the state combine to form a state in a singlet representation of group. Since only the net quantum numbers must vanish ZEO allows a symmetry breaking respecting a chosen Cartan algebra.

- (e) In order to speak about four-momenta for positive and negative energy parts of the states one must be able to define how the translations act on CDs. The most natural action is a shift of the upper (lower) tip of CD. In the scale of entire CD this transformation induced Lorentz boost fixing the other tip. The value of mass squared is identified as proportional to the average of conformal weight in p-adic thermodynamics for the scaling generator L_0 for either super-symplectic or Super Kac-Moody algebra.

4.4.2 Inclusion of HFFs as characterizer of finite measurement resolution at the level of S -matrix

The inclusion $\mathcal{N} \subset \mathcal{M}$ of factors characterizes naturally finite measurement resolution. This means following things.

- (a) Complex rays of state space resulting usually in an ideal state function reduction are replaced by \mathcal{N} -rays since \mathcal{N} defines the measurement resolution and takes the role of complex numbers in ordinary quantum theory so that non-commutative quantum theory results. Non-commutativity corresponds to a finite measurement resolution rather than something exotic occurring in Planck length scales. The quantum Clifford algebra \mathcal{M}/\mathcal{N} creates physical states modulo resolution. The fact that \mathcal{N} takes the role of gauge algebra suggests that it might be necessary to fix a gauge by assigning to each element of \mathcal{M}/\mathcal{N} a unique element of \mathcal{M} . Quantum Clifford algebra with fractal dimension $\beta = \mathcal{M} : \mathcal{N}$ creates physical states having interpretation as quantum spinors of fractal dimension $d = \sqrt{\beta}$. Hence direct connection with quantum groups emerges.
- (b) The notions of unitarity, hermiticity, and eigenvalue generalize. The elements of unitary and hermitian matrices and \mathcal{N} -valued. Eigenvalues are Hermitian elements of \mathcal{N} and thus correspond entire spectra of Hermitian operators. The mutual non-commutativity of eigenvalues guarantees that it is possible to speak about state function reduction for quantum spinors. In the simplest case of a 2-component quantum spinor this means that second component of quantum spinor vanishes in the sense that second component of spinor annihilates physical state and second acts as element of \mathcal{N} on it. The non-commutativity of spinor components implies correlations between them and thus fractal dimension is smaller than 2.
- (c) The intuition about ordinary tensor products suggests that one can decompose Tr in \mathcal{M} as

$$\text{Tr}_{\mathcal{M}}(X) = \text{Tr}_{\mathcal{M}/\mathcal{N}} \times \text{Tr}_{\mathcal{N}}(X) . \quad (4.4)$$

Suppose one has fixed gauge by selecting basis $|r_k\rangle$ for \mathcal{M}/\mathcal{N} . In this case one expects that operator in \mathcal{M} defines an operator in \mathcal{M}/\mathcal{N} by a projection to the preferred elements of \mathcal{M} .

$$\langle r_1 | X | r_2 \rangle = \langle r_1 | \text{Tr}_{\mathcal{N}}(X) | r_2 \rangle . \quad (4.5)$$

- (d) Scattering probabilities in the resolution defined by \mathcal{N} are obtained in the following manner. The scattering probability between states $|r_1\rangle$ and $|r_2\rangle$ is obtained by summing over the final states obtained by the action of \mathcal{N} from $|r_2\rangle$ and taking the analog of spin average over the states created in the similar from $|r_1\rangle$. \mathcal{N} average requires a division by $\text{Tr}(P_{\mathcal{N}}) = 1/\mathcal{M} : \mathcal{N}$ defining fractal dimension of \mathcal{N} . This gives

$$p(r_1 \rightarrow r_2) = \mathcal{M} : \mathcal{N} \times \langle r_1 | \text{Tr}_{\mathcal{N}}(SP_{\mathcal{N}}S^\dagger) | r_2 \rangle . \quad (4.6)$$

This formula is consistent with probability conservation since one has

$$\sum_{r_2} p(r_1 \rightarrow r_2) = \mathcal{M} : \mathcal{N} \times \text{Tr}_{\mathcal{N}}(SS^\dagger) = \mathcal{M} : \mathcal{N} \times \text{Tr}(P_{\mathcal{N}}) = 1 . \quad (4.7)$$

- (e) Unitarity at the level of \mathcal{M}/\mathcal{N} can be achieved if the unit operator Id for \mathcal{M} can be decomposed into an analog of tensor product for the unit operators of \mathcal{M}/\mathcal{N} and \mathcal{N} and M decomposes to a tensor product of unitary M -matrices in \mathcal{M}/\mathcal{N} and \mathcal{N} . For HFFs of type II projection operators of \mathcal{N} with varying traces are present and one expects a weighted sum of unitary M -matrices to result from the tracing having interpretation in terms of square root of thermodynamics.
- (f) This argument assumes that \mathcal{N} is HFF of type II_1 with finite trace. For HFFs of type III_1 this assumption must be given up. This might be possible if one compensates the trace over \mathcal{N} by dividing with the trace of the infinite trace of the projection operator to \mathcal{N} . This probably requires a limiting procedure which indeed makes sense for HFFs.

4.4.3 Quantum M -matrix

The description of finite measurement resolution in terms of inclusion $\mathcal{N} \subset \mathcal{M}$ seems to boil down to a simple rule. Replace ordinary quantum mechanics in complex number field C with that in \mathcal{N} . This means that the notions of unitarity, hermiticity, Hilbert space ray, etc.. are replaced with their \mathcal{N} counterparts.

The full M -matrix in \mathcal{M} should be reducible to a finite-dimensional quantum M -matrix in the state space generated by quantum Clifford algebra \mathcal{M}/\mathcal{N} which can be regarded as a finite-dimensional matrix algebra with non-commuting \mathcal{N} -valued matrix elements. This suggests that full M -matrix can be expressed as M -matrix with \mathcal{N} -valued elements satisfying \mathcal{N} -unitarity conditions.

Physical intuition also suggests that the transition probabilities defined by quantum S -matrix must be commuting hermitian \mathcal{N} -valued operators inside every row and column. The traces of these operators give \mathcal{N} -averaged transition probabilities. The eigenvalue spectrum of these Hermitian matrices gives more detailed information about details below experimental resolution. \mathcal{N} -hermicity and commutativity pose powerful additional restrictions on the M -matrix.

Quantum M -matrix defines \mathcal{N} -valued entanglement coefficients between quantum states with \mathcal{N} -valued coefficients. How this affects the situation? The non-commutativity of quantum spinors has a natural interpretation in terms of fuzzy state function reduction meaning that quantum spinor corresponds effectively to a statistical ensemble which cannot correspond to pure state. Does this mean that predictions for transition probabilities must be averaged over the ensemble defined by “quantum quantum states”?

4.4.4 Quantum fluctuations and inclusions

Inclusions $\mathcal{N} \subset \mathcal{M}$ of factors provide also a first principle description of quantum fluctuations since quantum fluctuations are by definition quantum dynamics below the measurement resolution. This gives hopes for articulating precisely what the important phrase “long range quantum fluctuations around quantum criticality” really means mathematically.

- (a) Phase transitions involve a change of symmetry. One might hope that the change of the symmetry group $G_a \times G_b$ could universally code this aspect of phase transitions. This need not always mean a change of Planck constant but it means always a leakage between sectors of embedding space. At quantum criticality 3-surfaces would have regions belonging to at least two sectors of H .
- (b) The long range of quantum fluctuations would naturally relate to a partial or total leakage of the 3-surface to a sector of embedding space with larger Planck constant meaning zooming up of various quantal lengths.
- (c) For M -matrix in \mathcal{M}/\mathcal{N} regarded as $calN$ module quantum criticality would mean a special kind of eigen state for the transition probability operator defined by the M -matrix. The properties of the number theoretic braids contributing to the M -matrix should characterize this state. The strands of the critical braids would correspond to fixed points for $G_a \times G_b$ or its subgroup.

4.4.5 M -matrix in finite measurement resolution

The following arguments relying on the proposed identification of the space of zero energy states give a precise formulation for M -matrix in finite measurement resolution and the Connes tensor product involved. The original expectation that Connes tensor product could lead to a unique M-matrix is wrong. The replacement of ω with its complex square root could lead to a unique hierarchy of M-matrices with finite measurement resolution and allow completely finite theory despite the fact that projectors have infinite trace for HFFs of type III₁.

- (a) In ZEO the counterpart of Hermitian conjugation for operator is replaced with $\mathcal{M} \rightarrow J\mathcal{M}J$ permuting the factors. Therefore $N \in \mathcal{N}$ acting to positive (negative) energy part of state corresponds to $N \rightarrow N' = JNJ$ acting on negative (positive) energy part of the state.
- (b) The allowed elements of N must be such that zero energy state remains zero energy state. The superposition of zero energy states involved can however change. Hence one must have that the counterparts of complex numbers are of form $N = JN_1J \vee N_2$, where N_1 and N_2 have same quantum numbers. A superposition of terms of this kind with varying quantum numbers for positive energy part of the state is possible.
- (c) The condition that N_{1i} and N_{2i} act like complex numbers in \mathcal{N} -trace means that the effect of $JN_{1i}J \vee N_{2i}$ and $JN_{2i}J \vee N_{1i}$ to the trace are identical and correspond to a multiplication by a constant. If \mathcal{N} is HFF of type II₁ this follows from the decomposition $\mathcal{M} = \mathcal{M}/\mathcal{N} \otimes \mathcal{N}$ and from $Tr(AB) = Tr(BA)$ assuming that M is of form $M = M_{\mathcal{M}/\mathcal{N}} \times P_{\mathcal{N}}$. Contrary to the original hopes that Connes tensor product could fix the M-matrix there are no conditions on $M_{\mathcal{M}/\mathcal{N}}$ which would give rise to a finite-dimensional M-matrix for Jones inclusions. One can replace the projector $P_{\mathcal{N}}$ with a more general state if one takes this into account in $*$ operation.
- (d) In the case of HFFs of type III₁ the trace is infinite so that the replacement of Tr_N with a state ω_N in the sense of factors looks more natural. This means that the counterpart of $*$ operation exchanging N_1 and N_2 represented as $SA\Omega = A^*\Omega$ involves Δ via $S = J\Delta^{1/2}$. The exchange of N_1 and N_2 gives altogether Δ . In this case the KMS condition $\omega_{\mathcal{N}}(AB) = \omega_{\mathcal{N}}(\Delta A)$ guarantees the effective complex number property [A2].
- (e) Quantum TGD more or less requires the replacement of ω with its “complex square root” so that also a unitary matrix U multiplying Δ is expected to appear in the formula for S and guarantee the symmetry. One could speak of a square root of KMS condition [A2] in this case. The QFT counterpart would be a cutoff involving path integral over the degrees of freedom below the measurement resolution. In TGD framework it would mean a cutoff in the functional integral over WCW and for the modes of the second quantized induced spinor fields and also cutoff in sizes of causal diamonds. Discretization in terms of braids replacing light-like 3-surfaces should be the counterpart for the cutoff.
- (f) If one has M -matrix in \mathcal{M} expressible as a sum of M -matrices of form $M_{\mathcal{M}/\mathcal{N}} \times M_{\mathcal{N}}$ with coefficients which correspond to the square roots of probabilities defining density matrix the tracing operation gives rise to square root of density matrix in M .

4.4.6 Is universal M-matrix possible?

The realization of the finite measurement resolution could apply only to transition probabilities in which \mathcal{N} -trace or its generalization in terms of state $\omega_{\mathcal{N}}$ is needed. One might however dream of something more.

- (a) Maybe there exists a universal M-matrix in the sense that the same M-matrix gives the M-matrices in finite measurement resolution for all inclusions $\mathcal{N} \subset \mathcal{M}$. This would mean that one can write

$$M = M_{\mathcal{M}/\mathcal{N}} \otimes M_{\mathcal{N}} \quad (4.8)$$

for any physically reasonable choice of \mathcal{N} . This would formally express the idea that M is as near as possible to M-matrix of free theory. Also fractality suggests itself in the sense that $M_{\mathcal{N}}$ is essentially the same as $M_{\mathcal{M}}$ in the same sense as \mathcal{N} is same as \mathcal{M} . It might be that the trivial solution $M = 1$ is the only possible solution to the condition.

- (b) $M_{\mathcal{M}/\mathcal{N}}$ would be obtained by the analog of $Tr_{\mathcal{N}}$ or $\omega_{\mathcal{N}}$ operation involving the “complex square root” of the state ω in case of HFFs of type III₁. The QFT counterpart would be path integration over the degrees of freedom below cutoff to get effective action.
- (c) Universality probably requires assumptions about the thermodynamical part of the universal M-matrix. A possible alternative form of the condition is that it holds true only for canonical choice of “complex square root” of ω or for the S-matrix part of M :

$$S = S_{\mathcal{M}/\mathcal{N}} \otimes S_{\mathcal{N}} \quad (4.9)$$

for any physically reasonable choice \mathcal{N} .

- (d) In TGD framework the condition would say that the M-matrix defined by the Kähler-Dirac action gives M-matrices in finite measurement resolution via the counterpart of integration over the degrees of freedom below the measurement resolution.

An obvious counter argument against the universality is that if the M-matrix is “complex square root of state” cannot be unique and there are infinitely many choices related by a unitary transformation induced by the flows representing modular automorphism giving rise to new choices. This would actually be a well-come result and make possible quantum measurement theory.

In the section “Handful of problems with a common resolution” it was found that one can add to both Kähler action and Kähler-Dirac action a measurement interaction term characterizing the values of measured observables. The measurement interaction term in Kähler action is Lagrange multiplier term at the space-like ends of space-time surface fixing the value of classical charges for the space-time sheets in the quantum superposition to be equal with corresponding quantum charges. The term in Kähler-Dirac action is obtained from this by assigning to this term canonical momentum densities and contracting them with gamma matrices to obtain Kähler-Dirac gamma matrices appearing in 3-D analog of Dirac action. The constraint terms would leave Kähler function and Kähler metric invariant but would restrict the vacuum functional to the subset of 3-surfaces with fixed classical conserved charges (in Cartan algebra) equal to their quantum counterparts.

4.4.7 Connes tensor product and space-like entanglement

Ordinary linear Connes tensor product makes sense also in positive/negative energy sector and also now it makes sense to speak about measurement resolution. Hence one can ask whether Connes tensor product should be posed as a constraint on space-like entanglement. The interpretation could be in terms of the formation of bound states. The reducibility of HFFs and inclusions means that the tensor product is not uniquely fixed and ordinary entanglement could correspond to this kind of entanglement.

Also the counterpart of p-adic coupling constant evolution would makes sense. The interpretation of Connes tensor product would be as the variance of the states with respect to some subgroup of $U(n)$ associated with the measurement resolution: the analog of color confinement would be in question.

4.4.8 2-vector spaces and entanglement modulo measurement resolution

John Baez and collaborators [A14] are playing with very formal looking formal structures obtained by replacing vectors with vector spaces. Direct sum and tensor product serve as the basic arithmetic operations for the vector spaces and one can define category of n-tuples of vectors spaces with morphisms defined by linear maps between vectors spaces of the tuple.

n -tuples allow also element-wise product and sum. They obtain results which make them happy. For instance, the category of linear representations of a given group forms 2-vector spaces since direct sums and tensor products of representations as well as n -tuples make sense. The 2-vector space however looks more or less trivial from the point of physics.

The situation could become more interesting in quantum measurement theory with finite measurement resolution described in terms of inclusions of hyper-finite factors of type II_1 . The reason is that Connes tensor product replaces ordinary tensor product and brings in interactions via irreducible entanglement as a representation of finite measurement resolution. The category in question could give Connes tensor products of quantum state spaces and describing interactions. For instance, one could multiply M -matrices via Connes tensor product to obtain category of M -matrices having also the structure of 2-operator algebra.

- (a) The included algebra represents measurement resolution and this means that the infinite-D sub-Hilbert spaces obtained by the action of this algebra replace the rays. Sub-factor takes the role of complex numbers in generalized QM so that one obtains non-commutative quantum mechanics. For instance, quantum entanglement for two systems of this kind would not be between rays but between infinite-D subspaces corresponding to sub-factors. One could build a generalization of QM by replacing rays with sub-spaces and it would seem that quantum group concept does more or less this: the states in representations of quantum groups could be seen as infinite-dimensional Hilbert spaces.
- (b) One could speak about both operator algebras and corresponding state spaces modulo finite measurement resolution as quantum operator algebras and quantum state spaces with fractal dimension defined as factor space like entities obtained from HFF by dividing with the action of included HFF. Possible values of the fractal dimension are fixed completely for Jones inclusions. Maybe these quantum state spaces could define the notions of quantum 2-Hilbert space and 2-operator algebra via direct sum and tensor production operations. Fractal dimensions would make the situation interesting both mathematically and physically.

Suppose one takes the fractal factor spaces as the basic structures and keeps the information about inclusion.

- (a) Direct sums for quantum vectors spaces would be just ordinary direct sums with HFF containing included algebras replaced with direct sum of included HFFs.
- (b) The tensor products for quantum state spaces and quantum operator algebras are not anymore trivial. The condition that measurement algebras act effectively like complex numbers would require Connes tensor product involving irreducible entanglement between elements belonging to the two HFFs. This would have direct physical relevance since this entanglement cannot be reduced in state function reduction. The category would defined interactions in terms of Connes tensor product and finite measurement resolution.
- (c) The sequences of super-conformal symmetry breakings identifiable in terms of inclusions of super-conformal algebras and corresponding HFFs could have a natural description using the 2-Hilbert spaces and quantum 2-operator algebras.

4.5 Questions about quantum measurement theory in Zero Energy Ontology

The following summary about quantum measurement theory in ZEO is somewhat out-of-date and somewhat sketchy. For more detailed view see [K17, K27, K2].

4.5.1 Fractal hierarchy of state function reductions

In accordance with fractality, the conditions for the Connes tensor product at a given time scale imply the conditions at shorter time scales. On the other hand, in shorter time scales

the inclusion would be deeper and would give rise to a larger reducibility of the representation of \mathcal{N} in \mathcal{M} . Formally, as \mathcal{N} approaches to a trivial algebra, one would have a square root of density matrix and trivial S -matrix in accordance with the idea about asymptotic freedom.

M -matrix would give rise to a matrix of probabilities via the expression $P(P_+ \rightarrow P_-) = \text{Tr}[P_+ M^\dagger P_- M]$, where P_+ and P_- are projectors to positive and negative energy \mathcal{N} -rays. The projectors give rise to the averaging over the initial and final states inside \mathcal{N} ray. The reduction could continue step by step to shorter length scales so that one would obtain a sequence of inclusions. If the U -process of the next quantum jump can return the M -matrix associated with \mathcal{M} or some larger HFF, U process would be kind of reversal for state function reduction.

Analytic thinking proceeding from vision to details; human life cycle proceeding from dreams and wild actions to the age when most decisions relate to the routine daily activities; the progress of science from macroscopic to microscopic scales; even biological decay processes: all these have an intriguing resemblance to the fractal state function reduction process proceeding to shorter and shorter time scales. Since this means increasing thermality of M -matrix, U process as a reversal of state function reduction might break the second law of thermodynamics.

The conservative option would be that only the transformation of intentions to action by U process giving rise to new zero energy states can bring in something new and is responsible for evolution. The non-conservative option is that the biological death is the U -process of the next quantum jump leading to a new life cycle. Breathing would become a universal metaphor for what happens in quantum Universe. The 4-D body would be lived again and again.

4.5.2 quantum classical correspondence is realized at parton level?

Quantum classical correspondence must assign to a given quantum state the most probable space-time sheet depending on its quantum numbers. The space-time sheet $X^4(X^3)$ defined by the Kähler function depends however only on the partonic 3-surface X^3 , and one must be able to assign to a given quantum state the most probable X^3 - call it X_{max}^3 - depending on its quantum numbers.

$X^4(X_{max}^3)$ should carry the gauge fields created by classical gauge charges associated with the Cartan algebra of the gauge group (color isospin and hypercharge and electromagnetic and Z^0 charge) as well as classical gravitational fields created by the partons. This picture is very similar to that of quantum field theories relying on path integral except that the path integral is restricted to 3-surfaces X^3 with exponent of Kähler function bringing in genuine convergence and that 4-D dynamics is deterministic apart from the delicacies due to the 4-D spin glass type vacuum degeneracy of Kähler action.

Stationary phase approximation selects X_{max}^3 if the quantum state contains a phase factor depending not only on X^3 but also on the quantum numbers of the state. A good guess is that the needed phase factor corresponds to either Chern-Simons type action or an action describing the interaction of the induced gauge field with the charges associated with the braid strand. This action would be defined for the induced gauge fields. YM action seems to be excluded since it is singular for light-like 3-surfaces associated with the light-like wormhole throats (not only $\sqrt{\det(g_3)}$ but also $\sqrt{\det(g_4)}$ vanishes).

The challenge is to show that this is enough to guarantee that $X^4(X_{max}^3)$ carries correct gauge charges. Kind of electric-magnetic duality should relate the normal components F_{ni} of the gauge fields in $X^4(X_{max}^3)$ to the gauge fields F_{ij} induced at X^3 . An alternative interpretation is in terms of quantum gravitational holography.

One is forced to introduce gauge couplings and also electro-weak symmetry breaking via the phase factor. This is in apparent conflict with the idea that all couplings are predictable. The essential uniqueness of M -matrix in the case of HFFs of type II_1 (at least) however means that their values as a function of measurement resolution time scale are fixed by internal consistency. Also quantum criticality leads to the same conclusion. Obviously a kind of bootstrap approach suggests itself.

4.5.3 Quantum measurements in ZEO

ZEO based quantum measurement theory leads directly to a theory of conscious entities. The basic idea is that state function reduction localizes the second boundary of CD so that it becomes a piece of light-cone boundary (more precisely $\delta M_{\pm}^4 \times CP_2$).

Repeated reductions are possible as in standard quantum measurement theory and leave the passive boundary of CD. Repeated reduction begins with U process generating a superposition of CDs with the active boundary of CD being de-localized in the moduli space of CDs, and is followed by a localization in this moduli space so that single CD is the outcome. This process tends to increase the distance between the ends of the CD and has interpretation as a space-time correlate for the flow of subjective time.

Self as a conscious entity corresponds to this sequence of repeated reductions on passive boundary of CD. The first reduction at opposite boundary means death of self and its re-incarnation at the opposite boundary of CD. Also the increase of Planck constant and generation of negentropic entanglement is expected to be associated with this state function reduction.

Weak form of NMP is the most plausible variational principle to characterize the state function reduction. It does not require maximal negentropy gain for state function reductions but allows it. In other words, the outcome of reduction is n -dimensional eigen space of density matrix space but this space need not have maximum possible dimension and even 1-D ray is possible in which case the entanglement negentropy vanishes for the final state and system becomes isolated from the rest of the world. Weak form of NMP brings in free will and can allow also larger negentropy gain than the strong form if n is a product of primes. The beauty of this option is that one can understand how the generalization of p-adic length scale hypothesis emerges.

4.6 Miscellaneous

The following considerations are somewhat out-of-date: hence the title “Miscellaneous”.

4.6.1 Connes tensor product and fusion rules

One should demonstrate that Connes tensor product indeed produces an M -matrix with physically acceptable properties.

The reduction of the construction of vertices to that for n -point functions of a conformal field theory suggest that Connes tensor product is essentially equivalent with the fusion rules for conformal fields defined by the Clifford algebra elements of $CH(CD)$ (4-surfaces associated with 3-surfaces at the boundary of causal diamond CD in M^4), extended to local fields in M^4 with gamma matrices acting on WCW spinor s assignable to the partonic boundary components.

Jones speculates that the fusion rules of conformal field theories can be understood in terms of Connes tensor product [A21] and refers to the work of Wassermann about the fusion of loop group representations as a demonstration of the possibility to formula the fusion rules in terms of Connes tensor product [A7].

Fusion rules are indeed something more intricate than the naïve product of free fields expanded using oscillator operators. By its very definition Connes tensor product means a dramatic reduction of degrees of freedom and this indeed happens also in conformal field theories.

- (a) For non-vanishing n -point functions the tensor product of representations of Kac Moody group associated with the conformal fields must give singlet representation.
- (b) The ordinary tensor product of Kac Moody representations characterized by given value of central extension parameter k is not possible since k would be additive.

- (c) A much stronger restriction comes from the fact that the allowed representations must define integrable representations of Kac-Moody group [A8]. For instance, in case of $SU(2)_k$ Kac Moody algebra only spins $j \leq k/2$ are allowed. In this case the quantum phase corresponds to $n = k + 2$. $SU(2)$ is indeed very natural in TGD framework since it corresponds to both electro-weak $SU(2)_L$ and isotropy group of particle at rest.

Fusion rules for localized Clifford algebra elements representing operators creating physical states would replace naïve tensor product with something more intricate. The naïvest approach would start from M^4 local variants of gamma matrices since gamma matrices generate the Clifford algebra Cl associated with $CH(CD)$. This is certainly too naïve an approach. The next step would be the localization of more general products of Clifford algebra elements elements of Kac Moody algebras creating physical states and defining free on mass shell quantum fields. In standard quantum field theory the next step would be the introduction of purely local interaction vertices leading to divergence difficulties. In the recent case one transfers the partonic states assignable to the light-cone boundaries $\delta M_{\pm}^4(m_i) \times CP_2$ to the common partonic 2-surfaces X_V^2 along $X_{L,i}^3$ so that the products of field operators at the same space-time point do not appear and one avoids infinities.

The remaining problem would be the construction an explicit realization of Connes tensor product. The formal definition states that left and right \mathcal{N} actions in the Connes tensor product $\mathcal{M} \otimes_{\mathcal{N}} \mathcal{M}$ are identical so that the elements $nm_1 \otimes m_2$ and $m_1 \otimes m_2n$ are identified. This implies a reduction of degrees of freedom so that free tensor product is not in question. One might hope that at least in the simplest choices for \mathcal{N} characterizing the limitations of quantum measurement this reduction is equivalent with the reduction of degrees of freedom caused by the integrability constraints for Kac-Moody representations and dropping away of higher spins from the ordinary tensor product for the representations of quantum groups. If fusion rules are equivalent with Connes tensor product, each type of quantum measurement would be characterized by its own conformal field theory.

In practice it seems safest to utilize as much as possible the physical intuition provided by quantum field theories. In [K7] a rather precise vision about generalized Feynman diagrams is developed and the challenge is to relate this vision to Connes tensor product.

4.6.2 Connection with topological quantum field theories defined by Chern-Simons action

There is also connection with topological quantum field theories (TQFTs) defined by Chern-Simons action [A10].

- (a) The light-like 3-surfaces X_l^3 defining propagators can contain unitary matrix characterizing the braiding of the lines connecting fermions at the ends of the propagator line. Therefore the modular S -matrix representing the braiding would become part of propagator line. Also incoming particle lines can contain similar S -matrices but they should not be visible in the M -matrix. Also entanglement between different partonic boundary components of a given incoming 3-surface by a modular S -matrix is possible.
- (b) Besides CP_2 type extremals MEs with light-like momenta can appear as brehmstrahlung like exchanges always accompanied by exchanges of CP_2 type extremals making possible momentum conservation. Also light-like boundaries of magnetic flux tubes having macroscopic size could carry light-like momenta and represent similar brehmstrahlung like exchanges. In this case the modular S -matrix could make possible topological quantum computations in $q \neq 1$ phase [K1]. Notice the somewhat counter intuitive implication that magnetic flux tubes of macroscopic size would represent change in quantum jump rather than quantum state. These quantum jumps can have an arbitrary long geometric duration in macroscopic quantum phases with large Planck constant [K10].

There is also a connection with topological QFT defined by Chern-Simons action allowing to assign topological invariants to the 3-manifolds [A10]. If the light-like CDs $X_{L,i}^3$ are boundary components, the 3-surfaces associated with particles are glued together somewhat

like they are glued in the process allowing to construct 3-manifold by gluing them together along boundaries. All 3-manifold topologies can be constructed by using only torus like boundary components.

This would suggest a connection with 2+1-dimensional topological quantum field theory defined by Chern-Simons action allowing to define invariants for knots, links, and braids and 3-manifolds using surgery along links in terms of Wilson lines. In these theories one consider gluing of two 3-manifolds, say 3-spheres S^3 along a link to obtain a topologically non-trivial 3-manifold. The replacement of link with Wilson lines in $S^3 \# S^3 = S^3$ reduces the calculation of link invariants defined in this manner to Chern-Simons theory in S^3 .

In the recent situation more general structures are possible since arbitrary number of 3-manifolds are glued together along link so that a singular 3-manifolds with a book like structure are possible. The allowance of CDs which are not boundaries, typically 3-D light-like throats of wormhole contacts at which induced metric transforms from Minkowskian to Euclidian, brings in additional richness of structure. If the scaling factor of CP_2 metric can be arbitrary large as the quantization of Planck constant predicts, this kind of structure could be macroscopic and could be also linked and knotted. In fact, topological condensation could be seen as a process in which two 4-manifolds are glued together by drilling light-like CDs and connected by a piece of CP_2 type extremal.

5 The idea of Connes about inherent time evolution of certain algebraic structures from TGD point of view

Jonathan Disckau asked me about what I think about the proposal of Connes represented in the summary of progress of noncommutative geometry in "Noncommutative Geometry Year 2000" [A6] (see <https://arxiv.org/abs/math/0011193>) that certain mathematical structures have inherent time evolution coded into their structure.

I have written years ago about Connes's proposal. At that time I was trying to figure out how to understand the construction of scattering amplitudes in the TGD framework and the proposal of Connes looked attractive. Later I had to give up this idea. However, the basic idea is beautiful. One should only replace the notion of time evolution from a one-parameter group of automorphisms to something more interesting. Also time evolution as increasing algebraic complexity is a more attractive interpretation.

The inclusion hierarchies of hyperfinite factors (HFFs) - closely related to the work of Connes - are a key element of TGD and crucial for understanding evolutionary hierarchies in TGD. Is it possible that mathematical structure evolves in time in some sense? The TGD based answer is that quantum jump as a fundamental evolutionary step - moment of subjective time evolution - is a necessary new element. The sequence of moments of consciousness as quantum jumps would have an interpretation as hopping around in the space of mathematical structures leading to increasingly complex structures.

The generalization of the idea of Connes is discussed in this framework. In particular, the inclusion hierarchies of hyper-finite factors, the extension hierarchies of rationals, and fractal inclusion hierarchies of subalgebras of supersymplectic algebra isomorphic with the entire algebra are proposed to be more or less one and the same thing in TGD framework.

The time evolution operator of Connes could corresponds to super-symplectic algebra (SSA) to the time evolution generated by $exp(iL_0\tau)$ so that the operator Δ of Connes would be identified as $\Delta = exp(L_0)$. This identification allows number theoretical universality if τ is quantized. Furthermore, one ends up with a model for the subjective time evolution by small state function reductions (SSFRs) for SSA with SSA_n gauge conditions: the unitary time evolution for given SSFR would be generated by a linear combination of Virasoro generators not annihilating the states. This model would generalize the model for harmonic oscillator in external force allowing exact S-matrix.

5.1 Connes proposal and TGD

In this section I develop in more detail the analog of Connes proposal in TGD framework.

5.1.1 What does Connes suggest?

One must first make clear what the automorphism of HFFs discovered by Connes is.

1. Tomita-Takesaki theory

Tomita-Takesaki theory is a vital part of the theory of factors. I have described the theory earlier [K18, K12].

First some definitions.

- (a) Let $\omega(x)$ be a faithful state of von Neumann algebra so that one has $\omega(xx^*) > 0$ for $x > 0$. Assume by Riesz lemma the representation of ω as a vacuum expectation value: $\omega = (\cdot\Omega, \Omega)$, where Ω is cyclic and separating state.
- (b) Let

$$L^\infty(\mathcal{M}) \equiv \mathcal{M} \ , \quad L^2(\mathcal{M}) = \mathcal{H} \ , \quad L^1(\mathcal{M}) = \mathcal{M}_* \ , \quad (5.1)$$

where \mathcal{M}_* is the pre-dual of \mathcal{M} defined by linear functionals in \mathcal{M} . One has $\mathcal{M}_*^* = \mathcal{M}$.

- (c) The conjugation $x \rightarrow x^*$ is isometric in \mathcal{M} and defines a map $\mathcal{M} \rightarrow L^2(\mathcal{M})$ via $x \rightarrow x\Omega$. The map $S_0; x\Omega \rightarrow x^*\Omega$ is however non-isometric.
- (d) Denote by S the closure of the anti-linear operator S_0 and by $S = J\Delta^{1/2}$ its polar decomposition analogous that for complex number and generalizing polar decomposition of linear operators by replacing (almost) unitary operator with anti-unitary J . Therefore $\Delta = S^*S > 0$ is positive self-adjoint and J an anti-unitary involution. The non-triviality of Δ reflects the fact that the state is not trace so that hermitian conjugation represented by S in the state space brings in additional factor $\Delta^{1/2}$.
- (e) What x can be is puzzling to physicists. The restriction fermionic Fock space and thus to creation operators would imply that Δ would act non-trivially only vacuum state so that $\Delta > 0$ condition would not hold true. The resolution of puzzle is the allowance of tensor product of Fock spaces for which vacua are conjugates: only this gives cyclic and separating state. This is natural in ZEO.

The basic results of Tomita-Takesaki theory are following.

- (a) The basic result can be summarized through the following formulas

$$\Delta^{it} M \Delta^{-it} = \mathcal{M} \ , \quad J \mathcal{M} J = \mathcal{M}' \ .$$

- (b) The latter formula implies that \mathcal{M} and \mathcal{M}' are isomorphic algebras. The first formula implies that a one parameter group of modular automorphisms characterizes partially the factor. The physical meaning of modular automorphisms is discussed in [A9, A16] Δ is Hermitian and positive definite so that the eigenvalues of $\log(\Delta)$ are real but can be negative. Δ^{it} is however not unitary for factors of type II and III. Physically the non-unitarity must relate to the fact that the flow is contracting so that hermiticity as a local condition is not enough to guarantee unitarity.
- (c) $\omega \rightarrow \sigma_t^\omega = Ad\Delta^{it}$ defines a canonical evolution -modular automorphism- associated with ω and depending on it. The Δ :s associated with different ω :s are related by a unitary inner automorphism so that their equivalence classes define an invariant of the factor.

Tomita-Takesaki theory gives rise to a non-commutative measure theory which is highly non-trivial. In particular the spectrum of Δ can be used to classify the factors of type II and III.

The definition of Δ^{it} reduces in eigenstate basis of Δ to the definition of complex function d^{it} . Note that d is positive so that the logarithm of d is real.

In TGD framework number theoretic universality poses additional conditions. In diagonal basis $e^{log(d)it}$ must exist. A simple manner to solve the conditions is $e = exp(m/r)$ existing p -adically for an extension of rational allowing r :th root of e . This requires also quantization of e as a root of unity so that the exponent reduces to a root of unity.

2. Modular automorphisms

Modular automorphisms of factors are central for their classification.

- (a) One can divide the automorphisms to inner and outer ones. Inner automorphisms correspond to unitary operators obtained by exponentiating Hermitian Hamiltonian belonging to the factor and connected to identity by a flow. Outer automorphisms do not allow a representation as a unitary transformations although $log(\Delta)$ is formally a Hermitian operator.
- (b) The fundamental group of the type II_1 factor defined as fundamental group of corresponding II_∞ factor characterizes partially a factor of type II_1 . This group consists of real numbers λ such that there is an automorphism scaling the trace by λ . Fundamental group typically contains all reals but it can be also discrete and even trivial.
- (c) Factors of type III allow a one-parameter group of modular automorphisms, which can be used to achieve a partial classification of these factors. These automorphisms define a flow in the center of the factor known as flow of weights. The set of parameter values λ for which ω is mapped to itself and the center of the factor defined by the identity operator (projector to the factor as a sub-algebra of $\mathcal{B}(\mathcal{H})$) is mapped to itself in the modular automorphism defines the Connes spectrum of the factor. For factors of type III_λ this set consists of powers of $\lambda < 1$. For factors of type III_0 this set contains only identity automorphism so that there is no periodicity. For factors of type III_1 Connes spectrum contains all real numbers so that the automorphisms do not affect the identity operator of the factor at all.

The modules over a factor correspond to separable Hilbert spaces that the factor acts on. These modules can be characterized by M-dimension. The idea is roughly that complex rays are replaced by the sub-spaces defined by the action of \mathcal{M} as basic units. M-dimension is not integer valued in general. The so called standard module has a cyclic separating vector and each factor has a standard representation possessing antilinear involution J such that $\mathcal{M}' = J\mathcal{M}J$ holds true (note that J changes the order of the operators in conjugation). The inclusions of factors define modules having interpretation in terms of a finite measurement resolution defined by \mathcal{M} .

3. Objections against the idea of Connes

One can represent objections against this idea.

- (a) Ordinary time evolution in wave mechanics is a unitary automorphism, so that in this framework they would not have physical meaning but act as gauge transformations. If outer automorphisms define time evolutions, they must act as gauge transformations. One would have an analog of gauge field theory in HFF. This would be of course highly interesting: when I gave up the idea of Connes, I did not consider this possibility. Super-symplectic algebras having fractal structure are however extremely natural candidate for defining HFF and there is infinite number of gauge conditions.
- (b) An automorphism is indeed in question so that the algebraic system would not be actually affected. Therefore one cannot say that HFF has inherent time evolution and

time. However, one can represent in HFF dynamical systems obeying this inherent time evolution. This possibility is highly interesting as a kind of universal gauge theory.

On the other hand, outer automorphisms affect the trace of the projector defining the identity matrix for a given factor. Does the scaling factor Λ represent some kind of renormalization operation? Could it relate to the action of scalings in the TGD framework where scalings replace time translations at the fundamental level? What the number theoretic vision of TGD could mean? Could this quantize the continuous spectrum of the scalings Λ for HFFs so that they belong to the extension? Could one have a spectrum of Λ for each extension of rationals? Are different extensions related by inclusions of HFFs?

- (c) The notion of time evolution itself is an essentially Newtonian concept: selecting a preferred time coordinate breaks Lorentz invariance. In TGD however time coordinate is replaced by scaling parameter and the situation changes.
- (d) The proposal of Connes is not general enough if evolution is interpreted as an increase of complexity.

For these reasons I gave up the automorphism proposed by Connes as a candidate for defining time evolution giving rise to scattering amplitudes in TGD framework.

5.1.2 Two views about TGD

The two dual views about what TGD is described briefly in [L25].

- (a) Physics as geometry of the world of "world of classical worlds" (WCW) identified as the space of space-time surfaces in $M^4 \times CP_2$ [K23]. Twistor lift of TGD [K25] implies that the space-time surfaces are minimal surfaces which can be also regarded as extremals of the Kähler action. This implies holography required by the general coordinate invariance in TGD framework.
- (b) TGD as generalized number theory forcing to generalize physics to adelic physics [L8] fusing real physics as correlate of sensory experience and various p-adic physics as correlates of cognition. Now space-times are naturally co-associative surfaces in complexified M^8 (complexified octonions) defined as "roots" of octonionic polynomials determined by polynomials with rational coefficients [L20, L21, L29]. Now holography extends dramatically: finite number of rational numbers/roots of rational polynomial/points of space-time region dictate it.

$M^8 - H$ duality relates these two views and is actually a generalization of Fourier transform and realizes generalization of momentum-position duality.

5.1.3 The notion of time evolution in TGD

Concerning various time evolutions in TGD, the general situation is now rather well understood.

There are two quantal time evolutions: geometric one assignable to single CD and subjective time evolution which reflects the generalization of point-like particle to a 3-surface and the introduction of CD as 4-D perceptive field of particle in ZEO [L17].

- (a) Geometric time evolution corresponds to the standard scattering amplitudes for which I have a general formula now in terms of zero energy ontology (ZEO) [L28, L20, L21, L29]. The analog of S-matrix corresponds to entanglement coefficients between members of zero energy state at opposite boundaries of causal diamond (CD).
- (b) Subjective time evolution of conscious entity corresponds to a sequence of "small" state function reductions (SSFRs) as moments of consciousness: each SSFR is preceded by an analog of unitary time evolution, call it U . SSFRs are the TGD counterparts of "weak" measurements.

$U(t)$ is generated by the scaling generator L_0 scaling light-like radial coordinate of light-cone boundary and is a generalization of corresponding operator in superconformal and string theories and defined for super-symplectic algebras acting as isometries of the world of classical worlds (WCW) [L29]. $U(t)$ is not the exponential of energy as a generator of time translation as in QFTs but an exponential of the mass squared operator and corresponds to the scaling of radial light-like coordinate r of the light-like boundary of CD: r is analogous to the complex coordinate z in conformal field theories.

Also "big" SFRs (BSFRs) are possible and correspond to "ordinary" SFRs and in TGD framework mean death of self in the universal sense and followed by reincarnation as time reversed subjective time evolution [L13].

- (c) There is also classical time evolution at the level of space-time surfaces. Here the assumption that X^4 belongs to $H = M^4 \times CP_2$ defines Minkowski coordinates of M^4 as almost unique space-time coordinates of X^4 is the M^4 projection of X^4 is 4-D. This generalizes also to the case of M^8 . Symmetries make it possible to identify an essentially a unique time coordinate.

This means enormous simplification. General coordinate invariance is a marvellous symmetry but it leads to the problem of specifying space-time coordinates that is finding preferred coordinates. This seems impossible since 3-metric is dynamical. M^4 provides a fixed reference system and the problem disappears. M^4 is dynamical by its Minkowskian signature and one can speak about classical signals.

- (d) There is also classical time evolution for the induced spinor fields. At the level of H the spinor field is a superposition of modes of the massless Dirac operator (massless in 8-D sense). This spinor field is free and second quantized. Second quantization of induced spinor trivializes and this is absolutely crucial for obtaining scattering amplitudes for fermions and avoiding the usual problems for quantization of fermions in curved background.

The induced spinor field is a restriction of this spinor field to the space-time surface and satisfies modified Dirac equation automatically. There is no need for second quantization at the level of space-time surface and propagators etc.... are directly calculable. This is an enormous simplification.

There are therefore as many as 4 time evolutions and subjective time evolution by BSFRs and possibly also by SSFRs is a natural candidate for time evolution as genuine evolution as emergence of more complex algebraic structures.

5.1.4 Could the inherent time evolution of HFF have a physical meaning in TGD after all?

The idea about inherent time evolution defined by HFF itself as one parameter group of outer automorphisms is very attractive by its universality: physics would become part of mathematics.

- (a) Thermodynamic interpretation, with inverse temperature identified as an analog of time coordinate, comes first in mind but need not be the correct interpretation.
- (b) Outer automorphisms should act at a very fundamental level analogous to the state space of topological field theories. Fundamental group is after all in question! The assignment of the S-matrix of particle physics to the outer automorphism does not look reasonable since the time evolution would be with respect to the linear Minkowski coordinate, which is not Lorentz invariant.

For these reasons I gave up the idea of Connes when considering it for the first time. However, TGD inspired theory of consciousness as a generalization of quantum measurement theory has evolved since then and the situation is different now.

The sequence of SSFRs defines subjective time evolution having no counterpart in QFTs. Each SSFR is preceded by a unitary time evolution, which however corresponds to the

scaling of the light-like radial coordinate of the light-cone boundary [L29] rather than time translation. Hamiltonian is replaced with the scaling generator L_0 acting as Lorentz invariant mass squared operator so that Lorentz invariance is not lost.

Could the time evolution assignable to L_0 correspond to the outer automorphism of Connes when one poses an infinite number of gauge conditions making inner automorphisms gauge transformations? The connection of Connes proposal with conformal field theories and with TGD is indeed suggestive.

- (a) Conformally invariant systems obey infinite number of gauge conditions stating that the conformal generators L_n , $n > 0$, annihilate physical states and carry vanishing Noether charges.

These gauge conditions bring in mind the condition that infinitesimal inner automorphisms do not change the system physically. Does this mean that Connes outer automorphism generates the time evolution and inner automorphisms act as gauge symmetries? One would have an analog of gauge field theory in HFF.

- (b) In TGD framework one has an infinite hierarchy of systems satisfying conditions analogous to the conformal gauge conditions. The generators of the super-symplectic algebra (SCA) acting as isometries of the "world of classical worlds" (WCW) are labelled by non-negative conformal weight n and it has infinite hierarchy of algebras SCA_k isomorphic to it with conformal weights given by k -multiple of those of the entire algebra, $k = 1, 2, \dots$

Gauge conditions state for SCA_k that the generators of SCA_k and its commutator with SCA annihilate physical states. The interpretation is in terms of a hierarchy of improving measurement resolutions with degrees of freedom below measurement resolution acting like gauge transformations.

The Connes automorphism would "see" only the time evolution in the degrees of freedom above measurement resolution and as k increases, their number would increase.

In the case of hyperfinite factors of type II_1 (HFFs) the fundamental group of corresponding factor II_∞ consists of all reals: I hope I am right here.

- (a) The hyperfinite factors of type II_1 and corresponding factors II_∞ are natural in the TGD context. Therefore the spectrum would consist of reals unless one poses additional conditions.
- (b) Could the automorphisms correspond to the scalings of the lightcone proper time, which replace time translations as fundamental dynamics. Also in string models scalings take the role of time translations.
- (c) In zero energy ontology (ZEO) the scalings would act in the moduli space of causal diamonds which is finite-dimensional. This moduli space defines the backbone of the "world of classical worlds". WCW itself consists of a union of sub-WCs as bundle structures over CDs [L31]. The fiber consists of space-time surfaces inside a given CD analogous to Bohr orbits and satisfying holography reducing to generalized holomorphy. The scalings as automorphisms scale the causal diamonds. The space of CDs is a finite-dimensional coset space and has also other symmetry transformations.
- (d) The number theoretic vision suggests a quantization of the spectrum of Λ so that for a given extension of rationals the spectrum would belong to the extension. HFFs would be labelled at least partially by the extensions of rationals. The recent view of $M^8 - H$ duality [L32] is dramatically simpler than the earlier view [L20, L21, ?] and predicts that the space-time regions are determined by a pair of analytic functions with rational coefficients forced by number theoretical universality meaning that the space-time surfaces have interpretation also as p-adic surfaces.

The simplest analytic functions are polynomials with integer coefficients and if one requires that the coefficients are smaller than the degree of the polynomial, the number of polynomials is finite for a given degree. This would give very precise meaning for the concept of number theoretic evolution.

There would be an evolutionary hierarchy of pairs of polynomials characterized by increasing complexity and one can assign to these polynomials extension of rationals characterized by ramified primes depending on the polynomials. The ramified primes would have interpretation as p-adic primes characterizing the space-time region considered. Extensions of rationals and ramified primes could also characterize HFFs. This is a rather dramatic conjecture at the level of pure mathematics.

- (e) Scalings define renormalization group in standard physics. Now they scale the size of the CD. Could the scalings as automorphisms of HFFs correspond to discrete renormalization operations?

5.1.5 Three views about finite measurement resolution

Evolution could be seen physically as improving finite measurement resolution: this applies to both sensory experience and cognition. There are 3 views about finite measurement resolution (FMR) in TGD.

1. Hyper finite factors (HFFs) and FMR

HFFs are an essential part of Connes's work and I encountered them about 15 years ago or so [K28, K12].

The inclusions of hyper-finite factors HFFs provide one of the three - as it seems equivalent - ways to describe finite measurement resolution (FMR) in TGD framework: the included factor defines an analog for gauge degrees of freedom which correspond to those below measurement resolution.

2. Cognitive representations and FMR

Another description for FMR in the framework of adelic physics would be in terms of cognitive representations [L14]. First some background about $M^8 - H$ duality.

- (a) There are number theoretic and geometric views about dynamics. In algebraic dynamics at the level of M^8 , the space-time surfaces are roots of polynomials. There are no partial differential equations like in the geometric dynamics at the level of H .
- (b) The algebraic "dynamics" of space-time surfaces in M^8 is dictated by co-associativity, which means that the normal space of the space-time surface is associative and thus quaternionic. That normal space rather than tangent space must be associative became clear last year [L20, L21].
- (c) $M^8 - H$ duality maps these algebraic surfaces in M^8 to $H = M^4 \times CP_2$ and the one obtains the usual dynamics based on variational principle giving minimal surfaces which are non-linear analogs for the solutions of massless field equations. Instead of polynomials the natural functions at the level of H are periodic functions used in Fourier analysis [L29].

At level of complexified M^8 cognitive representation would consist of points of co-associative space-time surface X^4 in complexified M^8 (complexified octonions), whose coordinates belong to extension of rationals and therefore make sense also p-adically for extension of p-adic numbers induced by extension of rationals. $M^8 - H$ duality maps the cognitive representations to H .

Cognitive representations form a hierarchy: the larger the extension of rationals, the larger the number of points in the extension and in the unique discretization of space-time surface. Therefore also the measurement resolution improves.

The surprise was that the cognitive representations which are typically finite, are for the "roots" of octonionic polynomials infinite [L20, L21]. Also in this case the density of points of cognitive representation increases as the dimension of extensions increases.

The understanding of the physical interpretation of $M^8 - H$ duality increased dramatically during the last half year.

- (a) X^4 in M^8 is highly analogous to momentum space (4-D analog of Fermi ball one might say) and H to position space. Physical states correspond to discrete sets of points - 4-momenta - in X^4 . This is just the description used in particle physics for physical states. Time and space in this description are replaced by energy and 4-momentum. At the level of H one space-time and classical fields and one talks about frequencies and wavelengths instead of momenta.
- (b) $M^8 - H$ duality is a generalization of Fourier transform. Hitherto I have assumed that the space-time surface in M^8 is mapped to H . The momentum space interpretation at the level of M^8 however requires that the image must be a superposition of translates of the image in plane wave with some momentum: only the translates inside some bigger CD are allowed - this means infrared cutoff.

The total momentum as sum of momenta for two half-cones of CD in M^8 is indeed well-defined. One has a generalization of a plane wave over translational degrees of freedom of CD and restricted to a bigger CD.

At the limit of infinitely large size for bigger CD, the result is non-vanishing only when the sum of the momenta for two half-cones of CD vanishes: this corresponds to conservation of 4-momentum as a consequence of Poincare invariance rather than assumption as in the earlier approach [L29].

This generalizes the position-momentum duality of wave mechanics lost in quantum field theory. Point-like particle becomes a quantum superposition of space-time surfaces inside the causal diamond (CD). Plane wave is a plane wave for the superposition of space-time surfaces inside CD having the cm coordinates of CD as argument.

3. Inclusion hierarchy of supersymplectic algebras and FMR

The third inclusion hierarchy allowing to describe finite measurement resolution is defined by supersymplectic algebras acting as the isometries of the "world of classical worlds" (WCW) consisting of space-time surfaces are preferred extremals ("roots" of polynomials in M^8 and minimal surfaces satisfying infinite-D set of additional "gauge conditions" in H).

At a given level of hierarchy generators with conformal weight larger than n act like gauge generators as also their commutators with generators with conformal weight smaller than n correspond to vanishing Noether charges. This defines "gauge conditions".

To sum up, there are therefore 3 hierarchies allowing to describe finite measurement resolution and they must be essentially equivalent in TGD framework.

5.1.6 Three evolutionary hierarchies

There are three evolutionary hierarchies: hierarchies of extensions of extensions of... ofrationals...; inclusions of inclusions of of HFFs, and inclusions of isomorphic super symplectic algebras.

1. Extensions of rationals

The extensions of rationals become algebraically increasingly complex as their dimension increases. The co-associative space-time surfaces in M^8 are "roots" of real polynomials with rational coefficients to guarantee number theoretical universality and this means space-time surfaces are characterized by extension of rationals.

Each extension of rationals defines extensions for p-adic number fields and entire adèle. The interpretation is as a cognitive leap: the system's intelligence/algebraic complexity increases when the extension is extended further.

The extensions of extensions of define hierarchies with Galois groups in certain sense products of extensions involved. Exceptional extensions are those which do not allow this decomposition. In this case Galois group is a simple group. Simple groups are primes of finite groups and correspond to elementary particles of cognition. Kind of fundamental, non-decomposable ideas. Mystic might speak of pure states of consciousness with no thoughts.

In the evolution by quantum jumps the dimension of extension increases in statistical sense and evolution is unavoidable. This evolution is due to subjective time evolution by quantum jumps, something which is in spirit with Connes proposal but replaces time evolution by a sequence of evolutionary leaps.

2. Inclusions of HFFs as a hierarchy

HFFs are fractals. They have infinite inclusion hierarchies in which sub-HFF isomorphic to entire HFFs is included to HFF.

Also the hierarchies of inclusions define evolutionary hierarchies: HFF which is isomorphic with original becomes larger and in some sense more complex than the included factor. Also now one has sequences of inclusions of inclusions of... These sequences would correspond to sequences for extensions of extensions... of rationals. Note that the inclusion hierarchy would be the basic object: not only single HFF in the hierarchy.

3. Inclusions of supersymplectic algebras as an evolutionary hierarchy

The third hierarchy is defined by the fractal hierarchy of sub-algebras of supersymplectic algebra isomorphic to the algebra itself. At a given level of hierarchy generators with conformal weight larger than n correspond to gauge degrees of freedom. As n increases the number of physical degrees of freedom above measurement resolution increases which means evolution. This hierarchy should correspond rather concretely to that for the extensions of rationals. These hierarchies would be essentially one and the same thing in the TGD Universe.

5.1.7 TGD based model for subjective time development

The understanding of subjective time development as sequences of SSFRs preceded by unitary "time" evolution has improved quite considerably recently [L29]. The idea is that the subjective time development as a sequence of scalings at the light-cone boundary generated by the vibrational part \hat{L}_0 of the scaling generator $L_0 = p^2 - \hat{L}_0$ (L_0 annihilates the physical states). Also p-adic mass calculations use \hat{L}_0 .

For more than 10 years ago [K18, K12], I considered the possibility that Connes time evolution operator that he assigned with thermo-dynamical time could have a significant role in the definition of S-matrix in standard sense but had to give up the idea.

It however seems that for super-symplectic algebra \hat{L}_0 generates an outer automorphism since the algebra has only generators with conformal with $n > 0$ and its extension to included also generators with $n \leq 0$ is required to introduce L_0 : since L_0 contains annihilation operators, it indeed generates outer automorphism in SCA. The two views could be equivalent! Whereas Connes considered thermo-dynamical time evolution, in TGD framework the time evolution would be subjective time evolution by SSFRs.

- (a) The guess would be that the exponential of the scaling operator L_0 gives the time evolution. The problem is that L_0 annihilates the physical states. The solution of the problem would be the same as in p-adic thermodynamics. L_0 decomposes as $L_0 = p^2 - \hat{L}_0$ and the vibrational part \hat{L}_0 this gives mass spectrum as eigenvalues of p^2 . The thermo-dynamical state in p-adic thermodynamics is $p^{\hat{L}_0 \beta}$. This operator exists p-adically in the p-adic number field defined by prime p .
- (b) Could unitary subjective time development involve the operator $\exp(i2\pi L_0 \tau)$ $\tau = \log(T/T_0)$? This requires $T/T_0 = \exp(n/m)$ guaranteeing that exponential is a root of unity for an eigenstate of L_0 . The scalings are discretized and scalings come as powers of $e^{1/m}$. This is possible using extensions of rationals generated by a root of e . The unique feature of p-adics is that e^p is ordinary p-adic number. This alone would give periodic time evolution for eigenstates of L_0 with integer eigenvalues n .

5.1.8 SSA and SSA_n

Supersymplectic algebra SSA has fractal hierarchies of subalgebras SSA_n . The integers in a given hierarchy are of form $n_1, n_1 n_2, n_1 n_2 n_3, \dots$ and correspond naturally to hierarchies of inclusions of HFFs. Conformal weights are positive: $n > 0$. For ordinary conformal algebras also negative weights are allowed. Yangians have only non-negative weights. This is of utmost importance.

SSA_n with generators have radial light-like conformal weights coming as multiples of n . SSA_n annihilates physical states and $[SSA_n, SSA]$ does the same. Hence the generators with conformal weight larger than n annihilate the physical states.

What about generators with conformal weights smaller than n ? At least a subset of them need not annihilate the physical states. Since L_n are superpositions of creation operators, the idea that analogs of coherent states could be in question.

It would be nice to have a situation in which $L_n, n < m$ commute. $[L_k, L_l] = 0$ effectively for $k + l \geq m$.

The simplest way to obtain a set of effectively commuting operators is to take the generators $L_k, [m/2] < k < m$, where $[m/2]$ is nearest integer larger than $m/2$.

This raises interesting questions.

- (a) Could the Virasoro generators $O(\{c_k\}) = \sum_{k \in [m/2], m} c_k L_k$ as linear combinations of creation operators generate a set of coherent states as eigenstates of their Hermitian conjugates.
- (b) Some facts about coherent states are in order.
 - i. When one adds to quantum harmonic oscillator Hamiltonian oscillator a time dependent perturbation which lasts for a finite the vacuum state evolves to an oscillator vacuum whose position is displaced. The displacement is complex and is a Fourier component of the external force $f(t)$ corresponding to the harmonic oscillator frequency ω . Time evolution picks up only this component.
 - ii. Coherent state property means that the state is eigenstate of the annihilation creation operator with eigenvalue $\alpha = -ig(\omega)$ where $g(\omega) = \int f(u) \exp(-i\omega u) du$ is Fourier transform of $f(t)$.
 - iii. Coherent states are not orthogonal and form an overcomplete set. The overlaps of coherent states are proportional to a Gaussian depending on the complex parameters characterizing them. One can however develop any state in terms of coherent states as a unique expansion since one can represent unitary in terms of coherent states.
 - iv. Coherent state obtained from the vacuum state by time evolution in presence of $f(t)$ by a unitary displacement operator $D(\alpha) = \exp(\alpha a^\dagger - \bar{\alpha} a)$. (https://en.wikipedia.org/wiki/Displacement_operator).
The displacement operator is a unitary operator and in the general case the displacement is complex. The product of two displacement operators would be apart from a phase factor a displacement operator associated with the sum of displacements.
 - v. Harmonic oscillator coherent states are indeed maximally classical since wave packets have minimal width in both q and p space. Furthermore, the classical expectation values for q and p obey classical equations of motion.

These observations raise interesting questions about how the evolution by SSFRs could be modelled.

- i. Instead of harmonic oscillator in q -space, one would have time evolution in the space of scalings of causal diamond parameterized by the scaling parameter $\tau = \log(T/T_0)$, where T can be identified as the radial light-like coordinate of light-cone boundary.
The analogs of harmonic oscillator states would be defined in this space and would be essentially wave packets with ground state minimizing the width of the wave packet.

- ii. The role of harmonic oscillator Hamiltonian in absence of external force would be taken by the generator \hat{L}_0 ($L_0 = p^2 - \hat{L}_0$ acts trivially) and gives rise to mass squared quantization. The situation would be highly analogous to that in p-adic thermodynamics. The role of ω would be taken by the minimal conformal weight h_{min} such that the eigenvalues of L_0 are its multiples. It seems that this weight must be equal to $h_{min} = 1$.
The commutations of $\hbar L_0$ with L_k , $k > 0$ would be as for L_0 so what the replacement should not affect the situation.
- iii. The scaling parameter τ is analogous to the spatial coordinate q for the harmonic oscillator. Can one identify the analog of the external force $f(t)$ acting during unitary evolution between two SSFRs? Or is it enough to use only the analog of $g(\omega \rightarrow h_{min} = 1)$ - that is the coefficients C_k .
To identify $f(t)$, one needs a time coordinate t . This was already identified as τ . This one would have $q = t$, which looks strange. The space in which time evolution is the space of scalings and the time evolutions are scalings and thus time evolution means translation in this space. The analog for this would be Hamiltonian $H = i\hbar d/dq$.
Number theoretical universality allows only the values of $\tau = r/s$ whose exponents give roots of unity. Also $exp(n\tau)$ makes sense p-adically for these values. This would mean that the Fourier transform defining g would become discrete and be sum over the values $f(\tau = r/s)$.
- iv. What happens if one replaces \hat{L}_0 with L_0 . In this case one would have the replacement of ω with $h_{vac} = 0$. Also the analog of Fourier transform with zero frequency makes sense. $\hat{L}_0 = p^2 - L_0$ is the most natural choice for the Hamiltonian defining the time evolution operator but is trivial. Could $\Delta^{i\tau}$ describe the inherent time evolution. It would be outer automorphism since it is not defined solely in terms of SCA. So: could one have $\Delta = exp(\hat{L}_0)$ so that $\Delta^{i\tau}$ coincide with $exp(i\hat{L}_0\tau)$? This would mean the identification

$$\Delta = exp(\hat{L}_0) ,$$

which is a positive definite operator. The exponents coming from $exp(iL_0\tau)$ can be number theoretically universal if $\tau = log(T/T_0)$ is a rational number implying $T/T_0 = exp(r/s)$, which is possible number theoretically) and the extension of rationals contains some roots of e .

- v. Could one have $\Delta = L_0$? Also now that positivity condition would be satisfied if SSA conformal weights satisfy $n > 0$.

The problem with this operation is that it is not number theoretically universal since the exponents $exp(i log(n)\tau)$ do not exist p-adically without introducing infinite-D extension of p-adic number making $log(n)$ well-defined.

What is however intriguing is that the "time" evolution operator $\Delta^{i\tau}$ in the eigenstate basis would have trace equal to $Tr(\Delta^{i\tau}) \sum d(n)n^{i\tau}$, where $d(n)$ is the degeneracy of the state. This is a typical zeta function: for Riemann Zeta one has $d(n) = 1$.

For $\Delta = exp(L_0)$ option $Tr(\Delta^{i\tau}) = \sum d(n)exp(in\tau)$ exists for $\tau = r/s$ if r :th root of e belongs to the extension of p-adics.

To sum up, one would have Gaussian wave packet as harmonic oscillator vacuum in the space of scaled variants of CD. The unitary time evolution associated with SSFR would displace the peak of the wave packet to a larger scalings. The Gaussian wave function in the space of scaled CDs has been proposed earlier.

Could this time evolution make sense and be even realistic?

- (a) The analogs of harmonic oscillator states are defined in the space of scalings as Gaussians and states obtained from them using oscillator operators. There would be a wave function in the moduli space of CDs analogous to a state of harmonic oscillator.

- (b) SSFR following the time evolutions would project to an eigenstate of harmonic oscillator having in general displaced argument. The unitary displacement operator D should commute with the operators having the members of zero energy states at the passive boundary of CD as eigenstates. This poses strong conditions. At least number theoretic measurements could satisfy these conditions.
- (c) SSFRs are identified as weak measurements as near as possible to classical measurements. Time evolution by the displacement would be indeed highly analogous to classical time evolution for the harmonic oscillator.
- (d) The unitary displacement operator corresponds to the arbitrary external force on the harmonic oscillator and it seems that it would be selected in SSFR for the unitary evolution after SSFR. This means fixing the coefficients C_k in the operator $\sum C_k L_k$.

What is the subjective "time" evolution operator when in the case of SSA_n ?

- (a) The scaling analog of the unitary displacement operator D as $D = \sum \exp(\sum C_k L_k - \bar{C}_k L_{-k})$ is highly suggestive and would take the oscillator vacuum to a coherent state. Coefficients C_k would be proportional to τ . There would be a large number of choices for the unitary displacement operator. One can also consider complex values of τ since one has complexified M^8 .
- (b) There should be a normalization for the coefficients: without this it is not possible to talk about a special value of τ does not make sense. For instance, the sum of their moduli squared could be equal to 1. This would give interpretation as a quantum state in the degrees of freedom considered. The width of the Gaussian would increase slowly during the unitary time evolution and be proportional to $\log(T/T_0)$.
The width of the Gaussian would increase slowly as a function of T during the unitary time evolution and be proportional to $\log(T/T_0)$. The condition that c_k are proportional the same complex number times τ is too strong.
- (c) The arbitrariness in the choice of C_k would bring in a kind of non-determinism as a selection of this superposition. The ability to engineer physical systems is in conflict with the determinism of classical physics and also difficult to understand in standard quantum physics. Could one interpret this choice as an analog for engineering a Hamiltonian as in say quantum computation or build-up of an electric circuit for some purpose? Could goal directed action correspond to this choice?
If so engineerable degrees of freedom would correspond to intermediate degrees of freedom associated with $L_k, [m/2] \leq k \leq m$. They would be totally absent for $k = 1$ and this would correspond to a situation analogous to the standard physics without any intentional action.

6 The dynamics of SSFRs as quantum measurement cascades in the group algebra of Galois group

Adelic physics [L7, L9] is a proposal for the physics of both sensory experience having real physics as correlate and cognition having various p-adic physics as correlates. Adele is a book-like structure formed by real numbers and the extensions of p-adic number fields induced by a given extension of rationals with the pages of the book glued together along its back consisting of numbers belonging to the extension of rationals. This picture generalizes to space-time level. Adelic physics relies on the notion of cognitive representation as unique number theoretic discretization of the space-time surface. This discretization has also fermionic analog in terms of spinor structure associated with the group algebra of the Galois group of extension.

Adelic physics, $M^8 - H$ duality, and zero energy ontology lead (ZEO) to a proposal that the dynamics involved with "small" state function reductions (SSFRs) as counterparts of weak measurements could be basically number theoretical dynamics with SSFRs identified as reduction cascades leading to completely un-entangled state in the space of wave functions in

Galois group of extension of rationals identifiable as wave functions in the space of cognitive representations. As a side product a prime factorization of the order of Galois group is obtained.

The result looks even more fascinating if the cognitive dynamics is a representation for the dynamics in real degrees of freedom in finite resolution characterized by the extension of rationals. If cognitive representations represent reality approximately, this indeed looks very natural and would provide an analog for adèle formula expressing the norm of a rational as the inverse of the product of its p-adic norms.

6.1 Adelic physics very briefly

Number theoretic vision leading to adelic physics [L7] provides a general formulation of TGD complementary to the vision [K23] (<http://tinyurl.com/sh42dc2>) about physics as geometry of “world of classical words” (WCW).

- (a) p-Adic number fields and p-adic space-time sheets serve as correlates of cognition. Adele is a Cartesian product of reals and extensions of all p-adic number fields induced by given extension of rationals. Adeles are thus labelled by extensions of rationals, and one has an evolutionary hierarchy labelled by these extensions. The larger the extension, the more complex the extension which can be regarded as $n - D$ space in K sense, that is with K -valued coordinates.
- (b) Evolution is assigned with the increase of algebraic complexity occurring in statistical sense in BSFRs, and possibly also during the time evolution by unitary evolutions and SSFRs following them. Indeed, in [L24] (<http://tinyurl.com/quoftt1>) I considered the possibility that the time evolution of self in this manner could be induced by an iteration of polynomials - at least in approximate sense. Iteration is a universal manner to produce fractals as Julia sets and this would lead to the emergence of Mandelbrot and Julia fractals and their 4-D generalizations. In the sequel will represent an argument that the evolution as iterations could hold true in exact sense.

Cognitive representations are identified as intersection of reality and various p-adicities (cognition). At space-time level they consist of points of embedding space $H = M^4 \times CP_2$ or M^8 ($M^8 - H$ duality [L4, L5, L6] allows to consider both as embedding space) having preferred coordinates - M^8 indeed has almost unique linear M^8 coordinates for a given octonion structure.

- (c) Given extension of given number field K (rationals or extension of rationals) is characterized by its Galois group leaving K - say rationals - invariant and mapping products to products and sums to sums. Given extension E of rationals decomposes to extension E_N of extension E_{N-1} of ... of extension E_1 - denote it by $E \equiv H_N = E_N \circ E_{N-1} \dots \circ E_1$. It is represented at the level of classical space-time dynamics in M^8 (<http://tinyurl.com/quoftt1>) by a polynomial P which is functional composite $P = P_N \circ P_{N-1} \circ \dots \circ P_1$. with $P_i(0) = 0$. The Galois group of $G(E)$ has the Galois group $H_{N-1} = G(E_{N-1} \circ \dots \circ E_1)$ as a normal subgroup so that $G(E)/H_{N-1}$ is group. The elements of $G(E)$ allow a decomposition to a product $g = h_{N-1} \times h_{N-1} \times \dots$ and the order of $G(E)$ is given as the product of orders of H_k : $n = n_0 \times \dots \times n_{N-1}$. This factorization of prime importance also from quantum point of view. Galois groups with prime order do not allow this decomposition and the maximal decomposition and are actually cyclic groups Z_p of prime order so that primes appear also in this manner.

Second manner for primes to appear is as ramified primes p_{ram} of extension for which the p-adic dynamics is critical in a well-defined sense since the irreducible polynomial with rational coefficients defining the extension becomes reducible (decomposes into a product) in order $O(p) = 0$. The p-adic primes assigned to elementary particles in p-adic calculation have been identified as ramified primes but also the primes labelling prime extensions possess properties making them candidates for p-adic primes.

Iterations correspond to the sequence $H_k = G_0^{ok}$ of powers of generating Galois groups for the extension of K serving as a starting point. The order of H_k is the power n_0^k of

integer $n_0 = \prod p_{0i}^{k_i}$. Now new primes emerges in the decomposition of n_0 . Evolution by iteration is analogous to a unitary evolution as ex^{iHt} power of Hamiltonian, where t parameter takes the role of k .

- (d) The complexity of extension is characterized by the orders n and the orders n_k as also the number N of the factors. In the case of iterations of extension the limit of large N gives fractal.
- (e) Galois group acts in the space of cognitive representations and for Galois extensions for which Galois group has same order as extensions, it is natural do consider quantum states as wave functions in $G(E)$ forming n -D group algebra. One can assign to the group algebra also spinor structure giving rise to $D = 2^{M/2}$ fermionic states where one has $N = 2M$ or $N = 2M + 1$). One can also consider chirality constraints reducing D by a power of 2. An attractive idea is that this spinor structure represents many-fermion states consisting of $M/2$ fermion modes and providing representation of the fermionic Fock space in finite measurement resolution.

6.2 Number theoretical state function reductions as symmetry breaking cascades and prime factorizations

The proposed picture has very important quantal implications and allows to interpret number theoretic quantum measurement as a number theoretic analog for symmetric breaking cascade and also as a factorization of an integer into primes.

- (a) The wave functions in $G(E)$ - elements of group algebra of $G(E)$ can be decomposed to tensor products of wave functions in $G(E)/H_{N-1}$ and H_{N-1} : these wave functions in general represent entangled states. One can decompose the wave functions in H_{N-1} in similar manner and the process can be continued so that one obtains a maximal decomposition allowing no further decomposition for any factor. These non-decomposable Galois groups have prime order since its group algebra as Hilbert space of prime dimension has no decomposition into tensor product.
- (b) In state function reduction of wave function $G(E)$ the density matrices associated with pairs $G(E)/H_{N-1}$ and H_{N-1} are measured. The outcome is an eigenstate or eigen-space and gives rise to symmetry breaking from $G(E) \equiv H_N$ to $E_N \times H_{N-1}$. The sequence of state function reductions should lead to a maximal symmetry breaking corresponding to a wave function as a produce of those associaetd with Galois groups of prime order. This define a prime factorization of the dimension n of Galois group/extension to $n = \prod_{i=1}^N p_i^{k_i}!$ The moments of consciousness for self would correspond to prime factorizations! Self would be number theoretician quite universally!

Also also the fermionic cognitive representation based on finite-D Fock states defined by spinor components of $G(E)$ is involved. The interpretation of Fock state basis as a a basis of Boolean algebra in TGD: the spinor structure of WCW could be representation for Boolean logic as a “square root” of Kähler geometry of WCW. Cognition indeed involves also Boolean logic.

6.3 SSFR as number theoretic state function reduction cascade and factorization of integer

A highly interesting unanswered question is following. “Small” state function reductions (SSFRs) define the life cycle of self as their sequence. What are the degrees of freedom where SSFRs occur?

- (a) SSFRs take place at the active boundary of CD which shifts in statistical sense towards future in the sequence of state function reductions. State at the passive boundary is not changed.

- (b) The idea that quantum randomness could correspond to classical chaos (or complexity) associated with the iteration of polynomials (Mandelbrot and Julia fractals) [L24] led to reconsider the hypothesis that the polynomial representing space-time decomposes to a product $P = P_2(T - r) \times P_1(r)$. T corresponds to the distance between the tips of CD and $r = t$ to the radial coordinate of M^4 assignable to the passive boundary of CD and equal to time coordinate t . $P_i(0) = 0$ is assumed to hold true.

P_2 would change in SSFRs whereas P_1 and state at passive boundary would not. SSFRs (analogous to so called weak measurements) at active boundary would give rise to sensory input and various associations - Maya in Eastern terminology. P_1 would correspond to the unchanging part of self - "soul" or real self as one might say.

I was also led to consider a simplified hypothesis that P_2 is obtained as iteration $P_2 = Q_1^{\circ n}$ in n :th n unitary evolution preceding SSFR. One would start from some iterate $Q_1^{\circ k}$. This would reduce quantum dynamics to iteration of polynomials and to a deep connection with Mandelbrot and Julia fractals but it was quite clear why this would be true.

- (c) The mere factorization $P = P_2 \times P_1$ implies that the Galois groups associated with active and passive boundary of CD commute and number theoretic state function reduction cascade for the wave functions in $G(E)$ for the extension determined by P_2 at active boundary could correspond to SSFR. Or course, also other commuting degrees of freedom are possible but number theoretic degrees of freedom could be the most important degrees of freedom involved with SSFRs.

6.4 The quantum dynamics of dark genes as factorization of primes

Gene level provides a fascinating application of this picture.

This contribution was inspired by discussion with Bruno Marchal about his with title "Do the laws of physics apply to the mind?" (<https://tinyurl.com/yc1s2bpt>). Bruno Marchal is a representative of computationalism, which might be called idealistic and Bruno believes that physics follows from computationalism. The somewhat mystical notion of self-reference is believed to lead to consciousness. I do not share this view. The gist of the posting comes towards end where I describe how computationalism generalizes to quantum computationalism in TGD generalizing also the notion of quantum computation. What conscious problem solving is? This is the question to be discussed.

- (a) As found, dark photons and dark protons forming DNA codons as triplets could correspond to triplet representations for prime factor Z_3 of Galois group of Z_6 . Codon and conjugate codon could in turn correspond to the prime factor Z_2 of Galois group Z_6 so that double strand would correspond to Z_6 suggested by findings of Mills [L2] and TGD inspired model color vision [L11].
- (b) DNA codons could correspond to extension with Galois group Z_3 , and one can consider an entire hierarchy of extensions of extensions of .. extensions with dimensions n_i satisfying thus $n = \prod_{i=1}^N n_i$ and having Z_6 as subgroup at the lowest level of the hierarchy. The number N of factors would be the number of polynomials in the functional composition and thus define a kind of abstraction levels (abstractions are thoughts about thoughts about..., maps of maps of ...). N is expected to increase in evolution.
- (c) Could this abstraction hierarchy be realized at gene level? Genes decompose into transcribed regions - exons - and introns. Could different decomposition of genes to exons and introns correspond to different values of N and n_i and to different Galois groups. Could genes themselves form larger composites?

Could genomes form even large structures such as chromosomes with larger Galois groups. Years ago I considered the possibility of a collective gene expression based on the collective MB of organelle, organ, or even population: could this correspond to an extension associated with several genomes?

- (d) Could SSFR correspond to a sequence of symmetry breakings for the Galois groups of these structures decomposing them to sub-groups? Number theoretic interpretation

would in terms of decompositions of integers to primes! Genome would be a quantum computer performing number theory!

- (e) Metabolic energy feed would increase h_{eff} would also increase the orders $n_i = h_{eff}/h_0$ of the extensions appearing in the composition of extensions and thus the orders of polynomial factors P_i in the functional composite defining the extensions. Therefore the decompositions would be dynamical.

Metabolic energy feed requires BSFR changing the arrow of time if metabolic energy feed is actually feed of negative energy to environment. The emergence of a new prime factorization would require BSFR. That the time evolution by iterations would not require BSFR would support the proposal that time evolution by BSFRs could be induced by iteration dynamics for the polynomial P_2 assignable to the active boundary of CD.

6.5 The relationship of TGD view about consciousness to computationalism

This text was inspired by discussion with Bruno Marchal about his with title "Do the laws of physics apply to the mind?" (<https://tinyurl.com/yc1s2bpt>). Bruno Marchal is a representative of computationalism, which might be called idealistic and Bruno believes that physics follows from computationalism. The somewhat mystical notion of self-reference is believed to lead to consciousness.

I do not share this view. The gist of the posting comes towards end where I describe how computationalism generalizes to quantum computationalism in TGD generalizing also the notion of quantum computation. What conscious problem solving is? This is the question to be discussed.

To my view computationalism is one of the failed approaches to consciousness - it cannot cope with free will for instance. It however contains an essential aspect which is correct: the idea of deterministic program leading from A to B. Problem solving can be regarded as attempt to find this program. You fix A as initial data and try to find a program leading from A to a final state characterized by data B. The program has duration T and can be very long and it is not clear whether it exists at all. You try again and again and eventually you might find it. In the real conscious problem solving this process means making guesses so that the process cannot be deterministic.

What does this view about problem solving correspond to in ZEO? We have states A and B represented as quantum states and we try to find quantum analog of classical program leading from A to B in some time T which can be varied.

- (a) A and B are realized as superpositions of 3-surfaces and fermionic states at them - located at time values $t=0$ and $t=T$. T can vary. Can we find by varying T a (superposition of) deterministic time evolution(s) - preferred extremal(s) (PE) - connecting A and B?

In ZEO and for fixed A and T PE in general does not exist. In ideal situation (infinite measurement resolution) and for given A and T, B is unique if it exists at all. One has analog of Bohr orbit and the quantum analog of classical program as the superposition of Bohr orbits starting from A and hopefully leading to B as a solution of the problem.

Remark: These superpositions can be regarded as counterparts of functions in biology and behaviors in neuroscience. The big difference to standard physics is that time=constant snapshot in time evolution of say bio-system is replaced with quantum superposition of very special time evolutions - PEs. Darwinian selection of also behaviors in biology correlates strongly with this.

- (b) So: given A and B, we try to find a value of T for which superposition of PEs from A to B exists. This would be the quantum program leading from A to B, and solving our problem.

Actually, not only ours, universe is full of conscious entities solving problems at various levels of self hierarchy. This takes place by a sequences of "small" SFRs (SSFRs, weak

measurements) increasing T in statistical sense and replacing the state at B with a new one determined by state A for given value of T . At the level of conscious experience this is sensory perception and all that which is associated with it.

Finding the solution is analogous to the halting of quantum Turing machine by ordinary state function reduction, which corresponds in ZEO to a "big" (ordinary) SFR (BSFR). This would mean death in universal sense and reincarnation with reversed arrow of time in ZEO? Or is BSFR and death failure to solve the problem? I cannot answer.

Remark: The notion of self-reference is replaced with much more concrete notion of becoming conscious of what one was conscious of before SSFR. SSFR indeed gives rise to conscious experience and one avoids the infinite regress associated with genuine self-reference. As an additional bonus one obtains evolution since the extension of rationals characterizing space-time surfaces can increase meaning higher level of consciousness. At the limit algebraic numbers the cognitive representation is dense subset of space-time surface.

- (c) Also finite measurement resolution and discreteness characterizing computation emerge from number theory.

To be a solution classically means that the 3-surface(s) representing B to have fixed discrete cognitive representation given by finite number of embedding space points in the extension of rationals defining the adèle. Quantally, quantum superpositions of these points with fixed quantum numbers represent the desired final state.

Also Boolean logic emerges at fundamental level as square root of Kähler geometry one might say. Many-fermion state basis defines a Boolean algebra and time evolution for induced spinors is analogous to truth preserving Boolean map in which truths code for infinite number of conservation laws associated with symmetries of WCW.

- (d) How to find the possibly existing solution at given step (unitary evolution plus SSFR) with $t=T$? One performs cognitive quantum measurements at each step represented by SSFR. They reduce to cascades of quantum measurements for the states in the group algebra of Galois group - call it Gal - of Galois extension considered.

Gal has hierarchical decomposition to inclusion hierarchy of normal subgroups implying the representation of states in group algebra of Gal as entangled states in the tensor product of the group algebras of normal sub-groups of Gal. The hope is that this Galois cascade of SFRs produces desired state as an outcome and one can shout "Eureka!".

7 The relation between U-Matrix and M-matrices

S-matrix is the key notion in quantum field theories. In Zero Energy Ontology (ZEO) this notion must be replaced with the triplet U-matrix, M-matrix, and S-matrix. U-matrix realizes unitary time evolution in the space for zero energy states realized geometrically as dispersion in the moduli space of causal diamonds (CDs) leaving second boundary (passive boundary) of CD and states at it fixed.

This process can be seen as the TGD counterpart of repeated state function reductions leaving the states at passive boundary unaffected and affecting only the member of state pair at active boundary (Zeno effect) [K17]. In TGD inspired theory of consciousness self corresponds to the sequence of these state function reductions [K27, K2, K24]. M-matrix describes the entanglement between positive and negative energy parts of zero energy states and is expressible as a hermitian square root H of density matrix multiplied by a unitary matrix S , which corresponds to ordinary S-matrix, which is universal and depends only the size scale n of CD through the formula $S(n) = S^n$. M-matrices and H-matrices form an orthonormal basis at given CD and H-matrices would naturally correspond to the generators of super-symplectic algebra.

The first state function reduction to the opposite boundary corresponds to what happens in quantum physics experiments. The relationship between U- and S-matrices has remained poorly understood.

The original view about the relationship was a purely formal guess: M -matrices would define the orthonormal rows of U -matrix. This guess is not correct physically and one must consider in detail what U -matrix really means.

- (a) First about the geometry of CD [K18]. The boundaries of CD will be called passive and active: passive boundary correspond to the boundary at which repeated state function reductions take place and give rise to a sequence of unitary time evolutions U followed by localization in the moduli of CD each. Active boundary corresponds to the boundary for which U induces delocalization and modifies the states at it.

The moduli space for the CDs consists of a discrete subgroup of scalings for the size of CD characterized by the proper time distance between the tips and the sub-group of Lorentz boosts leaving passive boundary and its tip invariant and acting on the active boundary only. This group is assumed to be represented unitarily by matrices Λ forming the same group for all values of n .

The proper time distance between the tips of CDs is quantized as integer multiples of the minimal distance defined by CP_2 time: $T = nT_0$. Also in quantum jump in which the size scale n of CD increases the increase corresponds to integer multiple of T_0 . Using the logarithm of proper time, one can interpret this in terms of a scaling parametrized by an integer. The possibility to interpret proper time translation as a scaling is essential for having a manifest Lorentz invariance: the ordinary definition of S-matrix introduces preferred rest system.

- (b) The physical interpretation would be roughly as follows. M -matrix for a given CD codes for the physics as we usually understand it. M -matrix is product of square root of density matrix and S-matrix depending on the size scale of CD and is the analog of thermal S-matrix. State function at the opposite boundary of CD corresponds to what happens in the state function reduction in particle physics experiments. The repeated state function reductions at same boundary of CD correspond to TGD version of Zeno effect crucial for understanding consciousness. Unitary U -matrix describes the time evolution zero energy states due to the increase of the size scale of CD (at least in statistical sense). This process is dispersion in the moduli space of CDs: all possible scalings are allowed and localization in the space of moduli of CD localizes the active boundary of CD after each unitary evolution.

In the following I will proceed by making questions. One ends up to formulas allowing to understand the architecture of U -matrix and to reduce its construction to that for S-matrix having interpretation as exponential of the generator L_1 of the Virasoro algebra associated with the super-symplectic algebra.

7.1 What can one say about M-matrices?

- (a) The first thing to be kept in mind is that M -matrices act in the space of zero energy states rather than in the space of positive or negative energy states. For a given CD M -matrices are products of hermitian square roots of hermitian density matrices acting in the space of zero energy states and universal unitary S-matrix $S(CD)$ acting on states at the active end of CD (this is also very important to notice) depending on the scale of CD:

$$M^i = H^i \circ S(CD) .$$

Here “ \circ ” emphasizes the fact that S acts on zero energy states at active boundary only. H^i is hermitian square root of density matrix and the matrices H^i must be orthogonal for given CD from the orthonormality of zero energy states associated with the same CD. The zero energy states associated with different CDs are not orthogonal and this makes the unitary time evolution operator U non-trivial.

- (b) Could quantum measurement be seen as a measurement of the observables defined by the Hermitian generators H^i ? This is not quite clear since their action is on zero energy

states. One might actually argue that the action of this kind of observables on zero energy states does not affect their vanishing net quantum numbers. This suggests that H^i carry no net quantum numbers and belong to the Cartan algebra. The action of S is restricted at the active boundary of CD and therefore it does not commute with H^i unless the action is in a separate tensor factor. Therefore the idea that S would be an exponential of generators H^i and thus commute with them so that H^i would correspond to sub-spaces remaining invariant under S acting unitarily inside them does not make sense.

- (c) In TGD framework symplectic algebras isometries of WCW is analogous to a Kac-Moody algebra with finite-dimensional Lie-algebra replaced with the infinite-dimensional symplectic algebra with elements characterized by conformal weights [K9, K8]. There is a temptation to think that the H^i could be seen as a representation for this algebra or its sub-algebra. This algebra allows an infinite fractal hierarchy of sub-algebras of the super-symplectic algebra isomorphic to the full algebra and with conformal weights coming as n -ples of those for the full algebra. In the proposed realization of quantum criticality the elements of the sub-algebra characterized by n act as a gauge algebra. An interesting question is whether this sub-algebra is involved with the realization of M-matrices for CD with size scale n . The natural expectation is that n defines a cutoff for conformal weights relating to finite measurement resolution.

7.2 How does the size scale of CD affect M-matrices?

- (a) In standard quantum field theory (QFT) S-matrix represents time translation. The obvious generalization is that now scaling characterized by integer n is represented by a unitary S-matrix that is as n :th power of some unitary matrix S assignable to a CD with minimal size: $S(CD) = S^n$. $S(CD)$ is a discrete analog of the ordinary unitary time evolution operator with n replacing the continuous time parameter.
- (b) One can see M-matrices also as a generalization of Kac-Moody type algebra. Also this suggests $S(CD) = S^n$, where S is the S-matrix associated with the minimal CD. S becomes representative of phase $exp(i\phi)$. The inner product between CDs of different size scales can n_1 and n_2 can be defined as

$$\langle M^i(m), M^j(n) \rangle = Tr(S^{-m} \circ H^i H^j \circ S^n) \times \theta(n - m) , \quad (7.1)$$

$$\theta(n) = 1 \text{ for } n \geq 0 , \quad \theta(n) = 0 \text{ for } n < 0 .$$

Here I have denoted the action of S-matrix at the active end of CD by “ \circ ” in order to distinguish it from the action of matrices on zero energy states which could be seen as belonging to the tensor product of states at active and passive boundary.

It turns out that unitarity conditions for U-matrix are invariant under the translations of n if one assumes that the transitions obey strict arrow of time expressed by $n_j - n_i \geq 0$. This simplifies dramatically unitarity conditions. This gives orthonormality for M-matrices associated with identical CDs. This inner product could be used to identify U-matrix.

- (c) How do the discrete Lorentz boosts affecting the moduli for CD with a fixed passive boundary affect the M-matrices? The natural assumption is that the discrete Lorentz group is represented by unitary matrices λ : the matrices M^i are transformed to $M^i \circ \lambda$ for a given Lorentz boost acting on states at active boundary only.

One cannot completely exclude the possibility that S acts unitarily at both ends of zero energy states. In this case the scaling would be interpreted as acting on zero energy states rather than those at active boundary only. The zero energy state basis defined by M_i would depend on the size scale of CD in more complex manner. This would not affect the above formulas except by dropping away the “ \circ ”.

Unitary U must characterize the transitions in which the moduli of the active boundary of causal diamond (CD) change and also states at the active boundary (paired with unchanging states at the passive boundary) change. The arrow of the experienced flow of time emerges during the period as state function reductions take place to the fixed (“passive”) boundary of CD and do not affect the states at it. Note that these states form correlated pairs with the changing states at the active boundary. The physically motivated question is whether the arrow of time emerges statistically from the fact that the size of CD tends to increase in average sense in repeated state function reductions or whether the arrow of geometric time is strict. It turns out that unitarity conditions simplify dramatically if the arrow of time is strict.

7.3 What Can One Say About U -Matrix?

- (a) Just from the basic definitions the elements of a unitary matrix, the elements of U are between zero energy states (M-matrices) between two CDs with possibly different moduli of the active boundary. Given matrix element of U should be proportional to an inner product of two M -matrices associated with these CDs. The obvious guess is as the inner product between M-matrices

$$\begin{aligned} U_{m,n}^{ij} &= \langle M^i(m, \lambda_1), M^j(n, \lambda_2) \rangle \\ &= \text{Tr}(\lambda_1^\dagger S^{-m} \circ H^i H^j \circ S^n \lambda_2) \\ &= \text{Tr}(S^{-m} \circ H^i H^j \circ S^n \lambda_2 \lambda_1^{-1}) \theta(n-m) . \end{aligned} \quad (7.2)$$

Here the usual properties of the trace are assumed. The justification is that the operators acting at the active boundary of CD are special case of operators acting non-trivially at both boundaries.

- (b) Unitarity conditions must be satisfied. These conditions relate S and the hermitian generators H^i serving as square roots of density matrices. Unitarity conditions $UU^\dagger = U^\dagger U = 1$ is defined in the space of zero energy states and read as

$$\sum_{j_1 n_1} U_{m n_1}^{i j_1} (U^\dagger)_{n_1 n}^{j_1 j} = \delta^{i,j} \delta_{m,n} \delta_{\lambda_1, \lambda_2} \quad (7.3)$$

To simplify the situation let us make the plausible hypothesis contribution of Lorentz boosts in unitary conditions is trivial by the unitarity of the representation of discrete boosts and the independence on n .

- (c) In the remaining degrees of freedom one would have

$$\sum_{j_1, k \geq \text{Max}(0, n-m)} \text{Tr}(S^k \circ H^i H^{j_1}) \text{Tr}(H^{j_1} H^j \circ S^{n-m-k}) = \delta^{i,j} \delta_{m,n} . \quad (7.4)$$

The condition $k \geq \text{Max}(0, n-m)$ reflects the assumption about a strict arrow of time and implies that unitarity conditions are invariant under the proper time translation $(n, m) \rightarrow (n+r, m+r)$. Without this condition n back-wards translations (or rather scalings) to the direction of geometric past would be possible for CDs of size scale n and this would break the translational invariance and it would be very difficult to see how unitarity could be achieved. Stating it in a general manner: time translations act as semigroup rather than group.

- (d) Irreversibility reduces dramatically the number of the conditions. Despite this their number is infinite and correlates the Hermitian basis and the unitary matrix S . There is an obvious analogy with a Kac-Moody algebra at circle with S replacing the phase

factor $\exp(in\phi)$ and H^i replacing the finite-dimensional Lie-algebra. The conditions could be seen as analogs for the orthogonality conditions for the inner product. The unitarity condition for the analog situation would involve phases $\exp(ik\phi_1) \leftrightarrow S^k$ and $\exp(i(n-m-k)\phi_2) \leftrightarrow S^{n-m-k}$ and trace would correspond to integration $\int d\phi_1$ over ϕ_1 in accordance with the basic idea of non-commutative geometry that trace corresponds to integral. The integration of ϕ_i would give $\delta_{k,0}$ and $\delta_{m,n}$. Hence there are hopes that the conditions might be satisfied. There is however a clear distinction to the Kac-Moody case since S^n does not in general act in the orthogonal complement of the space spanned by H^i .

- (e) The idea about reduction of the action of S to a phase multiplication is highly attractive and one could consider the possibility that the basis of H^i can be chosen in such a way that H^i are eigenstates of S . This would reduce the unitarity constraint to a form in which the summation over k can be separated from the summation over j_1 .

$$\sum_{k \geq \text{Max}(0, n-m)} \exp(iks_i - (n-m-k)s_j) \sum_{j_1} \text{Tr}(H^i H^{j_1}) \text{Tr}(H^{j_1} H^j) = \delta^{i,j} \delta_{m,n} . \quad (7.5)$$

The summation over k should give a factor proportional to δ_{s_i, s_j} . If the correspondence between H^i and eigenvalues s_i is one-to-one, one obtains something proportional to $\delta(i, j)$ apart from a normalization factor. Using the orthonormality $\text{Tr}(H^i H^j) = \delta^{i,j}$ one obtains for the left hand side of the unitarity condition

$$\exp(is_i(n-m)) \sum_{j_1} \text{Tr}(H^i H^{j_1}) \text{Tr}(H^{j_1} H^j) = \exp(is_i(n-m)) \delta_{i,j} . \quad (7.6)$$

Clearly, the phase factor $\exp(is_i(n-m))$ is the problem. One should have Kronecker delta $\delta_{m,n}$ instead. One should obtain behavior resembling Kac-Moody generators. H^i should be analogs of Kac-Moody generators and include the analog of a phase factor coming visible by the action of S .

7.4 How to obtain unitarity correctly?

It seems that the simple picture is not quite correct yet. One should obtain somehow an integration over angle in order to obtain Kronecker delta.

- (a) A generalization based on replacement of real numbers with function field on circle suggests itself. The idea is to identify eigenvalues of generalized Hermitian/unitary operators as Hermitian/unitary operators with a spectrum of eigenvalues, which can be continuous. In the recent case S would have as eigenvalues functions $\lambda_i(\phi) = \exp(is_i\phi)$. For a discretized version ϕ would have discrete spectrum $\phi(n) = 2\pi k/n$. The spectrum of λ_i would have n as cutoff. Trace operation would include integration over ϕ and one would have analogs of Kac-Moody generators on circle.
- (b) One possible interpretation for ϕ is as an angle parameter associated with a fermionic string connecting partonic 2-surface. For the super-symplectic generators suitable normalized radial light-like coordinate r_M of the light-cone boundary (containing boundary of CD) would be the counterpart of angle variable if periodic boundary conditions are assumed.

The eigenvalues could have interpretation as analogs of conformal weights. Usually conformal weights are real and integer valued and in this case it is necessary to have generalization of the notion of eigenvalues since otherwise the exponentials $\exp(is_i)$ would be trivial. In the case of super-symplectic algebra I have proposed that the generating elements of the algebra have conformal weights given by the zeros of Riemann

zeta. The spectrum of conformal weights for the generators would consist of linear combinations of the zeros of zeta with integer coefficients. The imaginary parts of the conformal weights could appear as eigenvalues of S .

- (c) It is best to return to the definition of the U-matrix element to check whether the trace operation appearing in it can already contain the angle integration. If one includes to the trace operation appearing the integration over ϕ it gives $\delta_{m,n}$ factor and U-matrix has elements only between states assignable to the same causal diamond. Hence one must interpret U-matrix elements as functions of ϕ realized factors $\exp(i(s_n - s_m)\phi)$. This brings strongly in mind operators defined as distributions of operators on line encountered in the theory of representations of non-compact groups such as Lorentz group. In fact, the unitary representations of discrete Lorentz groups are involved now.
- (d) The unitarity condition contains besides the trace also the integrations over the two angle parameters ϕ_i associated with the two U-matrix elements involved. The left hand side of the unitarity condition reads as

$$\sum_{k \geq \text{Max}(0, n-m)} I(ks_i)I((n-m-k)s_j) \times \sum_{j_1} \text{Tr}(H^i H^{j_1}) \text{Tr}(H^{j_1} H^j) = \delta^{i,j} \delta_{m,n} ,$$

$$I(s) = \frac{1}{2\pi} \times \int d\phi \exp(is\phi) = \delta_{s,0} .$$
(7.7)

Integrations give the factor $\delta_{k,0}$ eliminating the infinite sum obtained otherwise plus the factor $\delta_{n,m}$. Traces give Kronecker deltas since the projectors are orthonormal. The left hand side equals to the right hand side and one achieves unitarity. It seems that the proposed ansatz works and the U-matrix can be reduced by a general ansatz to S-matrix.

- (e) It should be made clear that the use of eigenstates of S is only a technical trick, the physical states need not be eigenstates. If the active parts of zero energy states were eigenstates of S , U-matrix would not have matrix elements between different H^i and projection operator could not change during time evolution.

7.5 What about the identification of S ?

- (a) S should be exponential of time the scaling operator whose action reduces to a time translation operator along the time axis connecting the tips of CD and realized as scaling. In other words, the shift $t/T_0 = m \rightarrow m + n$ corresponds to a scaling $t/T_0 = m \rightarrow km$ giving $m + n = km$ in turn giving $k = 1 + n/m$. At the limit of large shifts one obtains $k \simeq n/m \rightarrow \infty$, which corresponds to QFT limit. nS corresponds to $(nT_0) \times (S/T_0) = TH$ and one can ask whether QFT Hamiltonian could corresponds to $H = S/T_0$.
- (b) It is natural to assume that the operators H^i are eigenstates of radial scaling generator $L_0 = ir_M d/dr_M$ at both boundaries of CD and have thus well-defined conformal weights. As noticed the spectrum for super-symplectic algebra could also be given in terms of zeros of Riemann zeta.
- (c) The boundaries of CD are given by the equations $r_M = m^0$ and $r_M = T - m_0$, m_0 is Minkowski time coordinate along the line between the tips of CD and T is the distance between the tips. From the relationship between r_M and m_0 the action of the infinitesimal translation $H \equiv i\partial/\partial_{m^0}$ can be expressed as conformal generator $L_{-1} = i\partial/\partial_{r_M} = r_M^{-1} L_0$. Hence the action is non-diagonal in the eigenbasis of L_0 and multiplies with the conformal weights and reduces the conformal weight by one unit. Hence the action of U can change the projection operator. For large values of conformal weight the action is classically near to that of L_0 : multiplication by L_0 plus small relative change of conformal weight.

- (d) Could the spectrum of H be identified as energy spectrum expressible in terms of zeros of zeta defining a good candidate for the super-symplectic radial conformal weights. This certainly means maximal complexity since the number of generators of the conformal algebra would be infinite. This identification might make sense in chaotic or critical systems. The functions $(r_M/r_0)^{1/2+iy}$ and $(r_M/r_0)^{-2n}$, $n > 0$, are eigenmodes of r_M/dr_M with eigenvalues $(1/2+iy)$ and $-2n$ corresponding to non-trivial and trivial zeros of zeta.

There are two options to consider. Either L_0 or iL_0 could be realized as a hermitian operator. These options would correspond to the identification of mass squared operator as L_0 and approximation identification of Hamiltonian as iL_1 as iL_0 making sense for large conformal weights.

- i. Suppose that $L_0 = r_M d/dr_M$ realized as a hermitian operator would give harmonic oscillator spectrum for conformal confinement. In p-adic mass calculations the string model mass formula implies that L_0 acts essentially as mass squared operator with integer spectrum. I have proposed conformal confinement for the physical states net conformal weight is real and integer valued and corresponds to the sum over negative integer valued conformal weights corresponding to the trivial zeros and sum over real parts of non-trivial zeros with conformal weight equal to $1/2$. Imaginary parts of zeta would sum up to zero.
- ii. The counterpart of Hamiltonian as a time translation is represented by $H = iL_0 = ir_M d/dr_M$. Conformal confinement is now realized as the vanishing of the sum for the real parts of the zeros of zeta: this can be achieved. As a matter of fact the integration measure dr_M/r_M implies that the net conformal weight must be $1/2$. This is achieved if the number of non-trivial zeros is odd with a judicious choice of trivial zeros. The eigenvalues of Hamiltonian acting as time translation operator could correspond to the linear combination of imaginary part of zeros of zeta with integer coefficients. This is an attractive hypothesis in critical systems and TGD Universe is indeed quantum critical.

7.6 What about Quantum Classical Correspondence?

Quantum classical correspondence realized as one-to-one map between quantum states and zero modes has not been discussed yet.

- (a) M -matrices would act in the tensor product of quantum fluctuating degrees of freedom and zero modes. The assumption that zero energy states form an orthogonal basis implies that the hermitian square roots of the density matrices form an orthonormal basis. This condition generalizes the usual orthonormality condition.
- (b) The dependence on zero modes at given boundary of CD would be trivial and induced by 1-1 correspondence $|m\rangle \rightarrow z(m)$ between states and zero modes assignable to the state basis $|m_{\pm}\rangle$ at the boundaries of CD, and would mean the presence of factors $\delta_{z_+,f(m_+)} \times \delta_{z_-,f(n_-)}$ multiplying M -matrix $M_{m,n}^i$.

To sum up, it seems that the architecture of the U-matrix and its relationship to the S-matrix is now understood and in accordance with the intuitive expectations the construction of U-matrix reduces to that for S-matrix and one can see S-matrix as discretized counterpart of ordinary unitary time evolution operator with time translation represented as scaling: this allows to circumvent problems with loss of manifest Poincare symmetry encountered in quantum field theories and allows Lorentz invariance although CD has finite size. What came as surprise was the connection with stringy picture: strings are necessary in order to satisfy the unitary conditions for U-matrix. Second outcome was that the connection with super-symplectic algebra suggests itself strongly. The identification of hermitian square roots of density matrices with Hermitian symmetry algebra is very elegant aspect discovered already earlier. A further unexpected result was that U-matrix is unitary only for strict arrow of time (which changes in the state function reduction to opposite boundary of CD).

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